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A New Advanced Class of Convex Functions with Related Results

Muhammad Adil Khan [1], Adnan [1], Tareq Saeed [2] and Eze R. Nwaeze [3,*]

[1] Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
[2] Financial Mathematics and Actuarial Science (FMAS)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[3] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA
* Correspondence: enwaeze@alasu.edu

Abstract: It is the purpose of this paper to propose a novel class of convex functions associated with strong η-convexity. A relationship between the newly defined function and an earlier generalized class of convex functions is hereby established. To point out the importance of the new class of functions, some examples are presented. Additionally, the famous Hermite–Hadamard inequality is derived for this generalized family of convex functions. Furthermore, some inequalities and results for strong η-convex function are also derived. We anticipate that this new class of convex functions will open the research door to more investigations in this direction.

Keywords: strongly convex function; strongly η-convex function; η-convex function

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1. Introduction

There is no doubt that the convex sets and convex functions are widely used in numerous areas of science, such as control theory, operations research, optimization, geometry, information theory, and functional analysis. Roughly speaking, convex functions are extensively used in applied and theoretical mathematics because of thei two basic features: any local minima of the convex function is the global one and the convex function achieves its upper bound at the boundary point. Furthermore, there is at most one minimum of the strictly convex functions [1–3]. The popular book of Littlewood, Hardy, and Pólya played a dominant role in the popularization of convexity [4]. Mathematically, convex function is defined as follows [5]:

A function $\Psi : [p, q] \to \mathbb{R}$ is convex if the inequality

$$\Psi(\eta x_1 + (1-\eta)x_2) \leq \eta\Psi(x_1) + (1-\eta)\Psi(x_2)$$

holds for all $\eta \in [0, 1]$ and $x_1, x_2 \in [p, q]$. There are several fruitful and interesting properties of convex functions which attract several mathematicians to this class of functions. In what follows, we present some important and basic properties of convex functions.

If the function $\Psi : (p, q) \to \mathbb{R}$ is convex, then Ψ must be continuous and the left and right derivative of Ψ will exist. Moreover, Ψ will be differentiable almost everywhere on (p, q). The right and left derivative of a convex function is increasing and for each point in the domain of the convex function, and there must be at least one support line. The epigraph of a convex function is always a convex set. Due to those useful properties, several mathematicians generalized and extended the convex function and gave numerous results for the generalized convex functions. Many results associated to Hermite–Hadamard inequality for the n-polynomial P-convexity can be found in [6]. In the following part of this section, we focus on some of the generalized families of convex functions. In [7],

Dragomir introduced the idea of convexity of the functions on the coordinate and introduced the following family of generalized convexity of the functions defined on the rectangular domain.

Let $\Psi : [p,q] \times [r,s] \to \mathbb{R}$ be a function and the functions $\Psi_y : [p,q] \to \mathbb{R}, \Psi_x : [r,s] \to \mathbb{R}$ be the partial functions defined as: $\Psi_y(u) = \Psi(u,y)$ and $\Psi_x(v) = \Psi(x,v)$ for all $x \in [p,q]$ and $y \in [r,s]$. Then Ψ is called coordinate convex on $[p,q] \times [r,s]$, if Ψ_y and Ψ_x both are convex on $[p,q]$ and $[r,s]$ respectively. The important feature this class of functions is that a function which is convex must be coordinate convex and with the help of the example the author explained that every coordinate convex function may not be convex. The Hermite–Hadamard (H-H) and Jensen's type inequalities are presented for this family of functions [7,8].

Gordji et al. [9] gave the idea of η-convexity. This generalized convex function is given below (also see [10]): a function $\Psi : [p,q] \to \mathbb{R}$ is known as η-convex with respect to the function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$\Psi((1-t)y + tx) \leq \Psi(y) + t\eta(\Psi(x), \Psi(y))$$

holds for all $t \in [0,1]$ and $x, y \in [p,q]$.

It is obvious that if Ψ is η-convex associated to the η defined by $\eta(x_1, x_2) = x_1 - x_2$, then Ψ will be convex. The authors elaborated with the help of examples that there are nonconvex functions which are η-convex functions. Various related results to convexity are also presented for this class of functions, such as boundedness, support line inequality, Hermite–Hadamard, determinantal, and Jensen's type inequalities. Using the idea of coordinate convex function, Zaheer Ullah et al. further generalized the family of η-convex functions and gave the idea of coordinate (η_1, η_2)-convex functions as follows ([11]):

A function $\Psi : [p,q] \times [r,s] \to \mathbb{R}$ is coordinate (η_1, η_2)-convex if the functions $\Psi_y : [p,q] \to \mathbb{R}$ defined by $\Psi_y(w) = \Psi(w,y)$ and $\Psi_x : [r,s] \to \mathbb{R}$ defined by $\Psi_x(z) = \Psi(x,z)$ are η_1-convex and η_2-convex for each $x \in [p,q]$ and $y \in [r,s]$, respectively. Particularly, if $\eta_2 = \eta_1 = \eta$, then Ψ is said to be coordinate η-convex function.

The authors proved that if Ψ is η-convex function defined on a rectangular domain, then Ψ will be coordinate η-convex but the converse is not valid in general. Some interesting examples are presented and Hermite–Hadamard inequality is proved for this class of functions. One of the most widely applicable and dominant families of functions is the family of strong convexity. This family of functions was originally introduced and studied by Polyak in 1966 [12]. The strongly convex function is defined as follows:

If $\Psi : [p,q] \to \mathbb{R}$ is a function such that the inequality

$$\Psi(\gamma x_1 + (1-\gamma)x_2) \leq \gamma\Psi(x_1) + (1-\gamma)\Psi(x_2) - c\gamma(1-\gamma)(x_1 - x_2)^2$$

is valid for all $\gamma \in [0,1]$ and $x_1, x_2 \in [p,q]$ and for some $c > 0$, then Ψ is called a strong convex function.

The simple and easy criteria for checking the strong convexity of a function are: A differentiable function Ψ is a strongly convex function if and only if

$$\Psi(x_1) - \Psi(x_2) \geq \Psi'(x_1)(x_1 - x_2) + c(x_1 - x_2)^2.$$

Moreover, if a function Ψ is twice differentiable, then Ψ is strongly convex associated to c if and only if

$$\Psi'' \geq 2c$$

It is remarkable to note that each strongly convex function is convex while it is not valid in general that every convex function will be strongly convex. Actually, the idea of strong convexity provides a strengthened form of the classical convexity. There are numerous applications of strong convexity in optimization theory. As Dragomir generalized the concept of convex function to the concept of coordinate convex function, in a similar fashion

Adil Khan et al. presented a novel class of coordinate strongly convex functions ([13]), which is stated below.

A function $\Psi : [p,q] \times [r,s] \to \mathbb{R}$ is coordinate strongly convex if $\Psi_y : [p,q] \to \mathbb{R}$ and $\Psi_x : [r,s] \to \mathbb{R}$ are strongly convex for each $x \in [p,q]$ and $y \in [r,s]$, where Ψ_y and Ψ_x are defined as above.

In [13] the authors noted that strong convexity implies coordinate strong convexity while the converse is not generally true. For more related interesting results we recommend [13]. Before finishing this part, we recall the strong η-convexity definition which has been presented in [14].

