

Article

The Frobenius Number for Jacobsthal Triples Associated with Number of Solutions

Takao Komatsu ^{1,*}  and Claudio Pita-Ruiz ²
¹ Department of Mathematical Sciences, School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China

² Facultad de Ingeniería, Universidad Panamericana, Augusto Rodin 498, Ciudad de México 03920, Mexico

* Correspondence: komatsu@zstu.edu.cn

Abstract: In this paper, we find a formula for the largest integer (p -Frobenius number) such that a linear equation of non-negative integer coefficients composed of a Jacobsthal triplet has at most p representations. For $p = 0$, the problem is reduced to the famous linear Diophantine problem of Frobenius, the largest integer of which is called the Frobenius number. We also give a closed formula for the number of non-negative integers (p -genus), such that linear equations have at most p representations. Extensions to the Jacobsthal polynomial and the Jacobsthal–Lucas polynomial give more general formulas that include the familiar Fibonacci and Lucas numbers. A basic problem with the Fibonacci triplet was dealt by Marin, Ramírez Alfonsín and M. P. Revuelta for $p = 0$ and by Komatsu and Ying for the general non-negative integer p .

Keywords: Jacobsthal numbers; linear Diophantine problem of Frobenius; Frobenius numbers; the number of representations; Apéry set; genus

MSC: 11D07; 05A15; 20M14; 11B34; 11B39



Citation: Komatsu, T.; Pita-Ruiz, C. The Frobenius Number for Jacobsthal Triples Associated with Number of Solutions. *Axioms* **2023**, *12*, 98. <https://doi.org/10.3390/axioms12020098>

Academic Editors: Mircea Merca, Sidney A. Morris and Oliver Stein

Received: 7 December 2022

Revised: 9 January 2023

Accepted: 14 January 2023

Published: 17 January 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The Jacobsthal sequence $\{J_n\}_{n \geq 0}$ is defined by

$$J_n = J_{n-1} + 2J_{n-2} \quad (n \geq 2) \quad \text{with} \quad J_0 = 0 \quad \text{and} \quad J_1 = 1$$

and the first terms are given by 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10,923, 21,845, 43,691, ... ([1] and ([2], A001045)). Jacobsthal [3] may never have seen the actual values of this sequence. However, Horadam [4] uses the name “Jacobsthal sequence”; such an important sequence needs a name. Many properties and identities can be found in ([1], Ch.44).

In this paper, for fixed integers $i, k \geq 3$, we consider the linear Diophantine equation of the form

$$J_i x_1 + J_{i+2} x_2 + J_{i+k} x_3 = N. \quad (1)$$

For example, when $(i, k) = (4, 5)$ and $N = 268$, the equation $5x_1 + 21x_2 + 171x_3 = 268$ has three different non-negative solutions (representations): $(11, 2, 1)$, $(20, 8, 0)$ and $(41, 3, 0)$. However, $N = 268$ is not the largest integer that has three solutions, but $N = 289$ is the largest. This implies that for all $N > 289$, the linear equation $5x_1 + 21x_2 + 171x_3 = N$ has at least four solutions. We are interested in an explicit form of such a largest integer N .

More generally, for positive integers $a_1, \dots, a_n$ with $\gcd(a_1, \dots, a_n) = 1$ and N , we shall consider the linear equation

$$a_1 x_1 + \dots + a_n x_n = N. \quad (2)$$

For a given non-negative integer p , the largest integer N , such that the linear Equation (2) has at most p different non-negative solutions (representations), is called the p -Frobenius number and denoted by $g_p(a_1, \dots, a_n)$ [5–7]. When $p = 0$, 0-Frobenius number is called the Frobenius number, and such a problem is called the *linear Diophantine problem of Frobenius*. This problem, also known as the stamp exchange problem or the chicken McNuggets problem, has long fascinated many people in various ways. Much research has been done on this issue from various perspectives, but what we are interested in is the expression of closed formulas. Indeed, when $p = 0$, for two variables, Sylvester obtained the explicit form $g(a_1, a_2) = g_0(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$. For three (or more) variables, however, no explicit form of $g(a_1, a_2, a_3)$ has been found, but the Frobenius number cannot be given by any set of closed formulas which can be reduced to a finite set of certain polynomials [8]. Only some special cases, explicit closed formulas have been found, including arithmetic, geometric-like, Fibonacci, Mersenne, and triangular (see [9–15] and references therein). When $p > 0$, the situation is even harder. For two variables, it is known that $g_p(a_1, a_2) = (p + 1)a_1a_2 - a_1 - a_2$. However, for three or more variables, no concrete example had been found. Most recently, we have finally succeeded in giving the p -Frobenius number as a closed-form expression of the triangular number triplet [5]. Results on p -Frobenius numbers for repunits [6] and Fibonacci triples [16] will also be available.

In this paper, we find closed forms of p -Frobenius number of the linear Equation (1). In detail, we give complete expressions of p -Frobenius numbers about p -Frobenius numbers for $p = 0, 1, 2, 3$, and for general p when $\lfloor (J_i - 1)/J_k \rfloor \geq p$.

The Sylvester number (or genus) is as important as the Frobenius number. That is, the number of positive integers N such that the linear Equation (2) has at most p non-negative integer solutions is called the p -Sylvester number (or p -genus), and is denoted by $n_p(a_1, \dots, a_n)$. When $p = 0$, 0-Sylvester number is the original Sylvester number. The concept of genus is very important in studying algebraic geometry, in particular, algebraic curves. Similarly to Frobenius numbers, Sylvester obtained the explicit form $n(a_1, a_2) = n_0(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$, but for three (or more) variables, no explicit form of $n(a_1, a_2, a_3)$ has been found. If $p > 0$, the problem becomes even harder too.

In this paper, we also find closed forms of p -genus of the linear Equation (1). In order to compute both p -Frobenius numbers and p -genus, we analyze the geometrical structure of p -Apéry set. This framework is partly similar to that in Fibonacci triples [16], but complete descriptions require delicate case-by-case comparisons.

Throughout the paper, some known identities on Jacobsthal numbers are used. Two of the most important ones are $J_n = 2J_{n-1} - (-1)^n$ (exercise 12 [1], [Chapter 44]) and $J_{i+k} = J_k J_{i+2} - 4J_{k-2}J_i$ (exercise 26 in [1], [Chapter 44]).

2. Initial Results

For integers $i, k \geq 3$, set the integers r and ℓ by

$$J_i - 1 = rJ_k + \ell, \quad 0 \leq \ell \leq J_k - 1.$$

Theorem 1. For $i, k \geq 3$, we have

$$g_0(J_i, J_{i+2}, J_{i+k}) = \begin{cases} (J_i - rJ_k - 1)J_{i+2} + rJ_{i+k} - J_i & \text{if } r = 0, \text{ or } r \geq 1 \text{ and} \\ & (J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i; \\ (J_k - 1)J_{i+2} + (r - 1)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i. \end{cases} \quad (3)$$

Remark 1. When $r = 0$, that is, $k \geq i$, we have $g_0(J_i, J_{i+2}, J_{i+k}) = g_0(J_i, J_{i+2})$.

Theorem 2. For $i, k \geq 3$, we have

$$g_1(J_i, J_{i+2}, J_{i+k}) = (2J_i - 1)J_{i+2} - J_i \quad (k \geq i + 2), \quad (4)$$

$$g_1(J_i, J_{i+2}, J_{2i+1}) = \begin{cases} (J_{i+1} - 1)J_{i+2} - J_i & \text{if } i \text{ is odd;} \\ (2J_i - 1)J_{i+2} - J_i & \text{if } i \text{ is even,} \end{cases} \quad (5)$$

$$g_1(J_i, J_{i+2}, J_{2i}) = (J_i - 1)J_{i+2} + J_{2i} - J_i. \quad (6)$$

When $r = \lfloor (J_i - 1)/J_k \rfloor \geq 1$, that is, $k \leq i - 1$ ($i \geq 4$), we have

$$\begin{aligned} & g_1(J_i, J_{i+2}, J_{i+k}) \\ &= \begin{cases} (J_i - rJ_k - 1)J_{i+2} + (r+1)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i; \\ (J_k - 1)J_{i+2} + rJ_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i. \end{cases} \end{aligned} \quad (7)$$

Remark 2. For example, when $k = i - 1$, we obtain

$$g_1(J_i, J_{i+2}, J_{2i-1}) = g_1(J_i, J_{i+2}) - \begin{cases} (J_6 + J_2) & \text{if } i = 3; \\ (J_{2i} + J_{i+2} + J_{i-1}) & \text{if } i \geq 5 \text{ is odd;} \\ (J_{2i} - J_{i-1}) & \text{if } i \text{ is even.} \end{cases}$$

Concerning the relation with $g_1(J_i, J_{i+2})$, the identities (4), (5) and (6) can be written as

$$\begin{aligned} & g_1(J_i, J_{i+2}, J_{i+k}) = g_1(J_i, J_{i+2}) \quad (k \geq i+2), \\ & g_1(J_i, J_{i+2}, J_{2i+1}) = g_1(J_i, J_{i+2}) - \begin{cases} J_{i+2} & \text{if } i \text{ is odd;} \\ 0 & \text{if } i \text{ is even,} \end{cases} \\ & g_1(J_i, J_{i+2}, J_{2i}) = g_1(J_i, J_{i+2}) - 4J_{i-2}, \end{aligned}$$

respectively.

