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On the Radial Solutions of the Dirac Equation in the Kerr-Newman Black Hole Surrounded by a Cloud of Strings

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Abstract: In this paper, we obtain the metric of the space-time generated by a charged and rotating gravitational body surrounded by a loud of strings, namely, the Kerr–Newman black hole space-time with the addition of a cloud of strings. In this background, we find the radial solutions of the Dirac equation for massive particles and show that they are given in terms of the Generalized Heun functions. The dependence of these solutions on the parameter that codifies the presence of the cloud of strings is pointed out.

Keywords: dirac equation in curved space-times ; black holes with a cloud of strings; heun functions

MSC: 81S99; 83C57; 33E99

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1. Introduction

Certainly motivated by the success of the Dirac equation to describe the behavior of the electron, a question was posed, initially, in the later 1920s, namely: "Is it possible to extend Dirac's equation to the context of the Theory of General Relativity?". In 1928, Tetrode [1] attempted to answer this question and proposed a general relativistic Dirac equation, in which the Minkowskian partial derivative was replaced by a general coordinate partial derivative and Dirac's original matrices were substituted by curved space-time gamma matrices. In early 1929, Fock [2] and Weyl [3] considered the same question and used different approaches based on the idea of a local spinor structure. In 1932, Schrödinger [4] obtained a generalization of the Dirac equation to curved space-time but adopted another line of research, as compared to the ones followed by Tetrode [1], Fock [2] and Weyl [3]. Finally, in 1958, Dirac [5] considered the question related to the generalization of his equation for electrons, to generally curved space-times, and thus, compatible with the Theory of General Relativity.

Spin-1/2 particles interacting with gravitational fields have been the subject of many investigations. Along this line of research, we can mention those connected with quantum mechanics in different background space-times [6–10], and, in particular, the ones that consider the hydrogen atom [11–16] in arbitrarily curved space-times.

The study of the single-particle states that are exact solutions of the generalized Dirac equation in the context of the Theory of General Relativity, constitutes an important and interesting subject. In particular, the interaction of these particles placed around black holes may reveal significant information, and thus, it should be important from the theoretical as well as observational point of view. This is one of the reasons justifying this kind of investigation about the behavior of relativistic particles placed in a gravitational background.

The pioneering work by Chandrasekhar [17], in the 1970s, showing that the Dirac equation can be separated in Kerr Geometry and written in terms of radial and angular equations opens up a field of research concerning the study of Dirac equation in different background gravitational fields. In the same direction as Chandrasekhar, Page [18] showed that the Dirac equation is also separable in the Kerr–Newman black hole space-time.

Since that time, more precisely, during the last few decades, the behavior of relativistic spin-1/2 particles in a class of black hole space-times was studied using different approaches [19–30]. Using the solutions of the Dirac equation in different black hole space-times, many investigations, such as for example, the scattering of Dirac waves [25,31–33] and the existence of quasi-normal modes [33–36] were performed and many interesting results on the physics of black holes were achieved.

The modern concept of a black hole can be dated back to the 1930s, with the studies about the gravitational collapse of neutron stars conducted by Oppenheimer and Volkoff [37]. They discussed the limits of the mass of a neutron star such that a collapse should occur. At this time, this collapsed star was termed a frozen star. Only three decades after, the term black hole was introduced by John Wheeler [38,39]. These objects are uniquely characterized by their mass, angular momentum, and charge. This statement is known as the "no hair conjecture" [39].

In 1916, Schwarzschild published a solution to Einstein's equations [40,41] that a half-century later was identified as describing a black hole [42,43]. It was the first and also the simplest solution of the Einstein vacuum equations describing a black hole and represents a space-time that is asymptotically flat, whose source is a static, spherically symmetric, uncharged gravitational body.

Later on, in 1963, Kerr generalized the static solution corresponding to the Schwarzschild black hole to a rotating one [44]. Two years later, E. T. Newman et al. [45,46] presented a new solution to the Einstein–Maxwell equations which represents a charged rotating gravitational body. This solution was obtained using the Newman–Janis method [46] based on a complex coordinate transformation, which permits to obtain, in principle, the rotating solution from the static one.

In the early days of the Theory of General Relativity, Einstein predicted the existence of gravitational waves [47], which travel with the speed of light, by working out his equations in the weak-field regime. Einstein concluded that gravitational-wave amplitudes would be very small. For a long time, there was some skepticism in relation to the physical reality of gravitational waves [48]. Presently, there is no doubt about the existence of black holes, which is well established [49–53] according to the observation of the orbital motion of the S-stars in the center of the Milky Way [54–56], and more recently, through the observations of the Laser Interferometer Gravitational-Wave Observatory (LIGO) [49–57] that observed the first gravitational-wave signal GW150914 from a binary black hole merger [50]. More recently, another confirmation came out from an observation made by the Event Horizon Telescope of a black hole shadow in M87, which again confirms the existence of black holes [58].

The black hole solutions of Einstein field equations of general relativity we have known for a long time, and recently, we also got to know that they exist according to the observations concerning the first detection of associated gravitational waves by LIGO–VIRGO collaboration [50].

More than four decades ago, Letelier [59] proposed a model with a cloud of strings in the framework of general relativity and used this cloud as a source of the gravitational field. Thus, he obtained a class of solutions of the Einstein equations with different symmetries, namely, plane-symmetric, spherically, and cylindrically symmetric [59]. In the case of spherical symmetry, which is the object of our interest, the obtained solution is essentially a generalization of the Schwarzschild solution, in the sense that the solutions are similar, but with the horizon enlarged as compared with the Schwarzschild solution. This solution corresponds to a black hole surrounded by a spherically symmetric cloud of strings, which we are calling Letelier space-time [60]. Later on, Letelier extended his model [61] in order to include the pressure, and thus, a fluid of strings is considered rather than a cloud. In this case, the general solution for a fluid of strings with spherical symmetry was obtained.

The main motivation presented at that time to construct those models was based on the fact that the universe can be better represented, in principle, by a collection of extended objects, such as one-dimensional strings, rather than by point particles.

Otherwise, the great advantage to consider extended objects is the fact that they are potentially the best alternative to be used as the fundamental elements to describe physical phenomena that occur in the universe. From the gravitational point of view, it is important to investigate, for example, a black hole immersed in a cloud of strings due to the fact that these sources have astrophysical observable consequences [62–64]. Along this line of research, it was shown the role played by a cloud of strings that surrounds a Reissner–Nordstrom-AdS black hole on its thermodynamics [65].

Moreover, the quasinormal frequencies of the Reissner–Nordstrom black hole surrounded by quintessence and a cloud of strings, were investigated. The results tell us that the real and imaginary parts of this quantity depend on the parameter that codifies the presence of the cloud, in such a way that when this parameter increases, both the real and imaginary parts of the modes decrease, which means that the frequency of emission is smaller compared to the same system without the cloud of strings [66]. This seems to be an interesting result due to the fact that, in the astrophysical scenario, the most important effects arise from the lowest quasinormal frequencies. Thus, the time of emission is enlarged by the presence of the cloud of strings. The influence of a cloud of strings on the radii and stability of circular orbits was also investigated. The results indicate that the radii of the orbits increase as the values of the parameter associated with the cloud increase [67].

