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Article
A Nonparametric Dual Control Algorithm of Multidimensional Objects with Interval-Valued Observations

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Abstract: We focus on the dual interval control problem of multidimensional objects with delay. We propose a new nonparametric algorithm. In such a case, it is not necessary to determine a parametric structure of the investigated object. Another difficulty lies in the complex nature of the decision-making field as it might not be flexible or convenient for decision-makers to exactly quantify their opinions with crisp numbers. Due to this fact, we introduce the interval-valued observations into the algorithm by means of the single-level constraint interval arithmetic. The results of computational experiments illustrate the effectiveness of the algorithm in the case of using intervals instead of crisp values.

Keywords: dual control; nonparametric uncertainty; interval observations; static system

MSC: 93C35; 93C40; 65G30

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1. Introduction

In this article, we consider a control problem of multidimensional objects. In [1], Feldbaum suggested dual control theory. Dual control algorithms combine control and object learning processes. This theory was extensively developed by Wittenmark and Astrom [2,3] who suggested applying dual control algorithms in two cases: a short time horizon and rapidly changing object parameters. In the first stages of theory development, it was used for linear stochastic systems with unknown parameters [4,5]. Dual control algorithms were developed for the case of parametric uncertainty [6]. Using this type of dual algorithm assumes that the structure of the true system is a priory known and the control task is to optimize its parameters. The dual control approach is widely used in the development of model predictive control (MPC) systems [7]. Thus, in [8,9], an adaptive MPC strategy was suggested for linear multi-input multi-output systems. In [10], the MPC approach for model-structure uncertainty is introduced. The authors highlight the beneficial effect of MPC with active learning under parametric or structural model uncertainty. In practice, dual control algorithms were applied in a wide variety of fields, such as diabetes investigation [11], a semi-batch reactor equipped with a cooling jacket modeling [12], energy hub modeling [13] and a penicillin fermentation process control [14].

In practice, there is not much a priori information about the control process. In [15], the author presents an overview of adaptive control methods based on how much model information is needed. Processes in the industry (metallurgy, chemical industry, mining industry, production of electronic components, etc.) are complex and the researcher has no data on the mathematical structure of the system. It could be difficult and time-consuming to build an accurate parametric model. Thus, a parametric approach for constructing control systems might be impractical [16]. Therefore, data-driven or modal-free methods for creating controllers have become widespread [16]. One of these [17] shows some results in constructing a control system on data-dependent matrices that can replace systems models. In [18], dual control algorithms based on neural networks are used to approximate a priory unknown functions. The neural approach is applicable for multidimensional dynamic systems with unknown structures [19]. Another method that allows modeling systems with unknown structures is the echo state networks [20]. A dual control algorithm for multidimensional dynamic objects using nonparametric estimation of a reverse regression function was suggested in [21].

In control systems, the value of desired output is set by an expert. It is not frequent that the knowledge of experts is precise and such imprecise knowledge of experts should be represented by interval numbers for reflecting the imprecision. To solve the dual control problem we will use interval arithmetic which is called single-level constraint interval arithmetic (SLCIA) [22]. There are also some articles devoted to a framework of interval-based data analysis in control problems [23–25]. In this study, we include interval values in a nonparametric dual control algorithm for multidimensional systems. Similar to the work [24], we use SLCIA to calculate control actions as it simplifies the process of calculations and computer implementation.

In the existing methods [26], traditional fuzzy control algorithms are used to deal with dynamic systems which can be described with differential equations. In real applications, sometimes it is impossible to take into account the dynamics of the process if the measurement interval of the output variable is more than the output time constant. So it is impossible to take into account the dynamics of the process in the control system. We study static objects with transport delay. In the presence of such delay, it is possible to make a shift in the observation matrix by the delay value to bring the one-to-one correspondence between the values of the input and output variables and not take it into account in further reasoning.

Thus, the main problem considered in the article is the construction of control algorithms under conditions of uncertainty. For this, a synthesis of the following approaches is proposed. The first is the dual control theory for control in the absence of a training sample for setting up the control device. Dual control theory is used to combine two competing goals: training and control. In [27,28], a dual control algorithm for stochastic systems with multiple uncertainties is suggested for crisp values. The second one is the theory of nonparametric control for objects whose mathematical description is a priori unknown up to parameters. It is a general-purpose algorithm, meaning it does not depend on the object's mathematical description. It can be applied to a wide class of objects with known qualitative properties (dynamic or static, for example). Moreover, SLCIA is introduced for working with interval values of setpoints. In [29], SLCIA was used for the fuzzy interval optimal control problem. We propose this approach to the dual control problems for interval variables under uncertainties. In this paper, for the first time, a nonparametric interval dual control algorithm is proposed.

