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The Stereographic Projection in Topological Modules

Francisco Javier García-Pacheco

Department of Mathematics, College of Engineering, University of Cadiz, 11519 Cádiz, Spain; garcia.pacheco@uca.es

Abstract: The stereographic projection is constructed in topological modules. Let A be an additively symmetric closed subset of a topological R-module M such that $0 \in \mathrm{int}(A)$. If there exists a continuous functional $m^* : M \to R$ in the dual module M^*, an invertible $s \in \mathcal{U}(R)$ and an element a in the topological boundary $\mathrm{bd}(A)$ of A in such a way that $(m^*)^{-1}(\{s\}) \cap \mathrm{int}(A) = \varnothing$, $a \in (m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)$, and $s + m^*(\mathrm{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$, then the following function $b \mapsto -a + 2s(m^*(b) + s)^{-1}(b + a)$, from $\mathrm{bd}(A) \setminus \{-a\}$ to $(m^*)^{-1}(\{s\})$, is a well-defined stereographic projection (also continuous if multiplicative inversion is continuous on R). Finally, we provide sufficient conditions for the previous stereographic projection to become a homeomorphism.

Keywords: topological module; topological ring; stereographic projection; strongly rotund point

MSC: 46H25; 16W80; 54H13

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1. Introduction

The stereographic projection [1,2] is an important tool in Algebraic Topology. Since $\mathbb{R}^n$ (with the Euclidean topology) is Hausdorff and locally compact, it is Hausdorff-compactifiable by one point, and its one-point compactification turns out to be S^n. The inverse problem to this is to remove one point from S^n (usually the north pole) to obtain again $\mathbb{R}^n$. This is known as the stereographic projection, that is, S^n minus the north pole is homeomorphic to $\mathbb{R}^n$.

Infinite-dimensional real or complex Banach spaces are Hausdorff but not locally compact, therefore it does not make sense to talk about their Hausdorff one-point compactification (there exists their one-point compactification but it will not be Hausdorff). However, the stereographic projection can still be constructed in infinite-dimensional real or complex Banach spaces, and it turns out that the unit sphere minus any one point is homeomorphic to a closed hyperplane. This is accomplished in [3], where the stereographic projection is transported to infinite-dimensional real Banach spaces to prove that the unit sphere of any infinite-dimensional real Banach space minus any one point is homeomorphic to a closed hyperplane. Many geometric techniques from the Geometry of the real Banach spaces are employed to accomplish the results of [3]. It is worth mentioning that neither the unit sphere nor the unit ball of an infinite dimensional real or complex Banach space is compact.

The purpose of this manuscript is to construct the stereographic projection in topological modules in order to find sufficient conditions for such stereographic projection to become a homeomorphism. This way, we will have nontrivial examples of bodies in a topological module such that, after removing one point from their boundary, this boundary minus the point becomes homeomorphic to a hyperplane (Theorem 4).

2. Methodology

The topological interior of a subset A of a topological space will be denoted by $\mathrm{int}(A)$. The closure and the boundary of A are respectively denoted by $\mathrm{cl}(A)$ and $\mathrm{bd}(A)$. If B is a subset of A, then the interior of B relative to A is denoted as $\mathrm{int}_A(B)$, the closure of B

relative to A is denoted as $\mathrm{cl}_A(B)$, and the boundary of B relative to A is denoted as $\mathrm{bd}_A(B)$. It is clear that $\mathrm{int}(B) \cap A \subseteq \mathrm{int}_A(B)$, $\mathrm{cl}(B) \cap A = \mathrm{cl}_A(B)$, and $\mathrm{bd}(B) \cap A \supseteq \mathrm{bd}_A(B)$. Recall that a body in a topological space is simply a closed subset with nonempty interior. We will be working with bodies in topological modules. Since translations are homeomorphisms in topological modules, which mostly preserve the geometrical structure, we will focus on bodies of topological modules which are neighborhoods of zero. Only nonzero left-modules over nonzero associative and unitary rings are considered throughout this manuscript. Refer to [4–10] for a wide perspective on topological rings, modules, and algebras.

An affine manifold in a module M over a ring R is a translation of a submodule of M, that is, $m + N$ where $m \in M$ and N is a submodule of M. Observe that an affine manifold $m + N$ contains 0 if and only if $m + N = N$. Alongside this manuscript, we will be working with the following affine manifolds: $\mathrm{st}(m, n) := m + R(n - m)$ for $m, n \in M$ and $(m^*)^{-1}(\{s\})$ for $m^* \in M^*$ and $s \in R$ in the range of m^*. Notice that $\mathrm{st}(m, n) = \mathrm{st}(n, m)$ for all $m, n \in M$ because $m + r(n - m) = n + (1 - r)(m - n)$ for all $r \in R$, and it geometrically represents a straight line passing through m, n. We will pay special attention to a notable subset of $\mathrm{st}(m, n)$ given by $\mathrm{st}_{\mathcal{U}}(m, n) := m + (\mathcal{U}(R) \cup \{0\})(n - m)$. Note that if $u \in \mathcal{U}(R)$ and $m \neq n$, then $m \neq m + u(n - m)$. Another trivial observation is the fact that $n \in \mathrm{st}_{\mathcal{U}}(m, p)$ for all $p \in \mathrm{st}_{\mathcal{U}}(m, n) \setminus \{m\}$ in view of the fact that if $p = m + u(n - m)$ for some $u \in \mathcal{U}(R)$, then $n = m + u^{-1}(p - m) \in \mathrm{st}_{\mathcal{U}}(m, p)$. On the other hand, $(m^*)^{-1}(\{s\}) = m + \ker(m^*)$ for all $m \in (m^*)^{-1}(\{s\})$ and it geometrically represents a hyperplane, which will be closed if R is Hausdorff.