Theorem 3. For $i, k \geq 3$, we have

$$g_2(J_i, J_{i+2}, J_{i+k}) = (3J_i - 1)J_{i+2} - J_i \quad (k \geq i+2), \quad (8)$$

$$g_2(J_i, J_{i+2}, J_{2i+1}) = \begin{cases} (J_i - 1)J_{i+2} + J_{2i+1} - J_i & \text{if } i \text{ is odd;} \\ (J_i - 2)J_{i+2} + J_{2i+1} - J_i & \text{if } i \text{ is even,} \end{cases} \quad (9)$$

$$g_2(J_i, J_{i+2}, J_{2i}) = (2J_i - 1)J_{i+2} - J_i, \quad (10)$$

$$g_2(J_i, J_{i+2}, J_{2i-1}) = (J_{i-1} - 3)J_{i+2} + 3J_{2i-1} - J_i \quad (\text{if } i \text{ is even}). \quad (11)$$

When $r = \lfloor (J_i - 1)/J_k \rfloor \geq 2$, that is, $k \leq i - 1$ ($i \geq 5$:odd) or $k \leq i - 2$ ($i \geq 6$:even), we have

$$\begin{aligned} & g_2(J_i, J_{i+2}, J_{i+k}) \\ &= \begin{cases} (J_i - rJ_k - 1)J_{i+2} + (r+2)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i; \\ (J_k - 1)J_{i+2} + (r+1)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i. \end{cases} \end{aligned} \quad (12)$$

Remark 3. Concerning the relation with $g_2(J_i, J_{i+2})$, the identities (8), (9), (10) and (11) can be written as

$$\begin{aligned} & g_2(J_i, J_{i+2}, J_{i+k}) = g_2(J_i, J_{i+2}) \quad (k \geq i+2), \\ & g_2(J_i, J_{i+2}, J_{2i+1}) = g_2(J_i, J_{i+2}) - \begin{cases} (J_{i-1}J_{i+2} + J_{i-1} + J_{i+2}) & \text{if } i \text{ is odd;} \\ (J_{i-1}J_{i+2} - J_{i-1}) & \text{if } i \text{ is even,} \end{cases} \\ & g_2(J_i, J_{i+2}, J_{2i}) = g_2(J_i, J_{i+2}) - J_{i+2}, \\ & g_2(J_i, J_{i+2}, J_{2i-1}) = g_2(J_i, J_{i+2}) - 11J_{i-1}J_i \quad (\text{if } i \text{ is even}), \end{aligned}$$

respectively.

Theorem 4. For $i, k \geq 3$, we have

$$g_3(J_i, J_{i+2}, J_{i+k}) = (4J_i - 1)J_{i+2} - J_i \quad (k \geq i + 2), \quad (13)$$

$$g_3(J_i, J_{i+2}, J_{2i+1}) = \begin{cases} (3J_i - 2)J_{i+2} - J_i & \text{if } i \text{ is odd;} \\ (3J_i - 1)J_{i+2} - J_i & \text{if } i \text{ is even,} \end{cases} \quad (14)$$

$$g_3(J_i, J_{i+2}, J_{2i}) = \begin{cases} (3J_i - 1)J_{i+2} - (J_{i+1} + 4)J_i & \text{if } i \text{ is odd;} \\ (3J_i - 1)J_{i+2} - (J_{i+1} - 2)J_i & \text{if } i \text{ is even,} \end{cases} \quad (15)$$

When $r = \lfloor (J_i - 1)/J_k \rfloor \geq 3$, that is, $k \leq i - 2$ ($i \geq 5$), we have

$$\begin{aligned} & g_3(J_i, J_{i+2}, J_{i+k}) \\ &= \begin{cases} (J_i - rJ_k - 1)J_{i+2} + (r + 3)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i; \\ (J_k - 1)J_{i+2} + (r + 2)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i. \end{cases} \end{aligned} \quad (16)$$

Remark 4. Concerning the relation with $g_3(J_i, J_{i+2})$, the identities (13), (14) and (15) can be written as

$$\begin{aligned} g_3(J_i, J_{i+2}, J_{i+k}) &= g_3(J_i, J_{i+2}) \quad (k \geq i + 2), \\ g_3(J_i, J_{i+2}, J_{2i+1}) &= g_3(J_i, J_{i+2}) - \begin{cases} (J_i + 1)J_{i+2} & \text{if } i \text{ is odd;} \\ J_i J_{i+2} & \text{if } i \text{ is even,} \end{cases} \\ g_3(J_i, J_{i+2}, J_{2i}) &= g_3(J_i, J_{i+2}) - \begin{cases} J_i(J_{i+2} + J_{i+1} + 3) & \text{if } i \text{ is odd;} \\ J_i(J_{i+2} + J_{i+1} - 3) & \text{if } i \text{ is even,} \end{cases} \end{aligned}$$

respectively.

3. Preliminaries

Without loss of generality, we assume that $a_1 = \min_{1 \leq j \leq k} \{a_j\}$. For each $0 \leq i \leq a_1 - 1$, we introduce the positive integer $m_i^{(p)}$ congruent to i modulo a_1 , such that the number of representations of $m_i^{(p)}$ is bigger than or equal to $p + 1$ and that of $m_i^{(p)} - a_1$ is less than or equal to p . Note that $m_0^{(0)}$ is defined to be 0. The set

$$\text{Ap}_p(A) = \text{Ap}_p(a_1, a_2, \dots, a_k) = \{m_0^{(p)}, m_1^{(p)}, \dots, m_{a_1-1}^{(p)}\}$$

is called the p -Apéry set of $A = \{a_1, a_2, \dots, a_k\}$ for a non-negative integer p (see, [17]), which is congruent, modulo a_1 , to the set

$$\{0, 1, \dots, a_1 - 1\}.$$

When $p = 0$, 0-Apéry set is the original Apéry set.

If one can know the structure of the elements of p -Apéry set, we can obtain the p -Frobenius numbers and p -gaps (p -Sylvester numbers) by the following formulas. A more general form can be seen in [7].

Lemma 1. Let k and p be integers with $k \geq 2$ and $p \geq 0$.

Assume that $\gcd(a_1, a_2, \dots, a_k) = 1$. We have

$$g_p(a_1, a_2, \dots, a_k) = \max_{0 \leq j \leq a_1-1} m_j^{(p)} - a_1, \quad (17)$$

$$n_p(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j^{(p)} - \frac{a_1 - 1}{2}. \quad (18)$$

By using GAP package `numericalsgps` [18] (the authors thank Professor Pedro A. García-Sánchez for his kind instruction), p -Frobenius numbers of Jacobsthal triples can be obtained systematically.

```
gap> jacob:= function(n)
> if n=0 then
> return 0;
> elif n in [1, 2] then
> return 1;
> else
> return jacob(n-1) + 2*jacob(n-2);
> fi;
> end;
function( n ) ... end
gap> jacob(15);
10923
gap> Set([3..10], i->FrobeniusNumber(DenominatorIdeal
(NumericalSemigroup (jacob(i), jacob(i+2), jacob(2*i-1)), 3)));
[118, 145, 718, 2960, 12430, 50064, 202638, 811920 ]
```

For example,

$$\{g_3(J_i, J_{i+2}, J_{2i-1})\}_{i=3}^{10} = 118, 145, 718, 2960, 12430, 50064, 202638, 811920$$

(It is quite immediate to obtain the values until 50064. However, even by using GAP, it takes several hours to obtain 811920.)