A function $\Psi : [p,q] \to \mathbb{R}$ is strongly η-convex associated to $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c > 0$, if

$$\Psi(\gamma x + (1-\gamma)y) \leq \Psi(y) + \gamma\eta(\Psi(x), \Psi(y)) - c\gamma(1-\gamma)(x-y)^2$$

holds for all $\gamma \in [0,1]$ and $x, y \in [p,q]$. The Hermite–Hadamard inequality for this class of functions is given as follows:

$$\begin{aligned} &\Psi\left(\frac{p+q}{2}\right) - \frac{\Gamma_\eta}{2} + \frac{c}{12}(q-p)^2 \\ &\leq \frac{1}{q-p}\int_p^q \Psi(t)\,dt \\ &\leq \frac{\Psi(p)+\Psi(q)}{2} + \frac{\eta(\Psi(p),\Psi(q)) + \eta(\Psi(q),\Psi(p))}{4} - \frac{c}{6}(q-p)^2 \\ &\leq \frac{\Psi(p)+\Psi(q)}{2} - \frac{\Gamma_\eta}{2} - \frac{c}{6}(q-p)^2. \end{aligned} \tag{1}$$

In a similar fashion, a strongly η-convex function defined on a rectangular domain is given below: Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c > 0$. Then the function $\Psi : [p,q] \times [r,s] \to \mathbb{R}$ is said to be strongly η-convex with respect to c, if

$$\begin{aligned} &\Psi(\gamma a_1 + (1-\gamma)a_2, \gamma a_3 + (1-\gamma)a_4) \\ &\quad \leq \Psi(a_2, a_4) + \gamma\eta(\Psi(a_1,a_3), \Psi(a_2,a_4)) - c\gamma(1-\gamma)\|(a_1,a_3) - (a_2,a_4)\|^2 \end{aligned} \tag{2}$$

holds for all $\gamma \in [0,1]$ and $(a_2,a_4), (a_1,a_3) \in [p,q] \times [r,s]$.

In [14], the authors investigated Fejér as well as as Hermite–Hadamard type inequalities for n-times differentiable strongly η-convex functions. The strong η-convexity of the nth derivative of certain functions has been utilized and has derived some interesting inequalities [15]. Moreover, further generalization of this class of functions has been presented in [16], which is known as a strongly (η,ω)-convex function. Some useful examples of strongly (η,ω)-convex functions have been constructed. Furthermore, they presented Hermite-Hadamard type inequalities for this class of generalized convex functions.

2. Coordinated Strongly (η_1,η_2)-Convex Functions

This section is devoted to proposing a novel important class of functions which is named coordinated strongly (η_1,η_2)-convex function.

Definition 1. *Let $[d_1,d_2]$,$[d_3,d_4] \subseteq \mathbb{R}$ be two intervals and $\Upsilon : [d_1,d_2] \times [d_3,d_4] \to \mathbb{R}$, η_1,$\eta_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be three functions, and the functions $\Upsilon_y : [d_1,d_2] \to \mathbb{R}$ and $\Upsilon_x : [d_3,d_4] \to \mathbb{R}$ be defined by*

$$\Upsilon_y(u) = \Upsilon(u,y)\ ,\ \Upsilon_x(v) = \Upsilon(x,v).$$

Then Υ is known as the coordinated strongly (η_1,η_2)-convex on $[d_1,d_2] \times [d_3,d_4]$ if Υ_y and Υ_x are strongly η_1-convex and strongly η_2-convex, respectively. In particular, if $\eta_2 = \eta_1 = \eta$, then Υ is said to be a coordinated strongly η-convex function.

Example 1. *Let $\Upsilon : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be defined by $\Upsilon(x,y) = x^2 + y^2$ and $\eta(x,y) = y + 2x$. Then Υ is a coordinated strongly η-convex on $[0,\infty)^2$.*

Proof. Consider the function $\mathrm{Y}_y : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ defined by $\mathrm{Y}_y(u) = \mathrm{Y}(u, y)$ and $\mathrm{Y}_x : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ defined by $\mathrm{Y}_x(v) = \mathrm{Y}(x, v)$. Then for any $\chi_1, \chi_2 \in \mathbb{R}^+ \cup \{0\}$ and $0 \leq \nu \leq 1$, we have

$$\begin{aligned}
\mathrm{Y}_y(\nu\chi_1 + (1-\nu)\chi_2) &= \mathrm{Y}(\nu\chi_1 + (1-\nu)\chi_2, y) \\
&= (\nu\chi_1 + (1-\nu)\chi_2)^2 + y^2 \\
&= \chi_2^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 - 2\nu\chi_2^2 + \nu^2\chi_2^2 + y^2 \\
&\leq \chi_2^2 + y^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 + \nu\chi_1^2 + \nu^2\chi_2^2 \\
&= \chi_2^2 + y^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 + 2\nu\chi_1^2 - \nu\chi_1^2 + \nu^2\chi_2^2 \\
&= \chi_2^2 + y^2 + 2\nu\chi_1^2 + \nu\chi_2^2 - \nu\chi_1^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 - \nu\chi_2^2 + \nu^2\chi_2^2 \\
&= \mathrm{Y}_y(\chi_2) + \nu(2\chi_1^2 + \chi_2^2) - \nu(1-\nu)(\chi_1^2 - 2\chi_1\chi_2 + \chi_2^2) \\
&\leq \mathrm{Y}_y(\chi_2) + \nu\left\{2(\chi_1^2 + y^2) + (\chi_2^2 + y^2)\right\} - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= \mathrm{Y}_y(\chi_2) + \nu(2\mathrm{Y}_y(\chi_1) + \mathrm{Y}_y(\chi_2)) - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= \mathrm{Y}_y(\chi_2) + \nu\eta(\mathrm{Y}_y(\chi_1), \mathrm{Y}_y(\chi_2)) - \nu(1-\nu)(\chi_1 - \chi_2)^2.
\end{aligned}$$

This represents that Y_y is strongly η-convex on $[0, \infty)$. In a similar fashion, we can show that Y_x is strongly η-convex on $[0, \infty)$.

Therefore, Y is coordinated strongly η-convex on $[0, \infty) \times [0, \infty)$. □

Example 2. *Let* $\mathrm{Y} : [1,2] \times [1,2] \to \mathbb{R}$ *be defined by* $\mathrm{Y}(x, y) = y^3 + x^2$, $\eta_2(x, y) = 10x + 9y$ *and* $\eta_1(x, y) = 2x + y$. *Then* Y *is coordinated strongly* (η_1,η_2)*-convex.*

Proof. The function $\mathrm{Y}_y : [1,2] \to \mathbb{R}$ defined by $\mathrm{Y}_y(u) = y^3 + u^2$ and $\mathrm{Y}_x : [1,2] \to \mathbb{R}$ defined by $\mathrm{Y}_x(v) = v^3 + x^2$. Then, by the technique as given in the solution of Example 1, we can derive that Y_y is strongly η_1-convex.