The rest of the paper is organized as follows. In Section 2, we present the formulation of the dual control problem. In Section 3, we propose a nonparametric dual control algorithm. In Section 4, the results of the numerical experiments of modeling multidimensional objects are described. We conclude our work in Section 5.

2. Problem Formulation under Interval Uncertainty

Consider a control system, whose general scheme is shown in Figure 1. The notation is as follows: $x = (x_1, x_2, ..., x_n) \in \Omega(x) \subset \mathbb{R}^n$ is an output variable of the process, $x^* = (x_1^*, x_2^*, ..., x_n^*) \in \Omega(x) \subset \mathbb{R}^n$ is a vector of set points, $u = (u_1, u_2, ..., u_m) \in \Omega(u) \subset \mathbb{R}^m$ is a control input vector, ξ is a vector random disturbances, $G^{x_1}, G^{x_2}, ..., G^{x_n}$ are the system response channels corresponding to output variables and including control tools, $g^x = (g^{x_1}, g^{x_2}, ..., g^{x_n}) \in \Omega(g^x) \subset \mathbb{R}^n$ is the random inaccuracy of measurements of output variables of the process with zero mathematic expectation and limited dispersion.

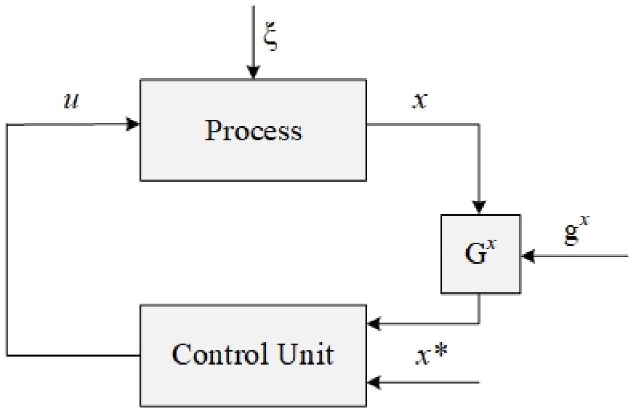


Figure 1. The general scheme of closed loop system.

The input and output variables are continuous because of the nature of the process but the measurements are made at discrete times due to control tools so we investigate discrete-continuous systems. Such systems are also called hybrids as the continuous part consists of multiple-operation technological chains and the discrete part consists of digital controllers [30]. The agreed notation is as follows: $u_{i,j}$, $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, s$ – the ith measurement of jth component of the control variable u; $x_{i,j}$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, s$ – the ith measurement of jth component of the output variable x. We have an initial sample of observations $\{u_i, x_i, i = 1, 2, \ldots, s\}$, where s is sample size.

The task of the control unit is to generate such a control action u that the difference between object output value x and the value x^* is minimal. In the previous paragraph, we commented on the case when all variables were considered crisp numbers. However, in practice, it is usual that decision information is uncertain. It might not be flexible or convenient for decision-makers to exactly quantify their opinions with crisp numbers. A possible solution to model and deal with such uncertainty is by means of interval values. In this regard, and following, we introduce and formulate a situation when values of reference variables are intervals. For that, we use intervals notation proposed by Stefanini and Bede in [31] to define the set of real intervals as

$$K_C = \{[\underline{a}, \overline{a}] : \underline{a} \leq \overline{a},\ \underline{a}, \overline{a} \in \mathbb{R}\},$$

where $[\underline{a}, \overline{a}]$ notes the classic real interval. In multidimensional case,

$$K_C^n = \underbrace{K_C \times \cdots \times K_C}_{n \text{ times}},$$

that is, K_C^n is the space of nonempty compact and convex sets of n-dimensional real numbers $\mathbb{R}^n$.

The value of desired output is set by an expert. So, we refer to new information given by an expert and due to this information, we deal with intervals. For this purpose, we make the transition from $x^* = (x_1^*, x_2^*, ..., x_n^*) \in \Omega(x) \subset \mathbb{R}^n$ to $y^* = (y_1^*, y_2^*, ..., y_n^*) \in K_C^n$. Under the assumption, we are going to obtain intervals instead of crisp values for control variables. We introduce a new notation of control variable $v = (v_1, \ldots, v_m) \in K_C^m$ for interval values. We use different notations $\{u_i, x_i, i = 1, 2, \ldots, s\}$, $x(t) \in \mathbb{R}^n$, $u \in R^m$ for observations, which we obtain by measuring input and output variables of the process and $\{v, y^*\}$, $y^* \in K_C^n$, $v \in K_C^m$ for approximation.