Other phenomena investigated are related to the accretion process onto the black hole with a string cloud background. It was found that the accretion is an explicit function of the black hole mass, as well as the gas boundary conditions and the string cloud. The mass accretion rate increases with the increasing of the values of the parameter which codifies the presence of the cloud of strings [68]. Thus, the accretion rate by the black hole surrounded by a cloud of strings is higher than that of a Schwarzschild black hole.

The influence of a cloud of strings in the black hole shadows, for example, was investigated in recent years and can be detected in astrophysical observations [69,70]. Studies concerning different aspects associated with the physics of a cloud of strings [64,71] and a fluid of strings [72] in the framework of general relativity have been performed during the last few decades.

It is worth emphasizing that strings have become a very important ingredient in many physical theories, and the idea of strings is fundamental in superstring theories [73]. The apparent relationship between counting string states and the entropy of the black hole horizon [63,74] suggests an association of strings with black holes. Furthermore, the intense level of activity in string theory has led to the idea that many of the classic vacuum scenarios, such as the static Schwarzschild black hole, may have atmospheres composed of strings [75].

In this framework, spin-1/2 particles are governed by the Dirac equation in the black hole space-times corresponding to a rotating and charged gravitating body surrounded by a cloud of strings, with the aim to understand what is the influence of the cloud of string on the particle and on the black hole itself.

In this paper, we restrict our investigation to the rotating and electrically charged body. The results corresponding to the limits of non-rotating and charged or uncharged cases can be obtained in a simple way, by taking the parameter that codifies the presence of the cloud of strings, equal to zero.

In Section 2, the solution of the Einstein equations corresponding to charged and rotating black holes surrounded by a cloud of strings is obtained. In Section 3, following straightforwardly the approach adopted by Kraniotis [23], we construct the Dirac equation, taking into account the massive case, in the Kerr–Newman black hole with a cloud of strings, and present the solution of the radial part, which is similar to the one obtained in [23]. In Section 4, the final remarks are presented.

Throughout this work we use units where $G = c = 1$ and a metric signature $(+,-,-,-)$. We use Greek letters μ, ν,... to denote space-time indices, Latin letters a, b, c ... to denote tetrad indices and Roman letters i, j, k,... to denote spatial indices $1, 2, 3$.

2. Black Holes with a Cloud of Strings

Different black hole models with the purpose of generalizing the current theoretic formulations of black holes for more elaborated scenarios, including electromagnetic field or a cloud of strings, were developed by Reissner–Nordstrom [76,77] and Letelier [59], respectively. Reissner–Nordstrom's black hole model considered a black hole in which the liquid charge was non-null and was surrounded by an electromagnetic field. Letelier, through a similar procedure to the one developed for the description of a cloud of particles approached by an incoherent perfect fluid invariant under reparametrization, developed a black hole model surrounded by a cloud of strings.

In order to incorporate Letelier's and Reissner–Nordstrom's models in our description, we should reproduce the obtaining of the energy-momentum tensors for the electromagnetic field, and for the cloud of strings, and then finally unify both of them in a unique formulation. We are going to treat each model individually in what follows.

Thus, let us consider writing the line element for the space-time in the neighborhood of a static and spherically symmetric source, without loss of generality, in the following form:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{1}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$.

2.1. Energy-Momentum Tensor for a Black Hole with a Cloud of Strings

The action that describes a string immersed in space-time according to the formulation proposed by Letelier [59] is:

$$s = \int L d\lambda^0 d\lambda^1, \tag{2}$$

$$L = \mathcal{M}\sqrt{-\gamma}, \tag{3}$$

where L is the lagrangian density, $\mathcal{M}$ is a dimensionless constant which features the string and

$$\gamma = \det \gamma_{AB}, \tag{4}$$

$$\gamma_{AB} = g_{\mu\nu}\frac{\partial x^{\mu}}{\partial \lambda^A}\frac{\partial x^{\nu}}{\partial \lambda^B}, \tag{5}$$

where $x^{\mu} = x^{\mu}(\lambda^A)$ describes the string world surface and $\lambda^A = (\lambda^0, \lambda^1)$ are the parameters of this world surface, with λ^0 being a time-like parameter and λ^1 a space-like parameter.

Associated with this string world surface, we have the bivector:

$$\Sigma^{\mu\nu} = \epsilon^{AB}\frac{\partial x^{\mu}}{\partial \lambda^A}\frac{\partial x^{\nu}}{\partial \lambda^B}, \tag{6}$$

where $\epsilon^{\alpha\beta}$ is the 2-dimensional Levi-Civita symbol, normalized as follows: $\epsilon^{01} = -\epsilon^{10} = 1$.

Combining Equations (3)–(5), for the Lagrangian density, we obtain the following result:

$$L = \mathcal{M}\left(-\frac{1}{2}\Sigma^{\mu\nu}\Sigma_{\mu\nu}\right)^{\frac{1}{2}}, \tag{7}$$

and therefore, the energy-momentum tensor for a single string is:

$$T^{\mu\nu} \equiv 2\frac{\partial}{\partial g_{\mu\nu}}L = \frac{\mathcal{M}}{\sqrt{-\gamma}}\Sigma^{\mu\beta}\Sigma^{\nu}_{\beta}. \tag{8}$$

Similarly to a cloud of particles [59], we will now consider the world surfaces for a cloud of strings as described by: $X^{\mu} = X^{\mu}(\lambda^A, \zeta, \eta)$, where ζ and η are the parameters that label a specific world surface and λ^A are the parameters that describe the evolution of this

world surface specifically. A cloud of strings is also featured by a proper density ρ_{cs}. The energy-momentum tensor for a cloud of strings in the most general case is accordingly:

$$T^{\mu\nu^{\mathrm{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}}\Sigma^{\mu\beta}\Sigma^{\nu}_{\beta}, \tag{9}$$

where the superscript "CS" refers to the cloud of strings.

However, the space-time symmetry considered in this framework narrows the density ρ_{cs} and the bivector $\Sigma_{\mu\nu}$ to be functions of r only. So, considering this symmetry, and the following invariance on reparametrization:

$$\lambda^0 \to \lambda^{0*} = \lambda^{0*}(\lambda^A, \zeta, \eta),$$
$$\lambda^1 \to \lambda^{1*} = \lambda^{1*}(\lambda^A, \zeta, \eta),$$

we concluded through an algebraic development made by Letelier [59], that the bivector $\Sigma_{\mu\nu}$ is constrained to have only Σ^{01} and Σ^{10} as its non-null components. This result also allows us to obtain the solution for the general conservation equation [59]:

$$\nabla_{\mu}(\rho\Sigma^{\mu\nu}) = 0, \tag{10}$$

as:

$$\Sigma^{01} = \frac{b}{\rho_{cs}r^2}e^{\frac{-(\lambda+\nu)}{2}}, \tag{11}$$

where b is an integration constant that will be associated with the presence of the cloud of strings. We also notice that the gauge invariant density $(-\gamma)^{\frac{1}{2}}\rho_{cs}$ has got the value:

$$(-\gamma)^{\frac{1}{2}}\rho_{cs} = \frac{b}{r^2}, \tag{12}$$

and therefore, b is a positive constant.