A topological ring R is said to be practical provided that $0 \in \mathrm{cl}(\mathcal{U}(R))$, that is, 0 belongs to the closure of the invertibles $\mathcal{U}(R)$ of R. Practical rings serve to extend the classical Operator Theory to the topological module setting. An extensive study on practical topological rings can be found in [11]. Let M be a topological module over a topological ring R. Let $A \subseteq M$. We say that A is bounded if for each 0-neighborhood U in M there is an invertible $u \in \mathcal{U}(R)$ such that $A \subseteq uU$. On the other hand, a point $a \in A$ is said to be an internal point of A provided for every $m \in M$ there exists a 0-neighborhood $V \subseteq R$ satisfying that $a + V(m - a) \subseteq A$. The set of internal points of A is denoted by $\mathrm{inter}(A)$. In this sense, A is called absorbing provided that $0 \in \mathrm{inter}(A)$.

An ordered ring is a ring endowed with a partial order that is compatible with the addition and the multiplication. In other words, a partial order $\leq$ in a ring R is a ring order provided that $r_1 \leq r_2 \Rightarrow r_1 + s \leq r_2 + s$ for all $s \in R$, and $r_1, r_2 \geq 0 \Rightarrow r_1 r_2 \geq 0$. The set of positive elements is denoted by $R^+ := \{r \in R : r \geq 0\}$. It is not hard to check that if $0, 1$ are comparable, then $0 < 1$ and the ring has null characteristic, that is, $\mathrm{char}(R) = 0$, because $1 + \overset{n}{\cdots} + 1 > 0$ for all $n \in \mathbb{N}$. Whenever we talk about an ordered topological ring we mean a ring that is endowed with a ring topology and a ring order, but there does not have to be any relation between the ring order and the ring topology. However, if we talk about topological ordered rings, then we mean a ring endowed with a ring order in such a way that the order topology is well-defined and turns out to be a ring topology.

Finally, we will make use of the Axiom of Choice, which states that for every nonempty set X there exists a choice function $\phi : \mathcal{P}(X) \setminus \{\varnothing\} \to X$, that is, $\phi(Y) \in Y$ for every $Y \in \mathcal{P}(X) \setminus \{\varnothing\}$.

3. Results

We will construct the stereographic projection in topological modules by relying on [3]. One of the geometrical principles upon which this construction is based is the fact that, in Euclidean spaces, straight lines passing through zero must intersect the boundary of bounded bodies whose interior contains zero. This is itself based upon the following topological fact: if X is a topological space and L is a connected subset of X such that $L \cap \mathrm{int}(A) \neq \varnothing$ and $L \cap \mathrm{int}(X \setminus A) \neq \varnothing$ for a subset $A \subseteq X$, then $L \cap \mathrm{bd}(A) \neq \varnothing$. Nevertheless, we will first study certain types of bodies in topological modules, fit bodies and Minkowski bodies, that will provide the necessary insight to properly construct the stereographic projection.

3.1. Fit Bodies

Let us begin by bearing in mind the trivial fact that if M is a module over a ring R, then $Ra = \mathrm{st}(0, a)$ for every $a \in M$.

Definition 1 (Fit body)**.** *Let M be a topological module over a topological ring R. A subset $A \subseteq M$ is said to be a fit body if A is closed, $0 \in \mathrm{int}(A)$, and every $a \in \mathrm{bd}(A)$ satisfies that $Ra \cap \mathrm{bd}(A) \subseteq \{-a, a\}$.*

Under the settings of the previous definition, it is immediate to realize that if a fit body A is additively symmetric, then $Ra \cap \mathrm{bd}(A) = \{-a, a\}$ for every $a \in \mathrm{bd}(A)$. Before stating and proving the main result of this subsection, we will provide nontrivial examples of fit bodies.

Proposition 1. *Let R be a topological totally ordered ring with no holes. Then $A := [-1, 1]$ is a fit body.*

Proof. By assumption, the ring topology of R is the order topology, thus $(-1, 1)$ is open. Let us show next that $\mathrm{bd}(A) = \{-1, 1\}$. Indeed, fix an arbitrary open interval (a, b) containing 1. Notice that $1 + 1 > 1$, so we may assume that $b \leq 1 + 1$. By hypothesis, A has no holes, therefore $(1, b)$ is not empty, meaning that $(a, b) \cap (R \setminus A) \neq \varnothing$. As a consequence, $1 \in \mathrm{bd}(A)$. In a similar way, $-1 \in \mathrm{bd}(A)$. On the other hand, $[-1, 1]$ is closed in the order topology, which assures that $\mathrm{bd}(A) = \{-1, 1\}$. Then A is an additively symmetric body whose interior contains 0. Finally, $R1 \cap \mathrm{bd}(A) = \{-1, 1\} = R(-1) \cap \mathrm{bd}(A)$, meaning that A is a fit body. $\square$

A nontrivial example of a ring satisfying the properties of Proposition 1 is provided next.

Example 1. *$R := \mathbb{Q}[\pi]$, endowed with the inherited topology from $\mathbb{R}$, is a topological totally ordered ring with no holes.*

We are now in the right position to state and prove the main theorem of this subsection.