3. SLCIA Basic Concepts

On the topic of interval arithmetic and analysis we can find discussions and notations by Stefanini and Bede in [31], Moore [32,33], and Alefeld and Herzberger [34], among others. In [22], it is proposed a variant of constraint interval arithmetic (CIA) that operates with a single parameter (level) in each interval operand of an expression, called single-level constraint interval arithmetic (SLCIA). This arithmetic was used in the discrete-time interval optimal control problem [29], and in the next section, we propose its extension to the evaluation of expressions in interval-valued dual control problems.

Let us bring the basic definitions of single-level constraint interval arithmetic [22].

Definition 1 ([22])**.** *Let $A = \{[\underline{a}, \overline{a}]\} \in K_C$ be any interval. Then*

1. *A continuous function $A : [0,1] \to \mathbb{R}$ such that*

$$\min_{0 \le \lambda \le 0} A(\lambda) = \underline{a}, \qquad \max_{0 \le \lambda \le 0} A(\lambda) = \overline{a},$$

will be called a constraint function associated with A.

2. *Associated with the interval A we define the decreasing convex constraint function $A : [0,1] \to \mathbb{R}$ by means*

$$A(\lambda) = \lambda \underline{a} + (1 - \lambda)\overline{a}, \qquad 0 \le \lambda \le 1,$$

or equivalently

$$A(\lambda) = (\underline{a} - \overline{a})\lambda + \overline{a}, \qquad 0 \le \lambda \le 1.$$

3. *Associated with the interval A we define the increasing convex constraint function A' : $[0,1] \to \mathbb{R}$ by means*

$$A'(\lambda) = (1 - \lambda)\underline{a} + \lambda \overline{a}, \qquad 0 \le \lambda \le 1,$$

or equivalently

$$A'(\lambda) = \underline{a} + \lambda(\overline{a} - \underline{a}), \qquad 0 \le \lambda \le 1.$$

For discussions and examples of SLCIA, we refer to [22], particularly, for the evaluation of expression with intervals. In this regard, we highlight the following definitions of expression in interval arithmetic, with a role for the calculus of interval-valued expressions.

Definition 2 ([22])**.** *An expression $E(A_1, \ldots, A_q)$ is a correct expression in interval arithmetic if $E(x_1, \ldots, x_q)$ is a correctly constructed expression in a formal language for arithmetic operations with real number operands $x_1, \ldots, x_q$ and usual arithmetic operations on real numbers.*

Definition 3 ([22])**.** *Let $A_1(\lambda), \ldots, A_q(\lambda)$ be the decreasing convex constraint functions associated to $A_1, \ldots, A_q \in K_C$, $E(A_1(\lambda), \ldots, A_q(\lambda))$. The evaluation of a correct expression is performed according to the following rule:*

$$E(A_1, \ldots, A_q) = \left[\min_{\lambda \in [0,1]} E\big(A_1(\lambda), \ldots, A_q(\lambda)\big), \max_{\lambda \in [0,1]} E\big(A_1(\lambda), \ldots, A_q(\lambda)\big) \right]. \tag{1}$$

This is the evaluation of the expression E with the given arguments provided that the min and max exist. A similar role exists for increasing convex constraint functions associated with $A_1, \ldots, A_q \in K_C$.

Let us observe that, as usual in computational calculus, it is interesting to explore how to evaluate the interval-valued expression E, by means of its corresponding real-valued expression given in (1), at a discrete set of values for λ. For such discretization, and given $p \in \mathbb{N}$, let us consider a classic partition of $[0,1]$ $\pi_p = \{\lambda_k : k = 1, \ldots, p+1\}$, with $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{p+1} = 1$; in particular, let us use $\lambda_k = \frac{k-1}{p}$, for $k = 1, \ldots, p+1$. As a result after computation on the partition, it is expected to obtain an approximation of the expression E. To this end, in the following, we present a useful result for computational calculus in the next section under continuity of the expression $E(x_1, \ldots, x_q)$.

Proposition 1. *Let us consider $A_1(\lambda), ..., A_q(\lambda)$ the decreasing convex constraint functions associated to $A_1, ..., A_q \in K_C$, and an expression in interval arithmetic $E(A_1, ..., A_q)$, in which the corresponding real-valued expression $E(x_1, \ldots, x_q)$ is continuous. For any $p \in \mathbb{N}$, consider the partition on $[0,1]$ $\pi_p = \{\lambda_k : k = 1, \ldots, p+1\}$, with $\lambda_k = \frac{k-1}{p}$, for $k = 1, \ldots, p+1$. Then, $E(A_1, ..., A_q)$ is a correct expression in interval arithmetic, and*

$$E(A_1, ..., A_q) = \left[\lim_{p\to\infty} \min_{\lambda_k \in \pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big), \lim_{p\to\infty} \max_{\lambda_k \in \pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big) \right]. \quad (2)$$