Now, we prove that Y_x is strongly η_2-convex. For this

$$\begin{aligned}
\mathrm{Y}_x(\nu\chi_1 + (1-\nu)\chi_2) &= \mathrm{Y}(x, \nu\chi_1 + (1-\nu)\chi_2) \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2)^3 \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2)(\nu\chi_1 + (1-\nu)\chi_2)^2 \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 - 2\nu\chi_2^2 + \nu^2\chi_2^2) \\
&\leq x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 + \nu\chi_1^2 + \nu^2\chi_2^2) \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 + 2\nu\chi_1^2 - \nu\chi_1^2 + \nu^2\chi_2^2) \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 + 2\nu\chi_1^2 - \nu\chi_1^2 + \nu\chi_2^2 - \nu\chi_2^2 + \nu^2\chi_2^2) \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + 2\nu\chi_1^2 + \nu\chi_2^2 - \nu\chi_1^2 + \nu^2\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 - \nu\chi_2^2 + \nu^2\chi_2^2) \\
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad (\chi_2^2 + \nu(2\chi_1^2 + \chi_2^2) - (\nu - \nu^2)\chi_1^2 + 2\nu(1-\nu)\chi_1\chi_2 - (\nu - \nu^2)\chi_2^2)
\end{aligned}$$

$$
\begin{aligned}
&= x^2 + (\nu\chi_1 + (1-\nu)\chi_2)(\chi_2^2 + \nu(2\chi_1^2 + \chi_2^2) - \nu(1-\nu)(\chi_1 - \chi_2)^2) \\
&\leq x^2 + (\nu\chi_1 + \chi_2)(\chi_2^2 + \nu(2\chi_1^2 + \chi_2^2) - \nu(1-\nu)(\chi_1 - \chi_2)^2) \\
&= x^2 + \nu\chi_1\chi_2^2 + \nu^2(2\chi_1^3 + \chi_1\chi_2^2) + \chi_2^3 + \nu(2\chi_1^2\chi_2 + \chi_2^3) \\
&\quad - \nu(1-\nu)(\chi_1 - \chi_2)^2(\nu\chi_1 + \chi_2).
\end{aligned}
$$

Noting that $\chi_2^2 \leq 4\chi_1^2$, $\chi_1^2 \leq 4\chi_2^2$ and $\nu\chi_1 + \chi_2 \geq 1$, we have

$$
\begin{aligned}
\mathrm{Y}_x(\nu\chi_1 + (1-\nu)\chi_2) &\leq x^2 + 4\nu\chi_1^3 + \nu^2(2\chi_1^3 + 4\chi_1^3) + \chi_2^3 + \nu(8\chi_2^3 + \chi_2^3) \\
&\quad - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&\leq x^2 + 4\nu\chi_1^3 + \nu(6\chi_1^3) + \chi_2^3 + \nu(9\chi_2^3) - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= x^2 + \chi_2^3 + \nu(10\chi_1^3 + 9\chi_2^3) - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&\leq x^2 + \chi_2^3 + \nu(10(x^2 + \chi_1^3) + 9(x^2 + \chi_2^3)) - \nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= \mathrm{Y}_x(\chi_2) + \nu\eta(\mathrm{Y}_x(\chi_1), \mathrm{Y}_x(\chi_2)) - \nu(1-\nu)(\chi_1 - \chi_2)^2,
\end{aligned}
$$

which shows that Y_x is strongly η_2-convex on $[1,2]$. Consequently, Y is coordinated strongly (η_1, η_2)-convex.

The next result gives an important relation among the coordinated strongly η-convex and strongly η-convex functions. □

Lemma 1. *Every strongly η-convex function defined on rectangular domain is coordinated strongly η-convex, while the converse is not generally true.*

Proof. Let Y be strongly η-convex on $[d_1, d_2] \times [d_3, d_4]$ and consider the partial mappings $\mathrm{Y}_y : [d_1, d_2] \to \mathbb{R}$ defined by $\mathrm{Y}_y(u) = \mathrm{Y}(u, y)$ and $\mathrm{Y}_x : [d_3, d_4] \to \mathbb{R}$ defined by $\mathrm{Y}_x(v) = \mathrm{Y}(x, v)$. For any $\chi_1, \chi_2 \in [d_1, d_2]$ and $\nu \in [0,1]$, we have

$$
\begin{aligned}
\mathrm{Y}_y(\nu\chi_1 + (1-\nu)\chi_2) &= \mathrm{Y}(\nu\chi_1 + (1-\nu)\chi_2, y) \\
&= \mathrm{Y}(\nu\chi_1 + (1-\nu)\chi_2, \nu y + (1-\nu)y)) \\
&\leq \mathrm{Y}(\chi_2, y) + \nu\eta(\mathrm{Y}(\chi_1, y), \mathrm{Y}(\chi_2, y)) - c\nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= \mathrm{Y}_y(\chi_2) + \nu\eta(\mathrm{Y}_y(\chi_1), \mathrm{Y}_y(\chi_2)) - c\nu(1-\nu)(\chi_1 - \chi_2)^2.
\end{aligned}
$$

The above inequality holds by using (2).

This concludes that Y_y is a strongly η-convex function. Similarly, we can show that Y_x is strongly η-convex. Hence, Y is a coordinated strongly η-convex function. □

For the converse of the above result, we describe the following example for an illustration.

Example 3. *Let* $\mathrm{Y} : [1,5] \times [1,5] \to \mathbb{R}$ *be defined by* $\mathrm{Y}(x, y) = x^2y^2$ *and* $\eta(x, y) = 2x + y$. *Then,* Y *is a coordinated strongly η-convex but not strongly η-convex function.*

Proof. Consider $\mathrm{Y}_y : [1,5] \to \mathbb{R}$ is defined by $\mathrm{Y}_y(u) = \mathrm{Y}(u, y)$ and $\mathrm{Y}_x : [1,5] \to \mathbb{R}$ is defined by $\mathrm{Y}_x(v) = \mathrm{Y}(x, v)$ be partial mapping of Y. For any $\chi_1, \chi_2 \in [1,5]$ and $\nu \in [0,1]$ we get

$$\begin{aligned}
\mathrm{Y}_y(\nu\chi_1+(1-\nu)\chi_2) &= \mathrm{Y}(\nu\chi_1+(1-\nu)\chi_2, y)\\
&= (\nu\chi_1+(1-\nu)\chi_2)^2y^2\\
&= (\chi_2^2+\nu^2\chi_1^2+2\nu(1-\nu)\chi_1\chi_2-2\nu\chi_2^2+\nu^2\chi_2^2)y^2\\
&\leq (\chi_2^2+\nu^2\chi_1^2+2\nu(1-\nu)\chi_1\chi_2+\nu\chi_1^2+\nu^2\chi_2^2)y^2\\
&= (\chi_2^2+\nu^2\chi_1^2+2\nu(1-\nu)\chi_1\chi_2+2\nu\chi_1^2-\nu\chi_1^2+\nu^2\chi_2^2)y^2\\
&= (\chi_2^2+2\nu\chi_1^2+\nu\chi_2^2-\nu\chi_1^2+\nu^2\chi_1^2+2\nu(1-\nu)\chi_1\chi_2-\nu\chi_2^2+\nu^2\chi_2^2)y^2\\
&= (\chi_2^2+\nu(2\chi_1^2+\chi_2^2)-\nu(1-\nu)(\chi_1-\chi_2)^2)y^2\\
&= \chi_2^2y^2+\nu(2\chi_1^2y^2+\chi_2^2y^2)-\nu(1-\nu)(\chi_1-\chi_2)^2y^2\\
&\leq \mathrm{Y}_y(\chi_2)+\nu\eta(\mathrm{Y}_y(\chi_1),\mathrm{Y}_y(\chi_2))-\nu(1-\nu)(\chi_1-\chi_2)^2.
\end{aligned}$$

This shows that Y_y is strongly η-convex with modulus 1. In a similar fashion, we may prove that Y_x is strongly η-convex on $[1,5]$ with modulus 1. Hence, Y is coordinated strongly η-convex on $[1,5]\times[1,5]$.