Theorem 1. *Let M be a torsionfree topological module over a Hausdorff topological-inversion integral domain ring R of $\mathrm{char}(R) \neq 2$ such that $\mathcal{U}(R)$ is open. Let $m^* \in M^*$, $A \subseteq M$ a fit body, and $s \in \mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\}) \cap \mathrm{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A) \neq \varnothing$. Then*

$$\mathrm{int}_{(m^*)^{-1}(\{s\})}\Big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\Big) \subseteq \mathrm{int}_{\mathrm{bd}(A)}\Big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\Big).$$

Proof. Fix an arbitrary $a \in \mathrm{int}_{(m^*)^{-1}(\{s\})}\big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\big)$. A 0-neighborhood $V_0 \subseteq M$ exists such that $(a + V_0) \cap (m^*)^{-1}(\{s\}) \subseteq \mathrm{bd}(A)$. Another 0-neighborhood $V_1 \subseteq M$ can be found in such a way that $V_1 + V_1 + V_1 \subseteq V_0$. Next, there are 0-neighborhoods $W_1 \subseteq R$ and $V_2 \subseteq M$ satisfying that $W_1 V_2 \subseteq V_1$. Take W_2 a 0-neighborhood in R so that $W_2 a \subseteq V_1$. Let W_3 be a 0-neighborhood in R such that $s + W_3 \subseteq \mathcal{U}(R)$. Since $\mathrm{char}(R) \neq 2$ and s is invertible, we have that $s + s \neq 0$, thus, since R is Hausdorff, there exists a 0-neighborhood $W_4 \subseteq R$ such that $-s - s \notin W_4$. Consider the continuous function (recall that R is a topological-inversion ring):

$$\begin{array}{rcl} V_3 & \to & R \\ m & \mapsto & s(s + m^*(m))^{-1}, \end{array} \tag{1}$$

where V_3 is a 0-neighborhood in M such that $V_3 \subseteq (m^*)^{-1}(W_3 \cap W_4)$. Note that (1) maps 0 to 1. Thus, we can find $V_4 \subseteq M$ a 0-neighborhood verifying that $s(s + m^*(m))^{-1} \in 1 + (W_1 \cap W_2)$ for all $m \in V_4 \cap V_3$. Finally, let $V_5 := V_4 \cap V_3 \cap V_2 \cap V_1$. We will show that

$(a+V_5)\cap \mathrm{bd}(A)\subseteq (m^*)^{-1}(\{s\})$. Indeed, fix an arbitrary $m\in V_5$ with $a+m\in \mathrm{bd}(A)$. Denote $t:=s(s+m^*(m))^{-1}$ and $v:=(t-1)a+(t-1)m+m$. Notice that $v\in W_2a+W_1V_2+V_1\subseteq V_1+V_1+V_1\subseteq V_0$. Therefore, $t(a+m)=a+v\in a+V_0$. Additionally, $t(a+m)\in (m^*)^{-1}(\{s\})$. As a consequence, $t(a+m)\in \mathrm{bd}(A)$. Since A is a fit body, either $t(a+m)=a+m$ or $t(a+m)=-(a+m)$. Next, $a+m\neq 0$ because $a+m\in \mathrm{bd}(A)$ and $0\in \mathrm{int}(A)$. Additionally, M is torsionfree and R is an integral domain, hence either $t=1$ or $t=-1$. In the latter case, $m^*(m)=-s-s$ which is impossible by bearing in mind that $m^*(m)\in W_4$ but $-s-s\notin W_4$. As a consequence, $t=1$ meaning that $m^*(m)=0$, so $a+m\in (m^*)^{-1}(\{s\})$. □

3.2. Minkowski Bodies

Let M be a topological module over a topological ring R. Let $A\subseteq M$ be an absorbing subset [12–14]. Notice that if $m\in M$, then there exists a 0-neighborhood $W\subseteq R$ such that $Wm\subseteq A$. If R is practical, then we can find an invertible $u\in W\cap \mathcal{U}(R)$. This fact motivates the following definition, based also upon the classical Minkowski functional [15] in real or complex topological vector spaces.

Definition 2 (Minkowski functional)**.** *Let M be a topological module over a practical topological-inversion ring R. Let $A\subseteq M$ be an absorbing subset. A Minkowski functional is a function*

$$\begin{array}{rccc} j_A: & M & \to & \mathcal{U}(R)\cup\{0\} \\ & m & \mapsto & j_A(m) \end{array} \qquad (2)$$

satisfying that $j_A(0)=0$ and, for every $m\in M\setminus\{0\}$, $j_A(m)\in \mathcal{U}(R)$ and $j_A(m)^{-1}m\in A$.

Under the settings of the previous definition, observe that if $u\in \mathcal{U}(R)$, then $g_A(m):=uj_A(u^{-1}m)$ for every $m\in M$ also defines a Minkowski functional.

Proposition 2. *Let M be a topological module over a practical topological-inversion ring R. Let $A\subseteq M$ be an absorbing subset. There exists a Minkowski functional j_A for A.*

