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Dynamic Behaviors of a COVID-19 and Influenza Co-Infection Model with Time Delays and Humoral Immunity

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Abstract: Co-infections with respiratory viruses were reported in hospitalized patients in several cases. Severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) and influenza A virus (IAV) are two respiratory viruses and are similar in terms of their seasonal occurrence, clinical manifestations, transmission routes, and related immune responses. SARS-CoV-2 is the cause of coronavirus disease 2019 (COVID-19). In this paper, we study the dynamic behaviors of an influenza and COVID-19 co-infection model in vivo. The role of humoral (antibody) immunity in controlling the co-infection is modeled. The model considers the interactions among uninfected epithelial cells (ECs), SARS-CoV-2-infected ECs, IAV-infected ECs, SARS-CoV-2 particles, IAV particles, SARS-CoV-2 antibodies, and IAV antibodies. The model is given by a system of delayed ordinary differential equations (DODEs), which include four time delays: (i) a delay in the SARS-CoV-2 infection of ECs, (ii) a delay in the IAV infection of ECs, (iii) a maturation delay of newly released SARS-CoV-2 virions, and (iv) a maturation delay of newly released IAV virions. We establish the non-negativity and boundedness of the solutions. We examine the existence and stability of all equilibria. The Lyapunov method is used to prove the global stability of all equilibria. The theoretical results are supported by performing numerical simulations. We discuss the effects of antiviral drugs and time delays on the dynamics of influenza and COVID-19 co-infection. It is noted that increasing the delay length has a similar influence to that of antiviral therapies in eradicating co-infection from the body.

Keywords: influenza; COVID-19; co-infection; time delay; global stability; Lyapunov function

MSC: 34D20; 34D23; 37N25; 92B05

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1. Introduction

Global health and economies have been severely affected since the emergence of the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) in December 2019. This virus causes coronavirus disease 2019 (COVID-19), which swept the whole world with its rapid spread. Based on an update given by the World Health Organization (WHO) on 18 December 2022 [1], about 649 million confirmed cases and over 6.6 million deaths were reported globally. Influenza is another infectious respiratory disease that can cause serious morbidity and death. Influenza viruses of types A, B, C, and D infect about 20% of the world's population in annual epidemics, resulting in 3–5 million severe illnesses and 290,000–650,000 deaths each year [2]. Influenza A virus (IAV) usually occurs in winter and is able to infect many species.

Epithelial cells (ECs) are the targets of both IAV and SARS-CoV-2 [3,4]. Both viruses have similar transmission paths. In addition, they have quite similar clinical manifestations, such as cough, myalgia, dyspnea, sore throat, headache, fever, and rhinitis [5]. Viral shedding often occurs within five to ten days in influenza, while it takes two to five weeks in COVID-19 [5]. Acute respiratory distress occurs more frequently in patients with COVID-19 than in those with influenza [5]. Less than 1% of influenza cases may die, while the death rate among COVID-19 patients is 3–4% [5]. The precautionary measures implemented by governments to limit the transmission of COVID-19 can play a role in reducing the transmission of influenza [6]. Currently, there are eleven vaccines for COVID-19 [7] and three types of influenza vaccines used worldwide [8].

One study by Zhu et al. [9] reported that 94.2% of COVID-19-infected individuals were also co-infected with many other types of microorganisms, such as viruses, fungi, and bacteria. Many co-infections of COVID-19 and influenza were reported in several studies [5,9–12] (see also the review articles [13–16]). Infection with multiple competitive respiratory viruses can cause the phenomenon of viral interference. It may happen that a certain type of virus has the ability to suppress the development and growth of another virus [17,18]. In [18,19], it was found that SARS-CoV-2 had a slower growth rate than that of IAV if the two infections started at the same time. If the influenza infection started after COVID-19, then influenza and COVID-19 co-infection could be detected. The progression and outcome of COVID-19 are highly dependent on a patient's immunity. The risk of co-infection may be increased for persons who are immunocompromised [17]. In addition, Hashemi et al. [20] conducted a study that reported that, in patients with co-infection of influenza and COVID-19, the presence of underlying diseases, such as chronic neurological pathologies, diabetes, asthma, and heart disease, may lead to an increase in mortality.

Mathematical models of mono-infection or co-infection of viruses are important for understanding in-host viral infections and for developing antiviral drugs and vaccines. Models of in-host influenza mono-infection were formulated in many works (see the review papers [21–24]). Baccam et al. [25] presented a basic target-cell-limited influenza infection model. Several extensions were made for this model by incorporating the impacts of innate immunity [25,26], adaptive immunity [27,28], both innate and adaptive immunities [3,29–32], drug therapies [33], and time delays [34].

The model presented in [25] was used to describe the in-host COVID-19 dynamics in [35]. Li et al. [36] considered the regeneration and death of susceptible ECs. A model that was limited to target cells and a model with the regeneration and death of susceptible ECs were presented, respectively, in [35,36], where they were modified and extended by taking into account the influences of immune response [37–44], drug therapies [45–47], time delays [48], and reaction diffusion [49]. In [50], a two-state mathematical model of within-host SARS-CoV-2-neutralizing antibody dynamics in response to vaccination was considered. The stability of in-host COVID-19 mono-infection models was investigated in [41–43,48,51,52].

Pinky and Dobrovolny [18] constructed a SARS-CoV-2/IAV co-infection model that was limited to target cells. The authors mentioned that some types of respiratory viruses may be able to inhibit the progression of SARS-CoV-2. In [18], the effect of the immune response was not included. Moreover, the production and death of susceptible ECs were not considered. Elaiw et al. [53,54] examined the global properties of a SARS-CoV-2/IAV co-infection model with antibody immune response. However, a time delay was not considered in these papers. Time delay is one of the key factors for studying innovative insights into viral dynamics. In the process of SARS-CoV-2 and IAV infections, it takes time for the viruses to infect susceptible ECs and then release new mature viral particles. Therefore, it is important to include a time delay in COVID-19 and influenza co-infection models. The aim of this article is to construct a system of delayed differential equations (DDEs) that describe the in-host co-dynamics of influenza and COVID-19. The model extends the model presented in [53] by incorporating four time delays: (i) a delay in the SARS-CoV-2 infection of ECs, (ii) a delay in the IAV infection of ECs, (iii) a maturation delay of newly released SARS-CoV-2 virions, and (iv) a maturation delay of newly released IAV virions. We first investigate the basic properties of the DDEs; then, we find all equilibria and examine their global stability. We illustrate the theoretical results via numerical simulations. The effects of time delays on the dynamics of COVID-19 and influenza co-infection are discussed.

2. Model Formulation

This section develops a system of DDEs that describe influenza and COVID-19 co-infection with four time delays. Let t represent the time and let $X(t)$, $Y(t)$, $I(t)$, $V(t)$, $P(t)$, $Z(t)$, and $M(t)$ represent the concentrations of susceptible ECs, SARS-CoV-2-infected ECs, IAV-infected ECs, SARS-CoV-2 particles, IAV particles, SARS-CoV-2 antibodies, and IAV antibodies. The following system of DDEs is to be studied:

$$\frac{dX(t)}{dt} = \overbrace{\delta}^{\text{ECs production}} - \overbrace{\varrho X(t)}^{\text{natural death}} - \overbrace{\xi_V X(t)V(t)}^{\text{SARS-CoV-2 infectious transmission}} - \overbrace{\xi_P X(t)P(t)}^{\text{IAV infectious transmission}}, \tag{1}$$

$$\frac{dY(t)}{dt} = \overbrace{e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1)}^{\text{SARS-CoV-2 infectious transmission}} - \overbrace{\beta_Y Y(t)}^{\text{natural death}}, \tag{2}$$

$$\frac{dI(t)}{dt} = \overbrace{e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3)}^{\text{IAV infectious transmission}} - \overbrace{\beta_I I(t)}^{\text{natural death}}, \tag{3}$$

$$\frac{dV(t)}{dt} = \overbrace{e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2)}^{\text{SARS-CoV-2 production}} - \overbrace{\lambda_V V(t)}^{\text{natural death}} - \overbrace{\rho_V V(t)Z(t)}^{\text{SARS-CoV-2 neutralization}}, \tag{4}$$

$$\frac{dP(t)}{dt} = \overbrace{e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4)}^{\text{IAV production}} - \overbrace{\lambda_P P(t)}^{\text{natural death}} - \overbrace{\rho_P P(t)M(t)}^{\text{IAV neutralization}}, \tag{5}$$

$$\frac{dZ(t)}{dt} = \overbrace{\eta_Z V(t)Z(t)}^{\text{proliferation SARS-CoV-2 antibodies}} - \overbrace{\gamma_Z Z(t)}^{\text{natural death}}, \tag{6}$$

$$\frac{dM(t)}{dt} = \overbrace{\eta_M P(t)M(t)}^{\text{proliferation IAV antibodies}} - \overbrace{\gamma_M M(t)}^{\text{natural death}}. \tag{7}$$

Here, τ_1 and τ_3 are the delays between the entries of SARS-CoV-2 and IAV into ECs and the start of production of immature SARS-CoV-2 and IAV virions, respectively. τ_2 and τ_4 are the maturation delays of newly released SARS-CoV-2 and IAV virions, respectively. The probabilities of SARS-CoV-2-infected ECs and IAV-infected ECs surviving to the ages of τ_1 and τ_3 are represented by $e^{-\alpha_1\tau_1}$ and $e^{-\alpha_3\tau_3}$, respectively. The probabilities of released SARS-CoV-2 and IAV virions surviving to the ages τ_2 and τ_4 are denoted by $e^{-\alpha_2\tau_2}$ and $e^{-\alpha_4\tau_4}$, respectively.

The initial states (conditions) for system (1)–(7) are given as:

$$\begin{aligned}
&X(u) = \psi_1(u), \quad Y(u) = \psi_2(u), \quad I(u) = \psi_3(u), \quad V(u) = \psi_4(u), \\
&P(u) = \psi_5(u), \quad Z(u) = \psi_6(u), \quad M(u) = \psi_7(u), \\
&\psi_i(u) \geq 0, \quad u \in [-\tau^*, 0], \\
&\psi_i(u) \in C([-\tau^*, 0], \mathbb{R}_{\geq 0}), \quad i = 1, 2, \ldots, 7,
\end{aligned} \tag{8}$$

where $\tau^* = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$, and C is the Banach space of continuous functions mapping the interval $[-\tau^*, 0]$ into $\mathbb{R}_{\geq 0}$ with $\|\psi_i\| = \sup_{-\tau^* \leq u \leq 0} |\psi_i(u)|$ for $\psi_i \in C$. We note that system (1)–(7), with initial conditions (8), has a unique solution [55].

3. Well-Posedness of the Solutions

Here, we investigate the non-negativity and ultimate boundedness of system (1)–(7).

Lemma 1. *The solutions of system (1)–(7) with initial states (8) are non-negative and ultimately bounded.*

Proof. We have that

$$\frac{dX}{dt}\mid_{X=0}=\delta>0,\quad \frac{dZ}{dt}\mid_{Z=0}=0,\quad \frac{dM}{dt}\mid_{M=0}=0.$$

Hence, $X(t), Z(t), M(t) \geq 0$ for all $t \geq 0$. Moreover, for all $t \in [0, \tau^*]$, we have:

$$Y(t)=\psi_2(0)e^{-\beta_Y t}+\xi_V e^{-\alpha_1\tau_1}\int_0^t e^{-\beta_Y(t-u)}X(u-\tau_1)V(u-\tau_1)du \geq 0,$$

$$I(t)=\psi_3(0)e^{-\beta_I t}+\xi_P e^{-\alpha_3\tau_3}\int_0^t e^{-\beta_I(t-u)}X(u-\tau_3)P(u-\tau_3)du \geq 0,$$

$$V(t)=\psi_4(0)e^{-\int_0^t(\lambda_V+\rho_V Z(r))dr}+\theta_V e^{-\alpha_2\tau_2}\int_0^t e^{-\int_u^t(\lambda_V+\rho_V Z(r))dr}Y(u-\tau_2)du \geq 0,$$

$$P(t)=\psi_5(0)e^{-\int_0^t(\lambda_P+\rho_P M(r))dr}+\theta_P e^{-\alpha_4\tau_4}\int_0^t e^{-\int_u^t(\lambda_P+\rho_P M(r))dr}I(u-\tau_4)du \geq 0.$$

Hence, $Y(t), I(t), V(t), P(t) \geq 0$ for all $t \in [0, \tau^*]$. Through recursive argumentation, we get $Y(t), I(t), V(t), P(t)$ for all $t \geq 0$. Therefore, X, Y, I, V, P, Z, and M are non-negative.

The non-negativity of the system's solution implies that:

$$\frac{dX(t)}{dt} \leq \delta-\varrho X \Longrightarrow \lim_{t\to\infty}\sup X(t)=\frac{\delta}{\varrho}.$$

Let us define

$$\Psi_1(t)=e^{-\alpha_1\tau_1}X(t-\tau_1)+Y(t).$$

Then,

$$\begin{aligned}\frac{d\Psi_1(t)}{dt}&=e^{-\alpha_1\tau_1}\delta-e^{-\alpha_1\tau_1}\varrho X(t-\tau_1)-e^{-\alpha_1\tau_1}\xi_P X(t-\tau_1)P(t-\tau_1)-\beta_Y Y(t)\\ &\leq \delta-e^{-\alpha_1\tau_1}\varrho X(t-\tau_1)-\beta_Y Y(t)\\ &\leq \delta-\varphi_1\big[e^{-\alpha_1\tau_1}X(t-\tau_1)+Y(t)\big]=\delta-\varphi_1\Psi_1(t),\end{aligned}$$

where $\varphi_1=\min\{\varrho,\beta_Y\}$. This implies that

$$\lim_{t\to\infty}\sup \Psi_1(t) \leq \frac{\delta}{\varphi_1}=A_1 \Longrightarrow \lim_{t\to\infty}\sup Y(t) \leq A_1.$$

Let

$$\begin{aligned}\Psi_2(t)&=e^{-\alpha_3\tau_3}X(t-\tau_3)+I(t)\\ \frac{d\Psi_2(t)}{dt}&=e^{-\alpha_3\tau_3}\delta-e^{-\alpha_3\tau_3}\varrho X(t-\tau_3)-e^{-\alpha_3\tau_3}\xi_V X(t-\tau_3)V(t-\tau_3)-\beta_I I(t)\\ &\leq \delta-e^{-\alpha_3\tau_3}\varrho X(t-\tau_3)-\beta_I I(t)\\ &\leq \delta-\varphi_2\big[e^{-\alpha_3\tau_3}X(t-\tau_3)+I(t)\big]=\delta-\varphi_2\Psi_2(t),\end{aligned}$$

where $\varphi_2=\min\{\varrho,\beta_I\}$. It follows that

$$\lim_{t\to\infty}\sup \Psi_2(t) \leq \frac{\delta}{\varphi_2}=A_2 \Longrightarrow \lim_{t\to\infty}\sup I(t) \leq A_2.$$

Let us define

$$\Psi_3(t) = V(t) + P(t) + \frac{\rho_V}{\eta_Z}Z(t) + \frac{\rho_P}{\eta_M}M(t).$$

$$\frac{d\Psi_3(t)}{dt} = e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V(t) + e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) - \lambda_P P(t) - \frac{\rho_V\gamma_Z}{\eta_Z}Z(t) - \frac{\rho_P\gamma_M}{\eta_M}M(t).$$

Since $Y(t) \le A_1, I(t) \le A_2$, then

$$\begin{aligned}\frac{d\Psi_3(t)}{dt} &\le \theta_V A_1 + \theta_P A_2 - \lambda_V V(t) - \lambda_P P(t) - \frac{\rho_V\gamma_Z}{\eta_Z}Z(t) - \frac{\rho_P\gamma_M}{\eta_M}M(t)\\ &\le \theta_V A_1 + \theta_P A_2 - \varphi_3\left[V(t) + P(t) + \frac{\rho_V}{\eta_Z}Z(t) + \frac{\rho_P}{\eta_M}M(t)\right]\\ &= \theta_V A_1 + \theta_P A_2 - \varphi_3\Psi_3(t)\end{aligned}$$

where $\varphi_3 = \min\{\lambda_V, \lambda_P, \gamma_Z, \gamma_M\}$. Then, we get

$$\lim_{t\to\infty}\sup \Psi_3(t) \le \frac{\theta_V A_1 + \theta_P A_2}{\varphi_3} = A_3.$$

Since $V(t) > 0, P(t) > 0, Z(t) > 0$ and $M(t) > 0$, then

$$\begin{aligned}&\lim_{t\to\infty}\sup V(t) \le A_3, \quad \lim_{t\to\infty}\sup P(t) \le A_3,\\ &\lim_{t\to\infty}\sup Z(t) \le \frac{\eta_Z}{\rho_V}A_3 = A_4 \quad \text{and} \quad \lim_{t\to\infty}\sup M(t) \le \frac{\eta_M}{\rho_P}A_3 = A_5.\end{aligned}$$

□

Based on Lemma 1, we can show that the domain

$$\Phi = \{(X,Y,I,V,P,Z,M) \in C^7_{\ge 0} : \|X\|, \|Y\| \le A_1,\ \|I\| \le A_2,\ \|V\|, \|P\| \le A_3,\ \|Z\| \le A_4,\ \|M\| \le A_5\}$$

is positively invariant for model (1)–(7).

4. Equilibria

Here, we calculate the system's equilibria and deduce the condition of their existence. Any equilibrium point $\Delta = (X,Y,I,V,P,Z,M)$ satisfies:

$$0 = \delta - \varrho X - \xi_V XV - \xi_P XP, \tag{9}$$
$$0 = e^{-\alpha_1\tau_1}\xi_V XV - \beta_Y Y, \tag{10}$$
$$0 = e^{-\alpha_3\tau_3}\xi_P XP - \beta_I I, \tag{11}$$
$$0 = e^{-\alpha_2\tau_2}\theta_V Y - \lambda_V V - \rho_V VZ, \tag{12}$$
$$0 = e^{-\alpha_4\tau_4}\theta_P I - \lambda_P P - \rho_P PM, \tag{13}$$
$$0 = \eta_Z VZ - \gamma_Z Z, \tag{14}$$
$$0 = \eta_M PM - \gamma_M M. \tag{15}$$

Solving Equations (9)–(15), we get eight equilibria.
(i) Infection-free equilibrium, $\Delta_0 = (X_0, 0, 0, 0, 0, 0, 0)$, where $X_0 = \delta/\varrho$.

(ii) COVID-19 mono-infection equilibrium with inactive antibody response $\Delta_1 = (X_1, Y_1, 0, V_1, 0, 0, 0)$, where

$$X_1 = \frac{\beta_Y \lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V \xi_V}, \quad Y_1 = \frac{\varrho\lambda_V}{e^{-\alpha_2\tau_2}\theta_V\xi_V}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\beta_Y\lambda_V} - 1\right],$$
$$V_1 = \frac{\varrho}{\xi_V}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\beta_Y\lambda_V} - 1\right].$$

Therefore, $Y_1 > 0$ and $V_1 > 0$ when

$$\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\beta_Y\lambda_V} > 1.$$

We define the basic COVID-19 mono-infection reproduction number as:

$$\Re_1 = \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\beta_Y\lambda_V}.$$

Thus, we can write:

$$X_1 = \frac{X_0}{\Re_1}, \ \ Y_1 = \frac{\varrho\lambda_V}{e^{-\alpha_2\tau_2}\theta_V\xi_V}(\Re_1 - 1), \ \ V_1 = \frac{\varrho}{\xi_V}(\Re_1 - 1).$$

Consequently, Δ_1 exists if $\Re_1 > 1$.

(iii) Influenza mono-infection equilibrium with inactive antibody response, $\Delta_2 = (X_2, 0, I_2, 0, P_2, 0, 0)$, where

$$X_2 = \frac{\beta_I \lambda_P}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P \xi_P}, \quad I_2 = \frac{\varrho\lambda_P}{e^{-\alpha_4\tau_4}\theta_P\xi_P}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\beta_I\lambda_P} - 1\right],$$
$$P_2 = \frac{\varrho}{\xi_P}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\beta_I\lambda_P} - 1\right].$$

Therefore, $I_2 > 0$ and $P_2 > 0$ when

$$\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\beta_I\lambda_P} > 1.$$

We define the basic influenza mono-infection reproduction number as:

$$\Re_2 = \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\beta_I\lambda_P}.$$

In terms of $\Re_2$, we write

$$X_2 = \frac{X_0}{\Re_2}, \ \ I_2 = \frac{\varrho\lambda_P}{e^{-\alpha_4\tau_4}\theta_P\xi_P}(\Re_2 - 1), \ \ P_2 = \frac{\varrho}{\xi_P}(\Re_2 - 1).$$

Therefore, Δ_2 exists if $\Re_2 > 1$.