Proof. On one hand, $A_1(\lambda), ..., A_q(\lambda)$ are decreasing convex constraint functions, and then, from Definition 1, they are continuous on $[0,1]$. Since the real-valued expression $E(x_1, \ldots, x_q)$ is continuous, it derives that $E\big(A_1(\lambda), ..., A_q(\lambda)\big)$ is continuous on the compact set $[0,1]$, what implies that there exist the minimum and maximum of $E\big(A_1(\lambda), ..., A_q(\lambda)\big)$ on $[0,1]$. Therefore, following Definitions 2 and 3, $E(A_1, ..., A_q)$ is a correct expression in interval arithmetic and can be calculated by the equality (1). On the other hand, from the continuity of $E\big(A_1(\lambda), ..., A_q(\lambda)\big)$ on the compact set $[0,1]$, it follows that $\min_{\lambda\in[0,1]} E\big(A_1(\lambda), ..., A_q(\lambda)\big) = \lim_{p\to\infty} \min_{\lambda_k\in\pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big)$, and $\max_{\lambda\in[0,1]} E\big(A_1(\lambda), ..., A_q(\lambda)\big) = \lim_{p\to\infty} \max_{\lambda_k\in\pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big)$, and then the equality (2) is fulfilled. □

As a consequence of the previous result, let us point out that given p and its associated partition π_p, the interval given by

$$\left[\min_{\lambda_k \in \pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big), \max_{\lambda_k \in \pi_p} E\big(A_1(\lambda_k), ..., A_q(\lambda_k)\big) \right]$$

can be used as an approximation to interval $E(A_1, ..., A_q)$, which is useful in the computational calculus in the next section.

4. Nonparametric Interval Dual Control Algorithm

The mathematical description of the investigated object can be as follows:

$$x = A(u), \quad (3)$$

where A is an unknown object operator. If there exists an inverse operator A^{-1}, $A^{-1}A = 1$, then

$$x = A^{-1}A < x = x^* >, u = A^{-1} < x > . \quad (4)$$

From now on, we assume that A^{-1} exists and it is a continuous function. It is an ill-posed problem [35]. The exact solution exists for the case of output noise absence. In the presence of noise, some regularization methods can be applied to obtain an exact solution for systems that could be modeled by a linear differential equation [36]. For the model-free case or situation of nonparametric uncertainty, it is advisable to use kernel estimations to obtain the estimation of the inverse operator [37].

The "ideal" regulator could have the form (4). The formula (4) could be used in order to obtain the desired trajectory $x = x^*$. In this case, we calculate the "ideal" value of the control variable u^*. The major problem is that in many cases it is impossible to construct such a scheme because the operator A is unknown. The estimation of the inverse operator $\widehat{A^{-1}}$ is used to obtain the estimation $\widehat{u^*}$. The idea is to estimate it directly from input u and output x [38].

Consider the dual control algorithms which were first proposed by Feldbaum [1]. The control aim of such algorithms has dual nature: caution and probing [3]. Feldbaum considered a situation when the structure of the model and the laws of the distribution of the random disturbances are known. In [39], the idea of applying the nonparametric estimation of regression function in control systems was first suggested for crisp values.

The method is robust to nonparametric uncertainty: the mathematical description of the object is unknown.

As a task of control unit to obtain control action $u(t)$, so the inverse function A_i^{-1} of (3) exists:

$$u_j = A_j^{-1}(x_i^*), i = 1,2,\ldots,n, j = 1,2,\ldots,m, \tag{5}$$

where A_j^{-1} is a continuous function. As y_i^* are compact sets and A_j^{-1} are continuous functions, then we find that v_j are also intervals.

Previously, the nonparametric algorithm of dynamic processes dual control for crisp values was suggested in [21]. Due to uncertain data context and the presence of interval-valued data, we propose the nonparametric interval dual control algorithm of a multidimensional object by means of SLCIA. The proposed algorithm includes the following steps:

Step 1. Under the new framework, we deal with intervals instead of points. For this purpose, taking into account SLCIA, we define each interval-valued variable of $y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ by means of their decreasing convex constraint functions associated, and give an initial value to p. Then, we consider the partition $\pi_p = \{\lambda_k : k = 1,\ldots,p+1\}$, with $\lambda_k = \frac{k-1}{p}$, for $k = 1,\ldots,p+1$, for the discretization for each interval variable $y_j^* = [\underline{y_j^*}, \overline{y_j^*}], j = 1,2,\ldots,n$, what provides the following discrete subsets:

$$\{y_j^{*k}\} \subset [\underline{y_j^*}, \overline{y_j^*}], k = 1,2,...,p+1,$$

with $y_j^{*k} = \lambda_k \underline{y_j^*} + (1-\lambda_k)\overline{y_j^*}$. Note that $y_j^{*1} = \underline{y_j^*}, y_j^{*p+1} = \overline{y_j^*}$.