From Equations (1) and (11) for the metric tensor, we can calculate the non-null bivector components in their mixed form, which are listed below:

$$\Sigma^1_0 = \frac{b}{\rho_{cs}r^2}e^{\frac{(\nu-\lambda)}{2}}, \tag{13}$$
$$\Sigma^0_1 = \frac{b}{\rho_{cs}r^2}e^{\frac{(\lambda-\nu)}{2}}. \tag{14}$$

Having obtained all this information, we can explicit the non-null components of the energy-momentum tensor from Equation (9), which in its contravariant form are:

$$T^{00^{\mathrm{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}}\Sigma^{01}\Sigma^0_1 = \frac{b}{r^2}e^{-\nu}, \tag{15}$$
$$T^{11^{\mathrm{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}}\Sigma^{10}\Sigma^1_0 = -\frac{b}{r^2}e^{-\lambda}. \tag{16}$$

Finally, we find that the non-null components of the energy-momentum tensor, associated with the cloud of strings, are given by

$$T^{\mathrm{CS}}_{00} = \frac{b}{r^2}e^{\nu}, \tag{17}$$
$$T^{\mathrm{CS}}_{11} = -\frac{b}{r^2}e^{\lambda}. \tag{18}$$

2.2. Energy-Momentum Tensor for a Charged Black Hole

We know the fact that, from the electromagnetic theory, the energy-momentum tensor at the external region to an electric source is not zero. In fact, its symmetric form is given by:

$$T_{\mu\nu} = 2\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{19}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \tag{20}$$

is the electromagnetic tensor that describes the electromagnetic field in the neighborhood of any charged source, and A_{μ} is the four-vector electromagnetic potential.

Since we are describing here in this context a source (the black hole) with spherical symmetry and assuming that this configuration is also static, then, we must have that:

$$A_2 = A_3 = 0; A_0 = A_0(r,t), A_1 = A_1(r,t). \tag{21}$$

Thus, due to this potential, the most general components of the Maxwell tensor $F_{\mu\nu}$ in the description of the electromagnetic field must have the form

$$F_{01} = -F_{10} = \alpha(r,t), \tag{22}$$

$$F_{\mu\nu} = 0, \text{ for other components}, \tag{23}$$

which means that we are not considering the magnetic field.

In view of the metric given by Equation (1), we can also find the non-null components of this tensor in its contravariant form and in its mixed form through the following calculations:

$$F^{0}_{1} = -e^{-\nu}\alpha(r,t), \tag{24}$$

$$F^{1}_{0} = -e^{-\lambda}\alpha(r,t), \tag{25}$$

$$F^{01} = -e^{-(\lambda+\nu)}\alpha(r,t), \tag{26}$$

$$F^{10} = -e^{-(\nu+\lambda)}\alpha(r,t). \tag{27}$$

From the results above, the scalar $F_{\mu\rho}F^{\rho\mu}$ is given by

$$F_{\mu\rho}F^{\rho\mu} = -2e^{-(\nu+\lambda)}[\alpha(r,t)]^2. \tag{28}$$

Then, we can obtain the expression for the energy-momentum tensor by replacing the result obtained above, for the scalar $F_{\mu\rho}F^{\rho\mu}$, into the Equation (19). The results are the following:

$$T^{\mathrm{EM}}_{00} = -e^{-\lambda}[\alpha(r,t)]^2; \tag{29}$$

$$T^{\mathrm{EM}}_{11} = e^{-\nu}[\alpha(r,t)]^2; \tag{30}$$

$$T^{\mathrm{EM}}_{22} = -r^2e^{-(\nu+\lambda)}[\alpha(r,t)]^2; \tag{31}$$

$$T^{\mathrm{EM}}_{33} = T^{\mathrm{EM}}_{22}\sin^2\theta, \tag{32}$$

where the superscript "EM" refers to the electromagnetic field.

It remains to determine the function $\alpha(r,t)$ in order to completely explicit the energy-momentum tensor, for the electromagnetic field produced by a static and spherically symmetric electric source, in terms of the functions $\nu(r)$, $\lambda(r)$ and of the variable r. For this purpose, we should solve the Maxwell equations at the neighborhood of the electric source, where $J^{\mu} = 0$. The covariant form of Maxwell equations is:

$$\nabla_{\mu}F^{\mu\nu} = 0. \tag{33}$$

which leads us to the following equations:

$$\nabla_\mu F^{\mu 0} = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\alpha(r,t)\Big) = 0, \tag{34}$$

$$\nabla_\mu F^{\mu 1} = -\frac{\partial \alpha}{\partial t}(r,t) = 0. \tag{35}$$

From this second equation we can conclude that:

$$\alpha(r,t) = \alpha(r).$$

Meanwhile, the first one, which is the generalized Gauss Law, can be integrated over the whole volume containing the electric charge Q, resulting in:

$$\alpha(r) = \frac{Q}{r^2}. \tag{36}$$

Substituting this result into the Equations (29)–(32), we can finally present the explicit form of the energy-momentum tensor associated with the electromagnetic field produced by a static and spherically symmetric charged black hole. The non-null energy-momentum tensor components are listed below:

$$\begin{aligned} T^{\mathrm{EM}}_{00} &= -\frac{Q^2}{r^4}\exp(-\lambda); \\ T^{\mathrm{EM}}_{11} &= \frac{Q^2}{r^4}\exp(-\nu); \\ T^{\mathrm{EM}}_{22} &= -r^2\exp[-(\nu+\lambda)]\frac{Q^2}{r^4}; \\ T^{\mathrm{EM}}_{33} &= sin^2\theta T^{\mathrm{TM}}_{22}. \end{aligned} \tag{37}$$

We can use these results to show that the trace of $T_{\mu\nu}$ is identically zero, as expected for the electromagnetic field in four dimensions.

2.3. Line Element for a Static and Charged Black Hole with a Cloud of Strings

We have that for a static charged black hole with a cloud of strings, the total energy-momentum tensor $T_{\mu\nu}$ must be a combination of the contributions from each of its sources considered separately, the contribution from the electromagnetic field $T^{EM}_{\mu\nu}$ and the contribution from the cloud of strings $T^{\mathrm{CS}}_{\mu\nu}$, so:

$$T_{\mu\nu} = T^{\mathrm{CS}}_{\mu\nu} + T^{\mathrm{EM}}_{\mu\nu}, \tag{38}$$

and its non-null components are:

$$T_{00} = \frac{b}{r^2}e^{\nu} - \frac{Q^2}{r^4}e^{-\lambda}; \tag{39}$$

$$T_{11} = -\frac{b}{r^2}e^{\nu} + \frac{Q^2}{r^4}e^{-\nu}; \tag{40}$$

$$T_{22} = -r^2 e^{-(\nu+\lambda)}\frac{Q^2}{r^4}; \tag{41}$$

$$T_{33} = sin^2\theta T_{22}. \tag{42}$$

Thus, the Einstein's field equations read as follows:

$$e^{\nu}\left[e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}\right] = -e^{\nu}\left[\frac{b}{r^2}-\frac{Q^2}{r^4}e^{-(\lambda+\nu)}\right]; \quad (43)$$

$$-\left[\frac{1}{r^2}+\frac{\nu'}{r}\right]+\frac{e^{\lambda}}{r^2} = -e^{\lambda}\left[-\frac{b}{r^2}+\frac{Q^2}{r^4}e^{-(\nu+\lambda)}\right]; \quad (44)$$

$$-\frac{1}{2}r^2e^{-\lambda}\left[\nu''+\frac{(\nu')^2}{2}-\frac{\nu'\lambda'}{2}+\frac{\nu'-\lambda'}{r}\right] = -\frac{r^2}{2}\left[-2e^{-(\nu+\lambda)}\frac{Q^2}{r^4}\right]. \quad (45)$$