(iv) COVID-19 mono-infection equilibrium with activated SARS-CoV-2-specific antibody response, $\Delta_3 = (X_3, Y_3, 0, V_3, 0, Z_3, 0)$, where

$$X_3 = \frac{\delta\eta_Z}{\xi_V\gamma_Z + \varrho\eta_Z}, \quad Y_3 = \frac{e^{-\alpha_1\tau_1}\delta\xi_V\gamma_Z}{\beta_Y(\xi_V\gamma_Z + \varrho\eta_Z)},$$
$$V_3 = \frac{\gamma_Z}{\eta_Z}, \quad Z_3 = \frac{\lambda_V}{\rho_V}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\eta_Z\theta_V}{\beta_Y\lambda_V(\xi_V\gamma_Z + \varrho\eta_Z)} - 1\right].$$

We note that Δ_3 exists when

$$\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\eta_Z\theta_V}{\beta_Y\lambda_V(\xi_V\gamma_Z+\varrho\eta_Z)}>1.$$

Let us define the SARS-CoV-2-specific antibody response activation number for COVID-19 mono-infection as:

$$\Re_3=\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\eta_Z\theta_V}{\beta_Y\lambda_V(\xi_V\gamma_Z+\varrho\eta_Z)}.$$

Thus, $Z_3=\frac{\lambda_V}{\rho_V}(\Re_3-1)$.

(v) Influenza mono-infection equilibrium with activation of of IAV-specific antibody response, $\Delta_4=(X_4,0,I_4,0,P_4,0,M_4)$, where

$$X_4=\frac{\delta\eta_M}{\xi_P\gamma_M+\varrho\eta_M},\quad I_4=\frac{e^{-\alpha_3\tau_3}\delta\xi_P\gamma_M}{\beta_I(\xi_P\gamma_M+\varrho\eta_M)},$$
$$P_4=\frac{\gamma_M}{\eta_M},\quad M_4=\frac{\lambda_P}{\rho_P}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\eta_M\theta_P}{\beta_I\lambda_P(\xi_P\gamma_M+\varrho\eta_M)}-1\right].$$

We note that Δ_4 exists when

$$\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\eta_M\theta_P}{\beta_I\lambda_P(\xi_P\gamma_M+\varrho\eta_M)}>1.$$

The IAV-specific antibody response activation number for influenza mono-infection is defined as:

$$\Re_4=\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\eta_M\theta_P}{\beta_I\lambda_P(\xi_P\gamma_M+\varrho\eta_M)}.$$

Thus, $M_4=\frac{\lambda_P}{\rho_P}(\Re_4-1)$.

(vi) Influenza and COVID-19 co-infection equilibrium with only the activated SARS-CoV-2-specific antibody response, $\Delta_5=(X_5,Y_5,I_5,V_5,P_5,Z_5,0)$, where

$$X_5=\frac{\beta_I\lambda_P}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P\xi_P},\quad Y_5=\frac{e^{-\alpha_1\tau_1}\xi_V\beta_I\lambda_P\gamma_Z}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P\xi_P\beta_Y\eta_Z},$$
$$I_5=\frac{\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}{e^{-\alpha_4\tau_4}\theta_P\xi_P\eta_Z}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_Z}{\beta_I\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}-1\right],\quad V_5=\frac{\gamma_Z}{\eta_Z},$$
$$P_5=\frac{\xi_V\gamma_Z+\varrho\eta_Z}{\xi_P\eta_Z}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_Z}{\beta_I\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}-1\right],$$
$$Z_5=\frac{\lambda_V}{\rho_V}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V\xi_V\beta_I\lambda_P}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P\xi_P\beta_Y\lambda_V}-1\right]=\frac{\lambda_V}{\rho_V}(\Re_1/\Re_2-1).$$

Hence, Δ_5 exists when

$$\frac{\Re_1}{\Re_2}>1\ \text{ and }\ \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_Z}{\beta_I\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}>1.$$

The influenza infection reproduction number in the presence of COVID-19 infection is stated as:

$$\Re_5=\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_Z}{\beta_I\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}.$$

Hence,

$$I_5=\frac{\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}{e^{-\alpha_4\tau_4}\theta_P\xi_P\eta_Z}(\Re_5-1),\quad P_5=\frac{\xi_V\gamma_Z+\varrho\eta_Z}{\xi_P\eta_Z}(\Re_5-1),$$

and then Δ_5 exists if $\frac{\Re_1}{\Re_2} > 1$ and $\Re_5 > 1$.

(vii) Influenza and COVID-19 co-infection equilibrium with only the activated IAV-specific antibody response, $\Delta_6 = (X_6, Y_6, I_6, V_6, P_6, 0, M_6)$, where

$$\begin{aligned}
X_6 &= \frac{\beta_Y \lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V \xi_V}, \quad Y_6 = \frac{\lambda_V(\xi_P\gamma_M + \varrho\eta_M)}{e^{-\alpha_2\tau_2}\theta_V\xi_V\eta_M}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M}{\beta_Y\lambda_V(\xi_P\gamma_M+\varrho\eta_M)} - 1\right],\\
I_6 &= \frac{e^{-\alpha_3\tau_3}\xi_P\gamma_M\beta_Y\lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V\xi_V\beta_I\eta_M}, \quad V_6 = \frac{\xi_P\gamma_M+\varrho\eta_M}{\xi_V\eta_M}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M}{\beta_Y\lambda_V(\xi_P\gamma_M+\varrho\eta_M)} - 1\right],\\
P_6 &= \frac{\gamma_M}{\eta_M}, \quad M_6 = \frac{\lambda_P}{\rho_P}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P\xi_P\beta_Y\lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V\xi_V\beta_I\lambda_P} - 1\right] = \frac{\lambda_P}{\rho_P}(\Re_2/\Re_1 - 1).
\end{aligned}$$

We note that Δ_6 exists when

$$\frac{\Re_2}{\Re_1} > 1 \text{ and } \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M}{\beta_Y\lambda_V(\xi_P\gamma_M+\varrho\eta_M)} > 1.$$

The COVID-19 infection reproduction number in the presence of influenza infection is stated as:

$$\Re_6 = \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M}{\beta_Y\lambda_V(\xi_P\gamma_M+\varrho\eta_M)}.$$

Thus,

$$Y_6 = \frac{\lambda_V(\xi_P\gamma_M+\varrho\eta_M)}{e^{-\alpha_2\tau_2}\theta_V\xi_V\eta_M}(\Re_6 - 1), V_6 = \frac{\xi_P\gamma_M+\varrho\eta_M}{\xi_V\eta_M}(\Re_6 - 1).$$

(viii) Influenza and COVID-19 co-infection equilibrium with activation of both SARS-CoV-2 and IAV antibody responses $\Delta_7 = (X_7, Y_7, I_7, V_7, P_7, Z_7, M_7)$, where

$$\begin{aligned}
X_7 &= \frac{\delta\eta_Z\eta_M}{\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M}, \qquad Y_7 = \frac{e^{-\alpha_1\tau_1}\delta\xi_V\gamma_Z\eta_M}{\beta_Y(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)},\\
I_7 &= \frac{e^{-\alpha_3\tau_3}\delta\xi_P\gamma_M\eta_Z}{\beta_I(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)}, \quad V_7 = \frac{\gamma_Z}{\eta_Z}, \quad P_7 = \frac{\gamma_M}{\eta_M},\\
Z_7 &= \frac{\lambda_V}{\rho_V}\left[\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M\eta_Z}{\beta_Y\lambda_V(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)} - 1\right],\\
M_7 &= \frac{\lambda_P}{\rho_P}\left[\frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_M\eta_Z}{\beta_I\lambda_P(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)} - 1\right].
\end{aligned}$$

It is obvious that Δ_7 exists when

$$\frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M\eta_Z}{\beta_Y\lambda_V(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)} > 1 \text{ and } \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_M\eta_Z}{\beta_I\lambda_P(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)} > 1.$$

Now, we define

$$\begin{aligned}
\Re_7 &= \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M\eta_Z}{\beta_Y\lambda_V(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)},\\
\Re_8 &= \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_M\eta_Z}{\beta_I\lambda_P(\xi_P\gamma_M\eta_Z + \xi_V\gamma_Z\eta_M + \varrho\eta_Z\eta_M)}.
\end{aligned}$$

Here, $\Re_7$ is the SARS-CoV-2-specific antibody response activation number for influenza and COVID-19 co-infection, and $\Re_8$ is the IAV-specific antibody response activation number for influenza and COVID-19 co-infection. Hence, $Z_7 = \frac{\lambda_V}{\rho_V}(\Re_7 - 1)$ and $M_7 = \frac{\lambda_P}{\rho_P}(\Re_8 - 1)$. If $\Re_7 > 1$ and $\Re_8 > 1$, then Δ_7 exists.

From what was stated above, we obtain eight threshold parameters that establish the existence of eight equilibria:

$$\begin{aligned}
\Re_1 &= \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\beta_Y\lambda_V}, & \Re_2 &= \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\beta_I\lambda_P}, \\
\Re_3 &= \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\eta_Z\theta_V}{\beta_Y\lambda_V(\xi_V\gamma_Z+\varrho\eta_Z)}, & \Re_4 &= \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\eta_M\theta_P}{\beta_I\lambda_P(\xi_P\gamma_M+\varrho\eta_M)}, \\
\Re_5 &= \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_Z}{\beta_I\lambda_P(\xi_V\gamma_Z+\varrho\eta_Z)}, & \Re_6 &= \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M}{\beta_Y\lambda_V(\xi_P\gamma_M+\varrho\eta_M)}, \\
\Re_7 &= \frac{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\delta\xi_V\theta_V\eta_M\eta_Z}{\beta_Y\lambda_V(\xi_P\gamma_M\eta_Z+\xi_V\gamma_Z\eta_M+\varrho\eta_Z\eta_M)}, & \Re_8 &= \frac{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\delta\xi_P\theta_P\eta_M\eta_Z}{\beta_I\lambda_P(\xi_P\gamma_M\eta_Z+\xi_V\gamma_Z\eta_M+\varrho\eta_Z\eta_M)}.
\end{aligned} \tag{16}$$

5. Global Stability

This section is devoted to the study of the global asymptotic stability of all equilibria. We configure the Lyapunov functions by following the way that was outlined in [56,57].

Let $\Lambda_k(X,Y,I,V,P,Z,M)$ be a Lyapunov function and let $\bar{\Theta}_k$ be the largest invariant subset of

$$\Theta_k = \left\{(X,Y,I,V,P,Z,M) : \frac{d\Lambda_k}{dt} = 0\right\}, \quad k = 0,1,2,\ldots,7.$$

We define a function $\digamma : (0,\infty) \longrightarrow [0,\infty)$ as $\digamma(u) = u - 1 - \ln u$. We denote $(X,Y,I,V,P,Z,M) = (X(t),Y(t),I(t),V(t),P(t),Z(t),M(t))$.

Theorem 1. *If $\Re_1 \leq 1$ and $\Re_2 \leq 1$, then Δ_0 is globally asymptotically stable (GAS).*

Proof. We define

$$\begin{aligned}
\Lambda_0 = {} & X_0\digamma\left(\frac{X}{X_0}\right) + e^{\alpha_1\tau_1}Y + e^{\alpha_3\tau_3}I + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}V + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}P \\
& + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}Z + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}M + \xi_V\int_{t-\tau_1}^{t}X(u)V(u)du \\
& + \xi_P\int_{t-\tau_3}^{t}X(u)P(u)du + \beta_Y e^{\alpha_1\tau_1}\int_{t-\tau_2}^{t}Y(u)du + \beta_I e^{\alpha_3\tau_3}\int_{t-\tau_4}^{t}I(u)du.
\end{aligned}$$

Clearly, $\Lambda_0 > 0$ for all $X,Y,I,V,P,Z,M > 0$, and $\Lambda_0(X_0,0,0,0,0,0,0) = 0$. We calculate $\frac{d\Lambda_0}{dt}$ along the solutions of model (1)–(7) as:

$$\begin{aligned}
\frac{d\Lambda_0}{dt} = {} & \left(1-\frac{X_0}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\frac{dY}{dt} + e^{\alpha_3\tau_3}\frac{dI}{dt} + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dV}{dt} + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dP}{dt} \\
& + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dZ}{dt} + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dM}{dt} + \xi_V[XV - X(t-\tau_1)V(t-\tau_1)] \\
& + \xi_P[XP - X(t-\tau_3)P(t-\tau_3)] + \beta_Y e^{\alpha_1\tau_1}[Y - Y(t-\tau_2)] + \beta_I e^{\alpha_3\tau_3}[I - I(t-\tau_4)].
\end{aligned}$$

Substituting from Equations (1)–(7), we obtain

$$\begin{aligned}
\frac{d\Lambda_0}{dt} = {} & \left(1-\frac{X_0}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP] + e^{\alpha_1\tau_1}\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right] \\
& + e^{\alpha_3\tau_3}\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3) - \beta_I I\right] \\
& + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V - \rho_V VZ\right] \\
& + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) - \lambda_P P - \rho_P PM\right] + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}[\eta_Z VZ - \gamma_Z Z] \\
& + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}[\eta_M PM - \gamma_M M] + \xi_V[XV - X(t-\tau_1)V(t-\tau_1)]
\end{aligned}$$

$$+\xi_P[XP - X(t-\tau_3)P(t-\tau_3)] + \beta_Y e^{\alpha_1\tau_1}[Y - Y(t-\tau_2)] + \beta_I e^{\alpha_3\tau_3}[I - I(t-\tau_4)]. \tag{17}$$

Simplifying Equation (17), we get:

$$\begin{aligned}\frac{d\Lambda_0}{dt} &= \left(1 - \frac{X_0}{X}\right)(\delta - \varrho X) + \left(\xi_V X_0 - \frac{\beta_Y \lambda_V}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\right)V \\ &+ \left(\xi_P X_0 - \frac{\beta_I \lambda_P}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\right)P - \frac{\rho_V \beta_Y \gamma_Z}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} Z - \frac{\rho_P \beta_I \gamma_M}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} M.\end{aligned}$$

Using the equilibrium condition $\delta = \varrho X_0$, we obtain:

$$\begin{aligned}\frac{d\Lambda_0}{dt} &= -\varrho\frac{(X - X_0)^2}{X} + \frac{\beta_Y \lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V}(\Re_1 - 1)V + \frac{\beta_I \lambda_P}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P}(\Re_2 - 1)P \\ &- \frac{\rho_V \beta_Y \gamma_Z}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} Z - \frac{\rho_P \beta_I \gamma_M}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} M.\end{aligned}$$

Since $\Re_1 \le 1$ and $\Re_2 \le 1$, then $\frac{d\Lambda_0}{dt} \le 0$ for all $X, V, P, Z, M > 0$. Further, $\frac{d\Lambda_0}{dt} = 0$ when $X = X_0$ and $V = 0$, $P = 0$, $Z = 0$, and $M = 0$. The solutions of system (1)–(7) converge to $\bar{\Theta}_0$ [55], which contains elements with $V = 0$ and $P = 0$. Hence, $\frac{dV}{dt} = 0$ and $\frac{dP}{dt} = 0$, and from Equations (4) and (5), we obtain

$$\begin{aligned}0 &= \frac{dV}{dt} = e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) \Longrightarrow Y(t) = 0, \text{ for all } t, \\ 0 &= \frac{dP}{dt} = e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) \Longrightarrow I(t) = 0, \text{ for all } t.\end{aligned}$$

Consequently, $\bar{\Theta}_0 = \{\Delta_0\}$, and by applying the Lyapunov–LaSalle asymptotic stability theorem (L-LAST) [58–60], we find that Δ_0 is GAS. $\square$

Theorem 2. *If $\Re_1 > 1$, $\Re_2/\Re_1 \le 1$, and $\Re_3 \le 1$, then Δ_1 is GAS.*

Proof. We formulate a Lyapunov function Λ_1 as:

$$\begin{aligned}\Lambda_1 &= X_1 F\left(\frac{X}{X_1}\right) + e^{\alpha_1\tau_1} Y_1 F\left(\frac{Y}{Y_1}\right) + e^{\alpha_3\tau_3} I + \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} V_1 F\left(\frac{V}{V_1}\right) \\ &+ \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} P + \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} Z + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} M \\ &+ \xi_V X_1 V_1 \int_{t-\tau_1}^{t} F\left(\frac{X(u)V(u)}{X_1 V_1}\right) du + \xi_P \int_{t-\tau_3}^{t} X(u)P(u) du \\ &+ \beta_Y e^{\alpha_1\tau_1} Y_1 \int_{t-\tau_2}^{t} F\left(\frac{Y(u)}{Y_1}\right) du + \beta_I e^{\alpha_3\tau_3} \int_{t-\tau_4}^{t} I(u) du.\end{aligned}$$

We calculate $\frac{d\Lambda_1}{dt}$ as:

$$\begin{aligned}\frac{d\Lambda_1}{dt} &= \left(1 - \frac{X_1}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\left(1 - \frac{Y_1}{Y}\right)\frac{dY}{dt} + e^{\alpha_3\tau_3}\frac{dI}{dt} \\ &+ \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1 - \frac{V_1}{V}\right)\frac{dV}{dt} + \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dP}{dt} \\ &+ \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dZ}{dt} + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dM}{dt} \\ &+ \xi_V X_1 V_1\left[\frac{XV}{X_1 V_1} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_1 V_1} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right]\end{aligned}$$

$$+\xi_P[XP - X(t-\tau_3)P(t-\tau_3)] + \beta_Y e^{\alpha_1\tau_1}Y_1\left[\frac{Y}{Y_1} - \frac{Y(t-\tau_2)}{Y_1} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right]$$
$$+\beta_I e^{\alpha_3\tau_3}[I - I(t-\tau_4)].$$

From Equations (1)–(7), we get

$$\begin{aligned}
\frac{d\Lambda_1}{dt} = &\left(1-\frac{X_1}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP] \\
&+ e^{\alpha_1\tau_1}\left(1-\frac{Y_1}{Y}\right)\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right] \\
&+ e^{\alpha_3\tau_3}\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3) - \beta_I I\right] \\
&+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_1}{V}\right)\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V - \rho_V VZ\right] \\
&+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) - \lambda_P P - \rho_P PM\right] \\
&+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}[\eta_Z VZ - \gamma_Z Z] + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}[\eta_M PM - \gamma_M M] \\
&+ \xi_V X_1V_1\left[\frac{XV}{X_1V_1} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_1V_1} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+ \xi_P[XP - X(t-\tau_3)P(t-\tau_3)] + \beta_Y e^{\alpha_1\tau_1}Y_1\left[\frac{Y}{Y_1} - \frac{Y(t-\tau_2)}{Y_1} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] \\
&+ \beta_I e^{\alpha_3\tau_3}[I - I(t-\tau_4)].
\end{aligned} \tag{18}$$

Simplifying Equation (18), we get

$$\begin{aligned}
\frac{d\Lambda_1}{dt} = &\left(1-\frac{X_1}{X}\right)(\delta - \varrho X) + \xi_V X_1 V + \xi_P X_1 P - \xi_V X(t-\tau_1)V(t-\tau_1)\frac{Y_1}{Y} \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_1 - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V - e^{\alpha_1\tau_1}\beta_Y Y(t-\tau_2)\frac{V_1}{V} + e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V_1 \\
&+ e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}V_1 Z - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M \\
&+ \xi_V X_1V_1\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right) + e^{\alpha_1\tau_1}\beta_Y Y_1\ln\left(\frac{Y(t-\tau_2)}{Y}\right).
\end{aligned}$$

Using the equilibrium conditions for Δ_1,

$$\delta = \varrho X_1 + \xi_V X_1V_1, \quad \xi_V X_1V_1 = e^{\alpha_1\tau_1}\beta_Y Y_1,$$
$$Y_1 = e^{\alpha_2\tau_2}\frac{\lambda_V}{\theta_V}V_1,$$

we obtain

$$\begin{aligned}
\frac{d\Lambda_1}{dt} = &\left(1-\frac{X_1}{X}\right)(\varrho X_1 - \varrho X) + 3\xi_V X_1V_1 - \xi_V X_1V_1\frac{X_1}{X} - \xi_V X_1V_1\frac{X(t-\tau_1)V(t-\tau_1)Y_1}{X_1V_1Y} \\
&- \xi_V X_1V_1\frac{Y(t-\tau_2)V_1}{Y_1V} + \xi_V X_1V_1\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right) + \xi_V X_1V_1\ln\left(\frac{Y(t-\tau_2)}{Y}\right) \\
&+ e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}\left[\frac{\xi_P X_1\theta_P}{\beta_I\lambda_P}e^{-\alpha_3\tau_3-\alpha_4\tau_4} - 1\right]P + e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}\left[\frac{\eta_Z}{\gamma_Z}V_1 - 1\right]Z \\
&- e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M.
\end{aligned} \tag{19}$$

