axioms

MDPI

Article

Geometrical Structure in a Relativistic Thermodynamical Fluid Spacetime

Mohd. Danish Siddiqi [1], Fatemah Mofarreh [2], Aliya Naaz Siddiqui [3,*] and Shah Alam Siddiqui [1]

1. Department of Mathematics, College of Science, Jazan University, Jazan 45142, Saudi Arabia; msiddiqi@jazanu.edu.sa (M.D.S.); sasiddiqui@jazanu.edu.sa (S.A.S.)
2. Mathematical Science Department, Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia; fyalmofarrah@pnu.edu.sa
3. Division of Mathematics, School of Basic & Applied Sciences, Galgotias University, Greater Noida 203201, Uttar Pradesh, India

* Correspondence: aliya.siddiqui@galgotiasuniversity.edu.in

Abstract: The goal of the present research paper is to study how a spacetime manifold evolves when thermal flux, thermal energy density and thermal stress are involved; such spacetime is called a *thermodynamical fluid spacetime (TFS)*. We deal with some geometrical characteristics of TFS and obtain the value of cosmological constant Λ. The next step is to demonstrate that a relativistic TFS is a generalized Ricci recurrent TFS. Moreover, we use TFS with thermodynamic matter tensors of Codazzi type and Ricci cyclic type. In addition, we discover the solitonic significance of TFS in terms of the Ricci metric (i.e., Ricci soliton RS).

Keywords: Lorentzian spacetime manifold; Einstein's field equation; thermal energy momentum tensor; curvatures; Ricci solitons

MSC: 53C44; 53B30; 53C50; 53C80

check for updates

Citation: Siddiqi, M.D.; Mofarreh, F.; Siddiqui, A.N.; Siddiqui, S.A. Geometrical Structure in a Relativistic Thermodynamical Fluid Spacetime. *Axioms* **2023**, *12*, 138. https://doi.org/10.3390/axioms12020138

Academic Editor: Igor V. Miroshnichenko

Received: 14 December 2022
Revised: 20 January 2023
Accepted: 22 January 2023
Published: 29 January 2023

1. Introduction

> *"I am convinced that the only physical theory with universal substance that can be utilized is thermodynamics, and that it will never be denied."*—**Albert Einstein**.

A fundamental thermodynamic system consists of a homogeneous macroscopic group of elements. The system is viewed as a "black box", and the state of the system is described by a handful of macroscopic parameters, often energy, entropy, volume, and particle number, which are determined by the environment in which the system lives. These are not all necessarily independent, though. Thermodynamic degrees of freedom are the number of n independent parameters in a basic one-phase system with $n-1$ components. Any more parameters will be dependent. A non-uniform thermodynamic system can be created through the interaction of simple thermodynamic systems.

A comprehensive function of state known as the internal energy exists for every thermodynamic system. Manifolds originally developed as a set of variables that were subject to equations. The creators of differential geometry thoroughly investigated the early examples, such as curves and surfaces [1]. Smooth is typically understood to mean a piecewise analytic for the manifold of equilibrium states of a thermodynamic system. Be aware that the overlap requirement must be satisfied if the manifold is a connected set, in which case all coordinate chart pictures must have the same n dimension. The dimension of the manifold is the name given to this value. The thermodynamic system's manifold of equilibrium states is the one we will focus on in the next sections. The number of degrees of freedom in the system is referred to as the dimension of this manifold. Therefore, the connected four-dimensional time-oriented Lorentzian manifold is modeled using the thermodynamical spacetime manifold (see [2]) of the general theory of relativity GTR

(general relativity is basically a theory of gravitation developed by A. Einstein between 1907 and 1915, which states that the observed gravitational effect between masses results from their wrapping of spacetime) and cosmological space.

Basically, both the spacetime of the GTR and cosmology are used to model a connected four-dimensional time-oriented Lorentzian manifold [3,4].

Definition 1. *If the Ricci tensor has a certain shape, then it is claimed that a Lorentzian manifold is a perfect, fluid spacetime [5].*

$$\mathcal{R}ic = ag + bu \otimes u, \tag{1}$$

where u is a 1-form metrically related to a vector field that resembles time and a and b are non-zero scalars; the spacetime manifold is a Lorentzian manifold.