The $(3-3)$ component of Einstein's field equations leads us to the same result obtained by the $(2-2)$ component. Summing the Equation (43) multiplied by $e^{-\nu}$, with (44) multiplied by $e^{-\lambda}$, we obtain:

$$e^{-\lambda}\left(\frac{\nu'+\lambda'}{r}\right)=0,$$

that leads us to:

$$\nu'=-\lambda',$$

and, therefore:

$$\nu=-\lambda, \quad (46)$$

since ν and λ are parameters and can be reparameterized in a convenient way in order to eliminate the integration constant. Following a procedure similar to the one proposed by Kiselev [78], we define, without loss of generality, the parameters $\nu(r)$ and $\lambda(r)$ in terms of a function $f(r)$ (still to be obtained) through the following expression:

$$\nu=-\lambda=ln(1+f). \quad (47)$$

Inserting the result of Equation (46) into the Equation (43) or Equation (44), we find that:

$$e^{\nu}\left[\frac{1}{r^2}+\frac{\nu'}{r}\right]-\frac{1}{r^2}=-\frac{b}{r^2}+\frac{Q^2}{r^4},$$

which together with (47) and through a relatively simple algebraic development, leads us to:

$$\frac{1}{r^2}(f+rf')=-\frac{b}{r^2}+\frac{Q^2}{r^4}. \quad (48)$$

On the other hand, using (47) into (45), we obtain:

$$f''+2\frac{f'}{r}=2\frac{Q^2}{r^4}. \quad (49)$$

Summing Equation (48) with Equation (49), we finally obtain, after multiplying the whole equation by r^2, the following second-order ordinary differential equation for $f(r)$:

$$r^2f''+3rf'+f=-b+\frac{Q^2}{r^2}. \quad (50)$$

This equation can easily be solved by the standard differential calculus procedures, and its solution can be shown to be:

$$f(r)=-\frac{2M}{r}-b+\frac{Q^2}{r^2}. \quad (51)$$

Hence, the parameters $\nu(r)$ and $\lambda(r)$ are:

$$\nu=-\lambda=ln\left(1-b-\frac{2M}{r}+\frac{Q^2}{r^2}\right). \quad (52)$$

Defining the function $h(r)$ as:

$$h(r)=e^{\nu(r)}, \quad (53)$$

we can write the line element in the neighborhood of a static and charged black hole surrounded by a cloud of string as

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{54}$$

with

$$h(r) = 1 - b - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{55}$$

The constants M and Q are identified with the mass and electric charge of the black hole.

2.4. Rotating Charged Black Hole with Cloud of Strings

In order to obtain the metric for the charged rotating black hole with a cloud of strings, we need to start from the metric of the static charged black hole with a cloud of strings, which we obtained in the previous section (Equations (54) and (55)). Having this information, we will only need to do the Newman and Janis procedure [79] with this metric. As result, we will obtain our aimed solution for the charged rotating black hole with a cloud of strings.

In order to do so, firstly, we rewrite this metric into the Eddington–Finkelstein coordinates, by means of the transformation:

$$dv = dt - \frac{dr}{h(r)}, \tag{56}$$

and, thus, we obtain

$$ds^2 = h(r)dv^2 - dvdr - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{57}$$

Now, we determine the null tetrad basis which describes this metric as:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1, -\frac{h(r)}{2}, 0, 0\right), \\ m^a &= \frac{1}{r\sqrt{2}}\left(0, 0, 1, \frac{i}{sin\theta}\right), \\ \bar{m}^a &= (m^a)^*. \end{aligned} \tag{58}$$

In the next step of the Newman–Janis algorithm [79], we allow the coordinate r to take complex values, and then we formally perform the complex coordinate transformations:

$$\begin{aligned} v &\rightarrow v' = v + iacos\theta, \\ r &\rightarrow r' = r + iacos\theta, \\ \theta &\rightarrow \theta', \\ \phi &\rightarrow \bar{\phi}, \end{aligned} \tag{59}$$

on the null tetrad.

We also use the changes $h(r) \rightarrow H(r,\theta)$, and $r \rightarrow \rho$. Thus, the null tetrad is rewritten in the following form:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1,-\frac{H(r,\theta)}{2},0,0\right), \\ m^a &= \frac{1}{\sqrt{2}(r'+iacos\theta)}\left(-iasin\theta,-iasin\theta,1,\frac{i}{sin\theta}\right), \\ \bar{m}^a &= \frac{1}{\sqrt{2}(r'-iacos\theta)}\left(iasin\theta,iasin\theta,1,-\frac{i}{sin\theta}\right), \end{aligned} \tag{60}$$

where

$$H(r,\theta) = \frac{r^2h(r)+a^2\cos^2\theta}{\rho^2}, \tag{61}$$

This null tetrad above is the right null tetrad for the charged rotating black hole with a cloud of strings in the Eddington–Finkelstein coordinates system. However, we are willing to obtain the metric for this black hole in the Boyer–Lindquist coordinates. Therefore, we use the following transformations:

$$dv = dt + \lambda(r)dr, \quad d\bar{\phi} = d\phi + \chi(r)dr, \tag{62}$$

where

$$\lambda(r) = \frac{r^2+a^2}{r^2h(r)+a^2}, \quad \chi(r) = \frac{a}{r^2h(r)+a^2}, \tag{63}$$

are chosen such that the non-diagonal components of the metric are null, excepting g_{03} and g_{30}. We can also write

$$\lambda(r) = \frac{r^2+a^2}{\Delta_r}, \quad \chi(r) = \frac{a}{\Delta_r}, \tag{64}$$

where

$$\Delta_r = (r^2+a^2) - br^2 - 2Mr + Q^2, \tag{65}$$

Finally, we can obtain the metric for the charged rotating black hole with a cloud of strings, which its line element is written, in Boyer–Lindquist coordinates, as:

$$ds^2 = \frac{\Delta_r}{\rho^2}(dt - asin^2\theta d\phi)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}(adt - (r^2+a^2)d\phi)^2, \tag{66}$$

where $\rho^2 = r^2 + a^2cos^2\theta$ and $\Delta_r = (r^2+a^2) - br^2 - 2Mr + Q^2$, where a, b, M, Q, denote the angular momentum per unit mass, the cloud of strings parameter, mass, and the electric charge of the black hole, respectively.

The black hole horizons are determined by the condition $\Delta_r = 0$, or equivalently,

$$\Delta_r = (1-b)[(r-r_+)(r-r_-)] = 0,$$

with r_+ and r_- being the event horizon and Cauchy horizon, respectively, which are given by:

$$\begin{aligned} r_+ &= \frac{M}{1-b} + \frac{\sqrt{M^2-(1-b)(a^2+Q^2)}}{1-b}, \\ r_- &= \frac{M}{1-b} - \frac{\sqrt{M^2-(1-b)(a^2+Q^2)}}{1-b}. \end{aligned}$$

Note the role played by the cloud of strings on the horizons. As $0 < b < 1$, these horizons are enlarged and therefore all quantities written in terms of these horizons, for example, in the thermodynamics of this black hole are affected as well.