We can have the 1-Apéry set and the representations of each element. For example, for $(J_5, J_7, J_9) = (11, 43, 171)$, we can obtain the 1-Apéry set $\text{Ap}_1(11, 43, 171) = \{385, 386, 387, 344, 301, 258, 215, 513, 514, 471, 428\}$, and each element has exactly two representations. For instance, $385 = 1 \cdot 43 + 2 \cdot 171 = 35 \cdot 11$.

```
gap> s:=NumericalSemigroup(11,43,171);
gap> s1:=DenominatorIdeal(s,1);
gap> ApéryList(s1,11);
[385, 386, 387, 344, 301, 258, 215, 513, 514, 471, 428 ]
gap> FactorizationsElementListWRTNumericalSemigroup(ApéryList(s1,11)
,s);
[[[0, 1, 2 ], [35, 0, 0 ]], [[0, 5, 1 ], [4, 0, 2 ]], [[0, 9, 0 ], [4, 4, 1 ]],
[[0, 8, 0 ], [4, 3, 1 ]], [[0, 7, 0 ], [4, 2, 1 ]], [[0, 6, 0 ], [4, 1, 1 ]],
[[0, 5, 0 ], [4, 0, 1 ]], [[0, 0, 3 ], [31, 4, 0 ]], [[0, 4, 2 ], [35, 3, 0 ]],
[[0, 3, 2 ], [35, 2, 0 ]], [[0, 2, 2 ], [35, 1, 0 ]]]
```

4. Proof of the Main Results

Put the linear representation

$$\begin{aligned} t_{x,y} &:= xJ_{i+2} + yJ_{i+k} \\ &= (x + yJ_k)J_{i+2} - 4yJ_{k-2}J_i \quad (x, y \geq 0). \end{aligned}$$

Given two Jacobsthal numbers J_i and J_k , integers r and ℓ are determined uniquely by

$$J_i - 1 = rJ_k + \ell \quad \text{with} \quad 0 \leq \ell \leq J_k - 1.$$

Proof of Theorem 1. Since $\gcd(J_i, J_{i+2}) = 1$ ($i \geq 3$), for $0 \leq x \leq J_k - 1$ and $0 \leq y \leq r - 1$ or $0 \leq x \leq \ell$ and $y = r$, we have each element of $t_{x,y}$ belongs to $\text{Ap}_0(J_i, J_{i+2}, J_{i+k})$. See Chart 1. Namely, $t_{x,y} \equiv t_{x',y'} \pmod{J_i}$ implies that $x = x'$ and $y = y'$. When $r = 0$, the largest possible element is $t_{\ell,r}$. When $r \geq 1$, the largest possible element is $t_{\ell,r}$ or $t_{J_k-1,r-1}$, and $t_{\ell,r} > t_{J_k-1,r-1}$ is equivalent to $(J_i - rJ_k)J_{i+2} > 4J_{k-2}J_i$. Hence, by (17), we obtain the desired result. $\square$

$t_{0,0}$	$\cdots$	$\cdots$	$t_{J_k-1,0}$
$t_{0,1}$	$\cdots$	$\cdots$	$t_{J_k-1,1}$
$\vdots$			$\vdots$
$t_{0,r-1}$	$\cdots$	$\cdots$	$t_{J_k-1,r-1}$
$t_{0,r}$	$\cdots$	$t_{\ell,r}$	

Chart 1. $\text{Ap}_0(J_i, J_{i+2}, J_{i+k})$.

Proof of Theorem 2. First, assume that $r = \lfloor (J_i - 1)/J_k \rfloor \geq 1$. Since there are three congruence relations

$$\begin{aligned} t_{x,y} &\equiv t_{J_k+x,y-1} \pmod{J_i} \quad (0 \leq x \leq J_k - 1, 1 \leq y \leq r - 1; \\ &\quad 0 \leq x \leq \ell, y = r), \\ t_{x,0} &\equiv t_{\ell+1+x,r} \pmod{J_i} \quad (0 \leq x \leq J_k - \ell - 2), \\ t_{J_k-\ell-1+x,0} &\equiv t_{x,r+1} \pmod{J_i} \quad (0 \leq x \leq \ell), \end{aligned}$$

we can obtain the 1-Apéry set $\text{Ap}_1(A)$ from the 0-Apéry set $\text{Ap}_0(A)$, where $A = \{J_i, J_{i+2}, J_{i+k}\}$. By $J_{i+k} = J_k J_{i+2} - 4J_{k-2}J_i$, we can see that each element of $\text{Ap}_1(A)$ has exactly two different expressions in terms of J_i, J_{i+2}, J_{i+k} :

$$\begin{aligned} t_{J_k+x,y-1} &= (J_k + x)J_{i+2} + (y - 1)J_{i+k} \\ &= 4J_{k-2}J_i + xJ_{i+2} + yJ_{i+k}, \\ t_{\ell+1+x,r} &= (\ell + 1 + x)J_{i+2} + rJ_{i+k} \\ &= (J_{i+2} - 4rJ_{k-2})J_i + xJ_{i+2}, \\ t_{x,r+1} &= xJ_{i+2} + (r + 1)J_{i+k} \\ &= (J_{i+2} - 4(r + 1)J_{k-2})J_i + (x + J_k - \ell - 1)J_{i+2}. \end{aligned}$$

Concerning the largest value in the set $\text{Ap}_1(A)$, as seen in Chart 2, there are four candidates:

$$t_{\ell,r+1}, t_{J_k-1,r}, t_{J_k+\ell,r-1}, t_{2J_k-1,r-2}.$$

Since $J_{i+2}/J_i \geq 3$ ($i \geq 1$), we can know that

$$t_{\ell,r+1} > t_{J_k+\ell,r-1} \quad \text{and} \quad t_{J_k-1,r} > t_{2J_k-1,r-2}.$$

Therefore, it is enough to compare $t_{\ell,r+1}$ and $t_{J_k-1,r}$. Since $t_{\ell,r+1} > t_{J_k-1,r}$ is equivalent to $(J_i - rJ_k)J_{i+2} > 4J_{k-2}J_i$, we have the identity (7).

The identities (4), (5) and (6) are yielded from the case $r = 0$. In this case, there are two different patterns for the arrangement of the elements of 1-Apéry set: $2\ell + 1 \leq J_k - 1$ or $2\ell + 1 > J_k - 1$.

$t_{0,0}$	$t_{1,0}$	$\cdots$	$\cdots$	$t_{J_k-1,0}$	$t_{J_k,0}$	$t_{J_k+1,0}$	$\cdots$	$\cdots$	$t_{2J_k-1,0}$
$t_{0,1}$	$t_{1,1}$	$\cdots$	$\cdots$	$t_{J_k-1,1}$	$t_{J_k,1}$	$t_{J_k+1,1}$	$\cdots$	$\cdots$	$t_{2J_k-1,1}$
$t_{0,2}$	$t_{1,2}$	$\cdots$	$\cdots$	$t_{J_k-1,2}$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\vdots$
$\vdots$	$\vdots$			$\vdots$	$t_{J_k,r-2}$	$t_{J_k+1,r-2}$	$\cdots$	$\cdots$	$t_{2J_k-1,r-2}$
$t_{0,r-1}$	$t_{1,r-1}$	$\cdots$	$\cdots$	$t_{J_k-1,r-1}$	$t_{J_k,r-1}$	$\cdots$	$t_{J_k+\ell,r-1}$		
$t_{0,r}$	$\cdots$	$t_{\ell,r}$	$t_{\ell+1,r}$	$\cdots$	$t_{J_k-1,r}$				
$t_{0,r+1}$	$\cdots$	$t_{\ell,r+1}$							

Chart 2. $\text{Ap}_1(A)$ from $\text{Ap}_0(A)$.