Formally, the effective energy-momentum tensor, defined in [6], can be used to recast the GTR. Then, in the presence of a competent time-like vector field, this tensor is described by isotropic pressure, energy density, an anisotropic pressure, and energy flow [3]. Moreover, this tensor also changes the shape of the Ricci tensor in Einstein's equation for perfect fluid spacetime (1).

For the thermodynamics aspect of the spacetime manifold from which Einstein's equation is derived, the fundamental relationship between the horizon area and entropy as well as their proportionality [7] is as follows:

$$\delta Q = \mathcal{T} dS, \tag{2}$$

where $\mathcal{T}$ is the temperature, S is the entropy and Q is the heat. The energy flux and temperature inside the horizon, as observed by an accelerating observer, are denoted by δQ and $\mathcal{T}$ in the relationship above, which holds for all local causal horizons via each spacetime point. For the Einstein equation to hold, gravitational lensing caused by matter-energy must specifically alter the causal structure of spacetime.

In 1995, Jacobson [8] proposed that the Einstein equation can be derived as a constitutive equation for the equilibrium (2) of a thermodynamical spacetime point of view with a cosmological constant, which can be given as:

$$\mathcal{R}ic - \frac{1}{2}\mathcal{R}g + \Lambda g = \frac{2\pi}{\hbar\eta}T. \tag{3}$$

Recent observations indicating an accelerating rate of cosmic expansion have led many cosmologists to believe that our universe is characterized by a positive value for cosmological constant (for more details, see [9]). The length $\eta^{-\frac{1}{2}}$ is twice the Planck length $(\hbar G)^{\frac{1}{2}}$, where Newton's constant, $G = (4\hbar\eta)^{-1}$, is determined by the constant of proportionality η between the entropy and the area (for additional information, see [7]). As mysterious as ever, the cosmological constant Λ is still unknown.

Inclusion of the cosmological constant Λ: A scientific hypothesis with amazing power and simplicity, GTR is a model example. Meanwhile, the cosmological constant serves as a prime example of an adjustment that, at least on the surface, seems unnecessary and unpleasant, but which actually helps match the facts. When the universe's expansion was discovered, its original purpose of enabling static homogeneous solutions to Einstein's equations in the presence of matter was shown to be unnecessary. Since then, there have been a number of instances in which a non-zero cosmological constant has been proposed as an explanation for a set of observations and later withdrawn when the observational case vanished [10]. In the meantime, researchers studying particle theory have discovered that the cosmological constant may be used to calculate the energy density of the vacuum. This energy density is the result of several, seemingly unrelated contributions that are all orders of magnitude greater than the cosmological constant's upper bounds (for further information, see [11]).

Other reviews of the cosmological constant's different elements include:

(i) Dark energy (DE), dark matter (DM) and regular composed of atoms matter were determined to be the universe's energy makeup. Understanding these observational data may require modifying the description offered by GTR. The inclusion of the cosmological constant in Einstein's field equations is one of the main models utilized for this goal, among the several techniques to describe the cosmic acceleration. The word DM refers to an unidentified kind of matter that exerts gravitational force but cannot be detected by its radiation. Dark energy, often known as DE, is an unidentified energy source and an unusual material with high negative pressure. The main and most pertinent candidate for DE, which offers the best rational explanation of the cosmos, is the cosmological constant.

(ii) In order to account for the state of the universe at the moment, cosmologists have recently become interested in modeling cosmological models for an alternative theory of gravity (modified theory). Numerous theories have been devised in accordance with the recognized rules of physics, but they are still unable to fully explain the enigma of the driving force behind the universe's expansion. Many cosmologists have researched the role of variables G and Λ to describe the current scenario of the accelerating universe as an alternative to changing the general theory of relativity by adding $f(R)$-gravity [12] and $f(G,T)$-gravity [13]. Newton's constant G may be regarded as a function of time or the scale factor, for example.

