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Levinson's Functional in Time Scale Settings

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Abstract: We introduce the Levinson functional on time scales using integral inequality of Levinson's type in the terms of Δ-integral for convex (concave) functions on time scale sets and investigate its properties such as superadditivity and monotonicity. The obtained properties are used to derive the bounds of the given Levinson's functional and those results provide a refinement and the converse of the known Levinson's inequality on time scales. Further, we define new types of functionals using weighted generalized and power means on time scales, and prove their properties which can be employed in future works to obtain refinements and converses of known integral inequalities on time scales.

Keywords: Levinson's inequality; Jensen's functional; time scale calculus

MSC: 26D15; 26A51; 28A25

1. Introduction

In [1], authors Baloch, Pečarić and Praljak introduced a knew class of functions, $\mathcal{K}_1^c(I)$ proving that $\mathcal{K}_1^c(I)$ is the largest class of functions for which Levinson's inequality hold under Mecer's assumptions. J. Barić, J. Pečarić and D. Radišić obtained in [2] Levinson's type inequalities in time scale settings by using the class $\mathcal{K}_1^c(I)$ and some known results regarding integral inequalites for convex (concave) functions on time scale sets. Constructing the Levinson functional on a time scale, as a difference between the right-hand side and left-hand side of the Levinson inequality on time scale, we get the opportunity to investigate known time scale integral inequalities in other directions using the properties and boundaries of new functionals.

The known Levinson's inequality ([3]) was proved in 1964 in the following theorem.

Theorem 1. *For $c \in \mathbf{R}$, $c > 0$, let $f : (0, 2c) \to \mathbb{R}$ satisfy $f''' \geq 0$ and let p_i, x_i, y_i, $i = 1, 2, \ldots, n$, be such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $0 \leq x_i \leq c$, and*

$$x_1 + y_1 = x_2 + y_2 = \cdots = x_n + y_n = 2c. \tag{1}$$

Then,

$$\sum_{i=1}^n p_i f(x_i) - f(\overline{x}) \leq \sum_{i=1}^n p_i f(y_i) - f(\overline{y}), \tag{2}$$

where $\overline{x} = \sum_{i=1}^n p_i x_i$ and $\overline{y} = \sum_{i=1}^n p_i y_i$ are the weighted arithmetic means.

In ([4]), Popoviciu generalized Levinson's inequality proving (2) is true if f is 3-convex function.

By rescaling axes, P. S. Bullen proved in [5], in 1973, that if $f : [a, b] \to \mathbb{R}$ is 3-convex and p_i, x_i, y_i, $i = 1, 2, \ldots, n$, are such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a \leq x_i, y_i \leq b$, $x_i + y_i = c$ and

$$\max\{x_1, \ldots, x_n\} \leq \min\{y_1, \ldots, y_n\}, \tag{3}$$

then (2) holds.

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In 2010, Mercer ([6]) improved the Levinson inequality proving that if $f : [a, b] \to \mathbb{R}$ satisfies $f''' \geq 0$ and p_i, x_i, y_i, $i = 1, 2, \ldots, n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leq x_i, y_i \leq b$ and $\max\{x_1, \ldots, x_n\} \leq \min\{y_1, \ldots, y_n\}$, inequality (2) holds if (1) is weakened by

$$\sum_{i=1}^{n} p_i (x_i - \overline{x})^2 = \sum_{i=1}^{n} p_i (y_i - \overline{y})^2. \tag{4}$$

In this paper, we base our main results on Levinson's type inequality on time scale ([2]) proved for the functions belonging in a new class of functions, $\mathcal{K}_1^c(I)$, defined as follows.

Definition 1. *Let $f \colon I \to \mathbb{R}$ and $c \in I^0$, where I^0 is the interior of the interval I. We say that $f \in \mathcal{K}_1^c(I)$, $(resp. f \in \mathcal{K}_2^c(I))$, if there exists a constant λ such that the function $F(x) = f(x) - \frac{\lambda}{2} x^2$ is concave (respectively, convex) on $(-\infty, c] \cap I$ and convex (respectively, concave) on $I \cap [c, \infty)$.*

The authors proved that $\mathcal{K}_1^c(I)$ is the largest class of functions for which Levinson's inequality holds under Mercer's assumptions. For function f, which belongs to class $\mathcal{K}_1^c(I)$, we say that it is 3-convex at point c. Therefore, the class $\mathcal{K}_1^c(I)$ generalizes 3-convex functions in the following sense: a function is 3-convex on I if and only if it is 3-convex at every $c \in I^0$.

Before citing the main results that are our motivation for the new results, let us briefly introduce some basic properties of the theory of time scale calculus in the next chapter.

2. Preliminaries

The theory of time scales is attributed to Stefan Hilger and was started in his PhD thesis [7]. It represents a unification of the theory of difference equations and the theory of differential equations, unifying integral and differential calculus with the calculus of finite differences. It has applications in any field that requires simultaneous modelling of discrete and continuous cases. Many interesting results, properties and applications regarding time scale calculus can be found in [8–11] and the books [12–14].

Now, we briefly introduce the basics on time scale calculus that we need in the rest of the article, using the same notations as in [13].

A *time scale* $\mathbb{T}$ is defined as any closed subset of the set of real numbers $\mathbb{R}$. Notice that the two most representative examples of time scales are $\mathbb{R}$ and $\mathbb{Z}$. In order to unify the approaches and theories for the sets that may or may not be connected, we introduce the concept of jump operators so, for $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The convention here is $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $\sigma(t) > t$, then we say that t is *right-scattered*, and if $\rho(t) < t$, then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Additionally, if $\sigma(t) = t$, then t is said to be *right-dense*, and if $\rho(t) = t$, then t is said to be *left-dense*. Points that are simultaneously right-dense and left-dense are called *dense*. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the *graininess function*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Definition 2. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. We define the* delta derivative *of f at t as a number $f^\Delta(t)$ (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* rd-continuous *if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in C^1_{\mathrm{rd}}$) if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta \in C_{\mathrm{rd}}$.*

Definition 4. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a* delta antiderivative *of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then, if $a \in \mathbb{T}$, the* delta integral *is defined by*

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad a, t \in \mathbb{T}.$$

Notice that *every rd-continuous function has a delta antiderivative*.