Proof. According to the Axiom of Choice, there exist choice functions $\phi:\mathcal{P}(\mathcal{P}(R))\setminus\{\varnothing\}\to \mathcal{P}(R)$ and $\chi:\mathcal{P}(\mathcal{U}(R))\setminus\{\varnothing\}\to \mathcal{U}(R)$. They satisfy that $\phi(D)\in D$ for all $D\in \mathcal{P}(\mathcal{P}(R))\setminus\{\varnothing\}$ and $\chi(Z)\in Z$ for all $Z\in \mathcal{P}(\mathcal{U}(R))\setminus\{\varnothing\}$. Let us then construct the Minkowski functional for A by relying on the choice functions ϕ and χ. Indeed, fix an arbitrary $m\in M\setminus\{0\}$. There exists a 0-neighborhood $W\subseteq R$ such that $Wm\subseteq A$. Therefore, the set $B_m:=\{W\in \mathcal{P}(R): W \text{ is a 0-neighborhood such that } Wm\subseteq A\}$ is not empty, hence $B_m\in \mathcal{P}(\mathcal{P}(R))\setminus\{\varnothing\}$. Observe that $\phi(B_m)\in B_m$ for all $m\in M\setminus\{0\}$. On the other hand, since R is practical, for every $m\in M\setminus\{0\}$ and every $W\in B_m$, we have that $W\cap \mathcal{U}(R)\neq\varnothing$, meaning that $W\cap \mathcal{U}(R)\in \mathcal{P}(\mathcal{U}(R))\setminus\{\varnothing\}$. Note that $\chi(W\cap \mathcal{U}(R))\in W\cap \mathcal{U}(R)$. Then it suffices to define $j_A(m):=\chi(\phi(B_m)\cap \mathcal{U}(R))^{-1}$ for $m\in M\setminus\{0\}$ and $j_A(0)=0$. □

For the next definition, bear in mind that every body of a topological module which is also a neighborhood of zero is absorbing.

Definition 3 (Minkowski body)**.** *Let M be a topological module over a practical topological-inversion ring R. A subset $A\subseteq M$ is said to be a Minkowski body if A is closed, $0\in \mathrm{int}(A)$, and there exists a continuous Minkowski functional j_A satisfying that $j_A(m)^{-1}m\in \mathrm{bd}(A)$ for every $m\in M\setminus\{0\}$ and $j_A(a)=1$ for all $a\in \mathrm{bd}(A)$.*

The simplest example of a Minkowski body is the unit ball of a real or complex normed space. The following is a nontrivial example of a Minkowski body.

Example 2. *Let $R:=\mathbb{Q}\left[\sqrt{2}\right]$ be endowed with the inherited topology from $\mathbb{R}$. Notice that R is a division ring because $\sqrt{2}$ is algebraic over $\mathbb{Q}$. In fact, R is a non-discrete topological division* *ring, so in particular it is a practical topological-inversion ring. Take $A := [-1, 1] \cap R$. Then A is a Minkowski body. Indeed, A is closed and $0 \in \mathrm{int}(A)$, and $j_A(x) := |x|$ is a continuous Minkowski functional satisfying that $j_A(x)^{-1}x \in \mathrm{bd}(A)$ for every $x \in R \setminus \{0\}$ and $j_A(a) = 1$ for all $a \in \mathrm{bd}(A)$.*

It is time now to state and prove the main result of this subsection.

Theorem 2. *Let M be a topological module over a practical topological-inversion ring R. Let $m^* \in M^*$, $A \subseteq M$ a Minkowski body, and $s \in R \setminus \mathtt{rd}(0)$ such that $(m^*)^{-1}(\{s\}) \cap \mathrm{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A) \neq \varnothing$. Then*

$$\mathrm{int}_{(m^*)^{-1}(\{s\})}\Big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\Big) \supseteq \mathrm{int}_{\mathrm{bd}(A)}\Big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\Big).$$

Proof. Fix an arbitrary $a \in \mathrm{int}_{\mathrm{bd}(A)}\big((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\big)$. There exists a 0-neighborhood $V_0 \subseteq M$ such that $(a + V_0) \cap \mathrm{bd}(A) \subseteq (m^*)^{-1}(\{s\})$. Another 0-neighborhood $V_1 \subseteq M$ can be found in such a way that $V_1 + V_1 + V_1 \subseteq V_0$. Next, there are 0-neighborhoods $W_1 \subseteq R$ and $V_2 \subseteq M$ satisfying that $W_1V_2 \subseteq V_1$. Take W_2 a 0-neighborhood in R so that $W_2a \subseteq V_1$. Notice that the map

$$\begin{array}{rcl} M \setminus \{0\} & \to & \mathcal{U}(R) \\ m & \mapsto & j_A(m)^{-1} \end{array} \tag{3}$$

is continuous and maps a to 1 (keep in mind that $a \neq 0$ because $a \in \mathrm{bd}(A)$ and $0 \in \mathrm{int}(A)$). Let V_3 be a 0-neighborhood in M such that $j_A((a + V_3) \cap (M \setminus \{0\}))^{-1} \subseteq 1 + (W_1 \cap W_2)$. Finally, let $V_4 := V_3 \cap V_2 \cap V_1$. We will show that $(a + V_4) \cap (m^*)^{-1}(\{s\}) \subseteq \mathrm{bd}(A)$. Indeed, fix an arbitrary $m \in V_4$ with $a + m \in (m^*)^{-1}(\{s\})$. Observe that $a + m \neq 0$ because $s \neq 0$, hence $j_A(a + m)$ is invertible. Denote $t := j_A(a + m)^{-1}$ and $v := (t - 1)a + (t - 1)m + m$. Notice that $v \in W_2a + W_1V_2 + V_1 \subseteq V_1 + V_1 + V_1 \subseteq V_0$. Therefore, $t(a + m) = a + v \in a + V_0$. Additionally, $t(a + m) \in \mathrm{bd}(A)$. As a consequence, $t(a + m) \in (m^*)^{-1}(\{s\})$. Then we obtain that $ts = s$. Since s is not a right 0-divisor, we conclude that $t = 1$ meaning that $a + m \in \mathrm{bd}(A)$. □

3.3. Exposed Faces

This section is devoted to construct nontrivial examples of topological modules and bodies for which there exists a hyperplane intersecting the boundary of the body but not the interior. We will rely on ordered topological rings and on the notion of exposed face [16], taken from the Geometry of (real) Banach Spaces.