The metric (66) above is the most important result for this second section. It is a solution to the Einstein field equations with an electromagnetic field and a cloud of strings as their sources. Our main objective for this work is to write and solve the Dirac equation in this metric space-time, i.e., in the neighborhood of a charged rotating black hole with a cloud of strings. In order to do so, we would need, first of all, to obtain this metric. This initial goal, we can make sure that we have already accomplished.

In the next section, we will write and solve the general relativistic Dirac equation for the specific space-time scenario associated with the metric (66). Finally, in the fourth section, we will obtain some results related to this equation and its radial solutions, corresponding to massive and massless particles.

3. Dirac Equation in Kerr–Newman Black Hole Surrounded by a Cloud of Strings

3.1. Introduction

In this section, we study the behavior of spin-$\frac{1}{2}$ particles in the Kerr–Newman space-time surrounded by a cloud of strings, obtain the solutions of the radial part of the Dirac equation, and analyze the dependence of these solutions with the presence of the cloud of strings. It is worth calling attention to the fact that these radial solutions were obtained following straightforwardly the approach adopted by Kraniotis [23], in such a way that in the absence of a cloud of strings, the results are the same as those obtained in [23]. Therefore, the motivation to do these calculations is to investigate the signature of a cloud of strings on the solutions of the Dirac equation in the space-time under consideration.

3.2. Dirac Equation in the Kerr–Newman Black Hole with a Cloud of Strings

Now, let us write the Dirac equation in the black hole space-time described by the metric (66). To do this, we follow what is performed by Kraniotis [23], due to the similarity between the solution we are considering and the Kerr–Newman solution considered in [23]. All we have to do is perform the appropriate changes.

The general relativistic Dirac equation is given by (in units where $\hbar = c = 1$):

$$(i\gamma^{\mu}(x)\nabla_{\mu} - m)\Psi = 0, \tag{67}$$

where

$$\nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu}, \tag{68}$$

$$\gamma^{\mu} = e^{\mu}_{(a)}(x)\gamma^{(a)}, \tag{69}$$

and $\gamma^{(a)}$ are the constant Dirac matrices.

In order for the Dirac equation above to remain consistent with the 2-spinor form of the Dirac equation as considered by Chandrasekhar [17,18,23], we shall use the complex version of the Weyl (chiral) representation [80].

Let us consider a local Newman-Penrose [81] null tetrad frame that is defined by:

$$l^{\mu} = \left(\frac{r^2+a^2}{\Delta_r}, 1, 0, \frac{a}{\Delta_r}\right), \tag{70}$$

$$n^{\mu} = \left(\frac{r^2+a^2}{2\rho^2}, -\frac{\Delta_r}{2\rho^2}, 0, \frac{a}{2\rho^2}\right), \tag{71}$$

$$m^{\mu} = \frac{1}{\sqrt{2}(r+iacos\theta)}\left(iasin\theta, 0, 1, \frac{i}{sin\theta}\right), \tag{72}$$

$$\bar{m}^{\mu} = \frac{1}{\sqrt{2}(r-iacos\theta)}\left(-iasin\theta, 0, 1, -\frac{i}{sin\theta}\right). \tag{73}$$

Thus, the Kinnersley null tetrad coefficients, for the Kerr–Newman space-time with a cloud of strings are given by:

$$e^{\mu}_{(a)} = \begin{bmatrix} \frac{r^2+a^2}{\Delta_r} & \frac{r^2+a^2}{2\rho^2} & \frac{iasin\theta}{\sqrt{2}(r+iacos\theta)} & \frac{-iasin\theta}{\sqrt{2}(r-iacos\theta)} \\ 1 & -\frac{\Delta_r}{2\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}(r+iacos\theta)} & \frac{1}{\sqrt{2}(r-iacos\theta)} \\ \frac{a}{\Delta_r} & \frac{a}{2\rho^2} & \frac{i}{(r+iacos\theta)sin\theta\sqrt{2}} & \frac{-i}{(r-iacos\theta)sin\theta\sqrt{2}} \end{bmatrix}, \tag{74}$$

with the inverse given by:

$$e^{(a)}_{\mu} = (e^{\mu}_{(a)})^{-1} = \begin{bmatrix} \frac{\Delta_r}{2\rho^2} & \frac{1}{2} & 0 & \frac{-asin^2\theta\Delta_r}{2\rho^2} \\ 1 & -\frac{\rho^2}{\Delta_r} & 0 & -asin^2\theta \\ \frac{iasin\theta}{\sqrt{2}(r-iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r-iacos\theta)} & -\frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r-iacos\theta)} \\ \frac{-iasin\theta}{\sqrt{2}(r+iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r+iacos\theta)} & \frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r+iacos\theta)} \end{bmatrix}. \tag{75}$$

Using the results concerning the tetrads, we can calculate the matrices γ^{μ} to be inserted in Equation (67).

Let us write the solution of the Dirac equation as

$$\Psi(x) = \begin{bmatrix} \chi^{A}(x) \\ \eta_{B}(x) \end{bmatrix}, \tag{76}$$

where χ^{A} and $\bar{\eta}_{B}$ are two dimensional spinors.

Assuming that the azimuthal and time-dependence of the components of the spinor are of the form $e^{i(m\phi-\omega t)}$, and taking into account the following ansatz:

$$\chi^{(0)} = e^{i(m\phi-\omega t)}\frac{S^{(-)}(\theta)R^{(-)}(r)}{\sqrt{2}\bar{\rho}*}, \tag{77}$$

$$\chi^{(1)} = e^{i(m\phi-\omega t)}\frac{S^{(+)}(\theta)R^{(+)}(r)}{\sqrt{\Delta_r}}, \tag{78}$$

$$\eta^{(0)} = -e^{i(m\phi-\omega t)}\frac{S^{(+)}(\theta)R^{(-)}(r)}{\sqrt{2}\bar{\rho}}, \tag{79}$$

$$\eta^{(1)} = e^{i(m\phi-\omega t)}\frac{S^{(-)}(\theta)R^{(+)}(r)}{\sqrt{\Delta_r}}, \tag{80}$$

where $\bar{\rho} = r + iacos\theta$ and $\bar{\rho}^2 = \bar{\rho}\bar{\rho}^*$, we obtain the following ordinary differential equations for the radial and angular parts of the general relativistic Dirac equation in the Kerr–Newman black hole with a cloud of strings space-time:

$$\frac{dR^{(+)}}{dr}(r) + i\left(\frac{(\omega(r^2+a^2)-ma)+qer}{\Delta_r}\right)R^{(+)}(r) = \frac{(\lambda-i\mu r)}{\sqrt{\Delta_r}}R^{(-)}(r), \tag{81}$$

$$\frac{dR^{(-)}}{dr}(r) - i\left(\frac{(\omega(r^2+a^2)-ma)+qer}{\Delta_r}\right)R^{(-)}(r) = \frac{(\lambda+i\mu r)}{\sqrt{\Delta_r}}R^{(+)}(r), \tag{82}$$

$$\frac{dS^{(+)}}{d\theta}(\theta) + \left[\frac{m}{sin\theta} - \omega asin\theta + \frac{1}{2}cot\theta\right]S^{(+)}(\theta) = (-\lambda + \mu acos\theta)S^{(-)}(\theta), \tag{83}$$

$$\frac{dS^{(-)}}{d\theta}(\theta) - \left[\frac{m}{sin\theta} - \omega asin\theta - \frac{1}{2}cot\theta\right]S^{(-)}(\theta) = (\lambda + \mu acos\theta)S^{(+)}(\theta), \tag{84}$$

where λ is a separation constant and $\mu = \mu_*\sqrt{2}$.