Next, suppose that Y is a strongly η-convex function. Then by (2), we have

$$\begin{aligned}
&\mathrm{Y}(\nu(\chi_1,\chi_2)+(1-\nu)(\chi_3,\chi_4))\\
&\quad\leq \mathrm{Y}(\chi_3,\chi_4)+\nu\eta(\mathrm{Y}(\chi_1,\chi_2),\mathrm{Y}(\chi_3,\chi_4))-c\nu(1-\nu)\|(\chi_1,\chi_2)-(\chi_3,\chi_4)\|^2\\
&\quad\leq \mathrm{Y}(\chi_3,\chi_4)+\nu(2\mathrm{Y}(\chi_1,\chi_2)+\mathrm{Y}(\chi_3,\chi_4))-c\nu(1-\nu)\|(\chi_1,\chi_2)-(\chi_3,\chi_4)\|^2\\
&\quad\leq \mathrm{Y}(\chi_3,\chi_4)+2\nu\mathrm{Y}(\chi_1,\chi_2)+\nu\mathrm{Y}(\chi_3,\chi_4)-c\nu(1-\nu)\|(\chi_1,\chi_2)-(\chi_3,\chi_4)\|^2.
\end{aligned}$$

Substituting $(\chi_1,\chi_2)=(5,1)$, $(\chi_3,\chi_4)=(1,5)$ and $\nu=\frac{1}{2}$, we deduce

$$\begin{aligned}
&\mathrm{Y}\left(\left(\frac{5}{2},\frac{1}{2}\right)+\left(\frac{1}{2},\frac{5}{2}\right)\right)\leq \mathrm{Y}(1,5)+\frac{1}{2}\times 2\times 25+\frac{1}{2}\times 25-\frac{c}{4}\|(5,1)-(1,5)\|^2\\
&\implies \mathrm{Y}\left(\frac{5}{2}+\frac{1}{2},\frac{1}{2}+\frac{5}{2}\right)\leq 25+25+12.5-\frac{c}{4}\|(4,-4)\|^2\\
&\qquad\implies 81\leq 62.5-\frac{c}{4}\|(4,-4)\|^2\\
&\qquad\implies 324\leq 250-c\|(4,-4)\|^2,
\end{aligned}$$

which is not true for all $c>0$. Thus, the condition of strongly η-convexity fails and hence Y is not a strongly η-convex function.

The H-H type inequality for coordinated strongly (η_1,η_2)-convex function is given in the following theorem. □

Theorem 1. *Let* $\mathrm{Y}:[d_1,d_2]\times[d_3,d_4]\to\mathbb{R}$ *be coordinated strongly* (η_1,η_2)*-convex with respect to* ϵ_1 *and* ϵ_2 *such that*

$$\eta_1(x,y)\leq\Gamma_{\eta_1},\qquad \eta_2(x,y)\leq\Gamma_{\eta_2}\ \textit{for}\ x,y\in\mathbb{R},$$

where Γ_{η_1} *and* Γ_{η_2} *are two positive real numbers. Then*

$$
\begin{aligned}
&\Upsilon\left(\frac{d_1+d_2}{2},\frac{d_3+d_4}{2}\right)-\frac{\Gamma_{\eta_1}+\Gamma_{\eta_2}}{2}+\frac{\epsilon_1}{12}(d_2-d_1)^2+\frac{\epsilon_2}{12}(d_4-d_3)^2\\
&\le\frac{1}{2}\left\{\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon\left(x,\frac{d_3+d_4}{2}\right)dx+\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon\left(\frac{d_1+d_2}{2},y\right)dy\right\}-\frac{\Gamma_{\eta_1}+\Gamma_{\eta_2}}{4}\\
&\quad+\frac{\epsilon_1}{24}(d_2-d_1)^2+\frac{\epsilon_2}{24}(d_4-d_3)^2\\
&\le\frac{1}{(d_4-d_3)(d_2-d_1)}\int_{d_1}^{d_2}\int_{d_3}^{d_4}\Upsilon(x,y)dxdy\\
&\le\frac{1}{4}\left\{\frac{1}{d_2-d_1}\int_{d_1}^{d_2}[\Upsilon(x,d_3)+\Upsilon(x,d_4)]dx+\frac{1}{d_4-d_3}\int_{d_3}^{d_4}[\Upsilon(d_1,y)+\Upsilon(d_2,y)]dy\right\}\\
&+\frac{\Gamma_{\eta_1}+\Gamma_{\eta_2}}{4}-\frac{\epsilon_1}{12}(d_2-d_1)^2-\frac{\epsilon_2}{12}(d_4-d_3)^2\\
&\le\frac{1}{4}\{\Upsilon(d_1,d_3)+\Upsilon(d_2,d_3)+\Upsilon(d_1,d_4)+\Upsilon(d_2,d_4)\}+\frac{\Gamma_{\eta_1}+\Gamma_{\eta_2}}{2}-\frac{\epsilon_1}{6}(d_2-d_1)^2-\frac{\epsilon_2}{6}(d_4-d_3)^2.
\end{aligned}
\tag{3}
$$

Proof. If Υ_x and Υ_y are two partial mappings of Υ, then by H-H inequality for the class of strongly η-convex functions given in (1), we may write

$$
\begin{aligned}
\Upsilon\left(x,\frac{d_3+d_4}{2}\right)-\frac{\Gamma_{\eta_2}}{2}+\frac{\epsilon_2}{12}(d_4-d_3)^2&\le\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon(x,y)dy\\
&\le\frac{\Upsilon(x,d_3)+\Upsilon(x,d_4)}{2}+\frac{\Gamma_{\eta_2}}{2}-\frac{\epsilon_2}{6}(d_4-d_3)^2.
\end{aligned}
\tag{4}
$$

Integrating (4) with respect to x on $[d_1,d_2]$, we get

$$
\begin{aligned}
&\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon\left(x,\frac{d_3+d_4}{2}\right)dx-\frac{\Gamma_{\eta_2}}{2}+\frac{\epsilon_2}{12}(d_4-d_3)^2\\
&\le\frac{1}{(d_2-d_1)(d_4-d_3)}\int_{d_1}^{d_2}\int_{d_3}^{d_4}\Upsilon(x,y)dxdy\\
&\le\frac{1}{2}\left\{\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon(x,d_3)dx+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon(x,d_4)dx\right\}+\frac{\Gamma_{\eta_2}}{2}-\frac{\epsilon_2}{6}(d_4-d_3)^2.
\end{aligned}
\tag{5}
$$

Similarly, for the partial mapping Υ_y we deduce

$$
\begin{aligned}
&\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon\left(\frac{d_1+d_2}{2},y\right)dy-\frac{\Gamma_{\eta_1}}{2}+\frac{\epsilon_1}{12}(d_2-d_3)^2\\
&\le\frac{1}{(d_2-d_1)(d_4-d_3)}\int_{d_1}^{d_2}\int_{d_3}^{d_4}\Upsilon(x,y)dxdy\\
&\le\frac{1}{2}\left\{\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon(d_1,y)dy+\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon(d_2,y)dy\right\}+\frac{\Gamma_{\eta_1}}{2}-\frac{\epsilon_1}{6}(d_2-d_1)^2.
\end{aligned}
\tag{6}
$$

Adding (5) and (6), we achieve the second and the third inequalities in (3).
Now, again by Hadamard inequality, we get

$$
\Upsilon\left(\frac{d_1+d_2}{2},\frac{d_3+d_4}{2}\right)-\frac{\Gamma_{\eta_1}}{2}+\frac{\epsilon_1}{12}(d_2-d_1)^2\le\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon\left(x,\frac{d_3+d_4}{2}\right)dx;\tag{7}
$$

$$
\Upsilon\left(\frac{d_1+d_2}{2},\frac{d_3+d_4}{2}\right)-\frac{\Gamma_{\eta_2}}{2}+\frac{\epsilon_2}{12}(d_4-d_3)^2\le\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon\left(\frac{d_1+d_2}{2},y\right)dy.\tag{8}
$$

Adding $-\frac{\Gamma_{\eta_2}}{2}+\frac{\epsilon_2}{12}(d_4-d_3)^2$ and $-\frac{\Gamma_{\eta_1}}{2}+\frac{\epsilon_1}{12}(d_2-d_1)^2$ to both sides of (7) and (8), respectively, and then summing the obtained inequalities, we derive the first inequality of (3).