In observational and relativistic cosmology, the cosmos is studied using equations of state (EoS), perfect fluid cosmological models and other tools. The gravitational constant G serves as a coupling constant in the general theory of gravity between the geometry of space and the matter content in Einstein's field equations. In general-relativistic quantum field theory, the cosmological constant Λ naturally appears and is stated in terms of the vacuum energy density. They are also thought of as basic constants. After identifying a potential resolution to the cosmological constant problem, a cosmological model with a dynamic cosmological constant that is free of the cosmological problems was created [14].

(iii) The key area of inquiry will be the thermodynamic features of cosmological models, where G and Λ are time-dependent variables. The cosmos as a whole is constrained by the second law of thermodynamics, which keeps the temperature law in its original form. Additionally, the study of heat, radiation and black holes uses thermodynamics. The evolution of our universe can be predicted by the large quantity of entropy that is present in the universe and is in the form of black-body radiation. Numerous cosmic facts indicate that matter was in a state where all portions of a system had the same temperature or amount of heat. It is further noticed that the universe is homogeneous in its early stages based on the isotropy of the cosmic data. As a result, thermodynamics can be used to study the early universe's behaviors. In the EoS $(p = \omega\rho)$, the parameter $\omega = 1$ describes dark energy, whereas $\omega < 1$ describes phantom energy. Symbolically, p and ρ signify the pressure and energy density, respectively [11].

(iv) The Lovelock theory of gravity reduces to the Gauss–Bonnet term in a four-dimensional connected spacetime manifold, and it emerges in a five-dimensional ($\geq$5) spacetime. Lovelock gravity also admits black hole solutions and the accompanying thermodynamics as expected in terms of the cosmological constant Λ and is ghost-free with second-order field equations.

These calculations, which can be referred to as non-extended phase spaces, were made with the cosmological constant present as a fixed quantity. Some values are thermodynamic variables in thermodynamic systems, while others are fixed parameters that cannot change.

Remark 1. *The energy density associated with dark energy gives rise to a negative pressure p_Λ and asymptotically approaches the constant known as the cosmological constant Λ after the 1998 discovery of the universe's accelerated expansion from the observation of supernovas.*

For $\Lambda > 0$, this cosmological constant Λ is crucial in understanding the universe's observed accelerated expansion. As a result, the energy density connected to the cosmological constant is known as the "vacuum energy density" or "dark energy density", ϵ_Λ, and it is defined as [15]:

$$\epsilon_\Lambda = \frac{c^4}{8\pi G}\Lambda, \tag{4}$$

where c is velocity and G is Newton's constant.

The formula for the mass density that corresponds to the vacuum energy density is [15]:

$$\rho_\Lambda = \frac{\epsilon_\Lambda}{c^4}. \tag{5}$$

Additionally, the definition of the dark energy equation of state is [15]:

$$p_\Lambda = -\epsilon_\Lambda. \tag{6}$$

The energy-momentum tensor is crucial in determining the amount of matter in spacetime, despite the fact that matter is typically thought of as a fluid with properties such as density, pressure and dynamic and kinematic characteristics, such as velocity, acceleration, shear and expansion [3]. In conventional cosmological models, the universe's matter composition is thought to behave as a certain fluid spacetime (perfect fluid spacetime). As a result, we split the effective energy-momentum tensor into two portions, the first of which is a perfect fluid that is pressureless, and the second of which is an imperfect fluid. The imperfect part is calculated to give an effective explanation of dark energy DE and the perfect fluid part is described to explain the dark matter DM [16].

Moreover, in the case of a perfect fluid spacetime, there is no existence of heat conduction and viscosity. A spacetime nature is dependent on the casting of stuff in it. Now, we may define the following:

Definition 2. *A four-dimensional Lorentzian spacetime manifold which includes thermal energy density, thermal flux and thermal stress, is called* thermodynamical fluid spacetime *TFS [17]. Heat is described by the GTR energy tensor T_H, often known as the thermal energy tensor [18].*