Then, collecting the terms of (19), we get

$$\begin{aligned}
\frac{d\Lambda_1}{dt} &= -\varrho\frac{(X-X_1)^2}{X} + 3\xi_V X_1 V_1 - \xi_V X_1 V_1 \frac{X_1}{X} - \xi_V X_1 V_1 \frac{X(t-\tau_1)V(t-\tau_1)Y_1}{X_1 V_1 Y} \\
&\quad - \xi_V X_1 V_1 \frac{Y(t-\tau_2)V_1}{Y_1 V} + e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I \lambda_P}{\theta_P}(\Re_2/\Re_1 - 1)P \\
&\quad + \xi_V X_1 V_1 \left[\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_1}{X_1 V_1 Y}\right) + \ln\left(\frac{X_1}{X}\right) + \ln\left(\frac{Y(t-\tau_2)V_1}{Y_1 V}\right)\right] \\
&\quad + \frac{\beta_Y \rho_V(\varrho\eta_Z + \xi_V\gamma_Z)}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\eta_Z\xi_V\theta_V}(\Re_3 - 1)Z - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M. \\
&= -\varrho\frac{(X-X_1)^2}{X} - \xi_V X_1 V_1 \left[F\left(\frac{X_1}{X}\right) + F\left(\frac{Y(t-\tau_2)V_1}{Y_1 V}\right) + F\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_1}{X_1 V_1 Y}\right)\right] \\
&\quad + \frac{\beta_I\lambda_P}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P}(\Re_2/\Re_1 - 1)P + \frac{\beta_Y\rho_V(\varrho\eta_Z+\xi_V\gamma_Z)}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\eta_Z\xi_V\theta_V}(\Re_3-1)Z - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M.
\end{aligned}$$

Since $\Re_2/\Re_1 \le 1$ and $\Re_3 \le 1$, then $\frac{d\Lambda_1}{dt} \le 0$ for all $X, Y, V, P, Z, M > 0$. Moreover, $\frac{d\Lambda_1}{dt} = 0$ when $X = X_1$, $Y = Y_1$, $V = V_1$, $P = 0$, $Z = 0$, and $M = 0$. The trajectories of system (1)–(7) converge to $\bar{\Theta}_1$, where $P = 0$. Thus, $\frac{dP}{dt} = 0$, and Equation (5) yields

$$0 = \frac{dP}{dt} = e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) \Longrightarrow I(t) = 0, \text{ fot all } t.$$

Then, $\bar{\Theta}_1 = \{\Delta_1\}$ and Δ_1 is GAS by utilizing the L-LAST. □

Theorem 3. *Let $\Re_2 > 1$, $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$; then, Δ_2 is GAS.*

Proof. Consider

$$\begin{aligned}
\Lambda_2 &= X_2 F\left(\frac{X}{X_2}\right) + e^{\alpha_1\tau_1}Y + e^{\alpha_3\tau_3} I_2 F\left(\frac{I}{I_2}\right) + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}V \\
&\quad + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}P_2 F\left(\frac{P}{P_2}\right) + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}Z + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}M \\
&\quad + \xi_V \int_{t-\tau_1}^{t} X(u)V(u)du + \xi_P X_2 P_2 \int_{t-\tau_3}^{t} F\left(\frac{X(u)P(u)}{X_2 P_2}\right)du \\
&\quad + e^{\alpha_1\tau_1}\beta_Y \int_{t-\tau_2}^{t} Y(u)du + e^{\alpha_3\tau_3}\beta_I I_2 \int_{t-\tau_4}^{t} F\left(\frac{I(u)}{I_2}\right)du.
\end{aligned}$$

We calculate $\frac{d\Lambda_2}{dt}$ as:

$$\begin{aligned}
\frac{d\Lambda_2}{dt} &= \left(1 - \frac{X_2}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\frac{dY}{dt} + e^{\alpha_3\tau_3}\left(1 - \frac{I_2}{I}\right)\frac{dI}{dt} + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dV}{dt} \\
&\quad + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1 - \frac{P_2}{P}\right)\frac{dP}{dt} + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dZ}{dt} + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dM}{dt} \\
&\quad + \xi_V[XV - X(t-\tau_1)V(t-\tau_1)] \\
&\quad + \xi_P X_2 P_2\left[\frac{XP}{X_2P_2} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_2P_2} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&\quad + \beta_Y e^{\alpha_1\tau_1}[Y - Y(t-\tau_2)] + \beta_I e^{\alpha_3\tau_3} I_2\left[\frac{I}{I_2} - \frac{I(t-\tau_4)}{I_2} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned}$$

From Equations (1)–(7), we have

$$\frac{d\Lambda_2}{dt} = \left(1 - \frac{X_2}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP] + e^{\alpha_1\tau_1}\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right]$$
$$+ e^{\alpha_3\tau_3}\left(1 - \frac{I_2}{I}\right)\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3) - \beta_I I\right]$$
$$+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V - \rho_V VZ\right]$$
$$+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1 - \frac{P_2}{P}\right)\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) - \lambda_P P - \rho_P PM\right]$$
$$+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}[\eta_Z VZ - \gamma_Z Z] + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}[\eta_M PM - \gamma_M M]$$
$$+ \xi_V[XV - X(t-\tau_1)V(t-\tau_1)]$$
$$+ \xi_P X_2P_2\left[\frac{XP}{X_2P_2} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_2P_2} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right]$$
$$+ \beta_Y e^{\alpha_1\tau_1}[Y - Y(t-\tau_2)] + \beta_I e^{\alpha_3\tau_3}I_2\left[\frac{I}{I_2} - \frac{I(t-\tau_4)}{I_2} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right]. \quad (20)$$

Then, simplifying Equation (20), we get

$$\frac{d\Lambda_2}{dt} = \left(1 - \frac{X_2}{X}\right)(\delta - \varrho X) + \xi_P X_2 P - \xi_P X(t-\tau_3)P(t-\tau_3)\frac{I_2}{I} + e^{\alpha_3\tau_3}\beta_I I_2$$
$$- e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P - e^{\alpha_3\tau_3}\beta_I I(t-\tau_4)\frac{P_2}{P} + e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P_2$$
$$+ \xi_P X_2P_2\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right) + e^{\alpha_3\tau_3}\beta_I I_2\ln\left(\frac{I(t-\tau_4)}{I}\right)$$
$$+ \frac{\beta_Y\lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V}\left(\frac{\xi_V X_2\theta_V e^{-\alpha_1\tau_1-\alpha_2\tau_2}}{\beta_Y\lambda_V} - 1\right)V + \frac{\beta_I\rho_P\gamma_M}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\theta_P\eta_M}\left(\frac{\eta_M}{\gamma_M}P_2 - 1\right)M$$
$$- e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\rho_V\beta_Y\gamma_Z}{\eta_Z\theta_V}Z.$$

Using the equilibrium conditions for Δ_2,

$$\delta = \varrho X_2 + \xi_P X_2P_2, \quad \xi_P X_2P_2 = e^{\alpha_3\tau_3}\beta_I I_2,$$
$$I_2 = e^{\alpha_4\tau_4}\frac{\lambda_P}{\theta_P}P_2,$$

we obtain

$$\frac{d\Lambda_2}{dt} = \left(1 - \frac{X_2}{X}\right)(\varrho X_2 - \varrho X) + 3\xi_P X_2P_2 - \xi_P X_2P_2\frac{X_2}{X} - \xi_P X_2P_2\frac{X(t-\tau_3)P(t-\tau_3)I_2}{X_2P_2I}$$
$$- \xi_P X_2P_2\frac{I(t-\tau_4)P_2}{I_2P} + \xi_P X_2P_2\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right) + \xi_P X_2P_2\ln\left(\frac{I(t-\tau_4)}{I}\right)$$
$$+ \frac{\beta_Y\lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V}\left(\frac{\Re_1}{\Re_2} - 1\right)V + \frac{\beta_I\rho_P(\varrho\eta_M + \gamma_M\xi_P)}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\xi_P\eta_M\theta_P}(\Re_4 - 1)M - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\rho_V\beta_Y\gamma_Z}{\eta_Z\theta_V}Z. \quad (21)$$

Then, simplifying Equation (21), we get:

$$\frac{d\Lambda_2}{dt} = -\varrho\frac{(X - X_2)^2}{X} + 3\xi_P X_2P_2 - \xi_P X_2P_2\frac{X_2}{X} - \xi_P X_2P_2\frac{I(t-\tau_4)P_2}{I_2P}$$
$$- \xi_P X_2P_2\frac{X(t-\tau_3)P(t-\tau_3)I_2}{X_2P_2I} + \frac{\beta_Y\lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V}(\Re_1/\Re_2 - 1)V$$

$$
\begin{aligned}
&+ \frac{\beta_I \rho_P(\varrho\eta_M + \gamma_M \xi_P)}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\xi_P \eta_M \theta_P}(\Re_4 - 1)M - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\rho_V \beta_Y \gamma_Z}{\eta_Z \theta_V} Z \\
&+ \xi_P X_2 P_2 \left[\ln\left(\frac{X_2}{X}\right) + \ln\left(\frac{I(t-\tau_4)P_2}{I_2 P}\right) + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)I_2}{X_2 P_2 I}\right)\right] \\
&= -\varrho \frac{(X - X_2)^2}{X} - \xi_P X_2 P_2 \left[F\left(\frac{X_2}{X}\right) + F\left(\frac{I(t-\tau_4)P_2}{I_2 P}\right) + F\left(\frac{X(t-\tau_3)P(t-\tau_3)I_2}{X_2 P_2 I}\right)\right] \\
&+ \frac{\beta_Y \lambda_V}{e^{-\alpha_1\tau_1-\alpha_2\tau_2}\theta_V}(\Re_1/\Re_2 - 1)V + \frac{\beta_I \rho_P(\varrho\eta_M + \gamma_M \xi_P)}{e^{-\alpha_3\tau_3-\alpha_4\tau_4}\xi_P \eta_M \theta_P}(\Re_4 - 1)M \\
&- e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\rho_V \beta_Y \gamma_Z}{\eta_Z \theta_V} Z.
\end{aligned}
$$

If $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$, then $\frac{d\Lambda_2}{dt} \le 0$ for all $X, I, V, P, Z, M > 0$. In addition, $\frac{d\Lambda_2}{dt} = 0$ when $X = X_2$, $I = I_2$, $P = P_2$, $V = 0$, $M = 0$, and $Z = 0$. The trajectories of system (1)–(7) converge to $\bar{\Theta}_2$, which includes solutions with $V = 0$, and thus, $\frac{dV}{dt} = 0$. Equation (4) implies that

$$
0 = \frac{dV}{dt} = e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) \Longrightarrow Y(t) = 0, \text{ for all } t.
$$

Hence, $\bar{\Theta}_2 = \{\Delta_2\}$, and the global stability of Δ_2 follows from applying the L-LAST. □

Theorem 4. *Let $\Re_3 > 1$ and $\Re_5 \le 1$; then, Δ_3 is GAS.*

Proof. We define

$$
\begin{aligned}
\Lambda_3 = {} & X_3 F\left(\frac{X}{X_3}\right) + e^{\alpha_1\tau_1} Y_3 F\left(\frac{Y}{Y_3}\right) + e^{\alpha_3\tau_3} I + \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} V_3 F\left(\frac{V}{V_3}\right) \\
&+ \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} P + \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} Z_3 F\left(\frac{Z}{Z_3}\right) + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} M \\
&+ \xi_V X_3 V_3 \int_{t-\tau_1}^{t} F\left(\frac{X(u)V(u)}{X_3 V_3}\right) du + \xi_P \int_{t-\tau_3}^{t} X(u)P(u) du \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_3 \int_{t-\tau_2}^{t} F\left(\frac{Y(u)}{Y_3}\right) du + e^{\alpha_3\tau_3}\beta_I \int_{t-\tau_4}^{t} I(u) du.
\end{aligned}
$$

We calculate $\frac{d\Lambda_3}{dt}$ as:

$$
\begin{aligned}
\frac{d\Lambda_3}{dt} = {} & \left(1 - \frac{X_3}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\left(1 - \frac{Y_3}{Y}\right)\frac{dY}{dt} + e^{\alpha_3\tau_3}\frac{dI}{dt} \\
&+ \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1 - \frac{V_3}{V}\right)\frac{dV}{dt} + \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dP}{dt} \\
&+ \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1 - \frac{Z_3}{Z}\right)\frac{dZ}{dt} + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dM}{dt} \\
&+ \xi_V X_3 V_3 \left[\frac{XV}{X_3 V_3} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_3 V_3} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+ \xi_P [XP - X(t-\tau_3)P(t-\tau_3)] \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_3 \left[\frac{Y}{Y_3} - \frac{Y(t-\tau_2)}{Y_3} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] + e^{\alpha_3\tau_3}\beta_I [I - I(t-\tau_4)].
\end{aligned}
$$

From Equations (1)–(7), we get

$$\begin{aligned}\frac{d\Lambda_3}{dt} &= \left(1-\frac{X_3}{X}\right)[\delta-\varrho X-\xi_V XV-\xi_P XP] \\ &+ e^{\alpha_1\tau_1}\left(1-\frac{Y_3}{Y}\right)\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1)-\beta_Y Y\right] \\ &+ e^{\alpha_3\tau_3}\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3)-\beta_I I\right] \\ &+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_3}{V}\right)\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2)-\lambda_V V-\rho_V VZ\right] \\ &+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4)-\lambda_P P-\rho_P PM\right] \\ &+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{Z_3}{Z}\right)[\eta_Z VZ-\gamma_Z Z]+\frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}[\eta_M PM-\gamma_M M] \\ &+ \xi_V X_3V_3\left[\frac{XV}{X_3V_3}-\frac{X(t-\tau_1)V(t-\tau_1)}{X_3V_3}+\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\ &+ \xi_P[XP-X(t-\tau_3)P(t-\tau_3)] \\ &+ e^{\alpha_1\tau_1}\beta_Y Y_3\left[\frac{Y}{Y_3}-\frac{Y(t-\tau_2)}{Y_3}+\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right]+e^{\alpha_3\tau_3}\beta_I[I-I(t-\tau_4)]. \end{aligned} \tag{22}$$

Then, simplifying Equation (22), we get:

$$\begin{aligned}\frac{d\Lambda_3}{dt} &= \left(1-\frac{X_3}{X}\right)(\delta-\varrho X)+\xi_V X_3V-\xi_V X(t-\tau_1)V(t-\tau_1)\frac{Y_3}{Y} \\ &+ e^{\alpha_1\tau_1}\beta_Y Y_3-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V-e^{\alpha_1\tau_1}\beta_Y Y(t-\tau_2)\frac{V_3}{V}+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V_3 \\ &+ e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}V_3Z-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}Z_3V \\ &+ e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z_3-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M \\ &+ e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}\left[\frac{\xi_P X_3\theta_P}{\beta_I\lambda_P}e^{-\alpha_3\tau_3-\alpha_4\tau_4}-1\right]P+\xi_V X_3V_3\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right) \\ &+ e^{\alpha_1\tau_1}\beta_Y Y_3\ln\left(\frac{Y(t-\tau_2)}{Y}\right). \end{aligned}$$

Using the equilibrium conditions for Δ_3,

$$\begin{aligned} &\delta=\varrho X_3+\xi_V X_3V_3, \quad \xi_V X_3V_3=e^{\alpha_1\tau_1}\beta_Y Y_3, \\ &Y_3=e^{\alpha_2\tau_2}\frac{\lambda_V}{\theta_V}V_3+e^{\alpha_2\tau_2}\frac{\rho_V}{\theta_V}Z_3V_3, \\ &V_3=\frac{\gamma_Z}{\eta_Z}, \end{aligned}$$

we obtain

$$\begin{aligned}\frac{d\Lambda_3}{dt} &= \left(1-\frac{X_3}{X}\right)(\varrho X_3-\varrho X)+3\xi_V X_3V_3-\xi_V X_3V_3\frac{X_3}{X} \\ &- \xi_V X_3V_3\frac{X(t-\tau_1)V(t-\tau_1)Y_3}{X_3V_3Y}-\xi_V X_3V_3\frac{Y(t-\tau_2)V_3}{Y_3V} \\ &+ \xi_V X_3V_3\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)+\xi_V X_3V_3\ln\left(\frac{Y(t-\tau_2)}{Y}\right) \\ &+ e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}(\Re_5-1)P-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M. \end{aligned} \tag{23}$$

Equation (23) can be written as:

$$
\begin{aligned}
\frac{d\Lambda_3}{dt} &= -\varrho\frac{(X-X_3)^2}{X} + 3\xi_V X_3 V_3 - \xi_V X_3 V_3 \frac{X_3}{X} - \xi_V X_3 V_3 \frac{Y(t-\tau_2)V_3}{Y_3 V} \\
&- \xi_V X_3 V_3 \frac{X(t-\tau_1)V(t-\tau_1)Y_3}{X_3 V_3 Y} + e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}(\Re_5-1)P - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M \\
&+ \xi_V X_3 V_3\left[\ln\left(\frac{X_3}{X}\right) + \ln\left(\frac{Y(t-\tau_2)V_3}{Y_3 V}\right) + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_3}{X_3 V_3 Y}\right)\right] \\
&= -\varrho\frac{(X-X_3)^2}{X} \\
&- \xi_V X_3 V_3\left[F\left(\frac{X_3}{X}\right) + F\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_3}{X_3 V_3 Y}\right) + F\left(\frac{Y(t-\tau_2)V_3}{Y_3 V}\right)\right] \\
&+ e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}(\Re_5-1)P - e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M.
\end{aligned}
$$

Obviously, $\frac{d\Lambda_3}{dt} \le 0$ for all $X, Y, V, P, M > 0$ when $\Re_5 \le 1$. Further, $\frac{d\Lambda_3}{dt} = 0$ when $X = X_3$, $Y = Y_3$, $V = V_3$, $P = 0$, and $M = 0$. Similarly to the proofs of the previous theorems, one can complete the proof. □

Theorem 5. *If $\Re_4 > 1$ and $\Re_6 \le 1$, then Δ_4 is GAS.*

Proof. We define a function Λ_4 as:

$$
\begin{aligned}
\Lambda_4 &= X_4 F\left(\frac{X}{X_4}\right) + e^{\alpha_1\tau_1}Y + e^{\alpha_3\tau_3}I_4 F\left(\frac{I}{I_4}\right) + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}V \\
&+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}P_4 F\left(\frac{P}{P_4}\right) + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}Z + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}M_4 F\left(\frac{M}{M_4}\right) \\
&+ \xi_V\int_{t-\tau_1}^{t} X(u)V(u)du + \xi_P X_4 P_4\int_{t-\tau_3}^{t} F\left(\frac{X(u)P(u)}{X_4 P_4}\right)du \\
&+ e^{\alpha_1\tau_1}\beta_Y\int_{t-\tau_2}^{t} Y(u)du + e^{\alpha_3\tau_3}\beta_I I_4\int_{t-\tau_4}^{t} F\left(\frac{I(u)}{I_4}\right)du.
\end{aligned}
$$

We calculate $\frac{d\Lambda_4}{dt}$ as:

$$
\begin{aligned}
\frac{d\Lambda_4}{dt} &= \left(1-\frac{X_4}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\frac{dY}{dt} + e^{\alpha_3\tau_3}\left(1-\frac{I_4}{I}\right)\frac{dI}{dt} + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dV}{dt} \\
&+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_4}{P}\right)\frac{dP}{dt} + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dZ}{dt} \\
&+ \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{M_4}{M}\right)\frac{dM}{dt} + \xi_V[XV - X(t-\tau_1)V(t-\tau_1)] \\
&+ \xi_P X_4 P_4\left[\frac{XP}{X_4 P_4} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_4 P_4} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+ e^{\alpha_1\tau_1}\beta_Y[Y - Y(t-\tau_2)] + e^{\alpha_3\tau_3}\beta_I I_4\left[\frac{I}{I_4} - \frac{I(t-\tau_4)}{I_4} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned}
$$