Then, for each level $\lambda_k, k = 1,2,...,p+1$ of the interval $(y_1^*(\lambda_k), y_2^*(\lambda_k), \ldots, y_n^*(\lambda_k))$ we calculate the level λ_k of the control variable $v = (v_1(\lambda_k), v_2(\lambda_k), \ldots, v_m(\lambda_k))$. We operate on all levels and then take the minimum and maximum of the operations in relation to λ for each $v_j, j = 1,2,\ldots,m$ to obtain the extremes of the new interval of the control variable $v = (v_1, v_2, \ldots, v_m)$. For each value of control variable $v = (v_1, v_2, \ldots, v_m)$ we use the following control algorithm.

Step 2. We use Nadaraya–Watson nonparametric estimation of inverse regression function [39,40]. For this purpose, it is necessary to define bandwidth parameters h^x and h^u. Bandwidth parameters for each component of the vector of variables u and x are determined due to the following algorithm.

(i) Calculate the value of bandwidth parameter $h_w^x, w = 1,2,\ldots,n$:

$$h_w^x = \beta|y_{w,s+1}^* - x_w^0|, \tag{6}$$

where coefficient $\beta > 1$, x_w^0 is the closest observation to the value $y_{w,s+1}^*$ of the sample $\{x_{w,i}, i = 1,2,\ldots,s\}, w = 1,2,\ldots,n$.

(ii) Determine the value of the coefficient h^u:

$$h^u = \gamma|u_{s+1} - u^0|, \tag{7}$$

where coefficient $\gamma > 1$, u^0 is the closest observation to the value u_{s+1} of the sample $\{u_i, i = 1,2,\ldots,s', s' < s\}$. The sampling points satisfy the following conditions:

$$|y_{w,s+1}^* - x_{w,i}|/h_w^x \leq 1, i = 1,\ldots,s', w = 1,2,\ldots,n. \tag{8}$$

Step 3. We calculate the component $v_{j,s}^*(\lambda_k), j = 1,2,\ldots,m$ which accumulates the knowledge about the object. The first variable $v_{1,s}^*(\lambda_k)$ can be calculated as a nonparametric estimation of the regression function for discrete observations $\{u_i, x_i, i = 1,2,\ldots,s\}$ in the following form:

$$v_{1,s}^*(\lambda_k) = \frac{\sum\limits_{i=1}^{s} u_{1,i} \prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)}{\sum\limits_{i=1}^{s} \prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)}, \tag{9}$$

where $\prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)$ is a kernel function. Kernel function $\prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)$ and bandwidth parameter h_w^x satisfies the following convergence conditions [39,41]:

$$\begin{aligned} & h_w^x > 0; \&\& 0 \le \Phi\big((y_{w,s+1}^*(\lambda_k) - x_{w,i})/h_w^x\big) < \infty; \\ & \lim_{s\to\infty} h_w^x = 0; \&\& \int\limits_{\Omega(x)} \Phi\big((y_{w,s+1}^*(\lambda_k) - x_{w,i})/h_w^x\big)\, dx_{w,i} = 1; \\ & \lim_{s\to\infty} s(h_w^x)^n = \infty; \&\& \frac{1}{h_w^x} \lim_{s\to\infty} \Phi\big((y_{w,s+1}^*(\lambda_k) - x_{w,i})/h_w^x\big) = \delta(x_{w,s+1}^*(\lambda_k) - x_{w,i}). \end{aligned} \tag{10}$$

The main idea is that each subsequent value $v_i(\lambda_k), i = 2, 3, \ldots, m$ depends on the value $v_i(\lambda_k), i = 1, 2, \ldots, m-1$ found in the previous step. The estimation of $v_{j,s}^*(\lambda_k)$, $j = 2, 3, \ldots, m$ is based on a Nadaraya–Watson estimation of inverse regression function which refers to the local approximation methods [39].

Step 4. So, for components $v_j(\lambda_k), j = 2, 3, \ldots, m$ addend $v_{j,s}^*(\lambda_k), j = 2, 3, \ldots, m$ is proposed to calculate due to the formula:

$$v_{j,s}^*(\lambda_k) = \frac{\sum\limits_{i=1}^{s} u_{j,i} \prod\limits_{w=1}^{j-1} \Phi\left(\frac{v_{j,s}^*(\lambda_k) - u_{w,i}}{h^u}\right) \prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)}{\sum\limits_{i=1}^{s} \prod\limits_{w=1}^{j-1} \Phi\left(\frac{u_{w,s+1} - u_{w,i}}{h^u}\right) \prod\limits_{w=1}^{n} \Phi\left(\frac{y_{w,s+1}^*(\lambda_k) - x_{w,i}}{h_w^x}\right)}. \tag{11}$$