Therefore, the entire cosmic foundation is also an imperfect fluid. Thermodynamics describes it. The only local sources of energy for a matter tensor's type of matter will be mass and heat [18]

$$T = T_M + T_H, \tag{7}$$

where the "*material energy tensor*" is an energy tensor denoted by the symbol T_M. This uses the following conventional form [18]:

$$T_M = \rho u \otimes u \tag{8}$$

for this kind of substance, often called *dust*. Additionally, as demonstrated by the relativistic kinetic theory of gases, T_H is the exact effective energy tensor created by the random motion of particles around the average flow denoted by u [16]. The standard form of T_H is given as [18]:

$$T_H = \varepsilon u \otimes u + 2u \otimes q + \tau. \tag{9}$$

In a TFS, the thermodynamic matter tensor T is of the following shape [18]:

$$T = (\rho + \varepsilon)u \otimes u + 2u \otimes q + \tau, \tag{10}$$

where ρ is an effective density, ε is the thermal energy density, τ is the thermal stress tensor and q is the thermal flux. In GR, one projection of a thermal energy tensor is the thermal flux. Furthermore, ξ and ζ are time-like and space-like orthogonal vector fields, respectively, such that $g(\xi,\xi) = -1$, $g(\zeta,\zeta) = 1$. They are corresponding orthogonal vector fields with 1-forms u and q, respectively, that is, $g(E,\xi) = u(E)$ and $g(E,\zeta) = q(E)$.

Chaki used a covariant constant energy momentum tensor to explore spacetimes [19]. Furthermore, a topic that is closely connected to this one and has been studied by a number of writers is the spacetime manifold with an energy momentum tensor (for more details, see ([3,20–24]).

Symmetries play a profound role in nature, and as such, physics. Different species of particles organize themselves in symmetric ways. Symmetry is also a mathematical explanation for conserved quantities such as momentum and energy. For example, there is even a whole field theoretical physics called "super-symmetry".The sort of symmetry varies on the matter and spacetime manifold geometry, and its Lorentzian metric frequently makes it easier to find solutions to a variety of problems, such as those posed by Einstein's field equations.

Physical matter symmetry in the GTR is directly applicable to spacetime geometry. A key symmetry is the soliton which is attached to the spacetime geometrical flow. In fact, the concept of kinematics and thermodynamics in GTR is understood via the Ricci flow. Curvatures maintain self-resemblance, which keeps RS concentrated.

RS, or self-similar solutions of the Ricci flow $\frac{\partial}{\partial t}g = -2\mathcal{R}ic$ [25], were proposed in Riemannian Geometry and play a crucial role in explaining its singularities.

Definition 3. *An RS is pseudo-Riemannian manifold (M,g), admitting a smooth vector field V, such that [25]*

$$\frac{1}{2}\mathfrak{L}_V g + \mathcal{R}ic + \theta g = 0, \tag{11}$$

where $\mathfrak{L}_V$, $\mathcal{R}ic$ and θ indicate the Lie derivative along the direction of V, the Ricci tensor and a real number, respectively. Referring to (11), an RS is known to be growing, stable or decreasing according to whether $\theta > 0$, $\theta = 0$ or $\theta < 0$, respectively.

RS have subsequently received a lot of attention in pseudo-Riemannian situations. One of the many factors contributing to the increased interest of theoretical physicists in RS is their connection to String theory. In terms of RS, Ahsan and Ali explored the spacetime manifold in the GTR [26]. In addition, the perfect fluid spacetimes were depicted by Blaga in [27], with η-RS and η-Einstein solitons. In [28], RS is also used by Venkatesha and Aruna to study perfect fluid spacetimes with a torse-forming vector field. Numerous authors conducted in-depth research on spacetimes with solitons in distinct manners; we may refer to ([27,29–34]) and references therein.

Therefore, the results of earlier research served as our motivation. We explore the behavior of TFS in Section 2. In Section 3, we examine a geometrical feature of TFS and demonstrate that the total density of space is not zero. In addition, we determine the cosmological constant Λ, whose value relies on the scalar curvature $\mathcal{R}$. With the help of the Codazzi and cyclic parallel Ricci conditions, we confine the curvature of TFS in Section 4.