Lemma 1. *Let M be a topological module over a topological ring R. If $m^* \in M^*$ is so that $m^*(M) \cap \mathcal{U}(R) \neq \varnothing$, then m^* is an open map.*

Proof. Let $m \in M$ such that $m^*(m) \in \mathcal{U}(R)$. We will show that $m^*(V)$ is a 0-neighborhood in R for every 0-neighborhood $V \subseteq M$. Indeed, there exists a 0-neighborhood $W \subseteq R$ such that $Wm \subseteq V$. Then $Wm^*(m) = m^*(Wm) \subseteq m^*(V)$. Finally, $Wm^*(m)$ is a 0-neighborhood in R because $m^*(m)$ is invertible. As a consequence, $m^*(V)$ is a 0-neighborhood in R. □

Definition 4 (Exposed face)**.** *Let M be a topological module over an ordered topological ring R. Let $m^* \in M^*$ and $A \subseteq M$ such that $\sup m^*(A)$ exists in R. The set $F(m^*, A) := \{m \in A : m^*(m) = \sup m^*(A)\}$ is called an exposed face.*

Under the settings of the previous definition, $F(m^*, A) = (m^*)^{-1}(\{\sup m^*(A)\}) \cap A$.

Theorem 3. *Let M be a topological module over an ordered topological ring R. Let $m^* \in M^*$ and $A \subseteq M$ such that $\sup m^*(A)$ exists in R. If m^* is an open map, $0 \in \mathrm{cl}(R^+ \setminus \{0\})$, and $(R^+ \setminus \{0\})(R^+ \setminus \{0\}) \subseteq R^+ \setminus \{0\}$, then $F(m^*, A) \cap \mathrm{inter}(A) = \varnothing$.*

Proof. Suppose on the contrary that there exists $a \in F(m^*, A) \cap \operatorname{inter}(A)$. Since m^* is open and $0 \in \operatorname{cl}(R^+ \setminus \{0\})$, we can find $m \in M$ such that $m^*(m) \in R^+ \setminus \{0\}$. Since a is an internal point of A, there exists a 0-neighborhood $V \subseteq R$ such that $a + Vm \subseteq A$. Take any $v \in V \cap (R^+ \setminus \{0\})$. Notice that $vm^*(m) > 0$, hence $m^*(a + vm) = m^*(a) + vm^*(m) > m^*(a)$ reaching a contradiction with the fact that $m^*(a) = \sup m^*(A)$. □

Under the settings of Theorem 3, observe that $F(m^*, A) \cap \operatorname{int}(A) = \varnothing$ because $\operatorname{int}(A) \subseteq \operatorname{inter}(A)$. Therefore, $F(m^*, A) \subseteq \operatorname{bd}(A)$. As a corollary of Theorem 3, we obtain the following final result.

Corollary 1. *Let R be a nondiscrete, integral domain, totally ordered topological ring R. Let M be a topological R-module. Let A be a subset of M. If $m^* \in M^* \setminus \{0\}$, then $F(m^*, A) \cap \operatorname{inter}(A) = \varnothing$.*

Proof. In the first place, since R is an integral domain, $(R^+ \setminus \{0\})(R^+ \setminus \{0\}) \subseteq R^+ \setminus \{0\}$. Next, let us show that $0 \in \operatorname{cl}(R^+ \setminus \{0\})$. Indeed, let V be any additively symmetric neighborhood of 0 in R. Since R is not discrete, $V \neq \{0\}$, so we can find $v \in V \setminus \{0\}$. Since V is additively symmetric, by choosing $-v$ if necessary, we may assume that $v > 0$. As a consequence, $0 \in \operatorname{cl}(R^+ \setminus \{0\})$. Nevertheless, with the hypotheses we have, we cannot guarantee that m^* will be an open map. Nonetheless, we can achieve the desired result. Indeed, fix an arbitrary $a \in \operatorname{inter}(A)$ such that $m^*(a) = \max m^*(A)$. Since $m^* \neq 0$, we can fix $m \in M$ such that $m^*(m) \neq 0$. By choosing $-m$ if necessary, we may assume that $m^*(m) > 0$. By hypothesis, there exists an additively symmetric neighborhood $V \subseteq R$ of 0 such that $a + Vm \subseteq A$. We have just shown the existence of $v \in V \cap (R^+ \setminus \{0\})$. We have also seen that $vm^*(m) > 0$. Next, $a + vm \in a + Vm \subseteq A$, thus

$$m^*(a) \geq m^*(a + vm) = m^*(a) + vm^*(m) > m^*(a),$$

which is a contradiction. □

3.4. The Stereographic Projection

This final section is aimed at constructing the stereographic projection. For this, we will strongly rely on the affine manifolds given by the straight lines and the hyperplanes.

Definition 5 (Stereographic projection)**.** *Let M be a topological module over a topological-inversion ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. Let $m^* \in M^*$ and $s \in \mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) \neq \varnothing$. Let $a \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$ such that $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$. The following map is known as a stereographic projection:*

$$\begin{array}{rcl} \operatorname{bd}(A) \setminus \{-a\} & \to & (m^*)^{-1}(\{s\}) \\ b & \mapsto & -a + 2s(m^*(b) + s)^{-1}(b + a). \end{array} \tag{4}$$

Observe that the stereographic projection (4) is well-defined and continuous (note that multiplicative inversion on R is continuous). In fact, notice also that $\phi(b) \in \operatorname{st}_{\mathcal{U}}(-a, b)$ for all $b \in \operatorname{bd}(A) \setminus \{-a\}$.