Next, by using Hadamard inequality again, we get

$$\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon(x,d_3)dx \leq \frac{\Upsilon(d_1,d_3)+\Upsilon(d_2,d_3)}{2}+\frac{\Gamma_{\eta_1}}{2}-\frac{\epsilon_1}{6}(d_2-d_1)^2;$$
$$\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Upsilon(x,d_4)dx \leq \frac{\Upsilon(d_1,d_4)+\Upsilon(d_2,d_4)}{2}+\frac{\Gamma_{\eta_1}}{2}-\frac{\epsilon_1}{6}(d_2-d_1)^2;$$
$$\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon(d_1,y)dy \leq \frac{\Upsilon(d-1,d_3)+\Upsilon(d_1,d_4)}{2}+\frac{\Gamma_{\eta_2}}{2}-\frac{\epsilon_2}{6}(d_4-d_3)^2;$$
$$\frac{1}{d_4-d_3}\int_{d_3}^{d_4}\Upsilon(d_2,y)dy \leq \frac{\Upsilon(d_2,d_3)+\Upsilon(d_2,d_4)}{2}+\frac{\Gamma_{\eta_2}}{2}-\frac{\epsilon_2}{6}(d_4-d_3)^2.$$

Summing all the above inequalities and then adding $\Gamma_{\eta_1}-\frac{\epsilon_1}{3}(d_2-d_3)^2$ and $\Gamma_{\eta_2}-\frac{\epsilon_2}{3}(s_4-d_3)^2$ to the both hand sides, we achieve the last required inequality. □

3. Results Arround Strongly η-Convex Functions

In the below mentioned lemma, it is presented that the convex combination of strongly η-convex functions is a strongly η-convex function.

Lemma 2. *Let $\zeta_j : [d_1,d_2] \to \mathbb{R}$ be strongly η-convex functions with modulus $c>0$ and η be non-negative homogeneous functions. If $\xi_{i's} \in \mathbb{R}^+ \cup \{0\}$ such that $\sum_{j=1}^n \xi_j = 1$, then the function $\Upsilon = \sum_{j=1}^n \xi_j\zeta_j$ is also a strongly η-convex function.*

Proof. Let

$$\Upsilon(\kappa) = \sum_{j=1}^{n}\xi_j\zeta_j(\kappa).$$

Set $\kappa = \nu\chi_1+(1-\nu)\chi_2$ with $\chi_1,\chi_2 \in [d_1,d_2]$. Then, we have:

$$\begin{aligned}
\Upsilon(\nu\chi_1+(1-\nu)\chi_2) &= \sum_{j=1}^{n}\xi_j\zeta_j(\nu\chi_1+(1-\nu)\chi_2)\\
&\leq \sum_{j=1}^{n}\xi_j\Big\{\zeta_j(\chi_2)+\nu\eta(\zeta_j(\chi_1),\zeta_j(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2\Big\}\\
&= \sum_{j=1}^{n}\xi_j\zeta_j(\chi_2)+\nu\sum_{j=1}^{n}\xi_j\eta(\zeta_j(\chi_1),\zeta_j(\chi_2))\\
&\quad -c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \sum_{j=1}^{n}\xi_j\zeta_j(\chi_2)+\nu\eta\left(\sum_{j=1}^{n}\xi_j\zeta_j(\chi_1),\sum_{j=1}^{n}\xi_j\zeta_j(\chi_2)\right)\\
&\quad -c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \Upsilon(\chi_2)+\nu\eta(\Upsilon(\chi_1),\Upsilon(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2.
\end{aligned}$$

It is clear from this that Υ is a strongly η-convex function. □

Lemma 3. *Let $\Upsilon = T_1+T_2$, where T_1 is a strongly η-convex function and T_2 is η-convex function such that η is additive. Then, prove that Υ is a strongly η-convex function.*

Proof. Consider

$$\begin{aligned}
\Upsilon(\nu\chi_1 + (1-\nu)\chi_2) &= T_1(\nu\chi_1 + (1-\nu)\chi_2) + T_2(\nu\chi_1 + (1-\nu)\chi_2) \\
&\leq T_1(\chi_2) + \nu\eta(T_1(\chi_1), T_1(\chi_2)) - c\nu(1-\nu)(\chi_1 - \chi_2)^2 + T_2(\chi_2) \\
&\quad + \nu\eta(T_2(\chi_1), T_2(\chi_2)) \\
&= T_1(\chi_2) + T_2(\chi_2) + \nu\eta(T_1(\chi_1) + T_2(\chi_1), T_1(\chi_2) + T_2(\chi_2)) \\
&\quad - c\nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= (T_1 + T_2)(\chi_2) + \nu\eta((T_1 + T_2)(\chi_1), (T_1 + T_2)(\chi_2)) \\
&\quad - c\nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&= \Upsilon(\chi_2) + \nu\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) - c\nu(1-\nu)(\chi_1 - \chi_2)^2.
\end{aligned}$$

This gives the strong η-convexity of Υ. □

Lemma 4. *Let* Υ *be a strongly* η*-convex function and* $\eta(\chi_1, \chi_2) = -\eta(\chi_2, \chi_1)$*. Then*

$$\Upsilon(y_1 + y_2 - \chi) \leq \Upsilon(y_1) + \Upsilon(y_2) - \Psi(\chi); \quad \textit{for all} \quad \chi \in [y_1, y_2]$$

with $\chi = \nu y_1 + (1-\nu)y_2$ *and* $\nu \in [0,1]$.