In what follows, we use the same notations and approaches as in [15] [Chapter 5] and recall briefly the introduction of Lebesgue Δ-integrals. For $[a, b] \subset \mathbb{T}$, we say it is a time scale interval if

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}, \quad a, b \in \mathbb{T},\ a \leq b.$$

Let μ_Δ be the Lebesgue Δ-measure on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be a μ_Δ-measurable function. Then, the Lebesgue Δ-integral of f on $[a, b]$ can be written as $\int\limits_{[a,b]} f d\mu_\Delta$, $\int\limits_{[a,b]} f(t)\, d\mu_\Delta(t)$ or $\int\limits_{[a,b]} f(t)\Delta(t)$. All theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on time scale $\mathbb{T}$ and the relation between the Lebesgue Δ-integral and the Riemann Δ-integral is given in the following way: if f is Riemann Δ-integrable from a to b, then f is Lebesgue Δ-integrable on $[a, b]$ and

$$R \int\limits_a^b f(t)\, \Delta t = L \int\limits_{[a,b]} f(t)\, \Delta t,$$

where R and L denote the Riemann and Lebesgue integrals, respectively, $[a, b]$ is a closed bounded interval in $\mathbb{T}$ and f is a bounded real valued function on $[a, b]$.

In this article, the integrals in our results are related to Lebesgue Δ-integrals and Lebesgue Δ-measure od $[a, b]$, but according to the properties of time scale theory, all results given here are true and can be rewritten for Cauchy delta integral, Cauchy nabla integral, α-diamond integral and multiple versions of Riemann and Lebesgue integrals.

Here are some properties of the Lebesgue delta integral.

Theorem 2. *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{\mathrm{rd}}$, then*

(i) $\int\limits_a^b (f(t) + g(t))\, d\mu_\Delta(t) = \int\limits_a^b f(t) d\mu_\Delta(t) + \int\limits_a^b g(t)\, d\mu_\Delta(t)$;

(ii) $\int\limits_a^b \alpha f(t)\, d\mu_\Delta(t) = \alpha \int\limits_a^b f(t)\, d\mu_\Delta(t)$;

(iii) $\int\limits_a^b f(t)\, d\mu_\Delta(t) = -\int\limits_b^a f(t)\, d\mu_\Delta(t)$;

(iv) $\int\limits_a^b f(t)\, d\mu_\Delta(t) = \int\limits_a^c f(t)\, d\mu_\Delta(t) + \int\limits_c^b f(t)\, d\mu_\Delta(t)$;

(v) $\int\limits_a^a f(t)\, d\mu_\Delta(t) = 0$;

(vi) *if* $f(t) \geq 0$ *for all* t, *then* $\int\limits_a^b f(t)\, d\mu_\Delta(t) \geq 0$.

The Jensen inequality on time scales via the Δ-integral is proved in [8] by Agarwal, Bohner and Peterson.

Applying weighted Jensen's inequality on time scales ([16]), the authors in [2] established weighted Levinson's type inequality in the settings of time scale calculus and proved the following theorem.

Theorem 3. *Let* $a_i, b_i \in \mathbb{T}$, $a_i < b_i$, $i = 1, 2$ *and suppose* $I \subset \mathbb{R}$ *is an interval. Assume* $w_i \in C_{\mathrm{rd}}([a_i, b_i], \mathbb{R})$, $i = 1, 2$, *are non-negative functions such that* $\int_{a_i}^{b_i} w_i(t)\, d\mu_\Delta(t) > 0$ *and* $[a_i, b_i]$, $i = 1, 2$, *are time scale intervals. Let* $f_i \in C_{\mathrm{rd}}([a_i, b_i], I)$, $i = 1, 2$, *and suppose there exists* $c \in I^0$ *such that*

$$\sup_{x \in [a_1, b_1]} f_1(x) \leq c \leq \inf_{x \in [a_2, b_2]} f_2(x). \tag{5}$$

If

$$\overline{\mathcal{D}}_{[a_1,b_1]}(w_1, f_1) = \overline{\mathcal{D}}_{[a_2,b_2]}(w_2, f_2) \tag{6}$$

then the inequality

$$\begin{aligned} &\Phi\left(\frac{\int_{a_2}^{b_2} w_2(t) f_2(t)\, d\mu_\Delta(t)}{\int_{a_2}^{b_2} w_2(t)\, d\mu_\Delta(t)}\right) - \frac{\int\limits_{a_2}^{b_2} w_2(t)\Phi(f_2(t))\, d\mu_\Delta(t)}{\int_{a_2}^{b_2} w_2(t)\, d\mu_\Delta(t)} \\ &\leq \Phi\left(\frac{\int_{a_1}^{b_1} w_1(t) f_1(t)\, d\mu_\Delta(t)}{\int_{a_1}^{b_1} w_1(t)\, d\mu_\Delta(t)}\right) - \frac{\int\limits_{a_1}^{b_1} w_1(t)\Phi(f_1(t))\, d\mu_\Delta(t)}{\int_{a_1}^{b_1} w_1(t)\, d\mu_\Delta(t)} \end{aligned} \tag{7}$$

holds for every function $\Phi \in \mathcal{K}_1^c(I)$, *where the term* $\overline{\mathcal{D}}_{[a,b]}(w, f)$ *denotes following expression*

$$\overline{\mathcal{D}}_{[a,b]}(w, f) = \frac{\int\limits_a^b w(t)(f(t))^2\, d\mu_\Delta(t)}{\int_a^b w(t)\, d\mu_\Delta(t)} - \left(\frac{\int\limits_a^b w(t) f(t)\, d\mu_\Delta(t)}{\int_a^b w(t)\, d\mu_\Delta(t)}\right)^2.$$

If the function Φ *is contained in* $\mathcal{K}_2^c(I)$, *then the sign of inequality* (7) *is reversed.*

This theorem will be our starting point in defining Levinson's functional.