Remark 1. *Let M be a topological module over a topological division ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. If $m^* \in M^*$ and $s \in \mathcal{U}(R)$ are such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$ is a singleton, then the only element $a \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$ satisfies that $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$. Indeed, if $b \in \operatorname{bd}(A) \setminus \{-a\}$ satisfies that $s + m^*(b) = 0$, then $-b \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$, meaning the contradiction that $b = -a$. As a consequence, the stereographic projection (4) is well-defined and continuous.*

The point is to determine under what circumstances the stereographic projection (4) is a homeomorphism. For this, the following definition, based upon the classical notion of strongly rotund point [17–22] from the Geometry of Banach Spaces, will be employed.

Definition 6 (Strongly rotund point)**.** *Let M be a topological module over a topological-inversion ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. An element $a \in \operatorname{bd}(A)$ is said to be a strongly rotund point of A provided that there exist $m^* \in M^*$ and $s \in \mathcal{U}(R)$, called the supporting functional and support value, respectively, satisfying the following conditions:*

- *$m^*(A)$ is relatively compact in R.*
- *$(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$.*
- *$(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$.*
- *$s + \operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\} \subseteq \mathcal{U}(R)$.*
- *$\operatorname{st}_{\mathcal{U}}(-a, b) \cap \operatorname{bd}(A) = \{-a, b\}$ for all $b \in \operatorname{bd}(A)$.*
- *$\operatorname{st}_{\mathcal{U}}(-a, c) \cap (\operatorname{bd}(A) \setminus \{-a\}) \neq \varnothing$ for all $c \in (m^*)^{-1}(\{s\})$.*
- *If $(c_j)_{j\in J} \subseteq \operatorname{bd}(A) \setminus \{-a\}$ is a net converging to $-a$, then $\left((m^*(c_j) + s)^{-1}(c_j + a)\right)_{j\in J}$ is not convergent.*

Notice that strongly rotund points are trivially preserved by isomorphisms of topological modules. The following remark gathers several considerations about strongly rotund points.

Remark 2. *Let M be a topological module over a topological ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. Let $a \in \operatorname{bd}(A)$ be a strongly rotund point of A with supporting functional $m^* \in M^*$ and support value $s \in \mathcal{U}(R)$. Observe the following:*

- *$s \neq 0$ because $0 \in \operatorname{int}(A)$ and $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$. Additionally, if $b \in \operatorname{bd}(A) \setminus \{-a\}$, then $m^*(b) \neq -s$ because otherwise we conclude that $b = -a$ in view of the fact that $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$. As a consequence, $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$. For these reasons, the condition $s + \operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\} \subseteq \mathcal{U}(R)$ is well-imposed in Definition 6.*
- *If $\operatorname{char}(R) \neq 2$, then $-a \notin (m^*)^{-1}(\{s\})$ or equivalently $-a \neq a$. Indeed, if $-a \in (m^*)^{-1}(\{s\})$, then $-s = s$, hence $(1+1)s = 0$, so $1+1 = 0$ because s is invertible, contradicting that $\operatorname{char}(R) \neq 2$.*

Recall that, given a real or complex Banach space X with unit sphere S_X and unit ball B_X, a closed subspace $Y \subseteq X$ is said to be an L^2-summand subspace of X if M is L^2-complemented in X, that is, there exists a closed subspace $Z \subseteq X$ such that $X = Y \oplus_2 Z$, in the sense that $\|y+z\|^2 = \|y\|^2 + \|z\|^2$ for all $y \in Y$ and all $z \in Z$. A point $x \in X$ is said to be an L^2-summand vector of X provided that $\mathbb{K}x$ is an L^2-summand subspace of X, where $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. By bearing in mind ([3], Lemma 2.1), if $x \in \mathsf{S}_X$ is an L^2-summand vector of a real Banach space X, then x is a strongly rotund point of the unit ball B_X of X in the sense of Definition 6. A couple of technical lemmas will be needed before constructing the stereographic projection.

Lemma 2. *Let X, Y be topological spaces. Let $f : X \to Y$ be bijective. Let $x \in X$. Suppose that for every net $(x_i)_{i\in I} \subseteq X$ such that $(f(x_i))_{i\in I}$ converges to $f(x)$, there exists a subnet $(z_j)_{j\in J}$ of $(x_i)_{i\in I}$ convergent to x. Then f^{-1} is continuous at $f(x)$.*

Proof. Assume to the contrary that f^{-1} is not continuous at $f(x)$. Then there exists a net $(x_i)_{i\in I} \subseteq X$ such that $(f(x_i))_{i\in I}$ converges to $f(x)$ but $(x_i)_{i\in I}$ does not converge to x. There exists a neighborhood U of x and a subnet $(z_j)_{j\in J}$ of $(x_i)_{i\in I}$ such that $z_j \notin U$ for all $j \in J$. Note that $(f(z_j))_{j\in J}$ converges to $f(x)$. Thus, by hypothesis, there exists a subnet $(p_k)_{k\in K}$ of $(z_j)_{j\in J}$ convergent to x, which is a contradiction in view of the fact that $z_j \notin U$ for all $j \in J$. $\square$

Lemma 3. *Let M be a topological module over a topological ring R. Let $(r_i)_{i\in I} \subseteq R$ and $(m_i)_{i\in I} \subseteq M$ be nets such that $(r_i)_{i\in I}$ is convergent to some $r \in R$ and $(r_i m_i)_{i\in I}$ is convergent to some $m \in M$. If $(m_i)_{i\in I}$ is bounded, then $(rm_i)_{i\in I}$ converges to m.*