In [28], the authors studied quasi-conformal flat perfect fluid spacetime, and in the present manuscript, we estimate a new and more general notion named the pseudo-quasi-conformal curvature tensor on TFS. Basically, a pseudo-Quasi-conformal curvature tensor is a generalization of a concircular curvature tensor, conformal curvature tensor, quasi-conformal curvature tensor and projective curvature tensor.

Furthermore, the authors analyze the behavior of distinct solitons on perfect fluid spacetimes using a torse-forming vector field, a Jacobi vector field, and a killing vector field in [27,28,35,36]. With a new sort of vector field called the ψQ or $\varphi(\mathcal{R}ic)$ vector field, we examine the RS on TFS in Section 7. Additionally, using a ψQ vector field and the RS, we determine the value of the cosmological constant Λ on TFS. In addition, we discover that an expanding universe, assuming the cosmological constant Λ is positive, is the condition for a TFS with an RS.

2. Relativistic TFS

In light of (3), for two vector fields E and F on M, Einstein's equation of state with the cosmological constant for TFS is given as:

$$\mathcal{R}ic(E,F) + (\Lambda - \frac{1}{2}\mathcal{R})g(E,F) = \frac{2\pi}{\hbar\eta}T(E,F), \tag{12}$$

where $\mathcal{R}ic$ and $\mathcal{R}$ stand for the Ricci tensor and scalar curvature of TFS, respectively.

We deduce Einstein's equation of state with the cosmological constant for a TFS via Equations (12) and (10):

$$\mathcal{R}ic(E,F) = \frac{(\mathcal{R} - 2\Lambda)}{2}g(E,F) + \frac{2\pi}{\hbar\eta}(\rho+\varepsilon)u(E)u(F) + \frac{4\pi}{\hbar\eta}u(E)q(F) + \frac{2\pi}{\hbar\eta}\tau(E,F), \tag{13}$$

where $E, F \in \chi(M^4, g)$, $\chi(M^4)$ denotes the collection of all C^∞-vector fields of M^4 and u, which is 1-form, and in this scenario, flux q is a 1-form or function over the spacetime and the thermal energy density τ is a $(0,2)$ type symmetric tensor. On contracting (13), we find the following result.

Theorem 1. *In a relativistic TFS with thermal energy density, thermal flux and thermal stress and satisfying the Einstein equation of state with cosmological constant, the scalar curvature is:*

$$\mathcal{R} = 4\Lambda + \frac{2\pi}{\hbar\eta}[\rho + \epsilon + J], \tag{14}$$

where $J = Tr(\tau)$.

The following corollary is deduced from Theorem 1 and Remark 1:

Corollary 1. *For a relativistic TFS with thermal energy density, thermal flux and thermal stress and satisfying the Einstein equation of state with positive cosmological constant, the TFS is an accelerating spacetime if and only if $\frac{\mathcal{R}}{4} > \frac{\pi}{2\hbar\eta}[\rho + \epsilon + J]$.*

With a time-like vector field ξ and space-like vector field ζ, we have $g(E,\xi) = u(E)$, $g(E,\zeta) = q(E)$, $u(\xi) = -1$, $q(\zeta) = 1$, $u(\zeta) = q(\xi) = 0$, $g(\xi,\zeta) = 0$ from (13), while ξ, ζ are orthogonal unit vector fields, respectively, we gain:

$$\mathcal{R}ic(E,\xi) = (\alpha - \beta)u(E), \tag{15}$$

$$\mathcal{R}ic(E,\zeta) = \alpha q(E) + \gamma q(E) + \omega\tau(E,\zeta), \tag{16}$$

$$\mathcal{R}ic(\xi,\xi) = \alpha - \beta, \tag{17}$$

$$\mathcal{R}ic(\zeta,\zeta) = \alpha + \omega I, \qquad \text{where} \quad \tau(\zeta,\zeta) = I, \tag{18}$$

$$\mathcal{R}ic(\xi,\zeta) = -\gamma, \tag{19}$$

where

$$\alpha = \frac{\mathcal{R}}{2} - \Lambda, \qquad \beta = \frac{2\pi}{\hbar\eta}(\rho+\varepsilon), \qquad \gamma = \frac{4\pi}{\hbar\eta}, \qquad \omega = \frac{2\pi}{\hbar\eta}. \tag{20}$$

Letting Q represent the symmetric endomorphism of the tangent space at any point on the spacetime manifold. Then, we have $\mathcal{R}ic(E,F) = g(QE,F)$ for all E, F, where Q is the Ricci operator.