Equations (81) and (82) determine the behavior of the radial functions $R^{+}(r)$ and $R^{-}(r)$, while Equations (83) and (84) describe the angular part of the spinor according to the ansatz expressed in Equations (77) to (80).

Let us consider, firstly, the equation corresponding to the angular part, given by Equations (83) and (84). Combining these two equations, we obtain a second-order equation, which is known as the Chandrasekhar–Page equation, and is given by:

$$\begin{aligned}&\left[\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d}{d\theta}\right)\mp\frac{a\mu sin\theta}{\lambda\mp\mu acos\theta}\frac{d}{d\theta}+\left(\frac{1}{2}\mp a\omega cos\theta\right)^2\right.\\&-\left(\frac{m\pm\frac{1}{2}cos\theta}{sin\theta}\right)^2+2a\omega m-a^2\omega^2\\&\left.\mp\frac{a\mu\left(\frac{1}{2}cos\theta\mp a\omega sin^2\theta\pm m\right)}{\lambda\mp a\mu cos\theta}-a^2\mu cos^2\theta+\lambda^2-\frac{3}{4}\right]S^{(\pm)}=0.\end{aligned}\tag{85}$$

As we can see, this differential equation, i.e., the angular part of the general relativistic Dirac equation in a Kerr–Newman black hole with a cloud of strings is analogous to the one obtained in the case of Kerr–Newman without a cloud of strings (Equation (40) in [23]). Therefore, all results obtained in Kraniotis [23] for the angular part of the general relativistic Dirac equation, whose solutions are given by Generalized Heun Functions, are exactly the same for the present case, due to the fact that there is no influence of the cloud of strings, in this case. Therefore, we will not consider the angular solutions due to the fact that they are the same obtained by Kraniotis [23] and have already been discussed in the literature [20].

With respect to the radial part of the Dirac equation it is affected by the presence of the cloud of strings surrounding the black hole, and for this reason, we will obtain the solutions and emphasize the role played by the cloud of strings as compared to the case in which this cloud is absent.

3.3. Solution of the Radial Equation

In this section, we follow straightforwardly the development made by Kraniotis [23] from the formal point of view. The qualitative analyzes associated with the presence of a cloud of strings for all black hole configurations are given from the obtained results.

Combining the Equations (81) and (82) in a similar way to what we have performed for the angular equation, we obtain the following radial equation for the $R^{(-)}$ mode:

$$\begin{aligned}&\Delta_r\frac{d^2R_{-\frac{1}{2}}(r)}{dr^2}+\left(\frac{1}{2}\frac{d\Delta_r}{dr}-\frac{i\mu\Delta_r}{\lambda+i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr}\\&+\left(K^2+\frac{i}{2}K\frac{d\Delta_r}{dr}\right)\frac{1}{\Delta_r}R_{-\frac{1}{2}}(r)-\frac{\mu K}{\lambda+i\mu r}R_{-\frac{1}{2}}(r)\\&+\left(-2i\omega r-iqe-\lambda^2-\mu^2r^2\right)R_{-\frac{1}{2}}(r)=0,\end{aligned}\tag{86}$$

where $K=K(r)\equiv(r^2+a^2)\omega-ma+eqr$, and $R_{-\frac{1}{2}}(r)\equiv R^{(-)}(r)$.

However, we know that Δ_r is defined in Equation (65), so if we differentiate that expression, we can rewrite (86) as follows:

$$\begin{aligned}&\frac{d^2R_{-\frac{1}{2}}(r)}{dr^2}+\left(\frac{(1-b)r-M}{\Delta_r}-\frac{i\mu}{\lambda+i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr}\\&+\left[\frac{K^2+iK((1-b)r-M)}{\Delta_r^2}\right]R_{-\frac{1}{2}}(r)-\frac{\mu K}{\Delta_r(\lambda+i\mu r)}R_{-\frac{1}{2}}(r)\\&+\frac{(-2i\omega r-iqe-\lambda^2-\mu^2r^2)}{\Delta_r}R_{-\frac{1}{2}}(r)=0.\end{aligned}\tag{87}$$

In terms of its roots, Δ_r can be written as:

$$\Delta_r=(1-b)[(r-r_+)(r-r_-)],\tag{88}$$

where

$$r_{\pm} = \frac{M}{(1-b)} \pm \sqrt{\frac{M^2}{(1-b)^2} - \frac{e^2+a^2}{(1-b)}}, \tag{89}$$

so finally the radial equation Equation (87) takes the form:

$$\begin{aligned} &\frac{d^2R_{-\frac{1}{2}}(r)}{dr^2} + \left(\frac{(1-b)r-M}{(1-b)[(r-r_+)(r-r_-)]} - \frac{i\mu}{\lambda+i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr} \\ &+ \left[\frac{K^2+iK((1-b)r-M)}{[(1-b)((r-r_+)(r-r_-))]^2}\right]R_{-\frac{1}{2}}(r) - \frac{\mu K}{(1-b)[(r-r_+)(r-r_-)](\lambda+i\mu r)}R_{-\frac{1}{2}}(r) \\ &+ \frac{(-2i\omega r - iqe - \lambda^2 - \mu^2r^2)}{(1-b)[(r-r_+)(r-r_-)]}R_{-\frac{1}{2}}(r) = 0. \end{aligned} \tag{90}$$

Now, we will apply the following transformation of variables to the Equation (90) [82]:

$$z = \frac{r-r_-}{r_+-r_-}, \tag{91}$$

which transforms the radii of the Cauchy horizon r_- and the event horizon r_+ to the points $z=0$ and $z=1$, respectively, and the singularity at $r_3 = \frac{i\lambda}{\mu}$ to the point $z_3 = \frac{r_3-r_-}{r_+-r_-}$.

With this transformation, the Equation (90) becomes:

$$R''(z) + \left(\sum_{i=1}^{2}\frac{\alpha_{i,b}}{(z-z_i)} - \frac{1}{z-z_3}\right)R'(z) + \left(\sum_{i=1}^{3}\frac{\beta_{i,b}}{(z-z_i)} + \sum_{i=1}^{2}\frac{\eta_{i,b}}{(z-z_i)^2} + \sigma_{0,b}\right)R(z) = 0, \tag{92}$$

where $R(z) \equiv R_{-\frac{1}{2}}(z)$, $z_1 = 0$ and $z_2 = 1$.