Case 1. When $2\ell + 1 \leq J_k - 1$, that is, $2J_i \leq J_k$, all the elements of 0-Apéry set and 1-Apéry set are in a unique line, as shown in Chart 3. There is the relation between each element in

0-Apéry set and 1-Apéry set as $t_{j,0} \equiv t_{\ell+j+1,0} \pmod{J_i}$. Each element in 1-Apéry set has exactly two representations because for $0 \leq j \leq \ell$

$$\begin{aligned} t_{j,0} &= jJ_{i+2} \equiv (J_i + j)J_{i+2} \\ &= (\ell + j + 1)J_{i+2} = t_{\ell+j+1,0} \pmod{J_i}. \end{aligned}$$

$t_{0,0}$	$\cdots$	$\cdots$	$t_{\ell,0}$	$t_{\ell+1,0}$	$\cdots$	$\cdots$	$t_{2\ell+1,0}$
-----------	----------	----------	--------------	----------------	----------	----------	-----------------

Chart 3. $\text{Ap}_1(A)$ from $\text{Ap}_0(A)$ when $r = 0$ and $2\ell + 1 \leq J_k - 1$.

Hence, the largest element of 1-Apéry set is

$$t_{2\ell+1,0} = (2\ell + 1)J_{i+2} = (2J_i - 1)J_{i+2}.$$

Notice that the condition $2\ell + 1 \leq J_k - 1$ is valid when $k \geq i + 2$ or $k = i + 1$ and i is even. By (17), we obtain the identity (4), and the identity (5) when i is even.

Case 2. When $2\ell + 1 > J_k - 1$, that is, $2J_i > J_k$, some elements of 1-Apéry set come to the second line, as shown in Chart 4. There are two corresponding relations between each element in 0-Apéry set and 1-Apéry set: $t_{j,0} \equiv t_{\ell+j+1,0} \pmod{J_i}$ ($0 \leq j \leq J_k - \ell - 2$) and $t_{J_k-\ell+j-1,0} \equiv t_{j,1} \pmod{J_j}$ ($0 \leq j \leq 2\ell + 1 - J_k$). Each element in 1-Apéry set has exactly two representations because for $0 \leq j \leq J_k - \ell - 2$

$$\begin{aligned} t_{j,0} &= jJ_{i+2} \equiv (J_i + j)J_{i+2} \\ &= (\ell + 1 + j)J_{i+2} = t_{\ell+1+j,0} \pmod{J_i} \end{aligned}$$

and for $0 \leq j \leq 2\ell + 1 - J_k$

$$\begin{aligned} t_{J_k-\ell-1+j,0} &= (J_k - \ell - 1 + j)J_{i+2} \\ &\equiv (j + J_k)J_{i+2} - 4J_{k-2}J_i = jJ_{i+2} + J_{i+k} = t_{j,1} \pmod{J_i}. \end{aligned}$$

$t_{0,0}$	$\cdots$	$\cdots$	$t_{\ell,0}$	$t_{\ell+1,0}$	$\cdots$	$t_{J_k-1,0}$
$t_{0,1}$	$\cdots$	$t_{2\ell+1-J_k,1}$				

Chart 4. $\text{Ap}_1(A)$ from $\text{Ap}_0(A)$ when $r = 0$ and $2\ell + 1 > J_k - 1$.

Hence, concerning the largest element of 1-Apéry set, there are two possibilities: $t_{J_k-1,0}$ and $t_{2\ell+1-J_k,1}$. Since $r = 0$ implies that $k \geq i$, the condition $2J_i > J_k$ is valid only when $k = i$ or $k = i + 1$ and i is odd. When $k = i$, only possibility is $t_{2\ell+1-J_k,1} = (J_i - 1)J_{i+2} + J_{2i}$, yielding the identity (6). When $k = i + 1$ (and i is odd), by $t_{J_k-1,0} > t_{2\ell+1-J_k,1}$, we obtain the identity (5) when i is odd. $\square$

Proof of Theorem 3. When $p = 2$, the elements of 2-Apéry set are determined from those of 1-Apéry set, by using the third block too (see Chart 5). Except for the elements in the top row, each element in the 1-Apéry set corresponds to the element in the 2-Apéry set with the same residue modulo J_i so that it shifts up one step and moves to the right block. Only the elements in the 1-Apéry in the top row correspond to the elements of the 2-Apéry set with the same residue so that they are arranged in order so as to fill the gap at the bottom of the leftmost block (first block).

1st block	2nd block	3rd block
$\underbrace{\hspace{1cm}}_{J_k}$	$\underbrace{\hspace{1cm}}_{J_k}$	$\underbrace{\hspace{1cm}}_{J_k}$

Chart 5. Three blocks.

Case 1. When $r \geq 2$, the elements of 2-Apéry set are placed below the first block, below the second block, and throughout the third block, as shown in Chart 6. Note that when $r = 2$, the third block consists of only one line with J_k or less elements.

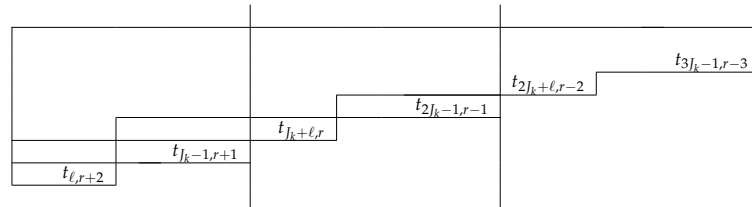


Chart 6. $\text{Ap}_p(J_i, J_{i+2}, J_{i+k})$ ($p = 0, 1, 2$) for $r \geq 2$.

Each element in the lower two lines of the first block (or just one line consisting of J_k elements) can be represented in only three ways in terms of J_i , J_{i+2} and J_{i+k} , respectively, and is congruent to an element in the top row of the second block modulo J_i . Each element in the lower two lines of the second block (or just one line consisting of J_k elements) can be represented in only three ways in terms of J_i , J_{i+2} and J_{i+k} , respectively, and is congruent to an element in the second row from the bottom of the first block modulo J_i . Every element of the third block has also only three ways of being represented in terms of J_i , J_{i+2} and J_{i+k} , and there is a congruence relation with each element (except the first row) of the second block shifted up by one step.

Concerning the largest element of $\text{Ap}_2(J_i, J_{i+2}, J_{i+k})$, there are six candidates:

$$t_{l,r+2}, t_{J_k-1,r+1}, t_{J_k+l,r}, t_{2J_k-1,r-1}, t_{2J_k+l,r-2}, t_{3J_k-1,r-3}.$$

Since $J_{i+2} \geq 3J_i$ ($j \geq 1$), $2J_{i+k} > J_k J_{i+2}$. Hence, we obtain

$$\begin{aligned} t_{l,r+2} &> t_{J_k+l,r} > t_{2J_k+l,r-2}, \\ t_{J_k-1,r+1} &> t_{2J_k-1,r-1} > t_{3J_k-1,r-3}. \end{aligned}$$

Therefore, the largest value must be $t_{l,r+2}$ or $t_{J_k-1,r+1}$. If $(J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i$, the largest value in $\text{Ap}_2(A)$ is $t_{l,r+2} = (J_i - rJ_k - 1)J_{i+2} + (r+2)J_{i+k}$. If $(J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i$, the largest value in $\text{Ap}_2(A)$ is $t_{J_k-1,r+1} = (J_k - 1)J_{i+2} + (r+1)J_{i+k}$. By (17), we obtain the identity (12).

Case 2. When $r = 1$, there are two patterns depending on $J_k \geq 2\ell + 2$ or $J_k < 2\ell + 2$.

If $J_k \geq 2\ell + 2$ then $3J_k \geq 2J_i$, so $k \geq i$.

However, by $r = 1$, $k \leq i - 1$. So, this case does not exist. If $J_k < 2\ell + 2$, by $r = 1$, the only possibility is $k = i - 1$ and i is even. As shown in Chart 7, there are four candidates for the largest value of $\text{Ap}_2(A)$:

$$\begin{aligned} t_{2\ell+1-J_k,3} &= (2J_i - 3J_{i-1} - 1)J_{i+2} + 3J_{2i-1}, \\ t_{J_k-1,2} &= (J_{i-1} - 1)J_{i+2} + 2J_{2i-1}, \\ t_{J_k+l,1} &= (J_i - 1)J_{i+2} + J_{2i-1}, \\ t_{2J_k-1,0} &= (2J_{i-1} - 1)J_{i+2}. \end{aligned}$$

Since $2J_{i-1} - J_i = 1$ when i is even, by $J_{2i-1} > 2J_{i-2}J_{i+2} \geq 2J_{i+2}$ we have

$$t_{2\ell+1-J_k,3} > t_{J_k-1,2} > t_{J_k+l,1} > t_{2J_k-1,0}.$$

By (17), we can obtain the identity (11).