Substituting from Equations (1)–(7), we obtain

$$
\frac{d\Lambda_4}{dt} = \left(1-\frac{X_4}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP] + e^{\alpha_1\tau_1}\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right]
$$

$$
\begin{aligned}
&+e^{\alpha_3\tau_3}\left(1-\frac{I_4}{I}\right)\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3)-\beta_I I\right] \\
&+\frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2)-\lambda_V V-\rho_V VZ\right] \\
&+\frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_4}{P}\right)\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4)-\lambda_P P-\rho_P PM\right] \\
&+\frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}[\eta_Z VZ-\gamma_Z Z]+\frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{M_4}{M}\right)[\eta_M PM-\gamma_M M] \\
&+\xi_V[XV-X(t-\tau_1)V(t-\tau_1)] \\
&+\xi_P X_4P_4\left[\frac{XP}{X_4P_4}-\frac{X(t-\tau_3)P(t-\tau_3)}{X_4P_4}+\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+e^{\alpha_1\tau_1}\beta_Y[Y-Y(t-\tau_2)]+e^{\alpha_3\tau_3}\beta_I I_4\left[\frac{I}{I_4}-\frac{I(t-\tau_4)}{I_4}+\ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned}
\tag{24}
$$

Collecting the terms of Equation (24), we get:

$$
\begin{aligned}
\frac{d\Lambda_4}{dt}=&\left(1-\frac{X_4}{X}\right)(\delta-\varrho X)+\xi_P X_4 P-\xi_P X(t-\tau_3)P(t-\tau_3)\frac{I_4}{I}+e^{\alpha_3\tau_3}\beta_I I_4 \\
&-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P-e^{\alpha_3\tau_3}\beta_I I(t-\tau_4)\frac{P_4}{P}+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P_4 \\
&+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}P_4M-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M \\
&-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}M_4P+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M_4 \\
&+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}\left[\frac{\xi_V X_4\theta_V}{\beta_Y\lambda_V}e^{-\alpha_1\tau_1-\alpha_2\tau_2}-1\right]V+\xi_P X_4P_4\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right) \\
&+e^{\alpha_3\tau_3}\beta_I I_4\ln\left(\frac{I(t-\tau_4)}{I}\right).
\end{aligned}
$$

Using the equilibrium conditions for Δ_4,

$$
\begin{aligned}
&\delta=\varrho X_4+\xi_P X_4P_4, \quad \xi_P X_4P_4=e^{\alpha_3\tau_3}\beta_I I_4, \\
&I_4=e^{\alpha_4\tau_4}\frac{\lambda_P}{\theta_P}P_4+e^{\alpha_4\tau_4}\frac{\rho_P}{\theta_P}P_4M_4, \quad P_4=\frac{\gamma_M}{\eta_M},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{d\Lambda_4}{dt}=&\left(1-\frac{X_4}{X}\right)(\varrho X_4-\varrho X)+3\xi_P X_4P_4-\xi_P X_4P_4\frac{X_4}{X}-\xi_P X_4P_4\frac{X(t-\tau_3)P(t-\tau_3)I_4}{X_4P_4I} \\
&-\xi_P X_4P_4\frac{I(t-\tau_4)P_4}{I_4P}+\xi_P X_4P_4\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)+\xi_P X_4P_4\ln\left(\frac{I(t-\tau_4)}{I}\right) \\
&+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}(\Re_6-1)V-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z.
\end{aligned}
\tag{25}
$$

Then, simplifying Equation (25), we get:

$$
\begin{aligned}
\frac{d\Lambda_4}{dt}=&-\varrho\frac{(X-X_4)^2}{X}+3\xi_P X_4P_4-\xi_P X_4P_4\frac{X_4}{X}-\xi_P X_4P_4\frac{I(t-\tau_4)P_4}{I_4P} \\
&-\xi_P X_4P_4\frac{X(t-\tau_3)P(t-\tau_3)I_4}{X_4P_4I}+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}(\Re_6-1)V-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z \\
&+\xi_P X_4P_4\left[\ln\left(\frac{X_4}{X}\right)+\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)I_4}{X_4P_4I}\right)+\ln\left(\frac{I(t-\tau_4)P_4}{I_4P}\right)\right]
\end{aligned}
$$

$$= -\varrho\frac{(X-X_4)^2}{X} - \xi_P X_4 P_4\left[F\left(\frac{X_4}{X}\right) + F\left(\frac{X(t-\tau_3)P(t-\tau_3)I_4}{X_4P_4I}\right) + F\left(\frac{I(t-\tau_4)P_4}{I_4P}\right)\right]$$
$$+ e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}(\Re_6-1)V - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z.$$

Since $\Re_6 \le 1$, then $\frac{d\Lambda_4}{dt} \le 0$ for all $X, I, V, P, Z > 0$. In addition, $\frac{d\Lambda_4}{dt} = 0$ when $X = X_4$, $I = I_4$, $P = P_4$, $V = 0$, and $Z = 0$. The proof can be completed similarly to the previous theorems. □

Theorem 6. *If $\Re_5 > 1$, $\Re_8 \le 1$, and $\Re_1/\Re_2 > 1$, then Δ_5 is GAS.*

Proof. We define

$$\Lambda_5 = X_5F\left(\frac{X}{X_5}\right) + e^{\alpha_1\tau_1}Y_5F\left(\frac{Y}{Y_5}\right) + e^{\alpha_3\tau_3}I_5F\left(\frac{I}{I_5}\right) + \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}V_5F\left(\frac{V}{V_5}\right)$$
$$+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}P_5F\left(\frac{P}{P_5}\right) + \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}Z_5F\left(\frac{Z}{Z_5}\right) + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}M$$
$$+ \xi_V X_5V_5\int_{t-\tau_1}^{t} F\left(\frac{X(u)V(u)}{X_5V_5}\right)du + \xi_P X_5P_5\int_{t-\tau_3}^{t} F\left(\frac{X(u)P(u)}{X_5P_5}\right)du$$
$$+ e^{\alpha_1\tau_1}\beta_Y Y_5\int_{t-\tau_2}^{t} F\left(\frac{Y(u)}{Y_5}\right)du + e^{\alpha_3\tau_3}\beta_I I_5\int_{t-\tau_4}^{t} F\left(\frac{I(u)}{I_5}\right)du.$$

We calculate $\frac{d\Lambda_5}{dt}$ as:

$$\frac{d\Lambda_5}{dt} = \left(1-\frac{X_5}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\left(1-\frac{Y_5}{Y}\right)\frac{dY}{dt} + e^{\alpha_3\tau_3}\left(1-\frac{I_5}{I}\right)\frac{dI}{dt}$$
$$+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_5}{V}\right)\frac{dV}{dt} + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_5}{P}\right)\frac{dP}{dt}$$
$$+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{Z_5}{Z}\right)\frac{dZ}{dt} + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{dM}{dt}$$
$$+ \xi_V X_5V_5\left[\frac{XV}{X_5V_5} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_5V_5} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right]$$
$$+ \xi_P X_5P_5\left[\frac{XP}{X_5P_5} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_5P_5} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right]$$
$$+ e^{\alpha_1\tau_1}\beta_Y Y_5\left[\frac{Y}{Y_5} - \frac{Y(t-\tau_2)}{Y_5} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right]$$
$$+ e^{\alpha_3\tau_3}\beta_I I_5\left[\frac{I}{I_5} - \frac{I(t-\tau_4)}{I_5} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right].$$

It follows from Equations (1)–(7) that

$$\frac{d\Lambda_5}{dt} = \left(1-\frac{X_5}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP]$$
$$+ e^{\alpha_1\tau_1}\left(1-\frac{Y_5}{Y}\right)\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right]$$
$$+ e^{\alpha_3\tau_3}\left(1-\frac{I_5}{I}\right)\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3) - \beta_I I\right]$$
$$+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_5}{V}\right)\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V - \rho_V VZ\right]$$

$$
\begin{aligned}
&+\frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_5}{P}\right)\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4)-\lambda_P P-\rho_P PM\right] \\
&+\frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{Z_5}{Z}\right)[\eta_Z VZ-\gamma_Z Z]+\frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}[\eta_M PM-\gamma_M M] \\
&+\xi_V X_5V_5\left[\frac{XV}{X_5V_5}-\frac{X(t-\tau_1)V(t-\tau_1)}{X_5V_5}+\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+\xi_P X_5P_5\left[\frac{XP}{X_5P_5}-\frac{X(t-\tau_3)P(t-\tau_3)}{X_5P_5}+\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+e^{\alpha_1\tau_1}\beta_Y Y_5\left[\frac{Y}{Y_5}-\frac{Y(t-\tau_2)}{Y_5}+\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] \\
&+e^{\alpha_3\tau_3}\beta_I I_5\left[\frac{I}{I_5}-\frac{I(t-\tau_4)}{I_5}+\ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned} \tag{26}
$$

Equation (26) can be simplified as:

$$
\begin{aligned}
\frac{d\Lambda_5}{dt}=&\left(1-\frac{X_5}{X}\right)(\delta-\varrho X)+\xi_V X_5V+\xi_P X_5P-\xi_V X(t-\tau_1)V(t-\tau_1)\frac{Y_5}{Y} \\
&+e^{\alpha_1\tau_1}\beta_Y Y_5-\xi_P X(t-\tau_3)P(t-\tau_3)\frac{I_5}{I}+e^{\alpha_3\tau_3}\beta_I I_5-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V \\
&-e^{\alpha_1\tau_1}\beta_Y Y(t-\tau_2)\frac{V_5}{V}+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V_5+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}ZV_5 \\
&-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P-e^{\alpha_3\tau_3}\beta_I I(t-\tau_4)\frac{P_5}{P}+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P_5 \\
&-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}Z_5V+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z_5 \\
&+\xi_V X_5V_5\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)+\xi_P X_5P_5\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right) \\
&+e^{\alpha_1\tau_1}\beta_Y Y_5\ln\left(\frac{Y(t-\tau_2)}{Y}\right)+e^{\alpha_3\tau_3}\beta_I I_5\ln\left(\frac{I(t-\tau_4)}{I}\right) \\
&+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}\left[\frac{\eta_M}{\gamma_M}P_5-1\right]M.
\end{aligned}
$$

Using the equilibrium conditions for Δ_5,

$$
\begin{gathered}
\delta=\varrho X_5+\xi_V X_5V_5+\xi_P X_5P_5,\quad \xi_V X_5V_5=e^{\alpha_1\tau_1}\beta_Y Y_5, \\
\xi_P X_5P_5=e^{\alpha_3\tau_3}\beta_I I_5,\quad Y_5=e^{\alpha_2\tau_2}\frac{\lambda_V}{\theta_V}V_5+e^{\alpha_2\tau_2}\frac{\rho_V}{\theta_V}V_5Z_5, \\
I_5=e^{\alpha_4\tau_4}\frac{\lambda_P}{\theta_P}P_5,\quad V_5=\frac{\gamma_Z}{\eta_Z},
\end{gathered}
$$

we obtain

$$
\begin{aligned}
\frac{d\Lambda_5}{dt}=&\left(1-\frac{X_5}{X}\right)(\varrho X_5-\varrho X)+3\xi_V X_5V_5+3\xi_P X_5P_5-\xi_V X_5V_5\frac{X_5}{X}-\xi_P X_5P_5\frac{X_5}{X} \\
&-\xi_V X_5V_5\frac{X(t-\tau_1)V(t-\tau_1)Y_5}{X_5V_5Y}-\xi_P X_5P_5\frac{X(t-\tau_3)P(t-\tau_3)I_5}{X_5P_5I} \\
&-\xi_V X_5V_5\frac{Y(t-\tau_2)V_5}{Y_5V}-\xi_P X_5P_5\frac{I(t-\tau_4)P_5}{I_5P}+\xi_V X_5V_5\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right) \\
&+\xi_P X_5P_5\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)+\xi_V X_5V_5\ln\left(\frac{Y(t-\tau_2)}{Y}\right) \\
&+\xi_P X_5P_5\ln\left(\frac{I(t-\tau_4)}{I}\right)+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P(\xi_P\gamma_M\eta_Z+\xi_V\gamma_Z\eta_M+\varrho\eta_Z\eta_M)}{\xi_P\theta_P\eta_M\eta_Z}(\Re_8-1)M.
\end{aligned} \tag{27}
$$

Then, simplifying Equation (27), we get:

$$
\begin{aligned}
\frac{d\Lambda_5}{dt} =& -\varrho\frac{(X-X_5)^2}{X} + 3\xi_V X_5 V_5 + 3\xi_P X_5 P_5 - \xi_V X_5 V_5 \frac{X_5}{X} - \xi_V X_5 V_5 \frac{Y(t-\tau_2)V_5}{Y_5 V} \\
& - \xi_V X_5 V_5 \frac{X(t-\tau_1)V(t-\tau_1)Y_5}{X_5 V_5 Y} - \xi_P X_5 P_5 \frac{X(t-\tau_3)P(t-\tau_3)I_5}{X_5 P_5 I} - \xi_P X_5 P_5 \frac{I(t-\tau_4)P_5}{I_5 P} \\
& + \xi_V X_5 V_5 \left[\ln\left(\frac{X_5}{X}\right) + \ln\left(\frac{Y(t-\tau_2)V_5}{Y_5 V}\right) + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_5}{X_5 V_5 Y}\right)\right] \\
& - \xi_P X_5 P_5 \frac{X_5}{X} + \xi_P X_5 P_5 \left[\ln\left(\frac{X_5}{X}\right) + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)I_5}{X_5 P_5 I}\right) + \ln\left(\frac{I(t-\tau_4)P_5}{I_5 P}\right)\right] \\
& + e^{\alpha_3\tau_3+\alpha_4\tau_4} \frac{\beta_I \rho_P(\xi_P \gamma_M \eta_Z + \xi_V \gamma_Z \eta_M + \varrho \eta_Z \eta_M)}{\xi_P \theta_P \eta_M \eta_Z} (\Re_8 - 1) M \\
=& -\varrho\frac{(X-X_5)^2}{X} \\
& - \xi_V X_5 V_5 \left[F\left(\frac{X_5}{X}\right) + F\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_5}{X_5 V_5 Y}\right) + F\left(\frac{Y(t-\tau_2)V_5}{Y_5 V}\right)\right] \\
& - \xi_P X_5 P_5 \left[F\left(\frac{X_5}{X}\right) + F\left(\frac{X(t-\tau_3)P(t-\tau_3)I_5}{X_5 P_5 I}\right) + F\left(\frac{I(t-\tau_4)P_5}{I_5 P}\right)\right] \\
& + e^{\alpha_3\tau_3+\alpha_4\tau_4} \frac{\beta_I \rho_P(\xi_P \gamma_M \eta_Z + \xi_V \gamma_Z \eta_M + \varrho \eta_Z \eta_M)}{\xi_P \theta_P \eta_M \eta_Z} (\Re_8 - 1) M.
\end{aligned}
$$

If $\Re_8 \leq 1$, then $\frac{d\Lambda_5}{dt} \leq 0$ for all $X, Y, I, V, P, M > 0$. Moreover, we have $\frac{d\Lambda_5}{dt} = 0$ when $X = X_5$, $Y = Y_5$, $V = V_5$, $I = I_5$, $P = P_5$, and $M = 0$. One can show that $\bar{\Theta}_5 = \{\Delta_5\}$, and then Δ_5 is GAS. $\square$

Theorem 7. *Let $\Re_6 > 1$, $\Re_7 \leq 1$ and $\Re_2/\Re_1 > 1$; then, Δ_6 is GAS.*

Proof. Consider

$$
\begin{aligned}
\Lambda_6 =& X_6 F\left(\frac{X}{X_6}\right) + e^{\alpha_1\tau_1} Y_6 F\left(\frac{Y}{Y_6}\right) + e^{\alpha_3\tau_3} I_6 F\left(\frac{I}{I_6}\right) + \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} V_6 F\left(\frac{V}{V_6}\right) \\
& + \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} P_6 F\left(\frac{P}{P_6}\right) + \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2} Z + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4} M_6 F\left(\frac{M}{M_6}\right) \\
& + \xi_V X_6 V_6 \int_{t-\tau_1}^{t} F\left(\frac{X(u)V(u)}{X_6 V_6}\right) du + \xi_P X_6 P_6 \int_{t-\tau_3}^{t} F\left(\frac{X(u)P(u)}{X_6 P_6}\right) du \\
& + e^{\alpha_1\tau_1} \beta_Y Y_6 \int_{t-\tau_2}^{t} F\left(\frac{Y(u)}{Y_6}\right) du + e^{\alpha_3\tau_3} \beta_I I_6 \int_{t-\tau_4}^{t} F\left(\frac{I(u)}{I_6}\right) du.
\end{aligned}
$$

We calculate $\frac{d\Lambda_6}{dt}$ as:

$$
\begin{aligned}
\frac{d\Lambda_6}{dt} =& \left(1 - \frac{X_6}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\left(1 - \frac{Y_6}{Y}\right)\frac{dY}{dt} + e^{\alpha_3\tau_3}\left(1 - \frac{I_6}{I}\right)\frac{dI}{dt} \\
& + \frac{\beta_Y}{\theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1 - \frac{V_6}{V}\right)\frac{dV}{dt} + \frac{\beta_I}{\theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1 - \frac{P_6}{P}\right)\frac{dP}{dt} \\
& + \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{dZ}{dt} + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1 - \frac{M_6}{M}\right)\frac{dM}{dt} \\
& + \xi_V X_6 V_6 \left[\frac{XV}{X_6 V_6} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_6 V_6} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right]
\end{aligned}
$$

$$+\xi_P X_6 P_6\left[\frac{XP}{X_6P_6}-\frac{X(t-\tau_3)P(t-\tau_3)}{X_6P_6}+\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right]$$
$$+e^{\alpha_1\tau_1}\beta_Y Y_6\left[\frac{Y}{Y_6}-\frac{Y(t-\tau_2)}{Y_6}+\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right]$$
$$+e^{\alpha_3\tau_3}\beta_I I_6\left[\frac{I}{I_6}-\frac{I(t-\tau_4)}{I_6}+\ln\left(\frac{I(t-\tau_4)}{I}\right)\right].$$

It follows from Equation (1)–(7) that

$$\begin{aligned}
\frac{d\Lambda_6}{dt} &= \left(1-\frac{X_6}{X}\right)[\delta-\varrho X-\xi_V XV-\xi_P XP] \\
&+e^{\alpha_1\tau_1}\left(1-\frac{Y_6}{Y}\right)\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1)-\beta_Y Y\right] \\
&+e^{\alpha_3\tau_3}\left(1-\frac{I_6}{I}\right)\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3)-\beta_I I\right] \\
&+\frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_6}{V}\right)\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2)-\lambda_V V-\rho_V VZ\right] \\
&+\frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_6}{P}\right)\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4)-\lambda_P P-\rho_P PM\right] \\
&+\frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}[\eta_Z VZ-\gamma_Z Z]+\frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{M_6}{M}\right)[\eta_M PM-\gamma_M M] \\
&+\xi_V X_6 V_6\left[\frac{XV}{X_6V_6}-\frac{X(t-\tau_1)V(t-\tau_1)}{X_6V_6}+\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+\xi_P X_6 P_6\left[\frac{XP}{X_6P_6}-\frac{X(t-\tau_3)P(t-\tau_3)}{X_6P_6}+\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+e^{\alpha_1\tau_1}\beta_Y Y_6\left[\frac{Y}{Y_6}-\frac{Y(t-\tau_2)}{Y_6}+\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] \\
&+e^{\alpha_3\tau_3}\beta_I I_6\left[\frac{I}{I_6}-\frac{I(t-\tau_4)}{I_6}+\ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned} \tag{28}$$

We collect the terms of Equation (28) as follows:

$$\begin{aligned}
\frac{d\Lambda_6}{dt} &= \left(1-\frac{X_6}{X}\right)(\delta-\varrho X)+\xi_V X_6 V+\xi_P X_6 P-\xi_V X(t-\tau_1)V(t-\tau_1)\frac{Y_6}{Y} \\
&+e^{\alpha_1\tau_1}\beta_Y Y_6-\xi_P X(t-\tau_3)P(t-\tau_3)\frac{I_6}{I}+e^{\alpha_3\tau_3}\beta_I I_6-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V \\
&-e^{\alpha_1\tau_1}\beta_Y Y(t-\tau_2)\frac{V_6}{V}+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V_6-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P \\
&-e^{\alpha_3\tau_3}\beta_I I(t-\tau_4)\frac{P_6}{P}+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P_6+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}P_6 M \\
&-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}M_6 P+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M_6 \\
&+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}\left[\frac{\eta_Z}{\gamma_Z}V_6-1\right]Z+\xi_V X_6 V_6\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right) \\
&+\xi_P X_6 P_6\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)+e^{\alpha_1\tau_1}\beta_Y Y_6\ln\left(\frac{Y(t-\tau_2)}{Y}\right) \\
&+e^{\alpha_3\tau_3}\beta_I I_6\ln\left(\frac{I(t-\tau_4)}{I}\right).
\end{aligned}$$