The coefficients appearing in (90) are given by

$$\alpha_{1,b} \equiv -\left(\frac{r_- - \frac{M}{(1-b)}}{r_+-r_-}\right), \tag{93}$$

$$\alpha_{2,b} \equiv \left(\frac{r_+ - \frac{M}{(1-b)}}{r_+-r_-}\right), \tag{94}$$

$$\begin{aligned} \beta_{1,b} \equiv &\left(iq\frac{e}{(1-b)} + 2ir_-\frac{\omega}{(1-b)} + \frac{\lambda^2}{(1-b)}\right) + \frac{r_-^2\mu^2}{(1-b)} - \frac{i\frac{\mathcal{H}_-}{(1-b)}}{z_3(r_- - r_+)} \\ &+ \frac{1}{(r_+-r_-)^2}\left[i\left(a^2(r_++r_-)\frac{\omega}{(1-b)} - 2(a^2+r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_-^2\frac{(3r_+-r_-)}{(1-b)}\omega\right)\right. \\ &\left.+i\left(\left(2a\frac{m}{(1-b)} - q(r_-+r_+)\frac{e}{(1-b)}\right)\frac{M}{(1-b)} - a(r_++r_-)\frac{m}{(1-b)} + 2qr_+r_-\frac{e}{(1-b)}\right)\right] \\ &+ \frac{1}{(r_+-r_-)^2}\left[2qa(r_++r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2q^2r_+r_-\frac{e^2}{(1-b)^2}\right. \\ &+ 2qr_-^2(3r_+-r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} + 4r_+r_-(a^2+r_-^2)\frac{\omega^2}{(1-b)^2} + 2a^2\frac{m^2}{(1-b)^2} \\ &\left.-4a(a^2+r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} + 2(a^4-r_-^4)\frac{\omega^2}{(1-b)^2}\right]; \end{aligned} \tag{95}$$

$$\begin{aligned}\beta_{2,b} \equiv & \left(-iq\frac{e}{(1-b)} - 2ir_+\frac{\omega}{(1-b)} - \frac{\lambda^2}{(1-b)}\right) - \frac{r_+^2\mu^2}{(1-b)} + \frac{i\frac{\mathcal{H}_+}{(1-b)}}{(z_3-1)(r_- - r_+)} \\ & + \frac{1}{(r_+-r_-)^2}\left[i\left(-a^2(r_++r_-)\frac{\omega}{(1-b)} + 2(a^2+r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_+^2\frac{(r_+-3r_-)}{(1-b)}\omega\right)\right. \\ & \left. + i\left(q(r_-+r_+)\frac{e}{(1-b)}\frac{M}{(1-b)} + a\left(r_++r_- - 2\frac{M}{(1-b)}\right)\frac{m}{(1-b)} - 2qr_+r_-\frac{e}{(1-b)}\right)\right] \\ & + \frac{1}{(r_+-r_-)^2}\left[-2qa(r_++r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) - 2q^2r_+r_-\frac{e^2}{(1-b)^2}\right. \\ & + 2qr_+^2(r_+-3r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} - 4r_+r_-(a^2+r_+^2)\frac{\omega^2}{(1-b)^2} - 2a^2\frac{m^2}{(1-b)^2} \\ & \left. + 4a(a^2+r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} - 2(a^4-r_+^4)\frac{\omega^2}{(1-b)^2}\right];\end{aligned} \tag{96}$$

$$\beta_{3,b} \equiv -\frac{\left[i\left(\frac{\omega}{(1-b)}(r_+-r_-)^2z_3^2 + 2r_-(r_+-r_-)\frac{\omega}{(1-b)}z_3 + q(r_+-r_-)\frac{e}{(1-b)}z_3 + \frac{\mathcal{H}_-}{(1-b)}\right)\right]}{(r_- - r_+)z_3(z_3-1)}; \tag{97}$$

$$\begin{aligned}\eta_{1,b} \equiv & \frac{1}{(r_+-r_-)^2}\left[i\left((a^2+r_-^2)\left(r_- - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} + \left(\frac{M}{(1-b)} - r_-\right)\left(a\frac{m}{(1-b)} - \frac{qr_-e}{(1-b)}\right)\right)\right. \\ & + qr_-\left(2a\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2r_-^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} + q^2r_-^2\frac{e^2}{(1-b)^2} \\ & + 2ar_-^2\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_-^4+a^4)\frac{\omega^2}{(1-b)^2} \\ & \left. -2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\right];\end{aligned} \tag{98}$$

$$\begin{aligned}\eta_{2,b} \equiv & \frac{1}{(r_+-r_-)^2}\left[i\left((a^2+r_+^2)\left(r_+ - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} + \left(\frac{M}{(1-b)} - r_+\right)\left(a\frac{m}{(1-b)} - \frac{qr_+e}{(1-b)}\right)\right)\right. \\ & + qr_+\left(2a\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2r_+^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} + q^2r_+^2\frac{e^2}{(1-b)^2} \\ & + 2ar_+^2\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_+^4+a^4)\frac{\omega^2}{(1-b)^2} \\ & \left. -2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\right];\end{aligned} \tag{99}$$

$$\sigma_{0,b} \equiv \left(\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}\right)(r_+-r_-)^2, \tag{100}$$

where $\mathcal{H}_\pm$ is given by

$$\mathcal{H}_\pm \equiv a^2\omega + eqr_\pm + \omega r_\pm^2 - am. \tag{101}$$

The indicial equation for the singularity at $z = z_1 = 0$ takes the form [82]:

$$\begin{aligned}F(s) &= s(s-1) + \alpha_{1,b}s + \eta_{1,b} \\ &= s^2 + \left(\frac{\frac{M}{(1-b)} - r_+}{r_+ - r_-}\right)s + \eta_{1,b}.\end{aligned} \tag{102}$$

Whose roots are:

$$\boldsymbol{\mu_1} \equiv s_{\pm}^{z=0} = \frac{-\left(\frac{\frac{M}{(1-b)}-r_+}{r_+-r_-}\right) \pm \sqrt{\left(\frac{\frac{M}{(1-b)}-r_+}{r_+-r_-}\right)^2 - 4\eta_{1,b}}}{2}. \tag{103}$$

Likewise, the indicial equation for the singularity at $z = z_2 = 1$ is [82]:

$$\begin{aligned} F(s) &= s(s-1) + \alpha_{2,b}s + \eta_{2,b} \\ &= s^2 + \left(\frac{r_- - \frac{M}{(1-b)}}{r_+ - r_-}\right)s + \eta_{2,b}, \end{aligned} \tag{104}$$

and its roots are:

$$\boldsymbol{\mu_2} \equiv s_{\pm}^{z=1} = \frac{\left(\frac{\frac{M}{(1-b)}-r_-}{r_+-r_-}\right) \pm \sqrt{\left(\frac{\frac{M}{(1-b)}-r_-}{r_+-r_-}\right)^2 - 4\eta_{2,b}}}{2}. \tag{105}$$

We apply now the F-homotopic transformation of $R(z)$ [82]:

$$R(z) = e^{\nu z} z^{\boldsymbol{\mu_1}} (z-1)^{\boldsymbol{\mu_2}} \bar{R}(z), \tag{106}$$

where $\nu = \pm i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)$.