3. Definition of Levinson's Functional on Time Scales and Its Properties

Recently, many authors investigated the concept of the Jensen functional as a difference between the right-hand side and the left-hand side of the Jensen inequality regarding different kinds of environments (discrete cases, integral cases, linear sets of real-valued functions, time scale sets, etc.) The benefit of investigating those new functionals lies in their properties, which yield to new generalizations of known inequalities.

In [17], authors defined Jensen's functional on time scales by

$$\mathcal{J}_{\mathbb{T}}(\Phi, f, p; \mu_\Delta) = \int_a^b p(\Phi \circ f)\, d\mu_\Delta - \int_a^b p\, d\mu_\Delta \Phi\left(\frac{\int_a^b pf\, d\mu_\Delta}{\int_a^b p\, d\mu_\Delta}\right), \tag{8}$$

where $\Phi \in C(I, \mathbb{R})$, $f : [a, b] \to \mathbb{R}$ is Δ-integrable and $p : [a, b] \to \mathbb{R}$ is non-negative and Δ-integrable such that $\int_a^b p\, d\mu_\Delta > 0$.

In this section, we define Levinson's functional on time scales and prove some of its properties.

Definition 5. *Let $a_i, b_i \in \mathbb{T}$, $a_i < b_i$, $i = 1, 2$ and suppose $I \subset \mathbb{R}$ is an interval. Assume $w_i \in C_{\rm rd}([a_i, b_i], \mathbb{R})$, $i = 1, 2$, are non-negative functions such that $\int_{a_i}^{b_i} w_i(t)\, d\mu_\Delta(t) > 0$. Let $f_i \in C_{\rm rd}([a_i, b_i], I)$, $i = 1, 2$, and suppose there exists $c \in I^0$ such that (5) is fulfilled. Then, we define Levinson's functional on time scales by*

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) = \mathcal{J}_{\mathbb{T}}(\Phi, f_2, w_2; \mu_\Delta) - \mathcal{J}_{\mathbb{T}}(\Phi, f_1, w_1; \mu_\Delta), \tag{9}$$

where $\mathcal{J}_{\mathbb{T}}(\Phi, f_i, w_i; \mu_\Delta)$, $i = 1, 2$, denotes Jensen's functionals on time scales defined by (8) and continuous function $\Phi \in \mathcal{K}_1^c(I)$ (or $\Phi \in \mathcal{K}_2^c(I)$).

Remark 1. *From the main statement of Theorem 3 it is obvious that*

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \geq 0, \tag{10}$$

for every continuous function $\Phi \in \mathcal{K}_1^c(I)$. If continuous function $\Phi \in \mathcal{K}_2^c(I)$, then the sing in (10) is reversed.

Remark 2. *Since the Levinson functional is related to a class $\mathcal{K}_1^c(I)$ or $\mathcal{K}_2^c(I)$, we will use Definition 1 to express it by the terms of convex or concave function as it will be substantial in proving our new results. In order to obtain that, we start with Jensen's functional on time scales $\mathcal{J}_{\mathbb{T}}(\Phi, f_i, w_i; \mu_\Delta)$, $i = 1, 2$. According to Definition 1, for $\Phi \in \mathcal{K}_1^c(I)$, there exists a constant λ such that $F(x) = \Phi(x) - \frac{\lambda}{2}x^2$ is concave on $(-\infty, c] \cap I$ and convex on $I \cap [c, \infty)$ so, for $i = 1, 2$, we can write*

$$\begin{aligned}
\mathcal{J}_{\mathbb{T}}(\Phi, f_i, w_i; \mu_\Delta) &= \int_{a_i}^{b_i} w_i\left(F \circ f_i + \frac{\lambda}{2} f_i^2\right) d\mu_\Delta \\
&\quad - \int_{a_i}^{b_i} w_i\, d\mu_\Delta \cdot \left[F\left(\frac{\int_{a_i}^{b_i} w_i f_i\, d\mu_\Delta}{\int_{a_i}^{b_i} w_i\, d\mu_\Delta}\right) + \frac{\lambda}{2}\left(\frac{\int_{a_i}^{b_i} w_i f_i\, d\mu_\Delta}{\int_{a_i}^{b_i} w_i\, d\mu_\Delta}\right)\right] \\
&= \mathcal{J}_{\mathbb{T}}(F, f_i, w_i; \mu_\Delta) + \frac{\lambda}{2}\left[\int_{a_i}^{b_i} w_i f_i^2\, d\mu_\Delta - \frac{\left(\int_{a_i}^{b_i} w_i \cdot f_i\, d\mu_\Delta\right)^2}{\int_{a_i}^{b_i} w_i\, d\mu_\Delta}\right].
\end{aligned}$$

For simplicity, let us denote

$$\sigma_i(f_i, w_i) = \int_{a_i}^{b_i} w_i {f_i}^2 \, d\mu_\Delta - \frac{\left(\int_{a_i}^{b_i} w_i \cdot f_i \, d\mu_\Delta\right)^2}{\int_{a_i}^{b_i} w_i \, d\mu_\Delta}.$$

Now, Jensen's functional on time scales can be rewritten in the following form

$$\mathcal{J}_{\mathbb{T}}(\Phi, f_i, w_i; \mu_\Delta) = \mathcal{J}_{\mathbb{T}}(F, f_i, w_i; \mu_\Delta) + \frac{\lambda}{2} \cdot \sigma_i(f_i, w_i) \tag{11}$$

and Levinson's functional on time scales, defined by (9), can be expressed via convex (concave) functions as

$$\begin{aligned} \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \quad = \quad & \mathcal{J}_{\mathbb{T}}(F, f_2, w_2; \mu_\Delta) - \mathcal{J}_{\mathbb{T}}(F, f_1, w_1; \mu_\Delta) \\ & + \frac{\lambda}{2} \cdot [\sigma_2(f_2, w_2) - \sigma_1(f_1, w_1)]. \end{aligned} \tag{12}$$

In the following theorem, we derive the superadditivity property of the Levinson functional on time scales.