Using the equilibrium conditions for Δ_6,

$$\delta = \varrho X_6 + \xi_V X_6 V_6 + \xi_P X_6 P_6, \quad \xi_V X_6 V_6 = e^{\alpha_1 \tau_1} \beta_Y Y_6, \quad \xi_P X_6 P_6 = e^{\alpha_3 \tau_3} \beta_I I_6,$$
$$Y_6 = e^{\alpha_2 \tau_2} \frac{\lambda_V}{\theta_V} V_6, \quad I_6 = e^{\alpha_4 \tau_4} \frac{\lambda_P}{\theta_P} P_6 + e^{\alpha_4 \tau_4} \frac{\rho_P}{\theta_P} P_6 M_6, \quad P_6 = \frac{\gamma_M}{\eta_M},$$

we obtain

$$\begin{aligned}
\frac{d\Lambda_6}{dt} &= \left(1 - \frac{X_6}{X}\right)(\varrho X_6 - \varrho X) + 3\xi_V X_6 V_6 + 3\xi_P X_6 P_6 - \xi_V X_6 V_6 \frac{X_6}{X} \\
&- \xi_P X_6 P_6 \frac{X_6}{X} - \xi_V X_6 V_6 \frac{X(t-\tau_1) V(t-\tau_1) Y_6}{X_6 V_6 Y} \\
&- \xi_P X_6 P_6 \frac{X(t-\tau_3) P(t-\tau_3) I_6}{X_6 P_6 I} - \xi_V X_6 V_6 \frac{Y(t-\tau_2) V_6}{Y_6 V} \\
&- \xi_P X_6 P_6 \frac{I(t-\tau_4) P_6}{I_6 P} + \xi_V X_6 V_6 \ln\left(\frac{X(t-\tau_1) V(t-\tau_1)}{XV}\right) \\
&+ \xi_P X_6 P_6 \ln\left(\frac{X(t-\tau_3) P(t-\tau_3)}{XP}\right) + \xi_V X_6 V_6 \ln\left(\frac{Y(t-\tau_2)}{Y}\right) \\
&+ \xi_P X_6 P_6 \ln\left(\frac{I(t-\tau_4)}{I}\right) + e^{\alpha_1 \tau_1 + \alpha_2 \tau_2} \frac{\beta_Y \rho_V (\xi_P \gamma_M \eta_Z + \xi_V \gamma_Z \eta_M + \varrho \eta_Z \eta_M)}{\xi_V \theta_V \eta_M \eta_Z} (\Re_7 - 1) Z.
\end{aligned} \tag{29}$$

Then, simplifying Equation (29), we get:

$$\begin{aligned}
\frac{d\Lambda_6}{dt} &= -\varrho \frac{(X - X_6)^2}{X} \\
&- \xi_V X_6 V_6 \left[F\left(\frac{X_6}{X}\right) + F\left(\frac{X(t-\tau_1) V(t-\tau_1) Y_6}{X_6 V_6 Y}\right) + F\left(\frac{Y(t-\tau_2) V_6}{Y_6 V}\right)\right] \\
&- \xi_P X_6 P_6 \left[F\left(\frac{X_6}{X}\right) + F\left(\frac{X(t-\tau_3) P(t-\tau_3) I_6}{X_6 P_6 I}\right) + F\left(\frac{I(t-\tau_4) P_6}{I_6 P}\right)\right] \\
&+ e^{\alpha_1 \tau_1 + \alpha_2 \tau_2} \frac{\beta_Y \rho_V (\xi_P \gamma_M \eta_Z + \xi_V \gamma_Z \eta_M + \varrho \eta_Z \eta_M)}{\xi_V \theta_V \eta_M \eta_Z} (\Re_7 - 1) Z.
\end{aligned}$$

If $\Re_7 \leq 1$, then $\frac{d\Lambda_6}{dt} \leq 0$ for all $X, Y, I, V, P, Z > 0$. In addition, $\frac{d\Lambda_6}{dt} = 0$ occurs at $X = X_6$, $Y = Y_6$, $I = I_6$, $V = V_6$, $P = P_6$, and $Z = 0$. The proof can be completed similarly to the previous theorems. □

Theorem 8. *If $\Re_7 > 1$ and $\Re_8 > 1$, then Δ_7 is GAS.*

Proof. We define a function Λ_7 as:

$$\begin{aligned}
\Lambda_7 &= X_7 F\left(\frac{X}{X_7}\right) + e^{\alpha_1 \tau_1} Y_7 F\left(\frac{Y}{Y_7}\right) + e^{\alpha_3 \tau_3} I_7 F\left(\frac{I}{I_7}\right) + \frac{\beta_Y}{\theta_V} e^{\alpha_1 \tau_1 + \alpha_2 \tau_2} V_7 F\left(\frac{V}{V_7}\right) \\
&+ \frac{\beta_I}{\theta_P} e^{\alpha_3 \tau_3 + \alpha_4 \tau_4} P_7 F\left(\frac{P}{P_7}\right) + \frac{\rho_V \beta_Y}{\eta_Z \theta_V} e^{\alpha_1 \tau_1 + \alpha_2 \tau_2} Z_7 F\left(\frac{Z}{Z_7}\right) + \frac{\rho_P \beta_I}{\eta_M \theta_P} e^{\alpha_3 \tau_3 + \alpha_4 \tau_4} M_7 F\left(\frac{M}{M_7}\right) \\
&+ \xi_V X_7 V_7 \int_{t-\tau_1}^{t} F\left(\frac{X(u) V(u)}{X_7 V_7}\right) du + \xi_P X_7 P_7 \int_{t-\tau_3}^{t} F\left(\frac{X(u) P(u)}{X_7 P_7}\right) du \\
&+ e^{\alpha_1 \tau_1} \beta_Y Y_7 \int_{t-\tau_2}^{t} F\left(\frac{Y(u)}{Y_7}\right) du + e^{\alpha_3 \tau_3} \beta_I I_7 \int_{t-\tau_4}^{t} F\left(\frac{I(u)}{I_7}\right) du.
\end{aligned}$$

We calculate $\frac{d\Lambda_7}{dt}$ as:

$$\begin{aligned}
\frac{d\Lambda_7}{dt} &= \left(1-\frac{X_7}{X}\right)\frac{dX}{dt} + e^{\alpha_1\tau_1}\left(1-\frac{Y_7}{Y}\right)\frac{dY}{dt} + e^{\alpha_3\tau_3}\left(1-\frac{I_7}{I}\right)\frac{dI}{dt} \\
&+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_7}{V}\right)\frac{dV}{dt} + \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_7}{P}\right)\frac{dP}{dt} \\
&+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{Z_7}{Z}\right)\frac{dZ}{dt} + \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{M_7}{M}\right)\frac{dM}{dt} \\
&+ \xi_V X_7 V_7\left[\frac{XV}{X_7V_7} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_7V_7} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+ \xi_P X_7 P_7\left[\frac{XP}{X_7P_7} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_7P_7} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_7\left[\frac{Y}{Y_7} - \frac{Y(t-\tau_2)}{Y_7} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] \\
&+ e^{\alpha_3\tau_3}\beta_I I_7\left[\frac{I}{I_7} - \frac{I(t-\tau_4)}{I_7} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned}$$

It follows from Equations (1)–(7) that

$$\begin{aligned}
\frac{d\Lambda_7}{dt} &= \left(1-\frac{X_7}{X}\right)[\delta - \varrho X - \xi_V XV - \xi_P XP] \\
&+ e^{\alpha_1\tau_1}\left(1-\frac{Y_7}{Y}\right)\left[e^{-\alpha_1\tau_1}\xi_V X(t-\tau_1)V(t-\tau_1) - \beta_Y Y\right] \\
&+ e^{\alpha_3\tau_3}\left(1-\frac{I_7}{I}\right)\left[e^{-\alpha_3\tau_3}\xi_P X(t-\tau_3)P(t-\tau_3) - \beta_I I\right] \\
&+ \frac{\beta_Y}{\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{V_7}{V}\right)\left[e^{-\alpha_2\tau_2}\theta_V Y(t-\tau_2) - \lambda_V V - \rho_V VZ\right] \\
&+ \frac{\beta_I}{\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{P_7}{P}\right)\left[e^{-\alpha_4\tau_4}\theta_P I(t-\tau_4) - \lambda_P P - \rho_P PM\right] \\
&+ \frac{\rho_V\beta_Y}{\eta_Z\theta_V}e^{\alpha_1\tau_1+\alpha_2\tau_2}\left(1-\frac{Z_7}{Z}\right)[\eta_Z VZ - \gamma_Z Z] \\
&+ \frac{\rho_P\beta_I}{\eta_M\theta_P}e^{\alpha_3\tau_3+\alpha_4\tau_4}\left(1-\frac{M_7}{M}\right)[\eta_M PM - \gamma_M M] \\
&+ \xi_V X_7 V_7\left[\frac{XV}{X_7V_7} - \frac{X(t-\tau_1)V(t-\tau_1)}{X_7V_7} + \ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\right] \\
&+ \xi_P X_7 P_7\left[\frac{XP}{X_7P_7} - \frac{X(t-\tau_3)P(t-\tau_3)}{X_7P_7} + \ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)\right] \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_7\left[\frac{Y}{Y_7} - \frac{Y(t-\tau_2)}{Y_7} + \ln\left(\frac{Y(t-\tau_2)}{Y}\right)\right] \\
&+ e^{\alpha_3\tau_3}\beta_I I_7\left[\frac{I}{I_7} - \frac{I(t-\tau_4)}{I_7} + \ln\left(\frac{I(t-\tau_4)}{I}\right)\right].
\end{aligned} \tag{30}$$

We collect the terms of Equation (30) as follows:

$$\begin{aligned}
\frac{d\Lambda_7}{dt} &= \left(1-\frac{X_7}{X}\right)(\delta - \varrho X) + \xi_V X_7 V + \xi_P X_7 P - \xi_V X(t-\tau_1)V(t-\tau_1)\frac{Y_7}{Y} \\
&+ e^{\alpha_1\tau_1}\beta_Y Y_7 - \xi_P X(t-\tau_3)P(t-\tau_3)\frac{I_7}{I} + e^{\alpha_3\tau_3}\beta_I I_7 - e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V \\
&- e^{\alpha_1\tau_1}\beta_Y Y(t-\tau_2)\frac{V_7}{V} + e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\lambda_V}{\theta_V}V_7 + e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}ZV_7 \\
&- e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P - e^{\alpha_3\tau_3}\beta_I I(t-\tau_4)\frac{P_7}{P} + e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\lambda_P}{\theta_P}P_7
\end{aligned}$$

$$
\begin{aligned}
&+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}P_7M-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z-e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V}{\theta_V}VZ_7\\
&+e^{\alpha_1\tau_1+\alpha_2\tau_2}\frac{\beta_Y\rho_V\gamma_Z}{\theta_V\eta_Z}Z_7-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M-e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P}{\theta_P}M_7P\\
&+e^{\alpha_3\tau_3+\alpha_4\tau_4}\frac{\beta_I\rho_P\gamma_M}{\theta_P\eta_M}M_7+\xi_VX_7V_7\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\\
&+\xi_PX_7P_7\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)+e^{\alpha_1\tau_1}\beta_YY_7\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\\
&+e^{\alpha_3\tau_3}\beta_II_7\ln\left(\frac{I(t-\tau_4)}{I}\right).
\end{aligned}
$$

Using the equilibrium conditions for Δ_7,

$$
\begin{aligned}
&\delta=\varrho X_7+\xi_VX_7V_7+\xi_PX_7P_7,\\
&\xi_VX_7V_7=e^{\alpha_1\tau_1}\beta_YY_7,\quad \xi_PX_7P_7=e^{\alpha_3\tau_3}\beta_II_7,\\
&Y_7=e^{\alpha_2\tau_2}\frac{\lambda_V}{\theta_V}V_7+e^{\alpha_2\tau_2}\frac{\rho_V}{\theta_V}V_7Z_7,\quad I_7=e^{\alpha_4\tau_4}\frac{\lambda_P}{\theta_P}P_7+e^{\alpha_4\tau_4}\frac{\rho_P}{\theta_P}P_7M_7,\\
&V_7=\frac{\gamma_Z}{\eta_Z},\quad P_7=\frac{\gamma_M}{\eta_M},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{d\Lambda_7}{dt}=&\left(1-\frac{X_7}{X}\right)(\varrho X_7-\varrho X)+3\xi_VX_7V_7+3\xi_PX_7P_7-\xi_VX_7V_7\frac{X_7}{X}-\xi_PX_7P_7\frac{X_7}{X}\\
&-\xi_VX_7V_7\frac{X(t-\tau_1)V(t-\tau_1)Y_7}{X_7V_7Y}-\xi_PX_7P_7\frac{X(t-\tau_3)P(t-\tau_3)I_7}{X_7P_7I}\\
&-\xi_VX_7V_7\frac{Y(t-\tau_2)V_7}{Y_7V}-\xi_PX_7P_7\frac{I(t-\tau_4)P_7}{I_7P}+\xi_VX_7V_7\ln\left(\frac{X(t-\tau_1)V(t-\tau_1)}{XV}\right)\\
&+\xi_PX_7P_7\ln\left(\frac{X(t-\tau_3)P(t-\tau_3)}{XP}\right)+\xi_VX_7V_7\ln\left(\frac{Y(t-\tau_2)}{Y}\right)\\
&+\xi_PX_7P_7\ln\left(\frac{I(t-\tau_4)}{I}\right).
\end{aligned}\tag{31}
$$

Then, simplifying Equation (31), we get:

$$
\begin{aligned}
\frac{d\Lambda_7}{dt}=&-\varrho\frac{(X-X_7)^2}{X}\\
&-\xi_VX_7V_7\left[F\left(\frac{X_7}{X}\right)+F\left(\frac{X(t-\tau_1)V(t-\tau_1)Y_7}{X_7V_7Y}\right)+F\left(\frac{Y(t-\tau_2)V_7}{Y_7V}\right)\right]\\
&-\xi_PX_7P_7\left[F\left(\frac{X_7}{X}\right)+F\left(\frac{X(t-\tau_3)P(t-\tau_3)I_7}{X_7P_7I}\right)+F\left(\frac{I(t-\tau_4)P_7}{I_7P}\right)\right].
\end{aligned}
$$

Clearly, $\frac{d\Lambda_7}{dt}\leq 0$ for all $X,Y,I,V,P>0$, where $\frac{d\Lambda_7}{dt}=0$ when $X=X_7$, $Y=Y_7$, $V=V_7$, $I=I_7$, and $P=P_7$. One can show that $\bar{\Theta}_7=\{\Delta_7\}$, and by using the L-LAST, we find that Δ_7 is GAS. $\square$

The existence and global stability conditions of the equilibria are summarized in Table 1.

Table 1. Existence and stability conditions of the equilibria.

Equilibrium Point	Existence Conditions	Global Stability Conditions
$\Delta_0 = (X_0, 0, 0, 0, 0, 0, 0)$	None	$\Re_1 \le 1$ and $\Re_2 \le 1$
$\Delta_1 = (X_1, Y_1, 0, V_1, 0, 0, 0)$	$\Re_1 > 1$	$\Re_1 > 1$, $\Re_2/\Re_1 \le 1$ and $\Re_3 \le 1$
$\Delta_2 = (X_2, 0, I_2, 0, P_2, 0, 0)$	$\Re_2 > 1$	$\Re_2 > 1$, $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$
$\Delta_3 = (X_3, Y_3, 0, V_3, 0, Z_3, 0)$	$\Re_3 > 1$	$\Re_3 > 1$ and $\Re_5 \le 1$
$\Delta_4 = (X_4, 0, I_4, 0, P_4, 0, M_4)$	$\Re_4 > 1$	$\Re_4 > 1$ and $\Re_6 \le 1$
$\Delta_5 = (X_5, Y_5, I_5, V_5, P_5, Z_5, 0)$	$\Re_5 > 1$ and $\Re_1/\Re_2 > 1$	$\Re_5 > 1$, $\Re_8 \le 1$ and $\Re_1/\Re_2 > 1$
$\Delta_6 = (X_6, Y_6, I_6, V_6, P_6, 0, M_6)$	$\Re_6 > 1$ and $\Re_2/\Re_1 > 1$	$\Re_6 > 1$, $\Re_7 \le 1$ and $\Re_2/\Re_1 > 1$
$\Delta_7 = (X_7, Y_7, I_7, V_7, P_7, Z_7, M_7)$	$\Re_7 > 1$ and $\Re_8 > 1$	$\Re_7 > 1$ and $\Re_8 > 1$

6. Numerical Simulations

We illustrate the global stability of the model's equilibria via numerical simulations. We use the values of the parameters presented in Table 2. In addition, we discuss the effects of antiviral treatments and time delays on the co-infection dynamics.

Table 2. Model parameters.

Parameter	Description	Value
δ	Production rate of susceptible ECs	0.5
ϱ	Death rate constant of susceptible ECs	0.05
β_Y	Death rate constant of SARS-CoV-2-infected ECs	0.11
β_I	Death rate constant of IAV-infected ECs	0.2
θ_V	Virus–cell incidence rate constant between SARS-CoV-2 particles and susceptible ECs	0.2
λ_V	Death rate constant of SARS-CoV-2 particles	0.2
ρ_V	Neutralization rate constant of SARS-CoV-2 by SARS-CoV-2-specific antibodies	0.05
θ_P	Virus–cell incidence rate constant between IAV particles and susceptible ECs	0.4
λ_P	Death rate constant of IAV particles	0.1
ρ_P	Neutralization rate constant of IAV by IAV-specific antibodies	0.04
γ_Z	Death rate constant of SARS-CoV-2-specific antibodies	0.05
γ_M	Death rate constant of IAV-specific antibodies	0.04
α_1	Constant	1
α_2	Constant	1
α_3	Constant	0.1
α_4	Constant	0.1

6.1. Stability of the Equilibria

Here, we fix the delay parameters as $\tau_1 = 0.1$, $\tau_2 = 0.1$, $\tau_3 = 0.2$, and $\tau_4 = 0.2$. In addition, we solve system (1)–(7) with the following initial states:

$$\begin{aligned}
\text{IS(I)} &: (X(u), Y(u), I(u), V(u), P(u), Z(u), M(u)) = (5, 1, 0.5, 0.03, 0.5, 1, 4),\\
\text{IS(II)} &: (X(u), Y(u), I(u), V(u), P(u), Z(u), M(u)) = (4, 1.5, 0.7, 0.06, 0.8, 2, 6),\\
\text{IS(III)} &: (X(u), Y(u), I(u), V(u), P(u), Z(u), M(u)) = (3, 2, 1, 0.3, 1.4, 3, 8),
\end{aligned}$$

where $u \in [-0.2, 0]$.

We use the values given in Table 2 and select eight sets of values of $(\xi_V, \xi_P, \eta_Z, \eta_M)$ for the following strategies.

First strategy (Stability of Δ_0): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.001, 0.001, 0.01, 0.02)$. These values gives $\Re_1 = 0.0744 < 1$ and $\Re_2 = 0.1922 < 1$. It is shown in Figure 1 that the trajectories starting with initials IS(I)-IS(III) tend to the equilibrium, $\Delta_0 = (10, 0, 0, 0, 0, 0, 0)$. This supports the global stability results given in Theorem 1. In this strategy, both influenza and COVID-19 will be cleared. In fact, making $\Re_1 \leq 1$ and $\Re_2 \leq 1$ can be achieved in one or more of the following ways: (i) applying two antiviral drugs for blocking SARS-CoV-2 and IAV infections with drug efficacies of ϵ_V and ϵ_P, respectively, where $0 \leq \epsilon_V \leq 1$ and $0 \leq \epsilon_P \leq 1$; then, the parameters ξ_V and ξ_P will be reduced to $(1-\epsilon_V)\xi_V$ and $(1-\epsilon_P)\xi_P$, respectively; (ii) applying two antiviral drugs for blocking the replication of SARS-CoV-2 and IAV with drug efficacies of ε_V and ε_P, respectively, where $0 \leq \varepsilon_V \leq 1$ and $0 \leq \varepsilon_P \leq 1$. Then, the parameters θ_V and θ_P will be reduced to $(1-\varepsilon_V)\theta_V$ and $(1-\varepsilon_P)\theta_P$, respectively.