Using Equation (106), the radial part of the Dirac equation in the curved space-time of the Kerr–Newman black hole with a cloud of strings (Equation (92)), turns into the following generalized Heun differential equation:

$$\left[\frac{d^2}{dz^2} + \left(\sum_{i=1}^{2}\frac{\alpha'_{i,b}}{(z-z_i)} - \frac{1}{z-z_3} + 2\nu\right)\frac{d}{dz} + \sum_{i=1}^{3}\frac{\beta'_{i,b}}{(z-z_i)}\right]\bar{R}(z) = 0, \tag{107}$$

where the coefficients in Equation (107) are:

$$\alpha'_{1,b} = 2\boldsymbol{\mu_1} + \left(\frac{\frac{M}{(1-b)} - r_-}{r_+ - r_-}\right), \tag{108}$$

$$\alpha'_{2,b} = 2\boldsymbol{\mu_2} + \left(\frac{r_+ - \frac{M}{(1-b)}}{r_+ - r_-}\right), \tag{109}$$

$$\begin{aligned} \beta'_{1,b} =& \beta_{1,b} + 2i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)\boldsymbol{\mu_1} + \alpha_{1,b}i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) - 2\boldsymbol{\mu_1}\boldsymbol{\mu_2} \\ &- \alpha_{1,b}\boldsymbol{\mu_2} - \alpha_{2,b}\boldsymbol{\mu_1} + \frac{\boldsymbol{\mu_1}}{z_3}, \end{aligned} \tag{110}$$

$$\begin{aligned} \beta'_{2,b} =& \beta_{2,b} + 2i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)\boldsymbol{\mu_2} + \alpha_{2,b}i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) + 2\boldsymbol{\mu_1}\boldsymbol{\mu_2} \\ &+ \alpha_{1,b}\boldsymbol{\mu_2} + \alpha_{2,b}\boldsymbol{\mu_1} - \frac{\boldsymbol{\mu_2}}{1-z_3}, \end{aligned} \tag{111}$$

$$\beta'_{3,b} = \beta_{3,b} - i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) - \frac{\boldsymbol{\mu_1}}{z_3} + \frac{\boldsymbol{\mu_2}}{1-z_3}. \tag{112}$$

With some algebra, Equation (107) can also be written in the general form of the generalized Heun differential equation as follows [82]:

$$\left[\frac{d^2}{dz^2}+\left(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{(z-1)}+\frac{1-\mu_2}{(z-z_3)}+\alpha\right)\frac{d}{dz}+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z(z-1)(z-z_3)}\right]\bar{R}(z)=0, \tag{113}$$

where:

$$\mu_0 = 1-2\boldsymbol{\mu_1}-\left(\frac{\frac{M}{(1-b)}-r_-}{r_+-r_-}\right), \tag{114}$$

$$\mu_1 = 1-2\boldsymbol{\mu_2}-\left(\frac{r_+-\frac{M}{(1-b)}}{r_+-r_-}\right), \tag{115}$$

$$\mu_2 = 2, \tag{116}$$

$$\beta_0 = \beta'_{1,b}z_3, \tag{117}$$

$$\beta_1 = -\beta'_{1,b}(1+z_3)-\beta'_{2,b}z_3-\beta'_{3,b}, \tag{118}$$

$$\beta_2 = \beta'_{1,b}+\beta'_{2,b}+\beta'_{3,b}, \tag{119}$$

$$\alpha = 2\nu. \tag{120}$$

Therefore, the solutions for the radial part, given by Equation (107), are written in terms of Generalized Heun Functions:

$$\begin{aligned} R_{-\frac{1}{2}}(z) =& e^{\alpha_{1,b}z}z^{\alpha_{2,b}}(z-1)^{\alpha_{3,b}}[\mathrm{HeunG}(z_3,\beta_{1,b},\beta_{2,b},\beta_{3,b},\alpha_{2,b},\alpha_{3,b};z) \\ &+\sigma_{0,b}z^{1-\alpha_{2,b}}\mathrm{HeunG}(z_3,\beta'_{1,b},\beta'_{2,b},\beta'_{3,b},\alpha'_{1,b},\alpha'_{2,b};z)\Big]. \end{aligned} \tag{121}$$

Let us close this section by examining the asymptotic behavior of the solutions at infinity $r\to\infty$. Firstly, let us write Equation (113) in the following canonical form

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{(z-1)}+\frac{\epsilon}{(z-z_3)}\right)\frac{du}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-z_3)}u=0, \tag{122}$$

where the parameters $\alpha,\beta,\gamma,\delta,\epsilon,q$ and z_3 obey the following relation:

$$\gamma+\delta+\epsilon=\alpha+\beta+1. \tag{123}$$

Consider γ as a not integer number. Then, mathematical results concerning the Heun equation given by (113), which is a differential equation with three regular singularities at z_i, with $(i=1,2,3)$, and an irregular singularity at infinity, assure that the Generalized Heun Function $\mathrm{HeunG}(a,q;\alpha,\beta,\gamma,\delta;z)$ exists in a region $|z|<1$, and can be expanded in a Maclaurin series given by:

$$\mathrm{HeunG}(a,q;\alpha,\beta,\gamma,\delta;z)=\sum_{j=0}^{\infty}b_jz^j, \tag{124}$$

where b_0 can be assumed as equal to 1 and

$$a\gamma b_1-qb_0=0, \tag{125}$$

$$X_jb_{j+1}-(Q_j+q)b_j+P_jb_{j-1}=0\,,\ j\geq 1. \tag{126}$$

with

$$P_j = (j-1+\alpha)(j-1+\beta), \tag{127}$$

$$Q_j = j[(j-1+\gamma)(1+a)+a\delta+\epsilon], \tag{128}$$

$$X_j = a(j+1)(j+\gamma). \tag{129}$$

Thus, following the discussions made by Kraniotis [23], we can write the following results

$$R_1^{\infty}(r) \sim e^{\pm i\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}(r-r_-)}\left(\frac{r-r_-}{r_+-r_-}\right)^{-\left(\frac{\beta_{1,b}+\beta_{2,b}+\beta_{3,b}}{\pm 2i(r_+-r_-)\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}}\right)}, \tag{130}$$

and

$$R_2^{\infty}(r) \sim e^{\mp i\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}(r-r_-)}\left(\frac{r-r_-}{r_+-r_-}\right)^{\left(\frac{\beta_{1,b}+\beta_{2,b}+\beta_{3,b}}{\pm 2i(r_+-r_-)\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}}\right)}. \tag{131}$$

Note that now, the solutions depend explicitly on the parameter that codifies the presence of the cloud of strings, and when $b = 0$ (absence of the cloud of strings) the results given by Kraniotis [23] are recovered, as expected.

As we have seen in this section, the radial Equation (86), coming from the Dirac Equation (67) for a spin-1/2 particle in the vicinity of a charged rotating black hole surrounded by a cloud of strings, have been reduced to a generalized Heun differential Equation (92). It is worth emphasizing that the analytical solutions given by Equation (121) are very complicated and therefore it is not so simple to work with them, nonetheless, it is possible to perform a qualitative analysis, especially, if we choose appropriately some of parameters which enter the metric considered.

4. Final Remarks

In this work, we obtained the solution corresponding to a charged and rotating black hole surrounded by a cloud of strings, which we are calling the Kerr–Newman black hole with a cloud of strings, and we investigated the behavior of spin-$\frac{1}{2}$ massive particles when interacting with this black hole.

We obtained the solutions of the radial part of the Dirac equation and showed, following the approach adopted by Kraniotis[23], that they are given in terms of the Generalized Heun Function. As a conclusion, we can say that the radial solutions are very complicated to be manipulated mathematically, but on the other hand, we can see clearly that they are strongly affected by the cloud of strings. As a consequence, all quantities associated to the spin-1/2 particle, such as the current would be affected by the presence of the cloud of strings.

With respect to the system, namely, a Dirac particle interacting with this black hole, all information arising from this interaction depends strongly on the cloud of stings, as for example, concerning the scattering, distribution of emitted particles, shadow, accretion and so on.

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