3. Geometrical Virtues of Relativistic TFS

Through (15), we have $\mathcal{R}ic(E,\xi) = (\alpha - \beta)g(E,\xi)$ for all E.

Theorem 2. *In a relativistic TFS, the generator ξ is an eigenvector of the Ricci tensor corresponding to the eigenvalue $\alpha - \beta$.*

Let us assume that in a relativistic TFS, ξ is the parallel velocity vector field. Then, $\nabla_E \xi = 0$ for all E, which argues that $R(E,F)G = 0$ is an outcome of $\mathcal{R}ic(E,\xi) = 0$ for all E. Again, from (15), we obtain $\mathcal{R}ic(E,\xi) = (\alpha - \beta)u(E)$. Thus, $\alpha - \beta = 0$ is required. As a result, we get the following outcome:

Theorem 3. *If the generator ξ of a relativistic TFS is a parallel velocity vector field, then the associated scalars α, β are linked by $\alpha - \beta = 0$.*

After all, both ξ and ζ are orthogonal to each other, thus, from (16), we obtain that:

$$g(Q\xi,\xi) = \alpha - \beta, \tag{21}$$

which signifies that $Q\xi$ is orthogonal to $\xi \Leftrightarrow \alpha - \beta = 0$. Thus, we can articulate the following:

Theorem 4. *In a relativistic TFS, $Q\xi$ is orthogonal to ξ if and only if $\alpha - \beta = 0$.*

Proof. From (55), we easily obtain our desired result. □

Therefore, from Equation (20), we notice that $\rho + \varepsilon = \frac{\hbar\eta}{2\pi}(\mathcal{R} - 4\Lambda) - J$.

Corollary 2. *If the velocity vector field is parallel of the relativistic TFS, then the sum of densities is $\rho + \varepsilon \neq 0$.*

Corollary 3. *In a relativistic TFS, $Q\xi$ is orthogonal to ξ with $\rho + \varepsilon \neq 0$.*

In addition, from (18) we find:

$$g(Q\zeta,\zeta) = \alpha + \gamma + \omega I,$$

which signifies that $Q\zeta$ is orthogonal to $\zeta \Leftrightarrow \alpha + \gamma + \omega I$. Thus, we can articulate:

Theorem 5. *In a relativistic TFS, $Q\zeta$ is orthogonal to ζ if and only if $\alpha + \gamma + \omega I = 0$.*

Likewise, in virtue of Equation (20) and the theorem 5, the following consequence is found:

Corollary 4. *If $Q\zeta$ is orthogonal to ζ in a relativistic TFS, then the scalar curvature is $\mathcal{R} = 4\Lambda + \frac{2\pi}{\hbar\eta}[(\rho + \epsilon) + J + 2]$ and the value of cosmological constant is $\Lambda = \frac{R}{4} - \frac{\pi}{2\hbar\eta}[(\rho + \epsilon) + I + 2]$.*

In light of Remark 1 and Corollary 4, the outcome is as follows:

Theorem 6. *If $Q\zeta$ is orthogonal to ζ in a relativistic TFS, then the relativistic TFS is an accelerating universe if and only if $\frac{R}{4} > \frac{\pi}{2\hbar\eta}[(\rho + \epsilon) + I + 2]$.*

Corollary 5. *If $Q\zeta$ is orthogonal to ζ in a relativistic TFS, then the relativistic TFS with $\Lambda > 0$ is a supernova if and only if $\frac{R}{4} > \frac{\pi}{2\hbar\eta}[(\rho + \epsilon) + I + 2]$.*

Furthermore, the relations (4), (15) and (6) entail that:

Theorem 7. *If $Q\zeta$ is orthogonal to ζ in a relativistic TFS with $\Lambda > 0$, then the dark energy, $\epsilon_\Lambda = \frac{c^4}{8\pi G}\frac{R}{4} - \frac{\pi}{2\hbar\eta}[(\rho+\epsilon)+I+2]$, pressure $p_\Lambda = -\frac{c^4}{8\pi G}\frac{R}{4} - \frac{\pi}{2\hbar\eta}[(\rho+\epsilon)+I+2]$ and vacuum energy density is $\rho_\Lambda = \frac{c^2}{8\pi G}\frac{R}{4} - \frac{\pi}{2\hbar\eta}[(\rho+\epsilon)+I+2]$.*

Next, let us assume that ξ and ζ are parallel vector field on TFS. Then, we have $\nabla_E\xi = 0$ and $\nabla_E\zeta = 0$, which imply that:

$$R(E,F)\xi = 0, \quad R(E,F)\zeta = 0.$$

Hence, it follows that:

$$\mathcal{R}ic(E,\xi) = 0, \quad \textit{and} \quad \mathcal{R}ic(E,\zeta) = 0.$$

Now, adopting (15) and (18) we obtain $\alpha - \beta = 0$, $\alpha + \gamma + \omega = 0$. Due to the parallel vector fields ξ and ζ's implication, $\gamma = \beta = -(\alpha + \omega I)$.

4. Codazzi and Cyclic Parallel Type Ricci Curvature Tensor on Relativistic *TFS*

The Codazzi type and Ricci cyclic type curvature explain the important limitation of the geometry of the spacetime manifold. These curvatures introduced the geometric or unconventional matter content in the Einstein equitation, depending on the point of the view, in a different way than other extended theories of gravity. The Codazzi condition and closed vector field determines a class of spacetime that host the tensor. The spacetime in turn determines the Ricci tensor and the way the Codazzi tensor interacts with the Ricci tensor. Finally, the Codazzi type and Ricci cyclic type tensor determine the energy-momentum tensor of the Einstein equation. The Codazzi type and Ricci cyclic type curvature property strongly restrict the spacetime they live in. For example, in a static spacetime, the Codazzi tensor determines the acceleration form under certain condition (for more details see [37]). The trivial Codazzi tensors give the Einstein equations without and with a cosmological constant. Derdzinski and Shen presented the idea of a non-trivial Codazzi tensor on a Riemannian manifold in the cited work [38]. The Codazzi tensors are exemplified by the parallel tensors.

Definition 4 ([38])**.** *If a non-vanishing Ricci tensor $\mathcal{R}ic$ of a semi-Riemannian manifold satisfies the condition:*

$$(\nabla_E\mathcal{R}ic)(F,G) = (\nabla_G\mathcal{R}ic)(E,F) \tag{22}$$

then the Ricci tensor $\mathcal{R}ic$ is said to be Codazzi type.

Definition 5 ([38])**.** *If the Ricci tensor $\mathcal{R}ic$ of a semi-Riemannian manifolds satisfies the condition:*

$$(\nabla_E\mathcal{R}ic)(F,G) + (\nabla_F\mathcal{R}ic)(E,G) + (\nabla_G\mathcal{R}ic)(E,F) = 0 \tag{23}$$

then Ricci tensor $\mathcal{R}ic$ is said to be cyclic parallel.

For the relativistic TFS, the thermodynamic matter tensor is Codazzi type. Then, the relativistic TFS satisfies:

$$(\nabla_E T)(F,G) = (\nabla_G T)(F,E), \qquad \textit{where} \quad T = T_M + T_H. \tag{24}$$

From (12) we attain:

$$\begin{aligned}(\nabla_E\mathcal{R}ic)(F,G) - (\nabla_G\mathcal{R}ic)(E,F) - \frac{1}{2}[d\mathcal{R}(E)g(F,G) - d\mathcal{R}g(E,F)]& \\ = \frac{2\pi}{\hbar\eta}[(\nabla_E T)(F,G) - (\nabla_G T)(F,E)].&\end{aligned} \tag{25}$$

Since, in light of (4), scalar curvature $\mathcal{R}$ is constant, which implies that $d\mathcal{R}(E) = 0$. Then, we find (24). Thus, we can state the following:

Theorem 8. *The thermodynamic matter tensor of a relativistic TFS is Codazzi type if and only if Ricci tensor is of Codazzi type.*