Second strategy (Stability of Δ_1): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.05, 0.001, 0.002, 0.02)$. This selection provides $\Re_1 = 3.7215 > 1$, $\Re_3 = 0.1431 < 1$, and $\Re_2/\Re_1 = 0.0516 < 1$. The equilibrium Δ_1 exists with $\Delta_1 = (2.69, 3.008, 0, 2.72, 0, 0, 0)$. Figure 2 shows that the trajectories initiated with IS(I)-IS(III) converge to Δ_1, and this result agrees with Theorem 2. This strategy suggests that a COVID-19 mono-infection with an inactive antibody response will be established.

Third strategy (Stability of Δ_2): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.005, 0.03, 0.01, 0.001)$. This gives $\Re_2 = 5.7647 > 1$, $\Re_4 = 0.2306 < 1$, and $\Re_1/\Re_2 = 0.0646 < 1$. The numerical solution confirms that $\Delta_2 = (1.73, 0, 2.03, 0, 7.94, 0, 0)$ exists. It can be observed from Figure 3 that the solutions initiated with IS(I)-IS(III) converge to Δ_2, and this result agrees with Theorem 3. This strategy suggests that an influenza mono-infection with an inactive antibody response will be established.

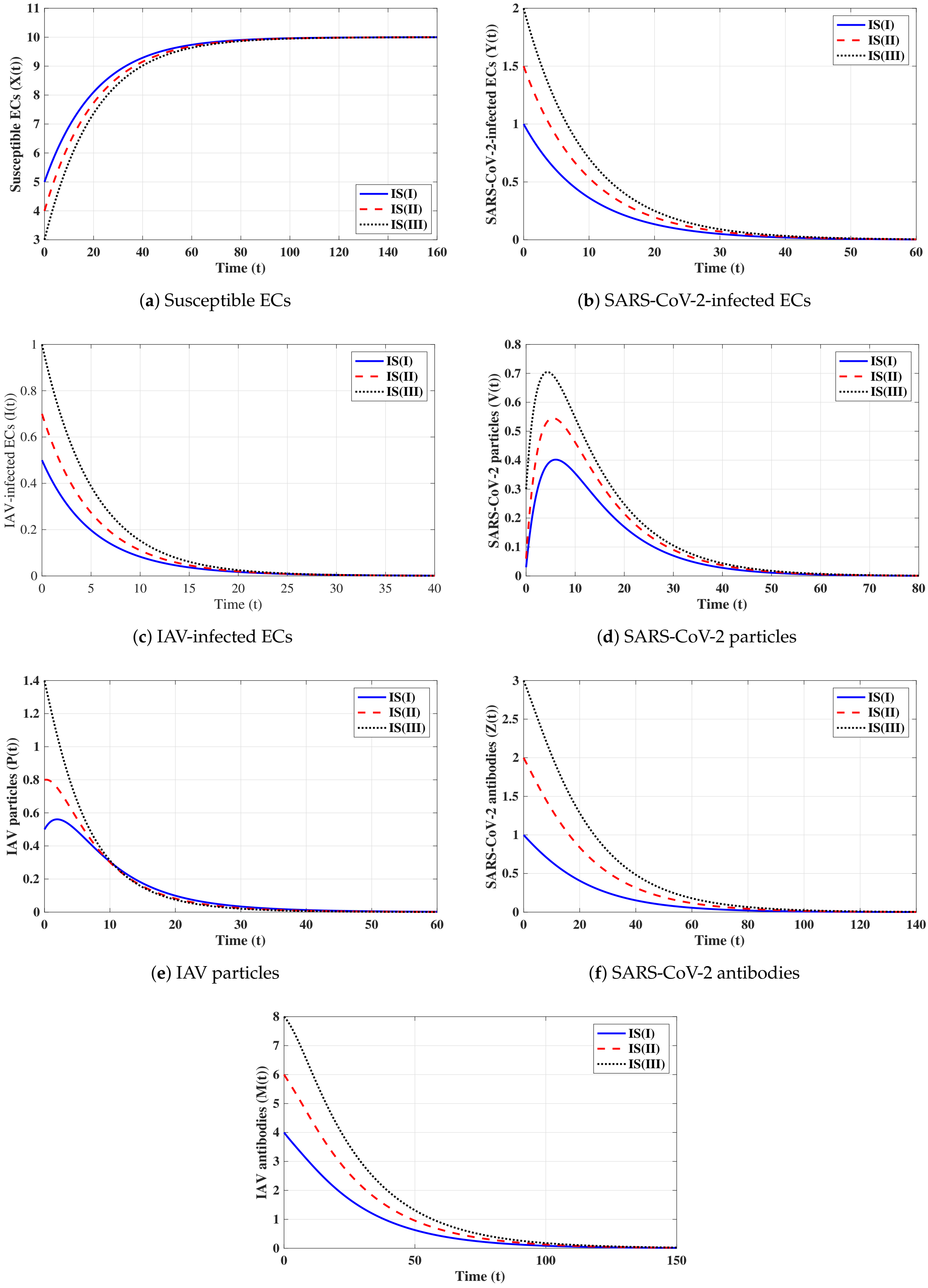
Fourth strategy (Stability of Δ_3): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.09, 0.002, 0.05, 0.05)$. This yields $\Re_3 = 2.3924 > 1$ and $\Re_5 = 0.1373 < 1$. Figure 4 illustrates that the solutions tend to $\Delta_3 = (3.57, 2.64, 0, 1, 0, 5.57, 0)$ regardless of the initial states. This result supports the global stability result given in Theorem 4. This strategy shows that a COVID-19 mono-infection with an activated SARS-CoV-2-specific antibody response will be attained.

Fifth strategy (Stability of Δ_4): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.01, 0.1, 0.01, 0.02)$. The values of $\Re_4$ and $\Re_6$ are computed as $\Re_4 = 3.8432 > 1$ and $\Re_6 = 0.1489 < 1$. Thus, Δ_4 exists with $\Delta_4 = (2, 0, 1.96, 0, 2, 0, 7.11)$. The numerical solutions with initials IS(I)-IS(III) tend to Δ_4 (see Figure 5). This shows the global stability of Δ_4 given in Theorem 5. In this strategy, an influenza mono-infection with a stimulated IAV-specific antibody response will be achieved.

Sixth strategy (Stability of Δ_5): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.09, 0.01, 0.9, 0.001)$. Then, we calculate $\Re_5 = 1.7469 > 1$, $\Re_8 = 0.2112 < 1$, and $\Re_1/\Re_2 = 3.486 > 1$. The numerical results displayed in Figure 6 establish that $\Delta_5 = (5.2, 0.21, 1.05, 0.06, 4.11, 9.94, 0)$ exists and that it is GAS; this agrees with the result of Theorem 6. This result suggests that a co-infection with influenza and COVID-19 with only an active SARS-CoV-2-specific antibody response will be attained.

Seventh strategy (Stability of Δ_6): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.04, 0.05, 0.01, 0.05)$. We compute $\Re_6 = 1.654 > 1$, $\Re_7 = 0.5133 < 1$, and $\Re_2/\Re_1 = 3.2272 > 1$. We find that the equilibrium $\Delta_6 = (3.36, 1.63, 0.66, 1.47, 0.8, 0, 5.57)$ exists. Figure 7 draws the numerical solutions of the DDEs with initials IS(I)-IS(III). It is shown that Δ_6 is GAS, and this supports the result of Theorem 7. This strategy leads to a co-infection with influenza and COVID-19 with only an active IAV-specific antibody response.

Eighth strategy 8 (Stability of Δ_7): $(\xi_V,\xi_P,\eta_Z,\eta_M) = (0.09, 0.09, 0.5, 0.5)$. This selection gives $\Re_7 = 5.0594 > 1$ and $\Re_8 = 13.0621 > 1$. Figure 8 shows that $\Delta_7 = (7.55, 0.56, 0.27, 0.1, 0.08, 16.24, 30.16)$ exists and that it is GAS according to Theorem 8. This strategy leads to the case of co-infection with influenza and COVID-19 in which both types of antibody responses are active.



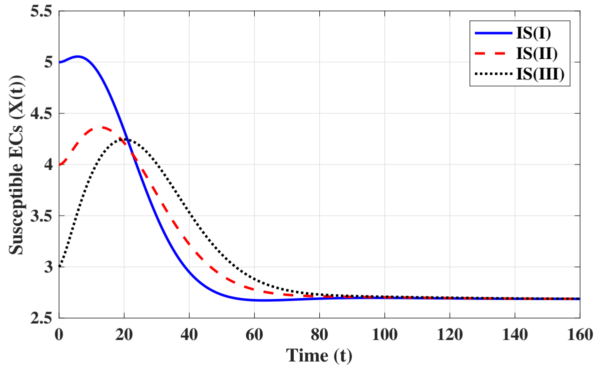
(**a**) Susceptible ECs (**b**) SARS-CoV-2-infected ECs

(**c**) IAV-infected ECs (**d**) SARS-CoV-2 particles

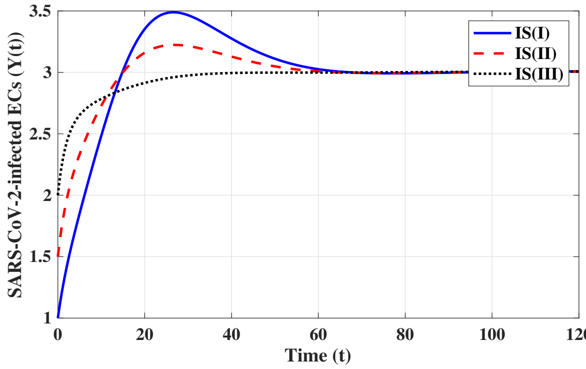
(**e**) IAV particles (**f**) SARS-CoV-2 antibodies

(**g**) IAV antibodies

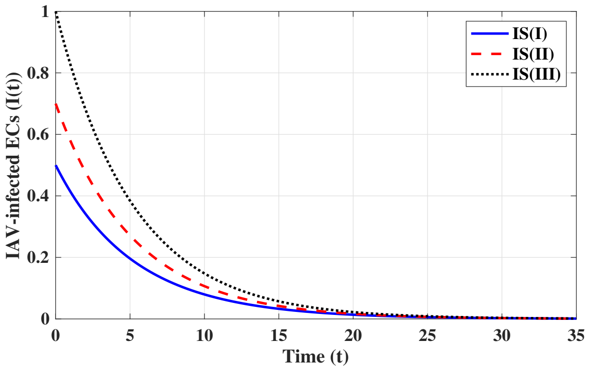
Figure 1. Solutions of system (1)–(7) when $\Re_1 \leq 1$ and $\Re_2 \leq 1$ (first strategy).



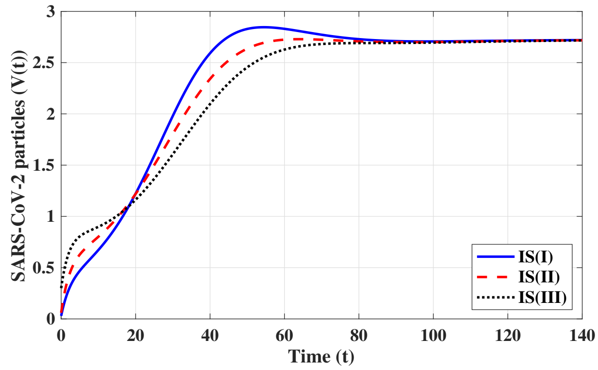
(**a**) Susceptible ECs



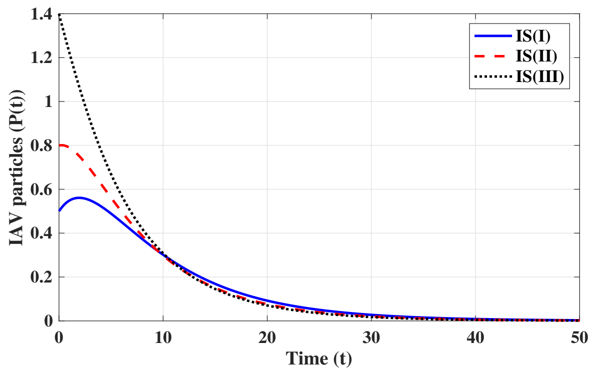
(**b**) SARS-CoV-2-infected ECs



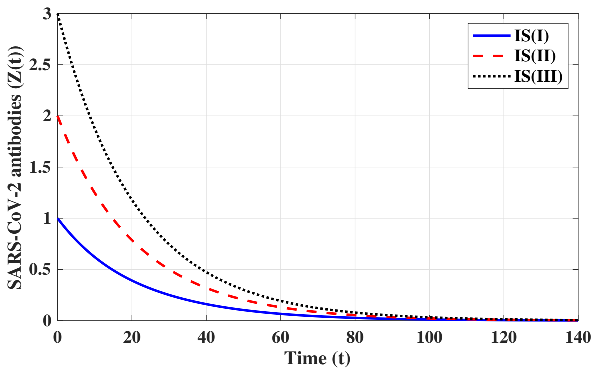
(**c**) IAV-infected ECs



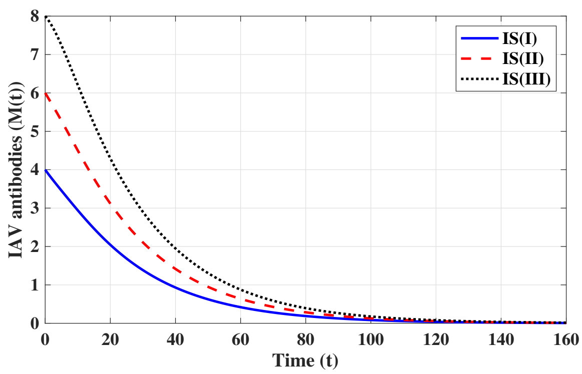
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

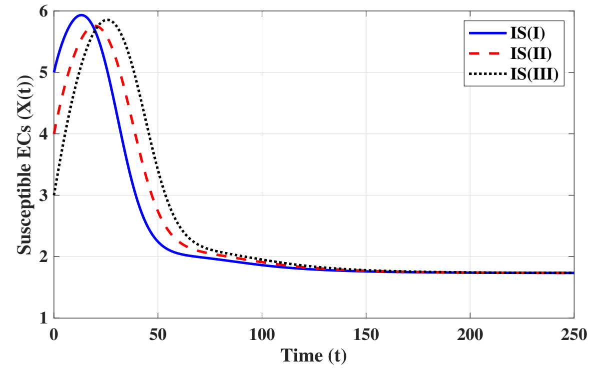


(**f**) SARS-CoV-2 antibodies

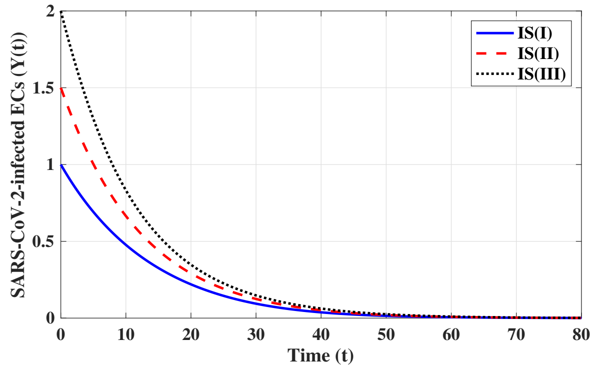


(**g**) IAV antibodies

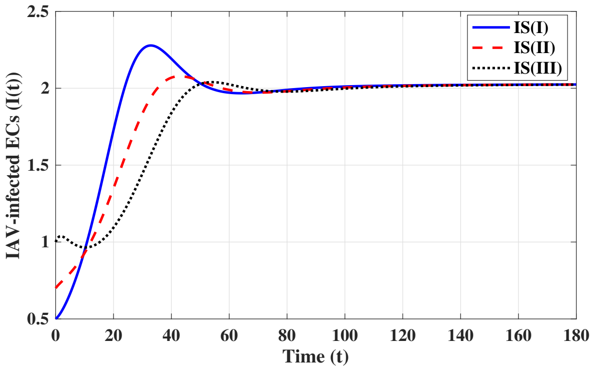
Figure 2. Solutions of system (1)–(7) when $\Re_1 > 1, \Re_2/\Re_1 \leq 1$, and $\Re_3 \leq 1$ (second strategy).



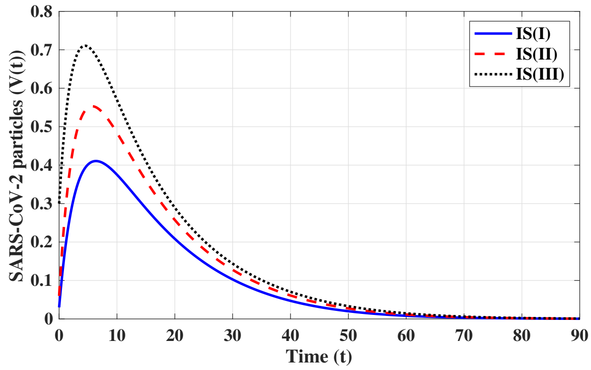
(**a**) Susceptible ECs



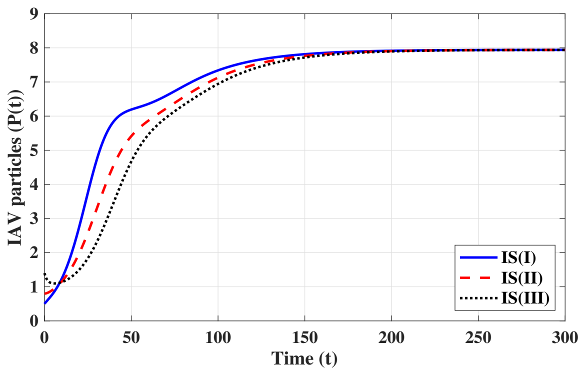
(**b**) SARS-CoV-2-infected ECs



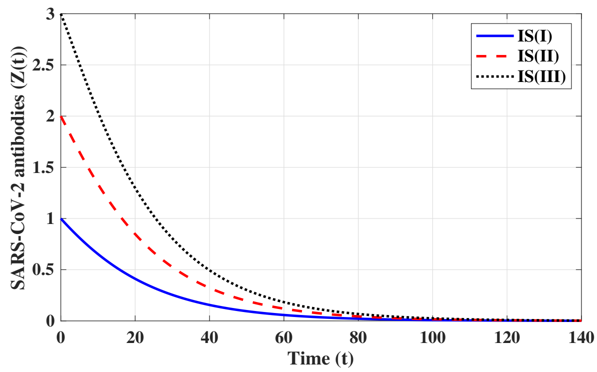
(**c**) IAV-infected ECs



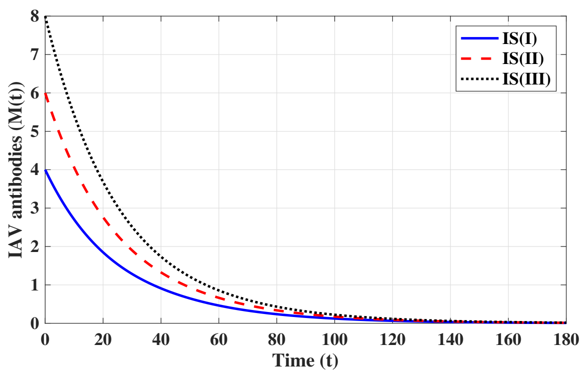
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

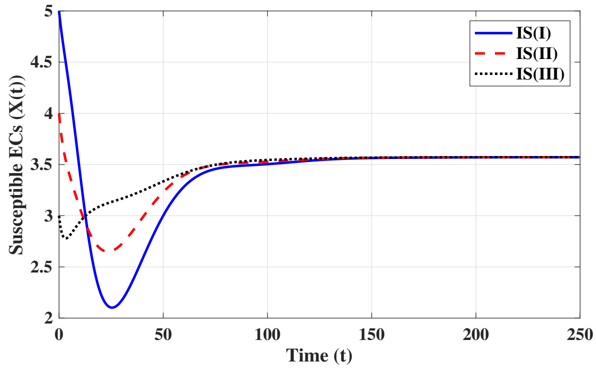


(**f**) SARS-CoV-2 antibodies

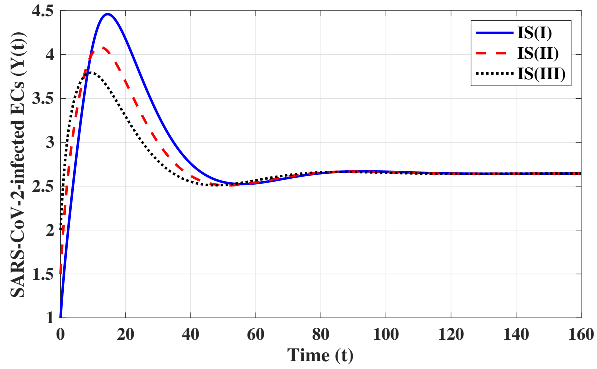


(**g**) IAV antibodies

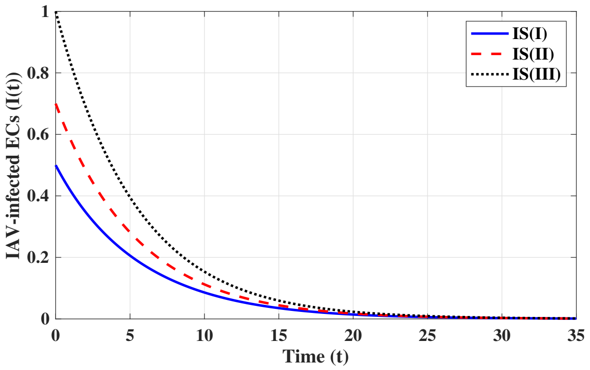
Figure 3. Solutions of system (1)–(7) when $\Re_2 > 1, \Re_1/\Re_2 \leq 1$, and $\Re_4 \leq 1$ (third strategy).