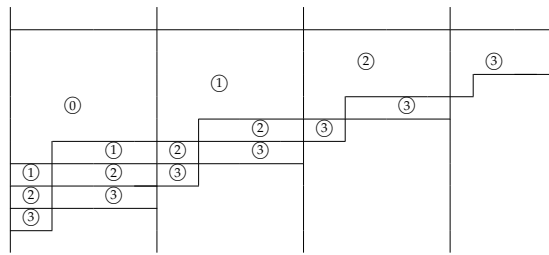


Chart 11. $\text{Ap}_p(A)$ ($p = 0, 1, 2, 3$) for $r \geq 3$.

Eventually, the largest element of $\text{Ap}_3(A)$ is from the first block, that is, $t_{\ell, r+3} = (J_i - rJ_k - 1)J_{i+2} + (r+3)J_{i+k}$ or $t_{J_k-1, r+2} = (J_k - 1)J_{i+2} + (r+2)J_{i+k}$. Hence, if $(J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i$, then $g_3(J_i, J_{i+2}, J_{i+k}) = t_{\ell, r+3} - J_i = (J_i - rJ_k - 1)J_{i+2} + (r+3)J_{i+k} - J_i$. If $(J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i$, then $g_3(J_i, J_{i+2}, J_{i+k}) = t_{J_k-1, r+2} - J_i = (J_k - 1)J_{i+2} + (r+2)J_{i+k} - J_i$. Notice that $r \geq 3$ implies that $k \leq i - 2$ with $i \geq 5$. $\square$

5. General p Case

Let p be a given non-negative integer. When $r \geq p$, by repeating the same process as those in previous theorems, we can obtain an explicit formula of p -Frobenius number.

Theorem 5. Let $i, k \geq 3$ and p be a non-negative integer. When $r = \lfloor (J_i - 1)/J_k \rfloor \geq p$ with $(r, p) \neq (0, 0)$, we have

$$g_p(J_i, J_{i+2}, J_{i+k}) = \begin{cases} (J_i - rJ_k - 1)J_{i+2} + (r+p)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} \geq 4J_{k-2}J_i; \\ (J_k - 1)J_{i+2} + (r+p-1)J_{i+k} - J_i & \text{if } (J_i - rJ_k)J_{i+2} < 4J_{k-2}J_i. \end{cases} \quad (19)$$

However, when $r < p$, the situation is not simple at all. We do need the case-by-case discussion. To see this depends on how regularly the elements of the p -Apéry set continues to appear. The elements of the 0-Apéry set represented by Chart 2 are continually arranged regularly to those of the 1-Apéry set, the 2-Apéry set, and until the r -Apéry set. However, when it reaches the time of the (p) -Apéry set when $p = r + 1$, the elements arranged at the lower left are insufficient, and the regularity shifts. That is, the positional relationship of the elements giving the largest value suddenly changes. Hereinafter, this deviation increases as $p = r + 2, p = r + 3, \dots$, and the regularity is greatly impaired.

Chart 12 shows the arrangement of the p -Apéry sets ($p = 0, 1, \dots, 5$) when $r = 5$. One can see that there will be a deficiency in the arrangement of the lower left for 6-Apéry set.

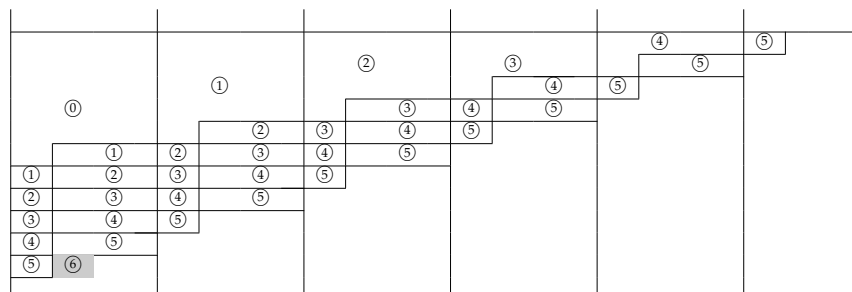


Chart 12. $\text{Ap}_p(A)$ ($p = r$).

6. p -Genus

In order to obtain an explicit form of p -genus (p -Sylvester number), we shall use the Formula (18). That is, in each case, it is necessary to find the sum of the elements within the areas of p -Apéry set. Calculating the sum is a bit cumbersome, but there is no need to

compare candidates as we would when finding the largest value. Therefore, the calculation method is similar to that of Fibonacci [16].

6.1. The Case $p = 0$

By Chart 1, we have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(0)} &= \sum_{x=0}^{J_k-1} \sum_{y=0}^{r-1} t_{x,y} + \sum_{x=0}^{\ell} t_{x,r} \\ &= \frac{r(J_k-1)J_k}{2} J_{i+2} + \frac{J_k(r-1)r}{2} J_{i+k} + \frac{\ell(\ell+1)}{2} J_{i+2} + (\ell+1)r J_{i+k} \\ &= \frac{J_i^2 + r(r+1)J_k^2 - J_i(2rJ_k+1)}{2} J_{i+2} \\ &\quad + \frac{2J_i - (r+1)J_k}{2} r(J_{i+2}J_k - 4J_iJ_{k-2}) \\ &= \frac{J_i(J_i-1)J_{i+2}}{2} - 2rJ_i(2J_i - (r+1)J_k)J_{k-2}. \end{aligned}$$

By (18), we have

$$\begin{aligned} n_0(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{J_i} \sum_{j=0}^{J_i-1} m_j^{(0)} - \frac{J_i-1}{2} \\ &= \frac{(J_i-1)J_{i+2}}{2} - 2r(2J_i - (r+1)J_k)J_{k-2} - \frac{J_i-1}{2} \\ &= \frac{(J_i-1)(J_{i+2}-1)}{2} - 2rJ_i(2J_i - (r+1)J_k)J_{k-2}. \end{aligned}$$

Theorem 6. We have

$$n_0(J_i, J_{i+2}, J_{i+k}) = \frac{(J_i-1)(J_{i+2}-1)}{2} - 2rJ_i(2J_i - (r+1)J_k)J_{k-2}.$$

Remark 5. For Fibonacci triples [19], we have

$$n_0(F_i, F_{i+2}, F_{i+k}) = \frac{(F_i-1)(F_{i+2}-1)}{2} - \frac{r(2F_i - (r+1)F_k)F_{k-2}}{2}.$$

6.2. The Case $p = 1$

When $r \geq 1$, by Chart 2, we have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(1)} &= \sum_{x=J_k}^{2J_k-1} \sum_{y=0}^{r-2} t_{x,y} + \sum_{x=J_k}^{J_k+\ell} t_{x,r-1} + \sum_{x=\ell+1}^{J_k-1} t_{x,r} + \sum_{x=0}^{\ell} t_{x,r+1} \\ &= \left(\frac{(2J_k-1)(2J_k)}{2} - \frac{(J_k-1)J_k}{2} \right) (r-1)J_{i+2} + \frac{(r-2)(r-1)}{2} J_k J_{i+k} \\ &\quad + \left(\frac{(J_k+\ell)(J_k+\ell+1)}{2} - \frac{(J_k-1)J_k}{2} \right) J_{i+2} + (\ell+1)(r-1)J_{i+k} \\ &\quad + \left(\frac{(J_k-1)(J_k)}{2} - \frac{\ell(\ell+1)}{2} \right) J_{i+2} + (J_k-\ell-1)rJ_{i+k} \\ &\quad + \frac{\ell(\ell+1)}{2} J_{i+2} + (\ell+1)(r+1)J_{i+k} \\ &= \frac{1}{2} ((r-1)(3J_k-1)J_k + (J_i-1-(r-1)J_k)(J_i-(r-1)J_k))J_{i+2} \\ &\quad + \left(rJ_i - \frac{(r-1)(r+2)}{2} J_k \right) (J_k J_{i+2} - 4J_{k-2}J_i) \end{aligned}$$

$$= \frac{1}{2}(J_i + 2J_k - 1)J_i J_{i+2} - 4\left(rJ_i - \frac{(r-1)(r+2)}{2}J_k\right)J_{k-2}J_i.$$