Step 5. The search step $\Delta v_{j,s+1}$ could have the form:

$$\Delta v_{j,s+1}(\lambda_k) = \sum_{i=1}^{n} \Theta_i \big(y_{i,s+1}^*(\lambda_k) - x_{i,s}\big), j = 1, 2, \ldots, m, \tag{12}$$

where $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n)$ could be found as a minimum of quadratic criterion:

$$\begin{aligned} R(\Theta_1, \Theta_2, \ldots, \Theta_n) = & \left(\sum_{i=1}^{s-1} \sum_{p=1}^{n} \left(y_{p,i+1}^*(\lambda_k) - \sum_{j=1}^{s-1} x_{p,j+1} \right.\right. \\ & \times \prod_{q=1}^{m} \Phi\left(\frac{v_{q,i}^* + \sum\limits_{w=1}^{n} \Theta_w \left(y_{w,i+1}^*(\lambda_k) - x_{w,i} \right) - u_{q,j+1}}{h^u} \right) \\ & \left.\left. \times \prod_{q=1}^{m} \Phi\left(\frac{v_{q,i}^*(\lambda_k) + \sum\limits_{w=1}^{n} \Theta_w \left(y_{w,i+1}^*(\lambda_k) - x_{w,i} \right) - u_{q,j+1}}{h^u} (\lambda_k) \right)^{-1} \right) \right)^2 \to \min_{(\Theta_1, \Theta_2, \ldots, \Theta_n)} \end{aligned} \tag{13}$$

The value of variable Θ belongs to the interval $[0, 1]$.

Step 6. In this case, the nonparametric dual control algorithm can be represented as follows:

$$v_{j,s+1}(\lambda_k) = v_{j,s}^*(\lambda_k) + \Delta v_{j,s+1}(\lambda_k), j = 1, 2, \ldots, m, \tag{14}$$

where the component $v_{j,s}^*$ accumulates the knowledge about the object, the component $\Delta v_{j,s+1}$ is the "learners" search step.

Step 7. For each value of reference variables $(y_1(\lambda_k), y_2(\lambda_k), \ldots, y_n(\lambda_k))$, we have calculated the value of control variable $v = (v_1(\lambda_k), v_2(\lambda_k), \ldots, v_m(\lambda_k))$, $\lambda_k = \frac{k-1}{p}$, for $k = 1, \ldots, p+1$. Then, we choose the minimum and the maximum value of the control variable. For example, for variable v_1: $\underline{v_1} = \min\{v_1(\lambda_k)\}, \overline{v_1} = \max\{v_1(\lambda_k)\}, k = 1, \ldots, p+1$. So, we obtain intervals for every control variable $v = (v_1, v_2, \ldots, v_m)$:

$$v_j = [\underline{v_j}, \overline{v_j}], j = 1, 2, \ldots, m.$$

We use the Gaussian kernel function as it is continuous and universal. From Proposition 1 it follows that the minimum and the maximum value of the control variable exist. So, we obtain an approximation of the interval values $v = (v_1, v_2, \ldots, v_m)$ using the proposed algorithm, as was concluded in Section 3.

5. Numerical Examples

At the initial stage of the control algorithm (14) search step $\Delta v_{j,s+1}(\lambda_k)$ (12) plays a key role. This component stands for the ability of control to lead the object to the desired output. A sample of observations $\{u_i, x_i, i = 1, 2, \ldots, s\}$ of input and output variables begins to accumulate from the first measurement and grows in the process of system control. The increased sample size leads to the growing role of the component $v^*_{j,s}(\lambda_k), j = 1, 2, \ldots, s$, this term contains the knowledge about the controlled object. This is the case of active data accumulation.

The combined method of data accumulation assumes that there is an initial sample of observations $\{u_i, x_i, i = 1, 2, \ldots, s\}$, but at the following times sample is supplemented with new elements (u_{s+1}, x_{s+1}), (u_{s+2}, x_{s+2}), ... In this case, the active and passive methods of data accumulation are associated. An available sample of observations is not sufficient to construct a high-quality system, but at the beginning, such a system is trained more than in the case of active data accumulation. This case is the most consistent with practice because the development of complex adaptive systems does not start from scratch.