Theorem 4. *Let $a_i, b_i \in \mathbb{T}$, $a_i < b_i$, $i = 1,2$ and suppose $I \subset \mathbb{R}$ is an interval. Assume $w_i, \phi_i \in \mathrm{C_{rd}}([a_i, b_i], \mathbb{R})$, $i = 1,2$, are non-negative functions such that $\int_{a_i}^{b_i} w_i \, d\mu_\Delta > 0$, $\int_{a_i}^{b_i} \phi_i \, d\mu_\Delta > 0$, $\int_{a_i}^{b_i} (w_i + \phi_i) \, d\mu_\Delta > 0$. Let $f_i \in \mathrm{C_{rd}}([a_i, b_i], I)$, $i = 1,2$, and suppose there exists $c \in I^0$ such that (5) is fulfilled. If*

$$\sigma_1(f_1, w_1) + \sigma_1(f_1, \phi_1) - \sigma_1(f_1, w_1 + \phi_1) = \sigma_2(f_2, w_2) + \sigma_2(f_2, \phi_2) - \sigma_2(f_2, w_2 + \phi_2) \tag{13}$$

then, for every continuous function $\Phi \in \mathcal{K}_1^c(I)$, we have

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 + \phi_1, w_2 + \phi_2; \mu_\Delta) \geq \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) + \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \tag{14}$$

If continuous function $\Phi \in \mathcal{K}_2^c(I)$, then the sing in (14) is reversed.

Proof. Suppose $\Phi \in \mathcal{K}_1^c(I)$ is continuous. According to Definition 1, there exists a constant λ such that $F(x) = \Phi(x) - \frac{\lambda}{2}x^2$ is convex on $I \cap [c, \infty)$. Using (11) and superadditivity of the Jensen functional on time scales, we obtain

$$\begin{aligned} & \mathcal{J}_{\mathbb{T}}(\Phi, f_2, w_2 + \phi_2; \mu_\Delta) = \mathcal{J}_{\mathbb{T}}(F, f_2, w_2 + \phi_2; \mu_\Delta) + \frac{\lambda}{2}\sigma_2(f_2, w_2 + \phi_2) \\ & \geq \mathcal{J}_{\mathbb{T}}(F, f_2, w_2; \mu_\Delta) + \mathcal{J}_{\mathbb{T}}(F, f_2, \phi_2; \mu_\Delta) + \frac{\lambda}{2}\sigma_2(f_2, w_2 + \phi_2) \qquad (15) \\ & = \mathcal{J}_{\mathbb{T}}(\Phi, f_2, w_2; \mu_\Delta) - \frac{\lambda}{2}\sigma_2(f_2, w_2) + \mathcal{J}_{\mathbb{T}}(\Phi, f_2, \phi_2; \mu_\Delta) \\ & - \frac{\lambda}{2}\sigma_2(f_2, \phi_2) + \frac{\lambda}{2}\sigma_2(f_2, w_2 + \phi_2) \qquad (16) \end{aligned}$$

Furthermore, according to Definition 1, for continuous function $\Phi \in \mathcal{K}_1^c(I)$ and taken constant λ, $F(x) = \Phi(x) - \frac{\lambda}{2}x^2$ is concave on $(-\infty, c] \cap I$, therefore, we can write

$$\begin{aligned} & \mathcal{J}_{\mathbb{T}}(\Phi, f_1, w_1 + \phi_1; \mu_\Delta) \leq \mathcal{J}_{\mathbb{T}}(\Phi, f_1, w_1; \mu_\Delta) + \mathcal{J}_{\mathbb{T}}(\Phi, f_1, \phi_1; \mu_\Delta) \\ & - \frac{\lambda}{2}\sigma_1(f_1, w_1) - \frac{\lambda}{2}\sigma_1(f_1, \phi_1) + \frac{\lambda}{2}\sigma_1(f_1, w_1 + \phi_1). \end{aligned} \tag{17}$$

Now, since

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 + \phi_1, w_2 + \phi_2; \mu_\Delta) = \mathcal{J}_{\mathbb{T}}(\Phi, f_2, w_2 + \phi_2; \mu_\Delta) - \mathcal{J}_{\mathbb{T}}(\Phi, f_1, w_1 + \phi_1; \mu_\Delta)$$

the required property (14) of superadditivity of Levinson's functional on time scales follows by adding up (15) and (17) and taking into account the assumption (13). If continuous function $\Phi \in \mathcal{K}_2^c(I)$, then the sing in (14) will be reversed since F is convex on $(-\infty, c] \cap I$ and concave on $I \cap [c, \infty)$ and the inequality signs in (15) and (17) are reversed. □

In the next corollary, we will use the property of superadditivity to prove the monotonicity of Levinson's functional on time scales.

Corollary 1. *Let w_i, ϕ_i, f_i satisfy the hypotheses of Theorem 4 for $i = 1, 2$. Suppose there exists $c \in I^0$ such that (5) is fulfilled. If*

$$\sigma_1(f_1, w_1 - \phi_1) = \sigma_2(f_2, w_2 - \phi_2) \tag{18}$$

$$\sigma_1(f_1, \phi_1) - \sigma_1(f_1, w_1) = \sigma_2(f_2, \phi_2) - \sigma_2(f_2, w_2) \tag{19}$$

then Levinson's functional on time scale is increasing, that is, $w_i \geq \phi_i$ for $i = 1, 2$, implies

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \geq \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, \phi_1, \phi_2; \mu_\Delta) \tag{20}$$

for continuous function $\Phi \in \mathcal{K}_1^c(I)$. If continuous function $\Phi \in \mathcal{K}_2^c(I)$, then (20) holds in reverse order.