Hence, by (18) we have

$$\begin{aligned} n_1(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{J_i} \sum_{j=0}^{J_i-1} m_j^{(0)} - \frac{J_i - 1}{2} \\ &= \frac{1}{2}(J_i + 2J_k - 1)J_{i+2} - 2(2rJ_i - (r-1)(r+2)J_k)J_{k-2} - \frac{J_i - 1}{2} \\ &= \frac{1}{2}((J_i + 2J_k - 1)J_{i+2} - J_i + 1) - 4J_k J_{k+2} - 2r(2J_i - (r+1)J_k)J_{k-2}. \end{aligned}$$

When $r = 0$ and $2\ell + 1 \leq J_k - 1$, that is, $k \geq i + 2$ or $k = i + 1$ and i is even, by using Chart 3, we have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(1)} &= \sum_{x=\ell+1}^{2\ell+1} t_{x,0} \\ &= \left(\frac{(2\ell+1)(2\ell+2)}{2} - \frac{\ell(\ell+1)}{2}\right)J_{i+2} \\ &= \frac{(3J_i - 1)J_i J_{i+2}}{2}. \end{aligned}$$

Hence, by (18), we have

$$\begin{aligned} n_1(J_i, J_{i+2}, J_{i+k}) &= \frac{(3J_i - 1)J_i J_{i+2}}{2} - \frac{J_i - 1}{2} \\ &= \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1), \end{aligned}$$

When $r = 0$ and $2\ell + 1 \geq J_k$, that is, $k = i$ or $k = i + 1$ and i is odd, by Chart 4, we have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(1)} &= \sum_{x=\ell+1}^{J_k-1} t_{x,0} + \sum_{x=0}^{2\ell+1-J_k} t_{x,1} \\ &= \left(\frac{(J_k - 1)J_k}{2} - \frac{\ell(\ell+1)}{2}\right)J_{i+2} \\ &\quad + \frac{(2\ell+1-J_k)(2\ell+2-J_k)}{2}J_{i+2} + (2\ell+2-J_k)J_{i+k} \\ &= \left(\frac{(3J_i - 1)J_i}{2} - 2J_i J_k + J_k^2\right)J_{i+2} + (2J_i - J_k)J_{i+k} \\ &= \left(\frac{(3J_i - 1)J_i J_{i+2}}{2} - 4(2J_i - J_k)J_{k-2}\right)J_i. \end{aligned}$$

Hence, by (18), we have

$$\begin{aligned} n_1(J_i, J_{i+2}, J_{i+k}) &= \frac{(3J_i - 1)J_i J_{i+2}}{2} - 4(2J_i - J_k)J_{k-2} - \frac{J_i - 1}{2} \\ &= \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(2J_i - J_k)J_{k-2}. \end{aligned}$$

Theorem 7. For $i, k \geq 3$, we have

$$\begin{aligned} n_1(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) \quad (k \geq i + 2), \\ n_1(J_i, J_{i+2}, J_{2i+1}) & \end{aligned}$$

$$= \begin{cases} \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(2J_i - J_{i+1})J_{i-1} & i \text{ is odd;} \\ \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) & i \text{ is even,} \end{cases}$$

$$n_1(J_i, J_{i+2}, J_{2i}) = \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) - 4J_i J_{i-2}.$$

When $r = \lfloor (J_i - 1)/J_k \rfloor \geq 1$, that is, $k \leq i - 1$ ($i \geq 4$), we have

$$n_1(J_i, J_{i+2}, J_{i+k}) = \frac{(J_i + 2J_k - 1)J_{i+2} - J_i + 1}{2} - 2(2rJ_i - (r + 2)(r - 1)J_k)J_{k-2}.$$

6.3. The Case $p = 2$

When $r \geq 2$, that is, $k \leq i - 1$ ($i \geq 5$) and i is odd or $k \leq i - 2$ ($i \geq 6$) and i is even, by Chart 6, we have

$$\begin{aligned} & \sum_{j=0}^{J_i-1} m_j^{(2)} \\ &= \sum_{x=2J_k}^{3J_k-1} \sum_{y=0}^{r-3} t_{x,y} + \sum_{x=2J_k}^{2J_k+\ell} t_{x,r-2} + \sum_{x=J_k+\ell+1}^{2J_k-1} t_{x,r-1} + \sum_{x=J_k}^{J_k+\ell} t_{x,r} \\ &+ \sum_{x=\ell+1}^{J_k-1} t_{x,r+1} + \sum_{x=0}^{\ell} t_{x,r+2} \\ &= (r-2) \left(\frac{(3J_k-1)(3J_k)}{2} - \frac{(2J_k-1)(2J_k)}{2} \right) J_{i+2} + \frac{(r-3)(r-2)}{2} J_k J_{i+k} \\ &+ \left(\frac{(2J_k+\ell)(2J_k+\ell+1)}{2} - \frac{(2J_k-1)(2J_k)}{2} \right) J_{i+2} + (\ell+1)(r-2) J_{i+k} \\ &+ \left(\frac{(2J_k-1)(2J_k)}{2} - \frac{(J_k+\ell)(J_k+\ell+1)}{2} \right) J_{i+2} + (J_k-\ell-1)(r-1) J_{i+k} \\ &+ \left(\frac{(J_k+\ell)(J_k+\ell+1)}{2} - \frac{(J_k-1)J_k}{2} \right) J_{i+2} + (\ell+1)r J_{i+k} \\ &+ \left(\frac{(J_k-1)J_k}{2} - \frac{\ell(\ell+1)}{2} \right) J_{i+2} + (J_k-\ell-1)(r+1) J_{i+k} \\ &+ \frac{\ell(\ell+1)}{2} J_{i+2} + (\ell+1)(r+2) J_{i+k} \\ &= \frac{1}{2}(J_i + 4J_k - 1)J_i J_{i+2} - 2(2rJ_i - (r+3)(r-2)J_k)J_{i+k-2}. \end{aligned}$$

Hence, by (18), we have

$$\begin{aligned} & n_2(J_i, J_{i+2}, J_{i+k}) \\ &= \frac{1}{2}(J_i + 4J_k - 1)J_i J_{i+2} - 2(2rJ_i - (r+3)(r-2)J_k)J_{i+k-2} - \frac{J_i - 1}{2} \\ &= \frac{1}{2}((J_i + 4J_k - 1)J_i J_{i+2} - J_i + 1) - 2(2rJ_i - (r+3)(r-2)J_k)J_{i+k-2}. \end{aligned}$$

When $r = 1$, that is, $k = i - 1$ and i is even, by Chart 7, we have

$$\begin{aligned} & \sum_{j=0}^{J_i-1} m_j^{(2)} \\ &= \sum_{x=J_k+\ell+1}^{2J_k-1} t_{x,0} + \sum_{x=J_k}^{J_k+\ell} t_{x,1} + \sum_{x=\ell+1}^{J_k-1} t_{x,2} + \sum_{x=0}^{2\ell+1-J_k} t_{x,3} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{(2J_k - 1)(2J_k)}{2} - \frac{(J_k + \ell)(J_k + \ell + 1)}{2} \right) J_{i+2} \\
&\quad + \left(\frac{(J_k + \ell)(J_k + \ell + 1)}{2} - \frac{(J_k - 1)J_k}{2} \right) J_{i+2} + (\ell + 1)J_{i+k} \\
&\quad + \left(\frac{(J_k - 1)J_k}{2} - \frac{\ell(\ell + 1)}{2} \right) J_{i+2} + 2(J_k - \ell - 1)J_{i+k} \\
&\quad + \frac{(2\ell + 1 - J_k)(2\ell + 2 - J_k)}{2} J_{i+2} + 3(2\ell + 2 - J_k)J_{i+k} \\
&= \frac{1}{2}(3J_i - 1)J_i J_{i+2} - 4(5J_i - 6J_k)J_i J_{k-2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2}(3J_i - 1)J_{i+2} - 4(5J_i - 6J_k)J_{k-2} - \frac{J_i - 1}{2} \\
&= \frac{1}{2}(3J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(5J_i - 6J_k)J_{k-2}.
\end{aligned}$$