Proof. Consider Υ is a strongly η-convex function, then

$$\begin{aligned}
\Upsilon(y_1 + y_2 - \chi) &= \Upsilon(y_1 + y_2 - (\nu y_1 + (1-\nu)y_2)) \\
&= \Upsilon(\nu y_2 + (1-\nu)y_1)) \\
&\leq \Upsilon(y_1) + \nu\eta(\Upsilon(y_2), \Upsilon(y_1)) - c\nu(1-\nu)(y_2 - y_1)^2 \\
&\leq \Upsilon(y_1) + \nu\eta(\Upsilon(y_2), \Upsilon(y_1)) + c\nu(1-\nu)(y_2 - y_1)^2 \\
&= \Upsilon(y_1) - \nu\eta(\Upsilon(y_1), \Upsilon(y_2)) + c\nu(1-\nu)(y_1 - y_2)^2 \\
&= \Upsilon(y_1) + \Upsilon(y_2) - \Upsilon(y_2) - \nu\eta(\Upsilon(y_1), \Upsilon(y_2)) \\
&\quad + c\nu(1-\nu)(y_1 - y_2)^2 \\
&= \Upsilon(y_1) + \Upsilon(y_2) \\
&\quad - \{\Upsilon(y_2) + \nu\eta(\Upsilon(y_1), \Upsilon(y_2)) - c\nu(1-\nu)(y_1 - y_2)^2\} \\
&\leq \Upsilon(y_1) + \Upsilon(y_2) - \Upsilon(\chi).
\end{aligned}$$

This proves the required inequality. □

Theorem 2. *Consider the nonempty set* $\{\zeta_j : [d_1, d_2] \to \mathbb{R}, j \in J\}$ *of strongly* η*-convex functions such that*

(a). *There exist* $\omega_1 \in [0, \infty]$ *and* $\omega_2 \in [-1, \infty]$ *with* $\eta(\chi_1, \chi_2) = \omega_1\chi_1 + \omega_2\chi_2$ *for all* $\chi_1, \chi_2 \in [d_1, d_2]$.

(b). *For each* $\chi \in [d_1, d_2]$, $\sup_{j \in J} \zeta_j(\chi)$ *exists in* $\mathbb{R}$.

Then, the function $\Upsilon : [d_1, d_2] \to \mathbb{R}$ *defined by* $\Upsilon(\chi) = \sup_{j \in J} \zeta_j(\chi)$ *for each* $\chi \in [d_1, d_2]$*, is a strongly* η*-convex function.*

Proof. For any $\chi_1, \chi_2 \in [d_1, d_2]$ and $\nu \in [0,1]$, we have

$$\begin{aligned}
\Upsilon(\nu\chi_1+(1-\nu)\chi_2) &= \sup_{j\in J}\zeta_j(\nu\chi_1+(1-\nu)\chi_2)\\
&\leq \sup_{j\in J}\Big\{\zeta_j(\chi_2)+\nu\eta(\zeta_j(\chi_1),\zeta_j(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2\Big\}\\
&\leq \sup_{j\in J}\{\zeta_j(\chi_2)+\nu\eta(\zeta_j(\chi_1),\zeta_j(\chi_2))\}-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \sup_{j\in J}\{\zeta_j(\chi_2)+\nu(\omega_1\zeta_j(\chi_1)+\omega_2\zeta_j(\chi_2))\}-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \sup_{j\in J}\{(1+\omega_2\nu)\zeta_j(\chi_2)+\nu\omega_1\zeta_j(\chi_1)\}-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&\leq (1+\nu\omega_2)\sup_{j\in J}\zeta_j(\chi_2)+\nu\omega_1\sup_{j\in J}\zeta_j(\chi_1)-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= (1+\nu\omega_2)\Upsilon(\chi_2)+\nu\omega_1\Upsilon(\chi_1)-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \Upsilon(\chi_2)+\nu(\omega_1\Upsilon(\chi_1)+\omega_2\Upsilon(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2\\
&= \Upsilon(\chi_2)+\nu\eta(\Upsilon(\chi_1),\Upsilon(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2.
\end{aligned}$$

This gives that Υ is a strongly η-convex function. □

Theorem 3. (Schur-type inequality). *Suppose that* $\Upsilon:[d_1,d_2]\to\mathbb{R}$ *is a strongly* η*-convex function. Then for* $\kappa_1,\kappa_2,\kappa_3\in[d_1,d_2]$ *such that* $\kappa_1<\kappa_2<\kappa_3$ *and* $\kappa_3-\kappa_1,\kappa_3-\kappa_2,\kappa_2-\kappa_1\in[d_1,d_2]$*, we have*

$$\begin{aligned}
&(\kappa_3-\kappa_1)\Upsilon(\kappa_2)\\
&\quad\leq(\kappa_3-\kappa_1)\Upsilon(\kappa_3)+(\kappa_3-\kappa_2)\eta(\Upsilon(\kappa_1),\Upsilon(\kappa_2))-c(\kappa_3-\kappa_2)(\kappa_2-\kappa_1)(\kappa_3-\kappa_1)
\end{aligned}\tag{9}$$

if and only if Υ *is a strongly* η*-convex function.*

Proof. With the help of strong η-convexity, we may write that

$$\Upsilon(\nu\chi_1+(1-\nu)\chi_2)\leq\Upsilon(\chi_2)+\nu\eta(\Upsilon(\chi_1),\Upsilon(\chi_2))-c\nu(1-\nu)(\chi_1-\chi_2)^2.\tag{10}$$

By setting $\nu=\frac{\kappa_2-\kappa_3}{\kappa_1-\kappa_3}$, $\chi_1=\kappa_1$ and $\chi_2=\kappa_3$ in (10), we have

$$\begin{aligned}
&\Upsilon\left\{\left(\frac{\kappa_2-\kappa_3}{\kappa_1-\kappa_3}\right)\kappa_1+\left(1-\frac{\kappa_3-\kappa_2}{\kappa_3-\kappa_1}\right)\kappa_3\right\}\\
&\quad\leq\Upsilon(\kappa_3)+\left(\frac{\kappa_2-\kappa_3}{\kappa_1-\kappa_3}\right)\eta(\Upsilon(\kappa_1),\Upsilon(\kappa_3))-c\left(\frac{v_3-\kappa_2}{\kappa_3-\kappa_1}\right)\left(1-\frac{\kappa_3-\kappa_2}{\kappa_3-\kappa_1}\right)(\kappa_1-\kappa_3)^2\\
&\quad=\Upsilon(\kappa_3)+\left(\frac{\kappa_3-\kappa_2}{\kappa_3-\kappa_1}\right)\eta(\Upsilon(\kappa_1),\Upsilon(\kappa_3))-c\frac{(\kappa_3-\kappa_2)(\kappa_2-\kappa_1)}{(\kappa_3-\kappa_1)^2}(\kappa_3-\kappa_1)^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(\kappa_3-\kappa_1)\Upsilon(\kappa_2)\\
&\quad\leq(\kappa_3-\kappa_1)\Upsilon(\kappa_3)+(\kappa_3-\kappa_2)\eta(\Upsilon(\kappa_1),\Upsilon(\kappa_3))-c(\kappa_3-\kappa_2)(\kappa_2-\kappa_1)(\kappa_3-\kappa_1).
\end{aligned}$$

Conversely, let (9) hold, and then by setting $\kappa_1 = \chi_1, \kappa_2 = \nu\chi_1 + (1-\nu)\chi_2$ and $\kappa_3 = \chi_2$ in (9) we deduce

$$\begin{aligned}
&(\chi_2 - \chi_1)\Upsilon(\nu\chi_1 + (1-\nu)\chi_2) \\
&\quad \leq (\chi_2 - \chi_1)\Upsilon(\chi_2) + (\chi_2 - \nu\chi_1 - (1-\nu)\chi_2)\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) \\
&\quad - c\nu(\chi_2 - \nu\chi_1 - (1-\nu)\chi_2)(\nu\chi_1 + (1-\nu)\chi_2 - \chi_1)(\chi_2 - \chi_1) \\
&\quad = (\chi_2 - \chi_1)\Upsilon(\chi_2) + \nu(\chi_2 - \chi_1)\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) - c\nu(\chi_2 - \chi_1)(-(1-\nu)\chi_1 \\
&\quad + (1-\nu)\chi_2)(\chi_2 - \chi_1) \\
&\quad = (\chi_2 - \chi_1)\Upsilon(y) + \nu(\chi_2 - \chi_1)\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) - c\nu(\chi_2 - \chi_1)(1-\nu)(\chi_2 - \chi_1)(\chi_2 - \chi_1),
\end{aligned}$$

which implies that

$$\Upsilon(\nu\chi_1 + (1-\nu)\chi_2) \leq \Upsilon(\chi_2) + \nu\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) - c\nu(1-\nu)(\chi_2 - \chi_1)^2.$$

This is the required result. □

In the upcoming result, an important relation among the strong η-convexity and η-convexity has been presented.