Proof. Taking into account condition (19), we can apply superadditivity of the Levinson functional in the following way.

$$\begin{aligned} &\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 - \phi_1 + \phi_1, w_2 - \phi_2 + \phi_2; \mu_\Delta) \\ &\geq \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 - \phi_1, w_2 - \phi_2; \mu_\Delta) + \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, \phi_1, \phi_2; \mu_\Delta), \end{aligned}$$

i.e.

$$\begin{aligned} & \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \\ \geq \; & \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 - \phi_1, w_2 - \phi_2; \mu_\Delta) + \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \end{aligned}$$

From (18) we have

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1 - \phi_1, w_2 - \phi_2; \mu_\Delta) \geq 0$$

so inequality (20) is true. If continuous function $\Phi \in \mathcal{K}_2^c(I)$, the sign in (20) is reversed according to Remark 1. □

In the next result, we obtain bounds for the Levinson functional on time scales assuming the positivity of the required functionals.

Theorem 5. *Suppose $a_i, b_i \in \mathbb{T}$, $a_i < b_i$, $i = 1, 2$, and $I \subset \mathbb{R}$ is an interval. Let $w_i \in \mathrm{C}_{\mathrm{rd}}([a_i, b_i], \mathbb{R})$, $i = 1, 2$ be non-negative functions such that $\int_{a_i}^{b_i} w_i \, d\mu_\Delta > 0$ and suppose w_i are bounded. Let $m_i = \inf\limits_{x \in [a_i, b_i]} w_i(x)$, $M_i = \sup\limits_{x \in [a_i, b_i]} w_i(x)$, $i = 1, 2$. Let $f_i \in \mathrm{C}_{\mathrm{rd}}([a_i, b_i], I)$ and suppose there exists $c \in I^0$ such that (5) holds. If*

$$\sigma_1(f_1, w_1) = \sigma_2(f_2, w_2), \tag{21}$$

$$\sigma_1(f_1, w_1 - min\{m_1, m_2\}) = \sigma_2(f_2, w_2 - min\{m_1, m_2\}) \tag{22}$$

and

$$\sigma_1(f_1, 1) = \sigma_2(f_2, 1), \tag{23}$$

then

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \geq min\{m_1, m_2\}\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, 1, 1; \mu_\Delta) \tag{24}$$

holds for continuous function $\Phi \in \mathcal{K}_1^c(I)$ *and* $\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, 1, 1; \mu_\Delta)$ *denotes the non-weighted Levinson's functional on time scales. If conditions (21), (23) are fulfilled and condition (22) is replaced by*

$$\sigma_1(f_1, max\{m_1, m_2\} - w_1) = \sigma_2(f_2, max\{m_1, m_2\} - w_2) \tag{25}$$

then

$$\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) \leq max\{m_1, m_2\}\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, 1, 1; \mu_\Delta). \tag{26}$$

If continuous function Φ *belongs to* $\mathcal{K}_2^c(I)$ *then (24) and (26) are reversed.*

Proof. In order to prove (24), we use the monotonicity property (20), taking $\phi_i(x) = \min\{m_1, m_2\}$, $x \in [a_i, b_i]$, $i = 1, 2$. Since $w_i(x) \geq \min\{m_1, m_2\}$, $i = 1, 2$, applying (20), we obtain

$$\begin{aligned}\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, w_1, w_2; \mu_\Delta) &\geq \mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, \min\{m_1, m_2\}, \min\{m_1, m_2\}; \mu_\Delta) \\ &= \min\{m_1, m_2\}\mathcal{L}_{\mathbb{T}}(\Phi, f_1, f_2, 1, 1; \mu_\Delta)\end{aligned}$$

thus, inequality (24) holds. Inequality (26) follows from the same reasoning. □

Remark 3. *Rewritting inequalities (24) and (7) in expanded forms, it can easily be seen that inequality (24) represents a refinement of the Levinson inequality (7) in Theorem 3 and inequality (26) yields a converse of (7).*

Example 1. *Taking in (9) that* $\mathbb{T} = \mathbb{Z}$, $[a_1, b_1] = \{1, 2, \ldots, n\}$, $[a_2, b_2] = \{1, 2, \ldots, m\}$, $n, m \in \mathbb{N}$ *and* $f_1(i) = x_i$, $w_1(i) = w_{1i}$, $f_2(j) = y_j$, $w_2(j) = w_{2j}$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, *we obtain the following discrete form of Levinson's functional (9)*

$$\begin{aligned}&\mathcal{L}(\Phi, x, y, w_1, w_2) = \mathcal{J}(\Phi, y, w_2) - \mathcal{J}(\Phi, x, w_1) \\ &= \sum_{j=1}^{m} w_{2j}\Phi(y_j) - \sum_{i=1}^{n} w_{1i}\Phi(x_i) + \sum_{i=1}^{n} w_{1i}\Phi\left(\frac{\sum_{i=1}^{n} w_{1i}x_i}{\sum_{i=1}^{n} w_{1i}}\right) - \sum_{j=1}^{m} w_{2j}\Phi\left(\frac{\sum_{j=1}^{m} w_{2j}y_j}{\sum_{j=1}^{m} w_{2j}}\right)\end{aligned}$$

where I *is an interval,* $c \in I^0$, $\Phi \in \mathcal{K}_1^c(I)$, $x = (x_1, \ldots, x_n) \in I \cap (-\infty, c]$, $y = (y_1, \ldots, y_m) \in I \cap [c, +\infty)$, $w_1 = (w_{11}, \ldots, w_{1n}) \in \mathbb{R}_+^n$, $w_2 = (w_{21}, \ldots, w_{2m}) \in \mathbb{R}_+^m$. *Under these notations, if*

$$\sum_{i=1}^{n} w_{1i}x_i^2 - \frac{\left(\sum_{i=1}^{n} w_{1i}x_i\right)^2}{\sum_{i=1}^{n} w_{1i}} \leq \sum_{j=1}^{m} w_{2j}y_j^2 - \frac{\left(\sum_{j=1}^{m} w_{2j}y_j\right)^2}{\sum_{j=1}^{m} w_{2j}}, \tag{27}$$