Proof. It suffices to show that $(rm_i - r_i m_i)_{i\in I}$ converges to 0. Fix an arbitrary 0-neighborhood $U \subseteq M$. There are 0-neighborhoods $V \subseteq R$ and $U_1 \subseteq M$ such that $VU_1 \subseteq U$. By hypothesis, there exists an invertible $u \in \mathcal{U}(R)$ such that $m_i \in uU_1$ for all $i \in I$. There exists $i_0 \in I$ such that $r - r_i \in Vu^{-1}$ for all $i \geq i_0$. Then, for all $i \geq i_0$, $rm_i - r_i m_i = (r - r_i)m_i \in Vu^{-1}uU_1 = VU_1 \subseteq U$. □

At this stage, we are finally in the right position to construct the stereographic projection.

Theorem 4. *Let M be a topological module over a topological-inversion ring R satisfying that $2 := 1 + 1 \in \mathcal{U}(R)$. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. If there exists a strongly rotund point $a \in \operatorname{bd}(A)$ with supporting functional $m^* \in M^*$ and support value $s \in \mathcal{U}(R)$, then the stereographic projection (4) is a homeomorphism.*

Proof. First off, let us denote by ϕ to the stereographic projection (4). We already know that ϕ is well-defined, $\phi(b) \in \operatorname{st}_{\mathcal{U}}(-a, b)$ for all $b \in \operatorname{bd}(A) \setminus \{-a\}$, and ϕ is trivially continuous due to the fact that R is a topological-inversion ring. Let us check now that ϕ is surjective. Fix an arbitrary $c \in (m^*)^{-1}(\{s\})$. If $c = a$, then $\phi(a) = a$. Thus let us assume that $c \neq a$. Since a is a strongly rotund point of A, by definition we have that $\operatorname{st}_{\mathcal{U}}(-a, c) \cap (\operatorname{bd}(A) \setminus \{-a\}) \neq \varnothing$. Let $u \in \mathcal{U}(R)$ such that $b := -a + u(c + a) \in \operatorname{bd}(A) \setminus \{-a\}$. We will show that $\phi(b) = c$. Indeed,

$$
\begin{aligned}
\phi(b) &= -a + 2s(m^*(b) + s)^{-1}(b + a) \\
&= -a + 2s(m^*(-a + u(c + a)) + s)^{-1}(-a + u(c + a) + a) \\
&= -a + 2s(-s + u2s + s)^{-1}u(c + a) \\
&= -a + (2s)(2s)^{-1}u^{-1}u(c + a) \\
&= -a + c + a \\
&= c.
\end{aligned}
$$

Next step is to prove that ϕ is one-to-one. Indeed, take $b_1, b_2 \in \operatorname{bd}(A) \setminus \{-a\}$ with $\phi(b_1) = \phi(b_2)$. Then $b_2 = -a + (m^*(b_2) + s)(m^*(b_1) + s)^{-1}(b_1 + a) \in \operatorname{st}_{\mathcal{U}}(-a, b_1) \cap \operatorname{bd}(A) = \{-a, b_1\}$, meaning that $b_1 = b_2$. Let us finally prove that ϕ^{-1} is continuous. We will rely on Lemma 2. Fix an arbitrary $b \in \operatorname{bd}(A) \setminus \{-a\}$. Take a net $(b_i)_{i\in I} \subseteq \operatorname{bd}(A) \setminus \{-a\}$ such that $(\phi(b_i))_{i\in I}$ converges to $\phi(b)$. We will show the existence of a subnet $(c_j)_{j\in J}$ of $(b_i)_{i\in I}$ convergent to b. Indeed, $m^*(A)$ is relatively compact in R, therefore there exists a subnet $(c_j)_{j\in J}$ of $(b_i)_{i\in I}$ such that $(m^*(c_j))_{j\in J}$ is convergent to some $r \in R$. Then $(\phi(c_j))_{j\in J}$ converges to $\phi(b)$. This is equivalent to saying that $\left((m^*(c_j) + s)^{-1}(c_j + a)\right)_{j\in J}$ converges to $(m^*(b) + s)^{-1}(b + a)$. Since $(m^*(c_j) + s)_{j\in J}$ is convergent to $r + s$, we conclude that $\left((m^*(c_j) + s)(m^*(c_j) + s)^{-1}(c_j + a)\right)_{j\in J}$ converges to $(r + s)(m^*(b) + s)^{-1}(b + a)$, in other words, $\left(c_j + a\right)_{j\in J}$ converges to $(r + s)(m^*(b) + s)^{-1}(b + a)$, which is equivalent to stating that $\left(c_j\right)_{j\in J}$ converges to $-a + (r + s)(m^*(b) + s)^{-1}(b + a)$. At this point, observe that $r \neq -s$ since otherwise we obtain that $(c_j)_{j\in J}$ converges to $-a$, reaching the contradiction that $\left((m^*(c_j) + s)^{-1}(c_j + a)\right)_{j\in J}$ is not convergent by bearing in mind Definition 6. As a consequence, $r + s \in s + \operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\} \subseteq \mathcal{U}(R)$, meaning that $-a + (r + s)(m^*(b) + s)^{-1}(b + a) \in \operatorname{st}_{\mathcal{U}}(-a, b)$. On the other hand, $\operatorname{bd}(A)$ is closed, hence $-a + (r + s)(m^*(b) + s)^{-1}(b + a) \in \operatorname{bd}(A)$. Since a is a strongly rotund point of A, $\operatorname{st}_{\mathcal{U}}(-a, b) \cap \operatorname{bd}(A) = \{-a, b\}$. As a consequence, either $-a + (r + s)(m^*(b) + s)^{-1}(b + a) = -a$ or $-a + (r + s)(m^*(b) + s)^{-1}(b + a) = b$. If $-a + (r + s)(m^*(b) + s)^{-1}(b + a) = -a$, then