Lemma 5. *Prove that a strongly η-convex function is η-convex, while the converse is not generally true.*

Proof. Let Υ be a strongly η-convex function, then by definition

$$\begin{aligned}
\Upsilon(\nu\chi_1 + (1-\nu)\chi_2) &\leq \Upsilon(\chi_2) + \nu\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)) - c\nu(1-\nu)(\chi_1 - \chi_2)^2 \\
&\leq \Upsilon(\chi_2) + \nu\eta(\Upsilon(\chi_1), \Upsilon(\chi_2)).
\end{aligned}$$

Implies that

$$\Upsilon(\nu\chi_1 + (1-\nu)\chi_2) \leq \Upsilon(\chi_2) + \nu\eta(\Upsilon(\chi_1), \Upsilon(\chi_2))$$

which shows that Υ is η-convex, □

For the converse of the above result, we demonstrate examples.

Example 4. *Let $\Upsilon : \mathbb{R} \to \mathbb{R}$ be a function defined as $\Upsilon(\vartheta) = -|\vartheta|, \vartheta \in \mathbb{R}$ and $\eta : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be defined as $\eta(\vartheta_1, \vartheta_2) = -2\vartheta_1 - \vartheta_2$. Then Υ is η-convex but not a strongly η-convex function.*

Proof. For $\vartheta_1, \vartheta_2 \in \mathbb{R}$ and $\nu \in [0,1]$, we have

$$\begin{aligned}
\Upsilon(\nu\vartheta_1 + (1-\nu)\vartheta_2) &= -|\nu\vartheta_1 + (1-\nu)\vartheta_2| \\
&= -|-(-\nu\vartheta_1) + (1-\nu)\vartheta_2| \\
&= -|(1-\nu)\vartheta_2 - (-\nu\vartheta_1)| \\
&\leq \nu|\vartheta_1| - (1-\nu)|\vartheta_2| \\
&= \nu|\vartheta_1| - |\vartheta_2| + \nu|\vartheta_2| \\
&= -|\vartheta_2| + \nu(|\vartheta_1| + |\vartheta_2|) \\
&\leq -|\vartheta_2| + \nu(2|\vartheta_1| + |\vartheta_2|) \\
&= -|\vartheta_2| + \nu(-2(-|\vartheta_1|) - (-|\vartheta_2|)) \\
&= \Upsilon(\vartheta_2) + \nu(-2\Upsilon(\vartheta_1) - \Upsilon(\vartheta_2)) \\
&= \Upsilon(\vartheta_2) + \nu\eta(\Upsilon(\vartheta_1), \Upsilon(\vartheta_2)).
\end{aligned}$$

Hence Υ is an η-convex function.

Next, If Y is a strongly η-convex function, then by definition

$$\mathrm{Y}(\nu\vartheta_1 + (1-\nu)\vartheta_2) \leq \mathrm{Y}(\vartheta_2) + \nu\eta(\mathrm{Y}(\vartheta_1), \mathrm{Y}(\vartheta_2)) - c\nu(1-\nu)(\vartheta_1 - \vartheta_2)^2$$

which implies that

$$-|\nu\vartheta_1 + (1-\nu)\vartheta_2| \leq -|\vartheta_2| + \nu(2|\vartheta_1| + |\vartheta_2|) - c\nu(1-\nu)(\vartheta_1 - \vartheta_2)^2.$$

Substitute $\vartheta_1 = 0$, we deduce

$$\begin{aligned} -|(1-\nu)|\vartheta_2| &\leq -|\vartheta_2| + \nu|\vartheta_2| - c\nu(1-\nu)\vartheta_2^2 \\ \text{that is } -|\vartheta_2| + \nu|\vartheta_2| &\leq -|\vartheta_2| + \nu|\vartheta_2| - c\nu(1-\nu)\vartheta_2^2. \end{aligned}$$

which is not true for all $\nu \in (0,1)$, $\vartheta_2 \in \mathbb{R} \setminus \{0\}$ and $c > 0$. Thus, Y is not a strongly η-convex function on $\mathbb{R}$. □

Example 5. *Let* $\mathrm{Y} : \mathbb{R} \to \mathbb{R}$ *be a function defined as* $\mathrm{Y}(\vartheta) = |\vartheta| + \vartheta$ *and* $\eta : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *be defined as* $\eta(\vartheta_1, \vartheta_2) = \vartheta_1 + 2\vartheta_2$. *Then* Y *is an* η*-convex but not strongly convex function.*

Proof. For any $\vartheta_1, \vartheta_2 \in \mathbb{R}$ and $\nu \in [0,1]$, we have

$$\begin{aligned} \mathrm{Y}(\nu\vartheta_1 + (1-\nu)\vartheta_2) &= |\nu\vartheta_1 + (1-\nu)\vartheta_2| + \nu\vartheta_1 + (1-\nu)\vartheta_2 \\ &\leq \nu|\vartheta_1| + (1-\nu)|\vartheta_2| + \nu\vartheta_1 + \vartheta_2 - \nu\vartheta_2 \\ &= \nu|\vartheta_1| + |\vartheta_2| - \nu|\vartheta_2| + \nu\vartheta_1 + \vartheta_2 - \nu\vartheta_2 \\ &= |\vartheta_2| + \vartheta_2 + \nu(|\vartheta_1| + \vartheta_1 - (|\vartheta_2| + \vartheta_2)) \\ &\leq |2| + \vartheta_2 + \nu(|\vartheta_1| + \vartheta_1 + 2(|\vartheta_2| + \vartheta_2)) \\ &= \mathrm{Y}(\vartheta_2) + \nu(\mathrm{Y}(\vartheta_1) + 2\mathrm{Y}(\vartheta_2)) \\ &= \mathrm{Y}(\vartheta_2) + \nu\eta(\mathrm{Y}(\vartheta_1), \mathrm{Y}(\vartheta_2)) \end{aligned}$$

This yields the η-convexity of Y on $\mathbb{R}$.
Now, suppose that Y is a strongly η-convex function. Then

$$\mathrm{Y}(\nu\vartheta_1 + (1-\nu)\vartheta_2) \leq \mathrm{Y}(\vartheta_2) + \nu\eta(\mathrm{Y}(\vartheta_1), \mathrm{Y}(\vartheta_2)) - c\nu(1-\nu)(\vartheta_1 - \vartheta_2)^2,$$

which implies that

$$\begin{aligned} |\nu\vartheta_1 + (1-\nu)\vartheta_2| + \nu\vartheta_1 + (1-\nu)\vartheta_2 &\leq |\vartheta_2| + \vartheta_2 + \nu(|\vartheta_1| + \vartheta_1 + 2(|\vartheta_2| + \vartheta_2)) \\ &\quad - c\nu(1-\nu)(\vartheta_1 - \vartheta_2)^2 \\ |\nu\vartheta_1 + (1-\nu)\vartheta_2| + \nu\vartheta_1 + (1-\nu)\vartheta_2 &\leq |\vartheta_2| + \vartheta_2 + \nu|\vartheta_1| + \nu\vartheta_1 + 2\nu|\vartheta_2| + 2\nu\vartheta_2 \\ &\quad - c\nu(1-\nu)(\vartheta_1 - \vartheta_2)^2. \end{aligned}$$