Let us consider the case of the combined method of data accumulation of simulation of the object which has three input $u = (u_1, u_2)$ and two output $x = (x_1, x_2)$ variables. Let the object be described by the following equations:

$$\begin{cases} x_1 = u_1 + 1.5u_2, \\ x_2 = 1.5u_1 + 2u_2. \end{cases} \tag{15}$$

As it was said, if reference variables are intervals, control input u is also interval variables, and expressions given in (15) are interpreted following the arithmetic given by SLCIA. To illustrate the situation when the reference variables are intervals we conduct computational experiments under the considered interval-valued arithmetic, by means of a discretization of the parameter λ. Then, in order to use the suggested algorithm for each level $\lambda_k, k = 1, 2, ..., p+1$ of the interval $(y^*_1(\lambda_k), y^*_2(\lambda_k), \ldots, y^*_n(\lambda_k))$ we calculate the level λ_k of the control variable $v = (v_1(\lambda_k), v_2(\lambda_k), \ldots, v_m(\lambda_k)$. In the experiment, the set points are $y^*_1 = [1.8, 2]$, $y^*_2 = [2.8, 3]$.

The object control system is constructed by using a nonparametric estimation. In this case, the dual control algorithm has the following form:

$$\begin{cases} v_{1,s+1}(\lambda_k) = v^*_{1,s}(\lambda_k) + \Delta v_{1,s+1}(\lambda_k), \\ v_{2,s+1}(\lambda_k) = v^*_{2,s}(\lambda_k) + \Delta v_{2,s+1}(\lambda_k). \end{cases} \tag{16}$$

Firstly, it is necessary to calculate the component $v^*_{1,s}$ of dual control algorithm (16):

$$v^*_{1,s}(\lambda_k) = \frac{\sum\limits_{i=1}^{s} u_{1,i} \prod\limits_{w=1}^{2} \Phi\left(\frac{y^*_{w,s+1}(\lambda_k) - x_{w,i}}{h^x_w}\right)}{\sum\limits_{i=1}^{s} \prod\limits_{w=1}^{2} \Phi\left(\frac{y^*_{w,s+1}(\lambda_k) - x_{w,i}}{h^x_w}\right)}. \tag{17}$$

The component $v^*_{2,s}$ is calculated as follows:

$$v^*_{2,s}(\lambda_k) = \frac{\sum\limits_{i=1}^{s} u_{2,i}\Phi\left(\frac{v^*_{1,s}(\lambda_k)-u_{1,i}}{h^u}\right)\prod\limits_{w=1}^{2}\Phi\left(\frac{y^*_{w,s+1}(\lambda_k)-x_{w,i}}{h^x_w}\right)}{\sum\limits_{i=1}^{s}\Phi\left(\frac{v^*_{1,s}(\lambda_k)-u_{1,i}}{h^u}\right)\prod\limits_{w=1}^{2}\Phi\left(\frac{y^*_{w,s+1}(\lambda_k)-x_{w,i}}{h^x_w}\right)}. \tag{18}$$

In numeric experiments, we use the Gaussian kernel function which is a popular and practical choice [42]. For instance, it has the following form for the variable x [43]:

$$\Phi\left(\frac{y^*_{w,s+1}(\lambda_k)-x_{w,i}}{h^x_w}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y^*_{w,s+1}(\lambda_k)-x_{w,i}}{h^x_w}\right)^2}. \tag{19}$$

To assess the results of the simulation of control algorithms using a nonparametric model, the quadratic relative error was used for each λ-level:

$$W(\lambda_k) = \sqrt{\frac{\frac{1}{s}\sum\limits_{i=1}^{s}\sum\limits_{j=1}^{n}\left(y^*_{j,i}(\lambda_k)-x_{j,i}\right)^2}{\frac{1}{s-1}\sum\limits_{i=1}^{s}\sum\limits_{j=1}^{n}\left(x_{j,i}-\hat{m}_{xj}\right)^2}}, \tag{20}$$

where $\hat{m}_{xj}$—the estimation of mathematic expectation of the j-th component of output variable x.

Then, the control error could have the following form:

$$W = \frac{1}{p+1}\sum_{i=1}^{p+1} W(\lambda_k). \tag{21}$$

The value of the relative error (21) belongs to the interval $[0,1]$. A small error value (close to zero) indicates the high accuracy of control algorithms.

There is a case of the combined method of information accumulation. The sample $\{u_{1,i}, u_{2,i}, x_i, i = 1,2,\ldots,s\}, s = 200$ was passively accumulated. Since the 200 step dual control (14) algorithm starts working on the next 500 steps. In the following experiment control, we calculate input 100 times and find the minimum and maximum values to obtain intervals $v_j = [\underline{v_j}, \overline{v_j}], j = 1,\ldots,3$. The results are presented in Table 1.