$b = -a$, which is impossible. Thus, $-a + (r+s)(m^*(b)+s)^{-1}(b+a) = b$. By relying on Lemma 2, we conclude that ϕ^{-1} is continuous at b. $\square$

4. Discussion

We will discuss a nontrivial example of a stereographic projection in a topological module other than a real topological vector space. Observe that if M is a module over a ring R, $S \subseteq R$ is a subring of R, and $m^* \in M^*$, then $(m^*)^{-1}(S)$ is an S-submodule of M.

Theorem 5. *Let M be a topological module over a topological division ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. Let $m^* \in M^*$ and $s \in \mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) \neq \varnothing$. Let $a \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$ such that $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$. Consider the stereographic projection (4). If $S \subseteq R$ is a dense subring of R such that $\mathcal{U}(R) \cap S = \mathcal{U}(S)$ and $s \in \mathcal{U}(S)$, then $B := A \cap (m^*)^{-1}(S)$ is an additively symmetric body of the S-module $(m^*)^{-1}(S)$ with $0 \in \operatorname{int}_{(m^*)^{-1}(S)}(B)$ and the following stereographic projection is well-defined and continuous:*

$$\begin{array}{rcl} \operatorname{bd}_{(m^*)^{-1}(S)}(B) \setminus \{-a\} & \to & (m^*)^{-1}(\{s\}) \\ b & \mapsto & -a + 2s(m^*(b)+s)^{-1}(b+a). \end{array} \tag{5}$$

Proof. Note that B is trivially closed in $(m^*)^{-1}(S)$ since A is closed in M. In fact, $0 \in \operatorname{int}(A) \cap (m^*)^{-1}(S) \subseteq \operatorname{int}_{(m^*)^{-1}(S)}(B)$. As a consequence, B is an additively symmetric body of the S-module $(m^*)^{-1}(S)$ with $0 \in \operatorname{int}_{(m^*)^{-1}(S)}(B)$. It only remains to show that

$$s + m^*\left(\operatorname{bd}_{(m^*)^{-1}(S)}(B) \setminus \{-a\}\right) \subseteq \mathcal{U}(S).$$

Indeed, $\operatorname{bd}_{(m^*)^{-1}(S)}(B) \subseteq \operatorname{bd}(A) \cap (m^*)^{-1}(S)$, hence

$$s + m^*\left(\operatorname{bd}_{(m^*)^{-1}(S)}(B) \setminus \{-a\}\right) \subseteq (s + m^*(\operatorname{bd}(A) \setminus \{-a\})) \cap S \subseteq \mathcal{U}(R) \cap S = \mathcal{U}(S).$$

Finally, notice that, since S is dense in R and R is a topological division ring, then $(m^*)^{-1}(S)$ is dense in M which implies, together with the fact that A is closed in M, that $\operatorname{int}(A) \cap (m^*)^{-1}(S) = \operatorname{int}_{(m^*)^{-1}(S)}(B)$ and $\operatorname{bd}(A) \cap (m^*)^{-1}(S) = \operatorname{bd}_{(m^*)^{-1}(S)}(B)$. $\square$

Under the settings of Theorem 5, observe that if S is a proper subring of R, then $(m^*)^{-1}(S)$ is a proper submodule of M, hence the stereographic projection (5) might not be necessarily surjective. Theorem 5 allows plenty of examples of nontrivial stereographic projections.

Example 3. *Let X be a real Banach space of dimension strictly greater than 1. Let $x \in \mathsf{S}_X$ and $x^* \in \mathsf{S}_{X^*}$ such that $\{x\} = (x^*)^{-1}(\{1\}) \cap \mathsf{S}_X$. According to Remark 1, the stereographic projection*

$$\begin{array}{rcl} \mathsf{S}_X \setminus \{-x\} & \to & (x^*)^{-1}(\{1\}) \\ y & \mapsto & -x + 2(x^*(y)+1)^{-1}(y+x) \end{array} \tag{6}$$

is well-defined and continuous. Here, $R := \mathbb{R}$, $M := X$, $s = 1$, $m^* = x^*$, and $A := \mathsf{B}_X$. Now, take $S := \mathbb{Q}\left[\sqrt{2}\right]$ and let N denote $(x^*)^{-1}(S)$ as S-module. Additionally, let $B := \mathsf{B}_X \cap N$. In view of Theorem 5, the stereograpahic projection

$$\begin{array}{rcl} \operatorname{bd}_N(B) \setminus \{-x\} & \to & (x^*)^{-1}(\{1\}) \\ y & \mapsto & -x + 2(x^*(y)+1)^{-1}(y+x) \end{array} \tag{7}$$

is well-defined and continuous.

5. Conclusions

The stereographic projection is possible in $\mathbb{R}^n$ with the Euclidean topology. Additionally, the setereographic projection is possible too in infinite-dimensional real Banach spaces. This work transports the stereographic projection to the very abstract setting of topological modules. Nontrivial examples have been discussed and provided. The category of topological modules is probably the most general category where the stereographic projection can be constructed. This is one step further in the study and comprehension of the Geometry of Topological Modules. Applications to Quantum Topology will be studied in a future work.

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