Substitute $\vartheta_2 = 0$, we get

$$\begin{aligned} |\nu\vartheta_1| + \nu\vartheta_1 &\leq \nu|\vartheta_1| + \nu\vartheta_1 - c\nu(1-\nu)\vartheta_1^2 \\ \nu|\vartheta_1| + \nu\vartheta_1 &\leq \nu|\vartheta_1| + \nu\vartheta_1 - c\nu(1-\nu)\vartheta_1^2. \end{aligned}$$

which is a contradiction for all $\nu \in (0,1)$, $\vartheta_1 \in \mathbb{R} \setminus \{0\}$ and $c > 0$. Thus, Y is not a strongly η-convex function on $\mathbb{R}$.

4. Brief History and Conclusions

One cannot ignore the great importance and scope of the class of convex functions in almost every branch of mathematics due to several features of convex functions [17–19]. The convex function has been refined, extended, and generalized in numerous angles, such

as p-convex, h-convex, α-convex, and Godunova–Levin functions [20–24]. We give a brief history of some generalizations of convex functions. In 2001, Dragomir gave the idea of convexity on the co-ordinates and proved that every convex function implies convex on the co-ordinates and they found some examples that there are functions which have the property of convex on the co-ordinate but are not convex [7]. In 2016, Gordji et al. extended further the family of convex functions to the family of η-convex functions [9]. Gordji's interesting work on η-convexity was devoted to pointing out the certain features and results for η-convex function. In 2019, Zaheer Ullah et al. further extended an η-convex function to (η_1, η_2)-convex on the co-ordinate [11]. They also presented some results and inequalities for this class of functions and elaborated this with the help of some interesting examples. There is another strengthened version of convex function which was introduced by Polyak in 1966 and which is known as strong convex function [12]. There is a rich body of literature devoted to strong convexity and numerous applications are devoted to strong convexity. In particular, it has a great role in the theory of optimization. As Dragomir gave the concept of convexity on the co-ordinate, in a similar fashion in 2019, Adil Khan et al. presented the idea of strong convexity on the co-ordinate and derived several important results and inequalities which clearly show the role of strong convexity on the co-ordinate [13]. In 2017, Awan et al. introduced an interesting generalized family of convex functions which is named strong η-convex function [14]. They also obtained several important results for this class of functions. Looking to the aforementioned history of different generalizations and extensions, in this article we presented a more advanced generalized (η_1, η_2)-convexity on the co-ordinate. Some examples and results are presented for this class of functions which show the relations and importance of this class of functions with earlier generalized convex functions. The article also presents some properties, examples, and results for the strong η-convex functions.

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References

1. Niculescu, C.P.; Persson, L.E. *Convex Functions and Their Applications. A Contemporary Approach*, 2nd ed.; CMS Books in Mathematics; Springer: New York, NY, USA, 2018; Volume 23.
2. Zhao, T.-H.; Shi, L.; Chu, Y.-M. Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2020**, *114*, 96. [CrossRef]
3. Chu, Y.-M.; Zhao, T.-H. Concavity of the error function with respect to Hölder means. *Math. Inequal. Appl.* **2016**, *19*, 589–595. [CrossRef]
4. Hardy, G.H.; Littlewood, J.E.; Pólya, G. *Inequalities*, 2nd ed.; Cambridge Mathematical Library: Cambridge, UK, 1952.
5. Pecarić, J.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press: New York, NY, USA, 1992.
6. Chen, S.-B.; Rashid, S.; Noor, M.A.; Hammouch, Z.; Chu, Y.-M. New fractional approaches for n-polynomial P-convexity with applications in special function theory. *Adv. Differ. Equ.* **2020**, *2020*, 543. [CrossRef]
7. Dragomir, S.S. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan J. Math.* **2001**, *5*, 775–778. [CrossRef]
8. Bakula, M.K.; Pecarić, J. On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan J. Math.* **2006**, *10*, 1271–1292. [CrossRef]
9. Gordji, M.E.; Delavar, M.R.; Sen, M.D.L. On ϕ-convex functions. *J. Math. Inequal.* **2016**, *10*, 173–183. [CrossRef]
10. Gordji, M.E.; Dragomir, S.S. On η-convexity. *Math. Inequal. Appl.* **2017**, *20*, 203–216.

11. Ullah, S.Z.; Khan, M.A.; Chu, Y.-M. A note on generalized convex functions. *J. Inequal. Appl.* **2019**, *2019*, 10.
12. Polyak, B.T. Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. *Sov. Math. Dokl.* **1966**, *7*, 72–75.
13. Khan, M.A.; Ullah, S.Z.; Chu, Y.-M. The concept of coordinate strongly convex functions and related inequalities. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2019**, *113*, 2235-2251. [CrossRef]
14. Awan, M.U.; Noor, M.A.; Noor, K.I.; Safdar, F. On strongly generalized convex functions. *Filomat* **2017**, *31*, 5783–5790. [CrossRef]
15. Kermausuor, S.; Nwaeze, E.R. A note on some new Hermite–Hadamard type inequalities for functions whose nth derivatives are strongly η-convex. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 179–187.
16. Tameru, A.M.; Nwaeze, E.R.; Kermausuor, S. Strongly (η, ω)-convex functions with nonnegative modulus. *J. Inequalities Appl.* **2020**, *2020*, 165. [CrossRef]
17. Zhao, T.-H.; He, Z.-Y.; Chu, Y.-M. Sharp bounds for the weighted Hölder mean of the zero-balanced generalized complete elliptic integrals. *Comput. Methods Funct. Theory* **2021**, *21*, 413–426. [CrossRef]
18. Zhao, T.-H.; Wang, M.-K.; Chu, Y.-M. Monotonicity and convexity involving generalized elliptic integral of the first kind. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2021**, *115*, 46. [CrossRef]
19. Kashuri, A.; Nwaeze, E.R. Some different types of parameterized inequalities pertaining to generalized (m, h_1, h_2)-preinvex functions via generalized fractional integral operators and their applications. *J. Interdiscip. Math.* **2021**, *24*, 821–852. [CrossRef]
20. Dragomir, S.S. Inequalities of Hermite-Hadamard type for λ-convex functions on linear spaces. *Preprint RGMIA Res. Rep. Coll.* **2014**, *18*, 13. Available online: http://rgmia.org/papers/v17/v17a13.pdf (accessed on 10 January 2022). [CrossRef]
21. Dragomir, S.S.; Fitzpatrick, S. The Jensen inequality for s-breckner convex functions in linear spaces. *Demonstr. Math.* **2000**, *XXXIII*, 8. [CrossRef]
22. Zhao, T.-H.; Yang, Z.-H.; Chu, Y.-M. Monotonicity properties of a function involving the psi function with applications. *J. Inequal. Appl.* **2015**, *2015*, 193. [CrossRef]
23. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
24. Roberts, A.W.; Varberg, D.E. *Convex Functions*; Academic Press: New York, NY, USA, 1973.

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