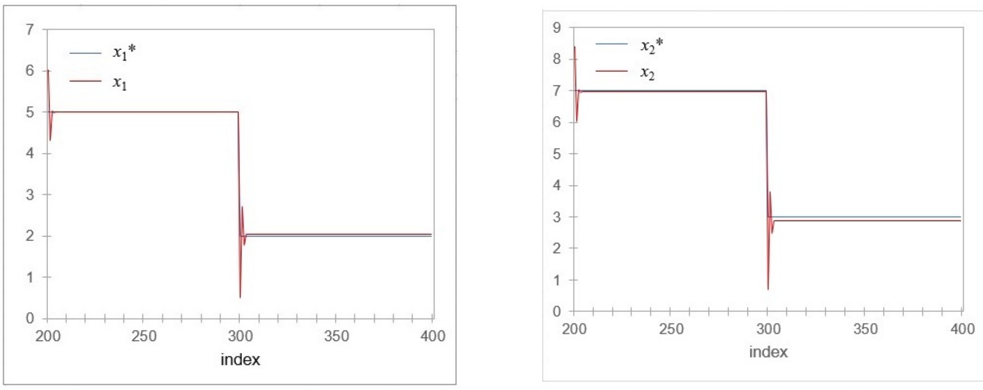
Table 1. The results of control for various values of the parameters.

s	v_1	v_2	y^*_1	y^*_2
200	[1.234, 1.690]	[0.164, 0.492]	[1.8, 2]	[2.8, 3]
300	[1.184, 1.681]	[0,032, 0.191]	[1.8, 2]	[2.8, 3]
400	[1.344, 1.712]	[0.116, 0.432]	[1.8, 2]	[2.8, 3]
500	[1.688, 1.808]	[0.082, 0.152]	[1.8, 2]	[2.8, 3]

Let us consider the results of the experiment when the desired output x^* has a stepwise form and is presented as crisp values:

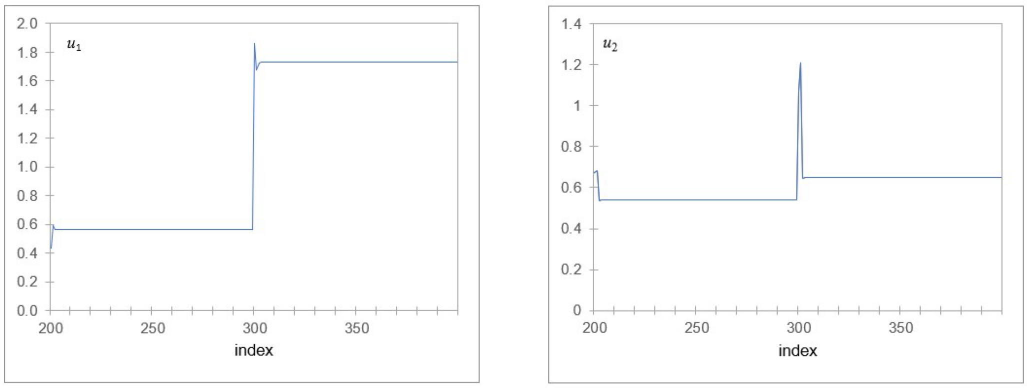
$$\begin{cases} x^*_{1,i} = 5, x^*_{2,i} = 7, i = \overline{200, 299}, \\ x^*_{1,i} = 2, x^*_{2,i} = 3, i = \overline{300, 400}. \end{cases} \tag{22}$$

In the first step, the algorithm is adjusted and then causes the object to the desired output. The simulation results in the absence of interference are shown in Figure 2. In Figures 2 and 3, the index means the number of a sample element. The first 200 sample elements from 0 to 199 were passively accumulated, and the control process started from the 200 elements, so the first index is 200 in Figure 2. With the new value of the reference variable, x_1^* or x_2^* algorithm tuning occurs, then the algorithm causes the object to the desired value. The graphics of control variables u_1, u_2, u_3 are shown in Figure 3. The control error is 0.08.



(**a**) (**b**)

Figure 2. The control results in the absence of interference, when the task control is a stepwise impact for variables: (**a**) x_1; (**b**) x_2.



(**a**) (**b**)

Figure 3. Control variables values: (**a**) u_1; (**b**) u_2.

6. Conclusions

In this paper, an interval dual control problem was proposed and the nonparametric algorithm was extended to the control theory using single-level constrained interval arithmetic. First, the training control algorithm is conducted at the same time as the control process. Second, the use of intervals allows for taking into account a variety of random factors, such as the inaccuracy of measurements. The proposed algorithm (Equation (14)) is effective in finding the interval solution of the control problem. A numerical example shows that the procedure to solve interval dual control problems is efficient. Moreover, a discretization method has a practical solution for a control problem decision. Future work will consider the insertion of the nonparametric dual control theory into the theory of fuzzy sets, i.e., we intend to study fuzzy control problems using single-level constrained fuzzy arithmetic. As another future research line, we will study the case when A^{-1} is not necessarily a function, and extend the method for the construction of the interval/fuzzy solution for such a new case.

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