$$\sum_{i=1}^{n}(w_{1i} - \gamma)x_i^2 - \frac{\left(\sum_{i=1}^{n}(w_{1i} - \gamma)x_i\right)^2}{\sum_{i=1}^{n}(w_{1i} - \gamma)} \leq \sum_{j=1}^{m}(w_{2j} - \gamma)y_j^2 - \frac{\left(\sum_{j=1}^{m}(w_{2j} - \gamma)y_j\right)^2}{\sum_{j=1}^{m}(w_{2j} - \gamma)} \tag{28}$$

and

$$\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n} \leq \sum_{j=1}^{m} y_j^2 - \frac{\left(\sum_{j=1}^{m} y_j\right)^2}{m}, \tag{29}$$

then

$$\mathcal{L}(\Phi, x, y, w_1, w_2) \geq \gamma \mathcal{L}(\Phi, x, y, 1, 1)$$

where $\gamma = min\{w_{1i}, w_{2j} : 1 = 1, \ldots, n; j = 1, \ldots, m\}$. *If conditions* (27) *and* (29) *are true and* (28) *is replaced by*

$$\sum_{i=1}^{n} (\Gamma - w_{1i}) x_i^2 - \frac{\left(\sum\limits_{i=1}^{n} (\Gamma - w_{1i}) x_i \right)^2}{\sum\limits_{i=1}^{n} (\Gamma - w_{1i})} \leq \sum_{j=1}^{m} (\Gamma - w_{2j}) y_j^2 - \frac{\left(\sum\limits_{j=1}^{m} (\Gamma - w_{2j}) y_j \right)^2}{\sum\limits_{j=1}^{m} (\Gamma - w_{2j})} \tag{30}$$

where $\Gamma = max\{w_{1i}, w_{2j} : 1 = 1, \ldots, n; j = 1, \ldots, m\}$, *then*

$$\mathcal{L}(\Phi, x, y, w_1, w_2) \leq \Gamma \mathcal{L}(\Phi, x, y, 1, 1)$$

In the case when $n = m$, $w_{1i} = w_{2i}$, $i = 1, \ldots, n$ *and the condition* (1) *is fulfilled, i.e., the distribution of points* x_i, y_i *around the* c *is symmetric, then* (27)–(30) *hold with equality signs.*

Example 2. *Suppose* $\mathbb{T} = \mathbb{R}$ *and* $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. *Then, the Levinson functional* (9) *becomes*

$$\int_{a_2}^{b_2} w_2(t)[\Phi(f_2(t))] \, d\mu(t) - \int_{a_2}^{b_2} w_2(t) \, d\mu(t) \cdot \Phi \left(\frac{\int_{a_2}^{b_2} w_2(t) f_2(t) \, d\mu(t)}{\int_{a_2}^{b_2} w_2(t) \, d\mu(t)} \right)$$
$$- \int_{a_1}^{b_1} w_1(t)[\Phi(f_1(t))] \, d\mu(t) + \int_{a_1}^{b_1} w_1(t) \, d\mu(t) \cdot \Phi \left(\frac{\int_{a_1}^{b_1} w_1(t) f_1(t) \, d\mu(t)}{\int_{a_1}^{b_1} w_1(t) \, d\mu(t)} \right),$$

where $w_i : [a_i, b_i] \to \mathbb{R}$ *are non-negative integrable functions such that* $\int_{a_i}^{b_i} w_i(t) \, d\mu(t) > 0$, $f_i : [a_i, b_i] \to \mathbb{R}$ *are integrable and* $\Phi \in \mathcal{K}_1^c(I)$.

4. Applications to Weighted Generalized Means

Weighted generalized mean is defined in [17] as follows.

Definition 6. *Let* $\chi \in C(I, \mathbb{R})$ *be strictly monotone,* $I \subset \mathbb{R}$ *is an interval. Assume* $f : [a, b] \to I$ *is* Δ*-integrable and* $w : [a, b] \to \mathbb{R}$ *is non-negative and* Δ*-integrable such that* $\int_a^b w \, d\mu_\Delta > 0$. *Weighted generalized mean on time scales is defined as*

$$\mathcal{M}_\chi(f, w; \mu_\Delta) = \chi^{-1} \left(\frac{\int_a^b w(\chi \circ f) \, d\mu_\Delta}{\int_a^b w \, d\mu_\Delta} \right)$$

provided that all integrals are well defined.

Applying the definition of Levinson's functional on time scales, in the following theorem, we establish new Levinson's type functional in the terms of generalized means and prove its properties. The obtained properties can be used to improve some known inequalities on time scales.

Theorem 6. *Assume $\chi, \psi \in C(I, \mathbb{R})$ are strictly monotone, $I \subset \mathbb{R}$ is an interval. Let w_i, ϕ_i, f_i satisfy the hypotheses of Theorem 4 for $i = 1, 2$ such that the functional*

$$\begin{aligned}
&\int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot \left[\chi(\mathcal{M}_\chi(f_2, w_2; \mu_\Delta)) - \chi\left(\mathcal{M}_\psi(f_2, w_2; \mu_\Delta)\right)\right] \\
&- \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot \left[\chi(\mathcal{M}_\chi(f_1, w_1; \mu_\Delta)) - \chi\left(\mathcal{M}_\psi(f_1, w_1; \mu_\Delta)\right)\right] \\
&+ \frac{\lambda}{2} \cdot [\sigma_2(\psi \circ f_2, w_2) - \sigma_1(\psi \circ f_1, w_1)]
\end{aligned} \tag{31}$$

is well defined. Denoting (31) by $\mathcal{G}(f_1, f_2, w_1, w_2; \mu_\Delta)$, we obtain that if $\chi \circ \psi^{-1}$ is convex and following conditions are fulfilled:

$$\sigma_1(\psi \circ f_1, w_1) = \sigma_2(\psi \circ f_2, w_2), \tag{32}$$

$$\sigma_1(\psi \circ f_1, \phi_1) - \sigma_1(\psi \circ f_1, w_1 + \phi_1) = \sigma_2(\psi \circ f_2, \phi_2) - \sigma_2(\psi \circ f_2, w_2 + \phi_2), \tag{33}$$

then (31) is superadditive, i.e.,

$$\mathcal{G}(f_1, f_2, w_1 + \phi_1, w_2 + \phi_2; \mu_\Delta) \geq \mathcal{G}(f_1, f_2, w_1, w_2; \mu_\Delta) + \mathcal{G}(f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \tag{34}$$

Moreover, if $\chi \circ \psi^{-1}$ is concave, then (31) is subadditive, that is, inequality (34) holds in reverse order.