When $r = 0$ and $J_k \geq 3\ell + 3$, that is, $k \geq i + 2$, by Chart 8, we have

$$\begin{aligned}
\sum_{j=0}^{J_i-1} m_j^{(2)} &= \sum_{x=2\ell+2}^{3\ell+2} t_{x,0} \\
&= \left(\frac{(3\ell + 2)(3\ell + 3)}{2} - \frac{(2\ell + 1)(2\ell + 2)}{2} \right) J_{i+2} \\
&= \frac{1}{2}(5J_i - 1)J_i J_{i+2}.
\end{aligned}$$

Hence, by (18), we have

$$\begin{aligned}
n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2}(5J_i - 1)J_{i+2} - \frac{J_i - 1}{2} \\
&= \frac{1}{2}(5J_i J_{i+2} - J_i - J_{i+2} + 1).
\end{aligned}$$

When $r = 0$ and $2\ell + 2 \leq J_k \leq 3\ell + 2$, that is, $k = i + 1$ and i is even, by Chart 9, we have

$$\begin{aligned}
\sum_{j=0}^{J_i-1} m_j^{(2)} &= \sum_{x=2\ell+2}^{J_k-1} t_{x,0} + \sum_{x=0}^{3\ell+2-J_k} t_{x,1} \\
&= \left(\frac{(J_k - 1)J_k}{2} - \frac{(2\ell + 1)(2\ell + 2)}{2} \right) J_{i+2} \\
&\quad + \frac{(3\ell + 2 - J_k)(3\ell + 3 - J_k)}{2} J_{i+2} + (3\ell + 3 - J_k)J_{i+k} \\
&= \frac{1}{2}(5J_i - 1)J_i J_{i+2} - 4(3J_i - J_k)J_i J_{k-2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2}(5J_i - 1)J_{i+2} - 4(3J_i - J_k)J_{k-2} - \frac{J_i - 1}{2} \\
&= \frac{1}{2}(5J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(3J_i - J_k)J_{k-2}.
\end{aligned}$$

When $r = 0$ and $J_k \leq 2\ell + 1$, that is, $k = i$ or $k = i + 1$ and i is odd, by Chart 10, we have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(2)} &= (t_{J_k,0} + \cdots + t_{2\ell+1,0}) + (t_{2\ell+2-J_k,1} + \cdots + t_{\ell,1}) \\ &= \left(\frac{(2\ell+1)(2\ell+2)}{2} - \frac{(J_k-1)J_k}{2} \right) J_{i+2} \\ &\quad + \left(\frac{\ell(\ell+1)}{2} - \frac{(2\ell+1-J_k)(2\ell+2-J_k)}{2} \right) J_{i+2} + (J_k - \ell - 1) J_{i+k} \\ &= \frac{1}{2} (J_i + 2J_k - 1) J_i J_{i+2} - 4(J_k - J_i) J_i J_{k-2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2} (J_i + 2J_k - 1) J_{i+2} - 4(J_k - J_i) J_{k-2} - \frac{J_i - 1}{2} \\ &= \frac{1}{2} ((J_i + 2J_k - 1) J_{i+2} - J_i + 1) - 4(J_k - J_i) J_{k-2}. \end{aligned}$$

Theorem 8. For $i, k \geq 3$, we have

$$\begin{aligned} n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2} (5J_i J_{i+2} - J_i - J_{i+2} + 1) \quad (k \geq i + 2), \\ n_2(J_i, J_{i+2}, J_{2i+1}) &= \begin{cases} \frac{1}{2} (J_i + 2J_{i+1} - 1) J_{i+2} - J_i + 1 - 4(J_{i+1} - J_i) J_{i+1} & \text{if } i \text{ is odd;} \\ \frac{1}{2} (5J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(3J_i - J_{i+1}) J_{i-1} & \text{if } i \text{ is even,} \end{cases} \\ n_2(J_i, J_{i+2}, J_{2i}) &= \frac{1}{2} ((3J_i - 1) J_{i+2} - J_i + 1), \\ n_2(J_i, J_{i+2}, J_{2i-1}) &= \begin{cases} \frac{1}{2} ((J_i + 4J_{i-1} - 1) J_{i+2} - J_i + 1) - 8J_i J_{i-3} & \text{if } i \text{ is odd;} \\ \frac{1}{2} (3J_i J_{i+2} - J_i - J_{i+2} + 1) - 4(5J_i - 6J_{i-1}) J_{i-3} & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

When $r = \lfloor (J_i - 1) / J_k \rfloor \geq 2$, that is, $k \leq i - 2$ ($i \geq 6$:even), we have

$$\begin{aligned} n_2(J_i, J_{i+2}, J_{i+k}) &= \frac{1}{2} ((J_i + 4J_k - 1) J_{i+2} - J_i + 1) - 2(2rJ_i - (r+3)(r-2)J_k) J_{k-2}. \end{aligned}$$

7. General p Case

We can continue the process with $p = 3, 4, \dots$. When $r < p$, it is necessary to consider complicated case classification, but when $r \geq p$, the closed formula for p -genus can be obtained according to the above-mentioned certain rule.

We have

$$\begin{aligned} \sum_{j=0}^{J_i-1} m_j^{(p)} &= \sum_{h=0}^{r-p-1} (t_{pJ_k,h} + \cdots + t_{(p+1)J_k-1,h}) \\ &\quad + (t_{pJ_k,r-p} + \cdots + t_{pJ_k+\ell,r-p}) \\ &\quad + (t_{(p-1)J_k+\ell+1,r-p+1} + \cdots + t_{pJ_k-1,r-p+1}) \end{aligned}$$

$$\begin{aligned}
& + (t_{(p-1)J_k, r-p+2} + \cdots + t_{(p-1)J_k, r-p+2}) \\
& + \cdots \\
& + (t_{\ell+1, r+p-1} + \cdots + t_{J_k-1, r+p-1}) + (t_{0, r+p} + \cdots + t_{\ell, r+p}) \\
& = \frac{1}{2} ((r-p)((2p+1)J_k-1)J_k + (J_i - (r-p)J_k - 1)(J_i - (r-p)J_k))J_{i+2} \\
& + \left(\frac{(r-p-1)(r-p)}{2} J_k + r(J_i - (r-p)J_k) \right) J_{i+k} \\
& = \frac{1}{2} (J_i + 2pJ_k - 1)J_i J_{i+2} - 2(2rJ_i - (r+p+1)(r-p)J_k)J_i J_{k-2}.
\end{aligned}$$

Hence, by (18), we have

$$\begin{aligned}
& n_p(J_i, J_{i+2}, J_{i+k}) \\
& = \frac{1}{2} (J_i + 2pJ_k - 1)J_{i+2} - 2(2rJ_i - (r+p+1)(r-p)J_k)J_{k-2} - \frac{J_i - 1}{2} \\
& = \frac{1}{2} ((J_i + 2pJ_k - 1)J_{i+2} - J_i + 1) - 2(2rJ_i - (r+p+1)(r-p)J_k)J_{k-2}.
\end{aligned}$$

Theorem 9. Let $i \geq 3$ and p be a non-negative integer. When $r = \lfloor (J_i - 1)/J_k \rfloor \geq p$, we have

$$\begin{aligned}
& n_p(J_i, J_{i+2}, J_{i+k}) \\
& = \frac{1}{2} ((J_i + 2pJ_k - 1)J_{i+2} - J_i + 1) - 2(2rJ_i - (r+p+1)(r-p)J_k)J_{k-2}. \quad (20)
\end{aligned}$$

Remark 6. Concerning Fibonacci triples [16], for $r = \lfloor (F_i - 1)/F_k \rfloor \geq p$ with $(r, p) \neq (0, 0)$, we have

$$\begin{aligned}
& n_p(F_i, F_{i+2}, F_{i+k}) \\
& = \frac{1}{2} ((F_i + 2pF_k - 1)F_{i+2} - F_i + 1) - \frac{1}{2} (2rF_i - (r+p+1)(r-p)F_k)F_{k-2}. \quad (21)
\end{aligned}$$

8. Jacobsthal Polynomials

Jacobsthal numbers J_n can be generalized as *Jacobsthal polynomials* $J_n(x)$, defined by the recurrence relation $J_n(x) = J_{n-1}(x) + xJ_{n-2}(x)$ ($n \geq 2$) with $J_0(x) = 0$ and $J_1(x) = 1$ ([1], [Ch.44]). When $x = 2$, $J_n = J_n(2)$ are the Jacobsthal numbers.