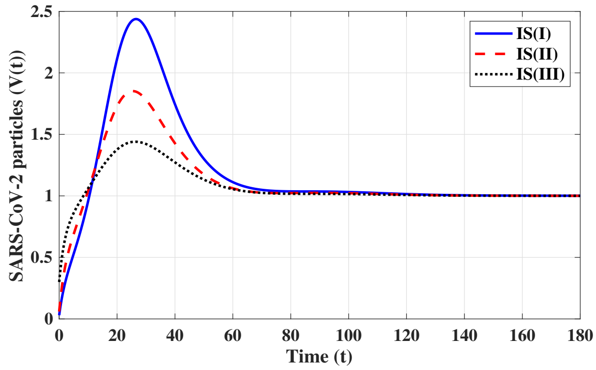
(**a**) Susceptible ECs



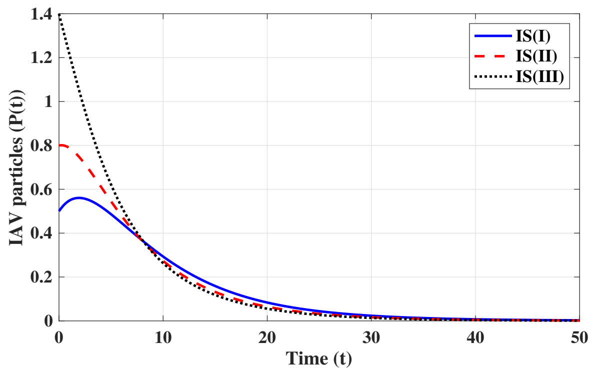
(**b**) SARS-CoV-2-infected ECs



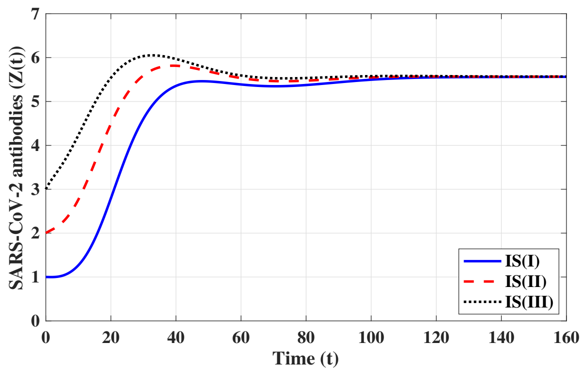
(**c**) IAV-infected ECs



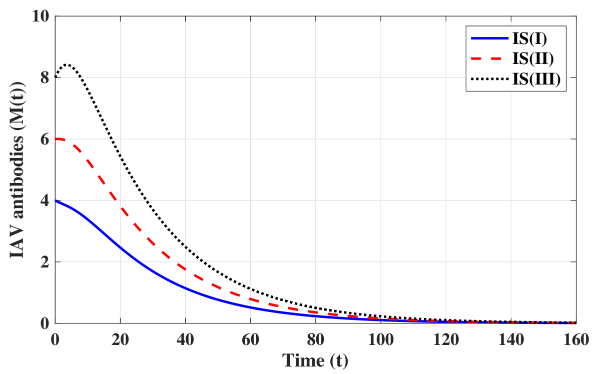
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

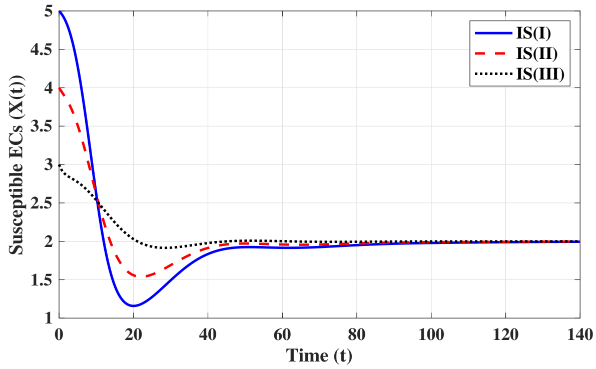


(**f**) SARS-CoV-2 antibodies

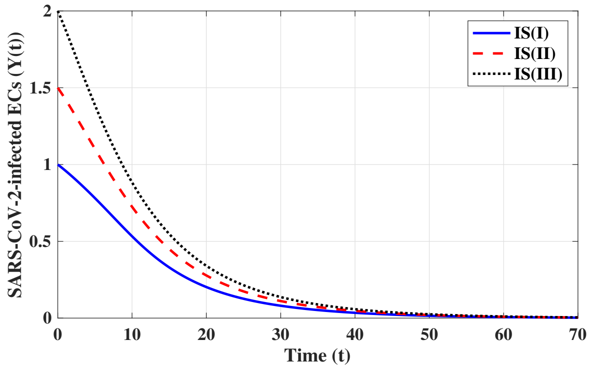


(**g**) IAV antibodies

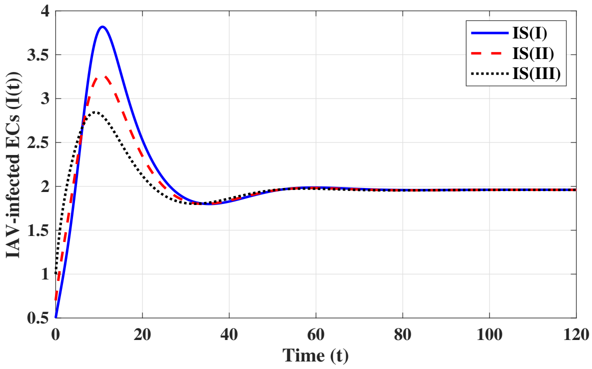
Figure 4. Solutions of system (1)–(7) when $\Re_3 > 1$ and $\Re_5 \leq 1$ (fourth strategy).



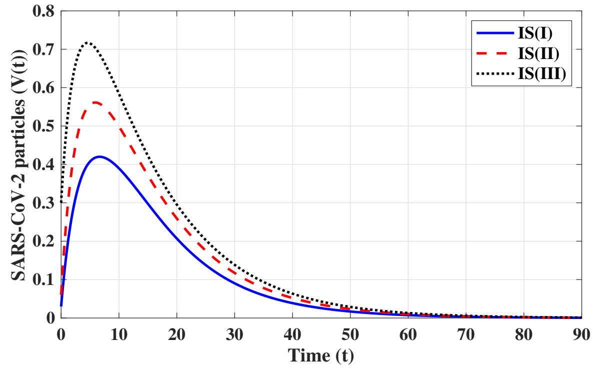
(**a**) Susceptible ECs



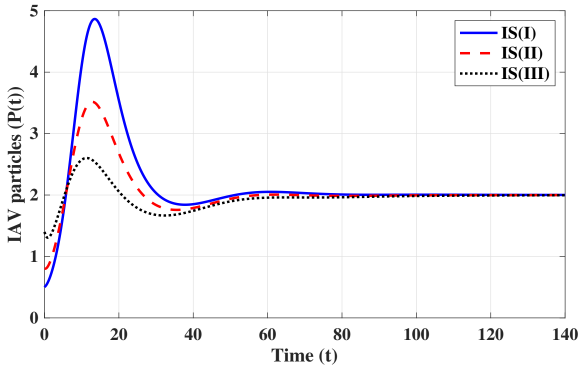
(**b**) SARS-CoV-2-infected ECs



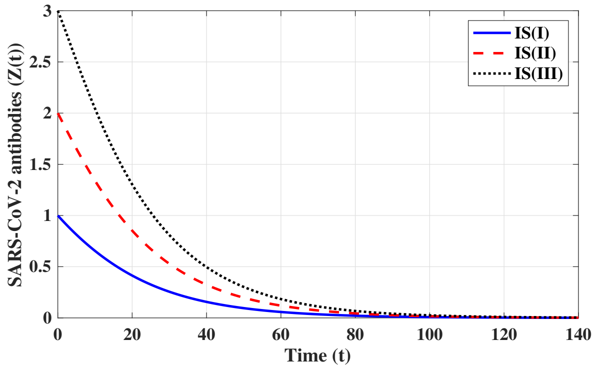
(**c**) IAV-infected ECs



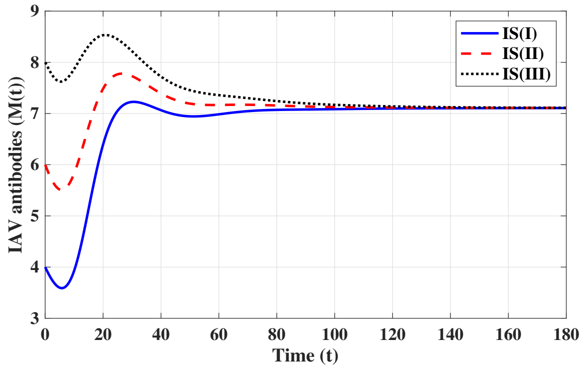
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

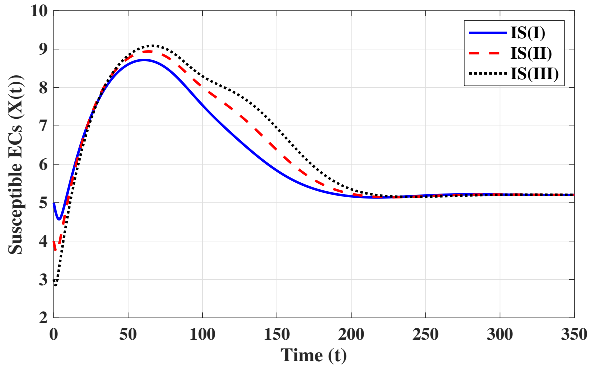


(**f**) SARS-CoV-2 antibodies

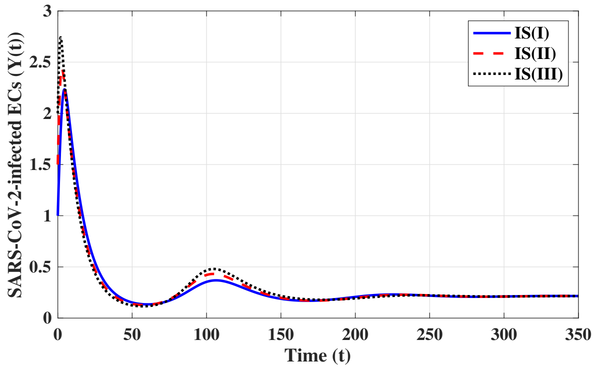


(**g**) IAV antibodies

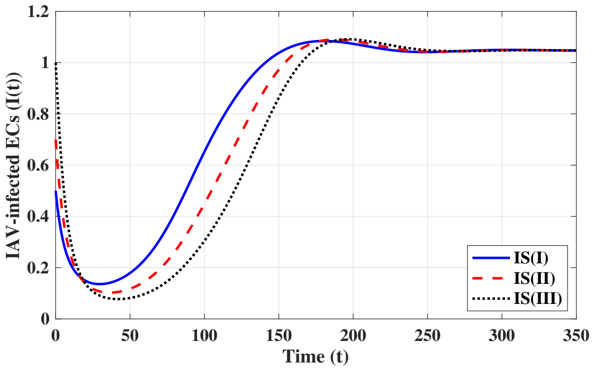
Figure 5. Solutions of system (1)–(7) when $\Re_4 > 1$ and $\Re_6 \leq 1$ (fifth strategy).



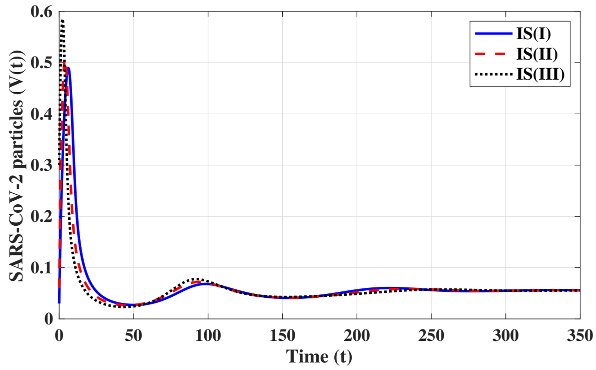
(**a**) Susceptible ECs



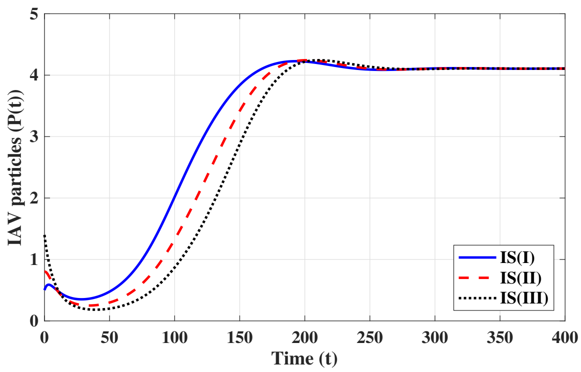
(**b**) SARS-CoV-2-infected ECs



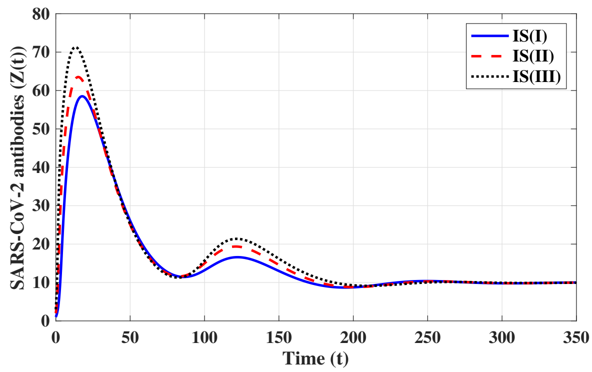
(**c**) IAV-infected ECs



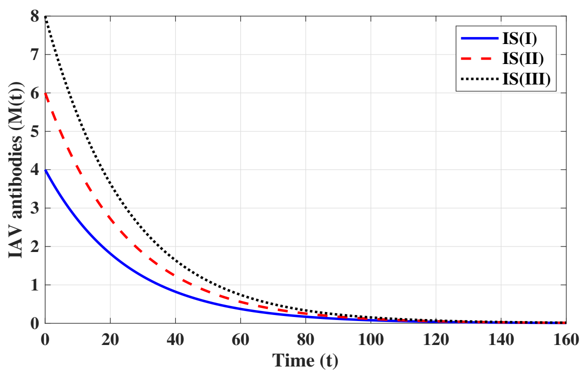
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

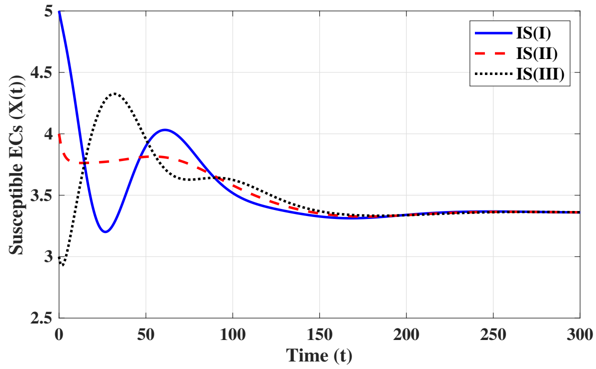


(**f**) SARS-CoV-2 antibodies

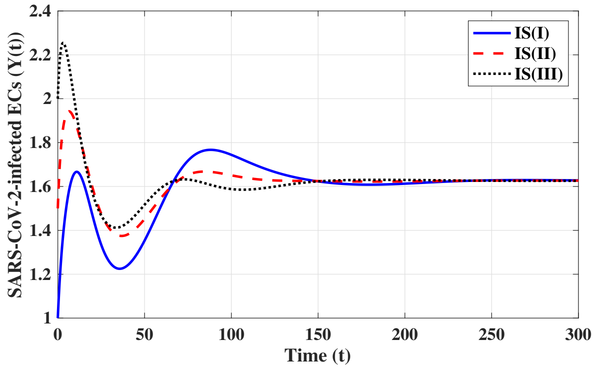


(**g**) IAV antibodies

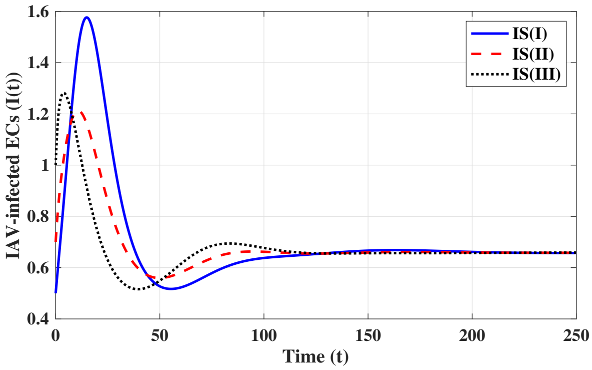
Figure 6. Solutions of system (1)–(7) when $\Re_5 > 1$, $\Re_1/\Re_2 > 1$, and $\Re_8 \leq 1$ (sixth strategy).