Proof. We start by replacing, in definition (12) of Levinson's functional on time scales, function F by $\chi \circ \psi^{-1}$ and function f_i by $\psi \circ f_i$, $i = 1, 2$. It follows that

$$\begin{aligned}
&\mathcal{L}_\mathbb{T}(\Phi, \psi \circ f_1, \psi \circ f_2, w_1, w_2; \mu_\Delta) = \mathcal{J}_\mathbb{T}\left(\chi \circ \psi^{-1}, \psi \circ f_2, w_2; \mu_\Delta\right) \\
&- \mathcal{J}_\mathbb{T}\left(\chi \circ \psi^{-1}, \psi \circ f_1, w_1; \mu_\Delta\right) + \frac{\lambda}{2} \cdot [\sigma_2(\psi \circ f_2, w_2) - \sigma_1(\psi \circ f_1, w_1)] \\
&= \int_{a_2}^{b_2} w_2 \cdot \left[\left(\chi \circ \psi^{-1}\right) \cdot (\psi \circ f_2)\right] d\mu_\Delta - \int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot \left(\chi \circ \psi^{-1}\right) \circ \left(\frac{\int_{a_2}^{b_2} w_2 \cdot (\psi \circ f_2)\, d\mu_\Delta}{\int_{a_2}^{b_2} w_2\, d\mu_\Delta}\right) \\
&- \int_{a_1}^{b_1} w_1 \left[\left(\chi \circ \psi^{-1}\right) \cdot (\psi \circ f_1)\right] d\mu_\Delta - \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot \left(\chi \circ \psi^{-1}\right) \circ \left(\frac{\int_{a_1}^{b_1} w_1 \cdot (\psi \circ f_1)\, d\mu_\Delta}{\int_{a_1}^{b_1} w_1\, d\mu_\Delta}\right) \\
&+ \frac{\lambda}{2} \cdot [\sigma_2(\psi \circ f_2, w_2) - \sigma_1(\psi \circ f_1, w_1)]
\end{aligned}$$

(according to Definition 6 and condition (32), we continue as follows)

$$
\begin{aligned}
&= \int_{a_2}^{b_2} w_2 \cdot (\chi \circ f_2)\, d\mu_\Delta - \int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot \chi(\mathcal{M}_\psi(f_2, w_2; \mu_\Delta)) \\
&- \int_{a_1}^{b_1} w_1 \cdot (\chi \circ f_1)\, d\mu_\Delta + \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot \chi(\mathcal{M}_\psi(f_1, w_1; \mu_\Delta)) \\
&= \int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot \chi(\mathcal{M}_\chi(f_2, w_2; \mu_\Delta)) - \int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot \chi(\mathcal{M}_\psi(f_2, w_2; \mu_\Delta)) \\
&- \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot \chi(\mathcal{M}_\chi(f_1, w_1; \mu_\Delta)) + \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot \chi(\mathcal{M}_\psi(f_1, w_1; \mu_\Delta)) \\
&= \int_{a_2}^{b_2} w_2\, d\mu_\Delta \cdot [\chi(\mathcal{M}_\chi(f_2, w_2; \mu_\Delta)) - \chi(\mathcal{M}_\psi(f_2, w_2; \mu_\Delta))] \\
&- \int_{a_1}^{b_1} w_1\, d\mu_\Delta \cdot [\chi(\mathcal{M}_\chi(f_1, w_1; \mu_\Delta)) - \chi(\mathcal{M}_\psi(f_1, w_1; \mu_\Delta))]
\end{aligned}
$$

Now, the superadditivity property (34) follows immediately from Theorem 4 and conditions (32) and (33). □

Corollary 2. *Assume $\chi, \psi \in C(I, \mathbb{R})$ are strictly monotone, $I \subset \mathbb{R}$ is an interval. Let w_i, ϕ_i, f_i satisfy the hypotheses of Theorem 4 for $i = 1, 2$ such that the functional $\mathcal{G}(f_1, f_2, w_1, w_2; \mu_\Delta)$ defined by (31) is well defined. If*

$$\sigma_1(\psi \circ f_1, w_1 - \phi_1) = \sigma_2(\psi \circ f_2, w_2 - \phi_2) \tag{35}$$

and

$$\sigma_1(\psi \circ f_1, \phi_1) - \sigma_1(\psi \circ f_1, w_1) = \sigma_2(\psi \circ f_2, \phi_2) - \sigma_2(\psi \circ f_2, w_2) \tag{36}$$

then, the Levinson's type functional (31) is increasing, that is, $w_i \geq \phi_i$ for $i = 1, 2$, implies

$$\mathcal{G}(f_1, f_2, w_1, w_2; \mu_\Delta) \geq \mathcal{G}(f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \tag{37}$$

Moreover, if $\chi \circ \psi^{-1}$ is concave, then (31) is decreasing, that is, inequality (37) holds in reverse order.

Proof. Monotonicity property (37) follows from the proof of Theorem 6, monotonicity properties of the Levinson functional on time scales obtained in Corollary 1 and conditions (35) and (36). □

In the next definition, we introduce weighted generalized power mean ([17]).