Let p be a given non-negative integer. It would be possible to give a complete explicit formula for a given $x = b$ and p , but it requires a very complicated case classification for each case, and it is more difficult to make a general statement. Nevertheless, it is possible to give a generalized formula only when $r \geq p$. The formula depends on the identity

$$J_{i+k}(b) = J_k(b)J_{i+2}(b) - b^2J_{k-2}(b)J_i(b).$$

Theorem 10. Let $i, k \geq 3$ and p be a non-negative integer, and b be a positive integer. When $r = \lfloor (J_i(b) - 1)/J_k(b) \rfloor \geq p$, we have

$$\begin{aligned}
& g_p(J_i(b), J_{i+2}(b), J_{i+k}(b)) \\
& = \begin{cases} (J_i(b) - rJ_k(b) - 1)J_{i+2}(b) + (r+p)J_{i+k}(b) - J_i(b) \\ \quad \text{if } (r, p) = (0, 0), \text{ or } (J_i(b) - rJ_k(b))J_{i+2}(b) \geq b^2J_{k-2}(b)J_i(b); \\ (J_k(b) - 1)J_{i+2}(b) + (r+p-1)J_{i+k}(b) - J_i(b) \\ \quad \text{if } (J_i(b) - rJ_k(b))J_{i+2}(b) < b^2J_{k-2}(b)J_i(b). \end{cases} \\
& n_p(J_i(b), J_{i+2}(b), J_{i+k}(b)) \\
& = \frac{1}{2} ((J_i(b) + 2pJ_k(b) - 1)J_{i+2}(b) - J_i(b) + 1)
\end{aligned}$$

$$-\frac{b^2}{2}(2rJ_i(b) - (r+p+1)(r-p)J_k(b))J_{k-2}(b).$$

Remark 7. Since $F_n = J_n(1)$ are Fibonacci numbers, when $b = 1$, Theorem 10 can be reduced to the Fibonacci results in [16].

It would be attractive to consider the numbers or polynomials with the same recurrence relations but different initial values. In [1, Ch.44], Jacobsthal–Lucas polynomials $j_n(x)$ are introduced as $j_n(x) = j_{n-1}(x) + xj_{n-2}(x)$ ($n \geq 2$) with $j_0(x) = 2$ and $j_1(x) = 1$. $j_n = j_n(2)$ are called Jacobsthal–Lucas numbers and $L_n = j_n(1)$ are well-known Lucas numbers. However, explicit formula cannot be expressed only by the Jacobsthal–Lucas polynomials (or numbers), and the help of the Jacobsthal polynomials (or numbers) are also needed. The formula depends on the identity

$$j_{i+k}(b) = J_k(b)j_{i+2}(b) - b^2J_{k-2}(b)j_i(b).$$

Theorem 11. Let $i, k \geq 3$ and p be a non-negative integer, and b be a positive integer. When $r = \lfloor (j_i(b) - 1)/J_k(b) \rfloor \geq p$, we have

$$\begin{aligned} &g_p(j_i(b), j_{i+2}(b), j_{i+k}(b)) \\ &= \begin{cases} (j_i(b) - rJ_k(b) - 1)j_{i+2}(b) + (r+p)j_{i+k}(b) - j_i(b) \\ \quad \text{if } (r, p) = (0, 0), \text{ or } (j_i(b) - rJ_k(b))j_{i+2}(b) \geq b^2J_{k-2}(b)j_i(b); \\ (J_k(b) - 1)j_{i+2}(b) + (r+p-1)j_{i+k}(b) - j_i(b) \\ \quad \text{if } (j_i(b) - rJ_k(b))j_{i+2}(b) < b^2J_{k-2}(b)j_i(b). \end{cases} \\ &n_p(j_i(b), j_{i+2}(b), j_{i+k}(b)) \\ &= \frac{1}{2}((j_i(b) + 2pJ_k(b) - 1)j_{i+2}(b) - j_i(b) + 1) \\ &\quad - \frac{b^2}{2}(2rj_i(b) - (r+p+1)(r-p)J_k(b))J_{k-2}(b). \end{aligned}$$

9. Open Questions

It would be interesting and possible to find an explicit form of p -Frobenius number about the so-called Pell-type numbers, satisfying the recurrence relation $P_n^* = aP_{n-1}^* + P_{n-2}^*$ for an integer $a \geq 2$. One should not think the situation is similar. In this case, as the value of a increases, the value of P_n^* increases even more rapidly, so it becomes more rapidly difficult to trace the actual situation. One of the difficulties is that there is no constant C that gives $P_{i+k}^* = P_k^*P_{i+2}^* - CP_{k-2}^*P_i^*$ for each fixed a , unlike in Jacobsthal.

Author Contributions: Writing—original draft preparation, T.K.; review and editing, C.P.-R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous referees for careful reading of this manuscript. Their comments and suggestions improved the quality of the paper. Most parts of this work were completed when the first author stayed in Universidad Panamericana in July 2022. The first author thanks the fruitful discussion with the second author and the cordial hospitality of his institute.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Koshy, T. *Fibonacci and Lucas Numbers with Applications*; Pure and Applied Mathematics (Hoboken); John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2019; Volume 2.
2. Sloane, N.J.A. The On-Line Encyclopedia of Integer Sequences. 2022. Available online: oeis.org (accessed on 1 December 2022).
3. Jacobsthal, E. Fibonaccische Polynome und Kreisteilungsgleichungen. *Sitzungsberichte Der Berl. Math. Ges.* **1919**, *17*, 43–57.
4. Horadam, A.F. Jacobsthal and Pell Curves. *Fib. Quart.* **1988**, *26*, 79–83.
5. Komatsu, T. The Frobenius number for sequences of triangular numbers associated with number of solutions. *Ann. Comb.* **2022**, *26*, 757–779. [\[CrossRef\]](#)
6. Komatsu, T. The Frobenius number associated with the number of representations for sequences of repunits. *Comptes Rendus. Mathématique* **2023**, *361*, 73–89. [\[CrossRef\]](#)
7. Komatsu, T. On p -Frobenius and related numbers due to p -Apéry set. *arXiv* **2022**, arXiv:2111.11021.
8. Curtis, F. On formulas for the Frobenius number of a numerical semigroup. *Math. Scand.* **1990**, *67*, 190–192. [\[CrossRef\]](#)
9. Ong, D.C.; Ponomarenko, V. The Frobenius number of geometric sequences. *Integers* **2008**, *8*, A33.
10. Rosales, J.C.; Branco, M.B.; Torráo, D. The Frobenius problem for Thabit numerical semigroups. *J. Number Theory* **2015**, *155*, 85–99. [\[CrossRef\]](#)
11. Rosales, J.C.; Branco, M.B.; Torráo, D. The Frobenius problem for repunit numerical semigroups. *Ramanujan J.* **2016**, *40*, 323–334. [\[CrossRef\]](#)
12. Rosales, J.C.; Branco, M.B.; Torráo, D. The Frobenius problem for Mersenne numerical semigroups. *Math. Z.* **2017**, *286*, 741–749. [\[CrossRef\]](#)
13. Robles-Pérez, A.M.; Rosales, J.C. The Frobenius number for sequences of triangular and tetrahedral numbers. *J. Number Theory* **2018**, *186*, 473–492. [\[CrossRef\]](#)
14. Shallit, J. Frobenius numbers and automatic sequences. *Integers* **2021**, *21*, A124.
15. Tripathi, A. On the Frobenius problem for geometric sequences. *Integers* **2008**, *8*, A43.
16. Komatsu, T.; Ying, H. The Frobenius number associated with the number of representations for Fibonacci triplets. *Math. Biosci. Eng.* **2023**, *20*, 3455–3481. [\[CrossRef\]](#)
17. Apéry, R. Sur les branches superlinéaires des courbes algébriques. *C. R. Acad. Sci. Paris* **1946**, *222*, 1198–1200.
18. Delgado, M.; García-Sánchez, P.A.; Morais, J. “Numericalsgps”: A GAP Package on Numerical Semigroups. Available online: <http://www.gap-system.org/Packages/numericalsgps.html> (accessed on 1 December 2022).
19. Marin, J.M.; Ramírez Alfonsín, J.L.; Revuelta, M.P. On the Frobenius number of Fibonacci numerical semigroups. *Integers* **2007**, *7*, A14.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.