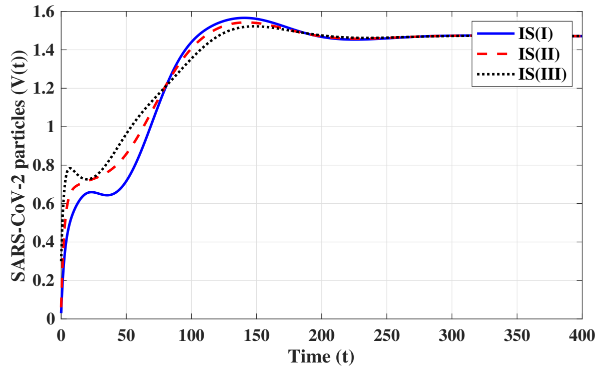
(**a**) Susceptible ECs



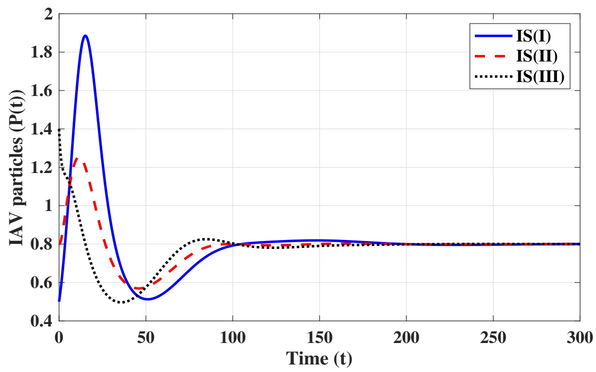
(**b**) SARS-CoV-2-infected ECs



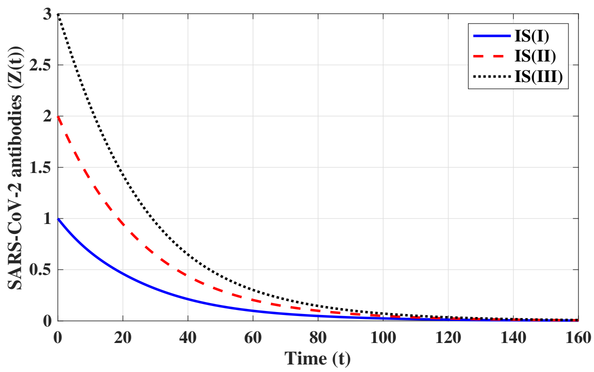
(**c**) IAV-infected ECs



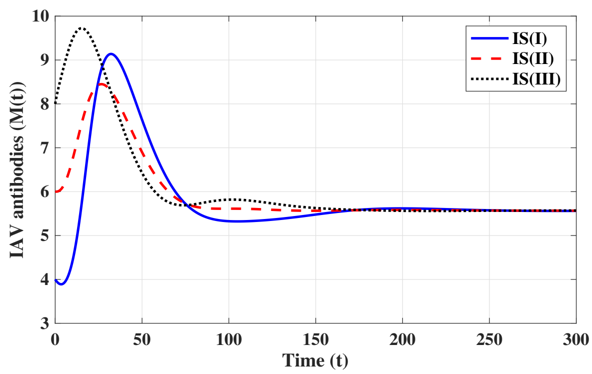
(**d**) SARS-CoV-2 particles



(**e**) IAV particles

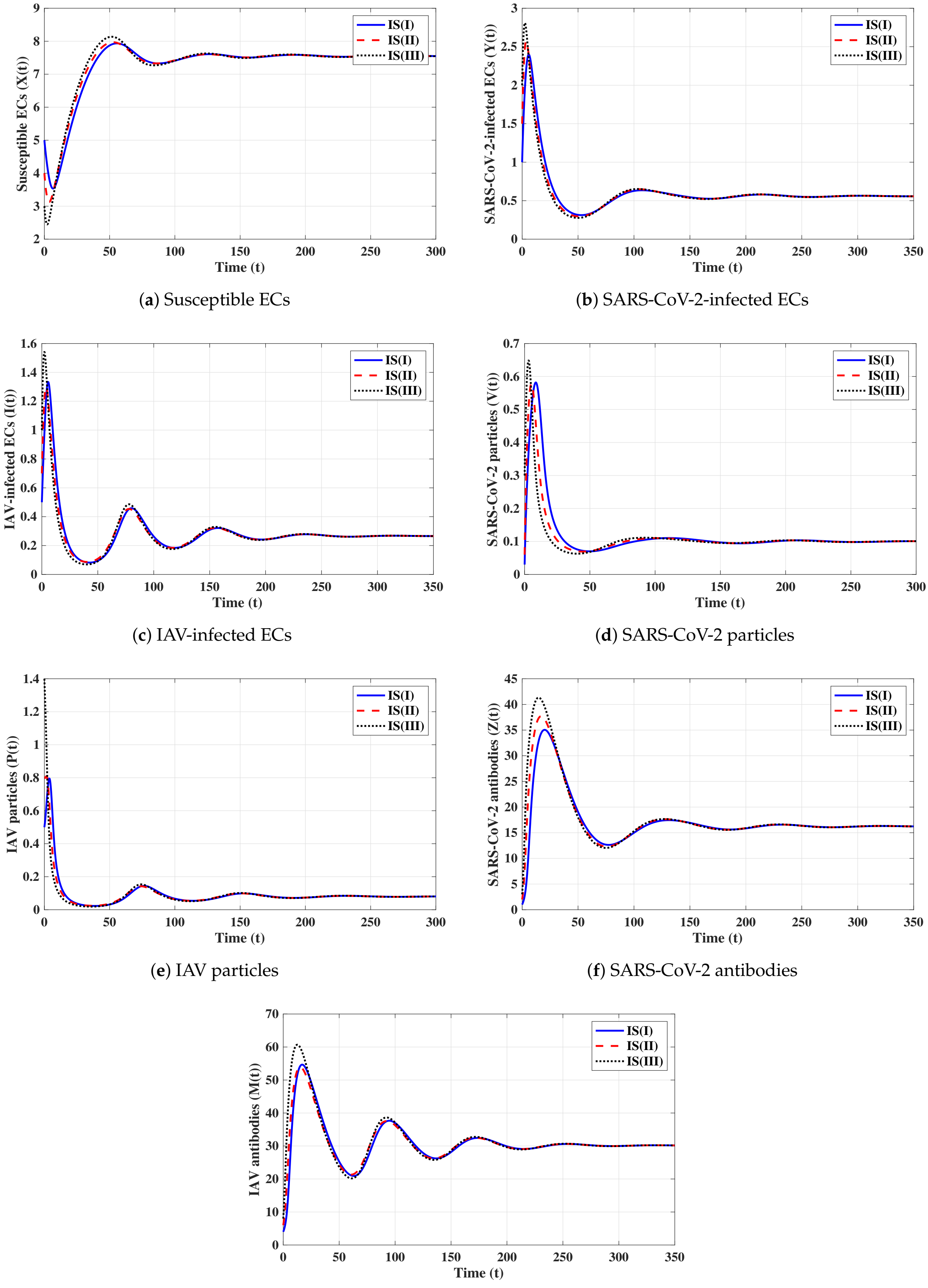


(**f**) SARS-CoV-2 antibodies



(**g**) IAV antibodies

Figure 7. Solutions of system (1)–(7) when $\Re_6 > 1, \Re_2/\Re_1 > 1$, and $\Re_7 \leq 1$ (seventh strategy).



(**a**) Susceptible ECs (**b**) SARS-CoV-2-infected ECs

(**c**) IAV-infected ECs (**d**) SARS-CoV-2 particles

(**e**) IAV particles (**f**) SARS-CoV-2 antibodies

(**g**) IAV antibodies

Figure 8. Solutions of system (1)–(7) when $\Re_7 > 1$ and $\Re_8 > 1$ (eighth strategy).

6.2. Effect of Antiviral Treatment on the Dynamics of Influenza and COVID-19 Co-Infection

We consider two antiviral drugs for SARS-CoV-2 and IAV with drug efficacies of ϵ_V and ϵ_P, respectively, where $0 \leq \epsilon_V \leq 1$ and $0 \leq \epsilon_P \leq 1$. Then, the parameters ξ_V and ξ_P will be changed to $(1-\epsilon_V)\xi_V$ and $(1-\epsilon_P)\xi_P$, respectively. Moreover, $\Re_1$ and $\Re_2$ become functions of ϵ_V and ϵ_P, respectively, when all other parameters are fixed:

$$\Re_1(\epsilon_V) = \frac{(1-\epsilon_V)e^{-\alpha_1\tau_1-\alpha_2\tau_2}X_0\theta_V\xi_V}{\lambda_V\beta_Y}, \quad \Re_2(\epsilon_P) = \frac{(1-\epsilon_P)e^{-\alpha_3\tau_3-\alpha_4\tau_4}X_0\theta_P\xi_P}{\lambda_P\beta_I}.$$

To make $\Re_1 \leq 1$ and $\Re_2 \leq 1$, the effectiveness of ϵ_V and ϵ_P has to satisfy

$$\begin{aligned} &\epsilon_V^{\min} \leq \epsilon_V \leq 1, \quad \epsilon_V^{\min} = \max\left\{0, 1 - \frac{e^{\alpha_1\tau_1+\alpha_2\tau_2}\lambda_V\beta_Y}{X_0\theta_V\xi_V}\right\}, \\ &\epsilon_P^{\min} \leq \epsilon_P \leq 1, \quad \epsilon_P^{\min} = \max\left\{0, 1 - \frac{e^{\alpha_3\tau_3+\alpha_4\tau_4}\lambda_P\beta_I}{X_0\theta_P\xi_P}\right\}. \end{aligned}$$

It follows that, if $\epsilon_V^{\min} \leq \epsilon_V \leq 1$ and $\epsilon_P^{\min} \leq \epsilon_P \leq 1$, then Δ_0 is GAS, and both influenza and COVID-19 are cleared. Therefore, if real data from patients co-infected with influenza and COVID-19 are used, the model's parameters can be estimated and the model can be used to determine the minimum drug efficacies required to eliminate both SARS-CoV-2 and IAV from the body.

6.3. Effects of Time Delays on the Dynamics of Influenza and COVID-19 Co-Infection

In this subsection, we analyze the impacts of time delays with various delay parameters τ_i, $i = 1,2,3,4$. We fix the parameters $\xi_V = 0.13$, $\xi_P = 0.1$, $\eta_Z = 0.3$, and $\eta_M = 0.5$. Let us consider the following scenarios:

$$\begin{array}{lllll} \text{S1:} & \tau_1 = 0.1, & \tau_2 = 0.3, & \tau_3 = 0.5, & \tau_4 = 0.8, \\ \text{S2:} & \tau_1 = 1, & \tau_2 = 0.9, & \tau_3 = 13, & \tau_4 = 14, \\ \text{S3:} & \tau_1 = 1.2348, & \tau_2 = 1.2348, & \tau_3 = 14.9787, & \tau_4 = 14.9787, \\ \text{S4:} & \tau_1 = 3, & \tau_2 = 4, & \tau_3 = 20, & \tau_4 = 25. \end{array}$$

From the above values, we solve the system (1)–(7) under the following initial condition:

$$\begin{aligned} \text{IS(IV)} \quad &: \quad (X(u), Y(u), I(u), V(u), P(u), Z(u), M(u)) = (7, 0.6, 0.5, 0.05, 0.05, 7, 8), \\ u \quad &\in \quad [-\tau^*, 0]. \end{aligned}$$

The numerical results are displayed in Figure 9. We note that time delays can significantly increase the concentration of susceptible ECs and reduce the concentrations of other factors. Since $\Re_1$ and $\Re_2$ are given in (16), they depend on τ_i, $i = 1,2,3,4$ when all other parameters are fixed. We observe from Table 3 that $\Re_1$ and $\Re_2$ decrease if τ_i increases; hence, the stability of Δ_0 will be changed.

Now, we need to calculate the critical value of the time delays that makes the system stable around the equilibrium point Δ_0. Let $\tau_{12} = \tau_1 = \tau_2$ and $\tau_{34} = \tau_3 = \tau_4$, and we write $\Re_1(\tau_{12})$ and $\Re_2(\tau_{34})$ as:

$$\Re_1(\tau_{12}) = \frac{e^{-(\alpha_1+\alpha_2)\tau_{12}}X_0\theta_V\xi_V}{\beta_Y\lambda_V}, \quad \Re_2(\tau_{34}) = \frac{e^{-(\alpha_3+\alpha_4)\tau_{34}}X_0\theta_P\xi_P}{\beta_I\lambda_P}.$$

Clearly, when all other parameters are fixed, $\Re_1$ and $\Re_2$ are decreasing functions of τ_{12} and τ_{34}, respectively. Let us calculate $\tau_{12}^{\min}$ and $\tau_{34}^{\min}$ such that $\Re_1(\tau_{12}^{\min}) = 1$ and $\Re_2(\tau_{34}^{\min}) = 1$ as:

$$\tau_{12}^{\min} = \max\left\{0, \frac{1}{\alpha_1 + \alpha_2} \ln\left(\frac{X_0 \theta_V \xi_V}{\beta_Y \lambda_V}\right)\right\},$$
$$\tau_{34}^{\min} = \max\left\{0, \frac{1}{\alpha_3 + \alpha_4} \ln\left(\frac{X_0 \theta_P \xi_P}{\beta_I \lambda_P}\right)\right\}.$$

Consequently,

$$\Re_1(\tau_{12}) \leq 1, \text{ for all } \tau_{12} \geq \tau_{12}^{\min},$$
$$\Re_2(\tau_{34}) \leq 1, \text{ for all } \tau_{34} \geq \tau_{34}^{\min}.$$

Therefore, Δ_0 is GAS when $\tau_{12} \geq \tau_{12}^{\min}$ and $\tau_{34} \geq \tau_{34}^{\min}$. Using the values of the parameters, we get $\tau_{12} = 1.2348$ and $\tau_{34} = 14.9787$. It follows that:

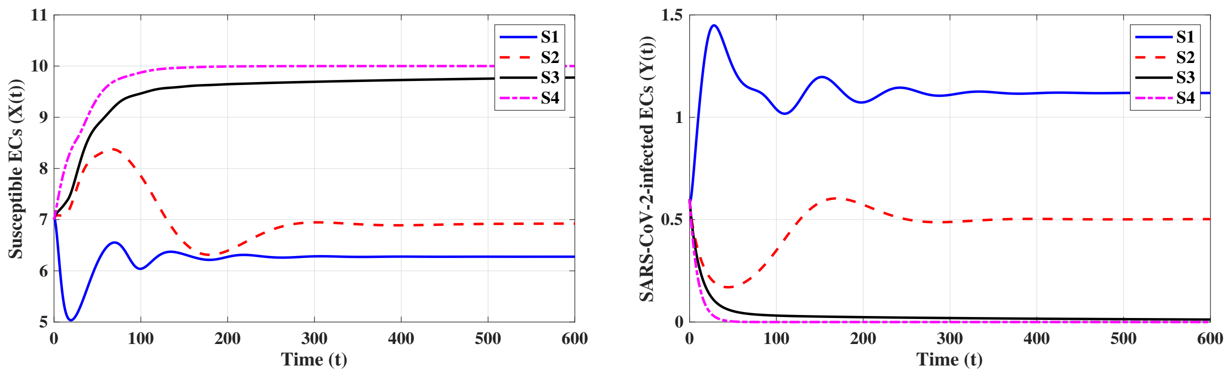
(i) If $\tau_{12} \geq 1.2348$ and $\tau_{34} \geq 14.9787$, then $\Re_1(\tau_{12}) \leq 1$, $\Re_2(\tau_{34}) \leq 1$, and Δ_0 is GAS.

(ii) If $\tau_{12} < 1.2348$ or $\tau_{34} < 14.9787$, then $\Re_1(\tau_{12}) > 1$ or $\Re_2(\tau_{34}) > 1$, and Δ_0 will lose its stability.

We note that time delays can play a similar role to that of antiviral drugs. This can guide researchers to create new treatments for influenza and COVID-19 co-infection that work to prolong time delays.

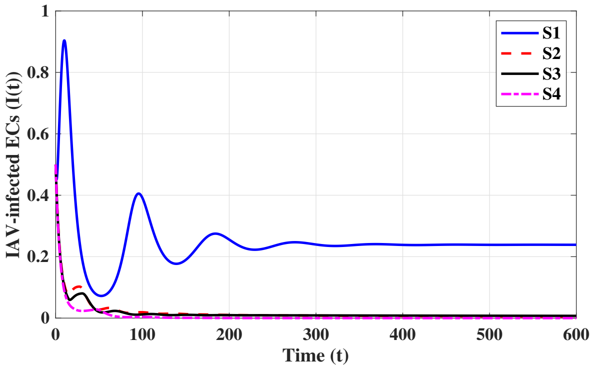
Table 3. The variation in $\Re_1$ and $\Re_2$ with respect to the delay parameters.

Delay Parameters	$\Re_1$	$\Re_2$
$\tau_1 = 0.1,\ \tau_2 = 0.3,\ \tau_3 = 0.5$ and $\tau_4 = 0.8$	7.92	17.56
$\tau_1 = 0.5,\ \tau_2 = 0.6,\ \tau_3 = 10$ and $\tau_4 = 11$	3.93	2.45
$\tau_1 = 1$, $\tau_2 = 0.9$, $\tau_3 = 13$ and $\tau_4 = 14$,	1.77	1.34
$\tau_1 = \tau_2 = 1.2348$ and $\tau_3 = \tau_4 = 14.9787$,	1.0	1.0
$\tau_1 = 2$, $\tau_2 = 3$, $\tau_3 = 15$ and $\tau_4 = 16$	0.08	0.9
$\tau_1 = 3$, $\tau_2 = 4$, $\tau_3 = 20$ and $\tau_4 = 25$.	0.011	0.22

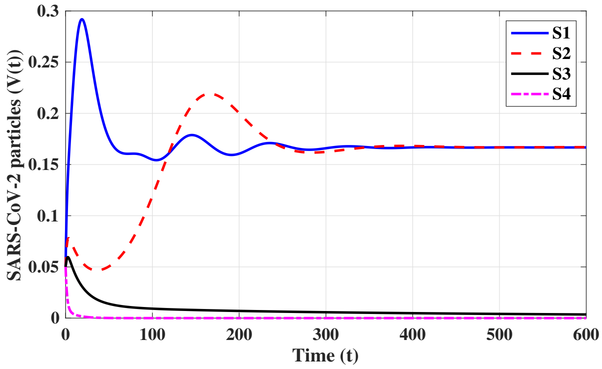


(**a**) Susceptible ECs (**b**) SARS-CoV-2-infected ECs.

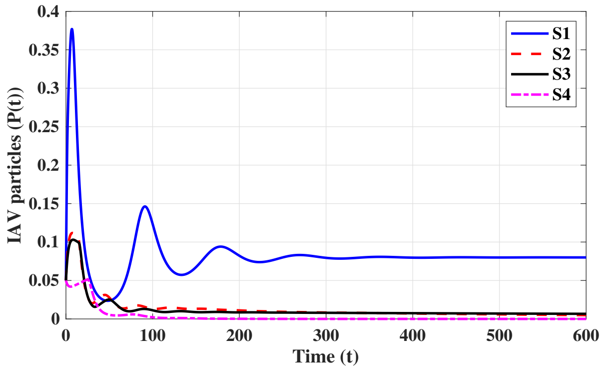
Figure 9. *Cont.*



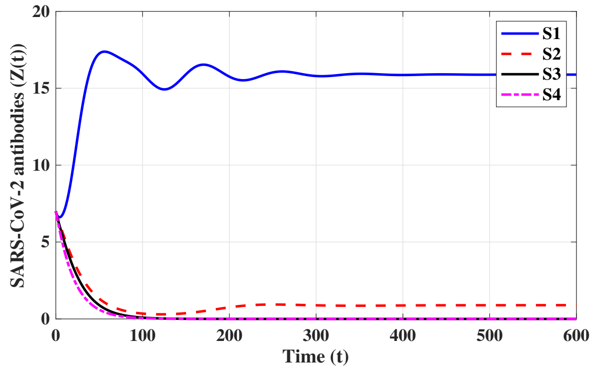
(**c**) IAV-infected ECs



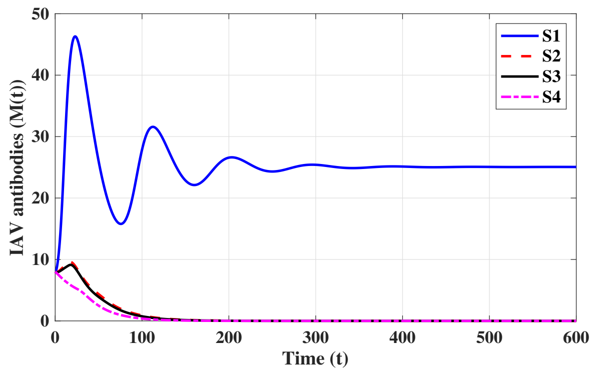
(**d**) SARS-CoV-2 particles



(**e**) IAV particles



(**f**) SARS-CoV-2 antibodies



(**g**) IAV antibodies

Figure 9. Effects of delay parameters τ_i, $i = 1, 2, 3, 4$, on the trajectories of system (1)–(7).

7. Conclusions

Influenza and COVID-19 co-infection cases were reported in recent works (see, e.g., [5,9–11]). Mathematical models can be helpful for understanding the dynamics of influenza and COVID-19 co-infection within a host. In this paper, we developed and examined a system of DDEs to describe the in-host dynamics of influenza and COVID-19 co-infection under the effects of humoral immunity. The model considered the interactions among susceptible ECs, SARS-CoV-2-infected ECs, IAV-infected ECs, SARS-CoV-2 particles, IAV particles, SARS-CoV-2 antibodies, and IAV antibodies. The model included four time delays: τ_1 and τ_3 for the delays between the entries of SARS-CoV-2 and IAV into ECs and the start of production of immature SARS-CoV-2 and IAV virions, respectively, and τ_2 and τ_4 for the maturation delays of newly released SARS-CoV-2 and IAV virions, respectively. We showed the non-negativity and ultimate boundedness of the solutions. We deduced that the system had eight equilibria, and their existence and stability were governed by eight threshold parameters ($\Re_i$, $i = 1, 2, \ldots, 8$). We used the Lyapunov method to prove the global stability of the equilibria. We carried out some numerical simulations and showed that they agreed with the theoretical results. We addressed the effects of antiviral drugs and time delays on the co-infection dynamics. We showed that both antiviral drugs and time delays had the same effect in eradicating co-infection from the body. This can guide scientists and pharmaceutical companies in synthesizing new drugs that prolong time delays. Our proposed model can be useful for determining the minimum drug doses that are required to eliminate both SARS-CoV-2 and IAV infections from the body. Moreover, the model can be used to describe the in-host dynamics of co-infection with two or more viral strains or co-infections with SARS-CoV-2 (or IAV) and other respiratory viruses.

The model presented in this article can be extended to include several biological aspects, such as viral mutation [61], stochastic interactions [62], and reaction diffusion [63].

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