Definition 7. *Suppose $r \in \mathbb{R}$, $I \subset \mathbb{R}$ is an interval. Assume $f : [a, b] \to I$ is positive and Δ-integrable and $w : [a, b] \to \mathbb{R}$ is non-negative and Δ-integrable such that $\int_a^b w\, d\mu_\Delta > 0$. Weighted generalized power mean on time scales is defined as*

$$\mathcal{M}^{[p]}(f,w;\mu_\Delta) = \begin{cases} \left(\frac{\int_a^b w f^p \, d\mu_\Delta}{\int_a^b w \, d\mu_\Delta} \right), & p \neq 0, \\ \exp\left(\frac{\int_a^b w \ln(f) \, d\mu_\Delta}{\int_a^b w \, d\mu_\Delta} \right), & p = 0. \end{cases}$$

provided that all integrals are well defined.

The following theorem establishes another Levinson's type functional in terms of generalized power mean and proves its properties using the functional obtained in Theorem 6.

Theorem 7. *Let $p, q \in \mathbb{R}$ and $p \neq 0$. Assume w_i, ϕ_i, f_i satisfy the hypotheses of Theorem 4 for $i = 1, 2$ such that the functional*

$$\begin{aligned} &\int_{a_2}^{b_2} w_2 \, d\mu_\Delta \cdot \left\{ \left[\mathcal{M}^{[q]}(f_2, w_2; \mu_\Delta)\right]^q - \left[\mathcal{M}^{[p]}(f_2, w_2; \mu_\Delta)\right]^q \right\} \\ &- \int_{a_1}^{b_1} w_1 \, d\mu_\Delta \cdot \left\{ \left[\mathcal{M}^{[q]}(f_1, w_1; \mu_\Delta)\right]^q - \left[\mathcal{M}^{[p]}(f_1, w_1; \mu_\Delta)\right]^q \right\} \\ &+ \frac{\lambda}{2} \cdot [\sigma_2({f_2}^p, w_2) - \sigma_1({f_1}^p, w_1)] \end{aligned} \tag{38}$$

is well defined. Denoting (38) by $\mathcal{P}(f_1, f_2, w_1, w_2; \mu_\Delta)$, we obtain that if $max\{0, p\} < q < min\{0, p\}$ and the following conditions are fulfilled:

$$\sigma_1({f_1}^p, w_1) = \sigma_2({f_2}^p, w_2), \tag{39}$$

$$\sigma_1({f_1}^p, \phi_1) - \sigma_1({f_1}^p, w_1 + \phi_1) = \sigma_2({f_2}^p, \phi_2) - \sigma_2({f_2}^p, w_2 + \phi_2), \tag{40}$$

then (38) is superadditive, i.e.,

$$\mathcal{P}(f_1, f_2, w_1 + \phi_1, w_2 + \phi_2; \mu_\Delta) \geq \mathcal{P}(f_1, f_2, w_1, w_2; \mu_\Delta) + \mathcal{P}(f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \tag{41}$$

If $0 < q < p$ or $p < q < 0$, then (38) is subadditive, that is, inequality (41) holds in reverse order.

Proof. Substituting in (31) that $\chi(x) = x^q$ and $\psi(x) = x^p$, $x > 0$, if $p \neq 0$, we obtain (38). Since now $(\chi \circ \psi^{-1})(x) = x^{\frac{q}{p}}$ and $(\chi \circ \psi^{-1})'' = \frac{g(q-p)}{p^2} x^{\frac{q}{p}-2}$, we conclude that, if $\max\{0, p\} < q < \min\{0, p\}$, then $\chi \circ \psi^{-1}$ is convex and if $0 < q < p$ or $p < q < 0$, then $\chi \circ \psi^{-1}$ is concave so property (41) follow from Theorem 6. If $p = 0$, then taking $\chi(x) = x^q$ and $\psi(x) = \ln(x)$, $x > 0$ in Theorem 6, we obtain $(\chi \circ \psi^{-1})(x) = x^{qx}$ so $\chi \circ \psi^{-1}$ is convex for $q \neq 0$ and results follow from Theorem 6. □

Corollary 3. *Assume $p, q \in \mathbb{R}$ and $p \neq 0$. Let w_i, ϕ_i, f_i satisfy the hypotheses of Theorem 4 for $i = 1, 2$ such that the functional $\mathcal{P}(f_1, f_2, w_1, w_2; \mu_\Delta)$ defined by (38) is well defined. If*

$$\sigma_1({f_1}^p, w_1 - \phi_1) = \sigma_2({f_2}^p, w_2 - \phi_2) \tag{42}$$

and

$$\sigma_1({f_1}^p, \phi_1) - \sigma_1({f_1}^p, w_1) = \sigma_2({f_2}^p, \phi_2) - \sigma_2({f_2}^p, w_2) \tag{43}$$

then the functional (38) is increasing, that is, $w_i \geq \phi_i$ for $i = 1, 2$, implies

$$\mathcal{P}(f_1, f_2, w_1, w_2; \mu_\Delta) \geq \mathcal{P}(f_1, f_2, \phi_1, \phi_2; \mu_\Delta). \tag{44}$$

If $0 < q < p$ *or* $p < q < 0$*, then* (38) *is decreasing, that is, inequality* (44) *holds in reverse order.*

Proof. Monotonicity property (44) follows from the proof of Theorem 7, monotonicity properties of the Levinson functional on time scales obtained in Corollary 1 and conditions (42) and (43). □

5. Conclusions

In this paper, we established the Levinson functional on time scales utilizing integral inequality of Levinson's type in the terms of Δ- integral for convex (concave) functions on time scale sets and proved the properties of superadditivity and monotonicity. Using obtained properties, we derived the bounds of the Levinson's functional on time scales. Applying the same methods in the rest of the article, we constructed new Levinson's types of functionals using weighted generalized and power means on time scales and proved their properties regarding superadditivity and monotonicity. In future investigations, using the same reasoning as in Theorem 5, the bounds for the functionals in Theorem 6 and Theorem 7 can be obtained. Furthermore, a new type of functionals can be constructed using the methods of Theorem 6 and Theorem 7 and some specific forms of functions χ and ψ, and then properties of superadditivity and monotonicity can be easily proved as well as the bounds of obtained new functionals. Derived properties can then be used to improve some known inequalities on time scales.

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