Let us assume that the time-like velocity vector field ξ and space-like thermal flux vector field ζ of a relativistic TFS are Killing vector fields.

$$(\mathfrak{L}_\xi g)(E,F) = 0, \qquad and \qquad (\mathfrak{L}_\zeta g)(E,F) = 0, \tag{26}$$

where $\mathfrak{L}$ indicates the Lie derivative along the direction of a vector field ξ and ζ, respectively. From (26) it follows that:

$$g(\nabla_E\xi, F) + g(E, \nabla_F\xi) = 0, \qquad g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta) = 0. \tag{27}$$

Since:

$$g(\nabla_E\xi, F) = (\nabla_E u)(F), \quad and \quad g(\nabla_E\zeta, F) = (\nabla_E q)(F), \tag{28}$$

we find a pair of relations from (27) that:

$$(\nabla_E u)(F) + (\nabla_F u)(E) = 0, \quad and \quad (\nabla_E q)(F) + (\nabla_F q)(E) = 0, \tag{29}$$

for all E, F.

In a similar fashion, we have:

$$(\nabla_E u)(G) + (\nabla_G u)(E) = 0, \quad and \quad (\nabla_E q)(G) + (\nabla_G q)(E) = 0, \tag{30}$$

$$(\nabla_G u)(F) + (\nabla_F u)(G) = 0, \quad and \quad (\nabla_G q)(F) + (\nabla_F q)(G) = 0, \tag{31}$$

for all E, F, G.

Furthermore, we consider that the associated scalars are constant. Then, from (13) and using (30), (31) we obtain:

$$\begin{aligned}(\nabla_E\mathcal{R}ic)(F,G) + (\nabla_F\mathcal{R}ic)(G,E) + (\nabla_G\mathcal{R}ic)(E,F)\\ = \frac{2\pi}{\hbar\eta}[(\nabla_E\tau)(F,G) + (\nabla_F\tau)(G,E) + \hbar\eta(\nabla_G\tau)(E,F)].\end{aligned} \tag{32}$$

Therefore, we can articulates the following:

Theorem 9. *If the time-like velocity vector field ξ and space-like thermal flux vector field ζ of a relativistic TFS are Killing vector fields and the associated scalars are constant, then the relativistic TFS is cyclic parallel if and only if the thermal stress tensor τ is cyclic parallel.*

5. Pseudo-Quasi-Conformal Curvature Tensor on Relativistic *TFS*

In [39], Shaikh and Jana introduced the notion of pseudo-quasi-conformal tensor and is defined as:

$$\begin{aligned}C^\sharp(E,F)G = (p+d)R(E,F)G + \left(q - \frac{d}{3}\right)[\mathcal{R}ic(F,G)E - \mathcal{R}ic(E,G)F]\\ + q[g(F,G)QE - g(E,G)QF]\\ - \frac{\mathcal{R}}{n}[\frac{p}{n-1} + 2q][g(F,G)F - g(E,G)F]\end{aligned} \tag{33}$$

where $E, F, G \in \chi(M)$, Q is a symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $\mathcal{R}ic$, i.e., $g(QX,Y) = \mathcal{R}ic(X,Y)$ and p, q, d are real constants such that $p^2 + q^2 + d^2 > 0$.

Remark 2. *In particular, C is reduced to the projective curvature tensor, quasi-conformal curvature tensor, conformal curvature tensor and concircular curvature tensor, respectively, if* (1) $p = q = 0$, $d = 1$; (2) $p \neq 0, q \neq 0, d = 0$; (3) $p = 1, q = -\frac{1}{n-2}, d = 0$; (4) $p = 1, q = d = 0$.

Now, Equation (4) signify that the scalar curvature $\mathcal{R}$ is a constant, i.e., $d\mathcal{R}(E) = 0$, for all E. Adopting $d\mathcal{R}(E) = 0$, we obtain from (33) that:

$$(\nabla_H C^\sharp)(E,F,G) = (p+d)(\nabla_H R)(E,F,G) + \left(q - \frac{d}{3}\right)[(\nabla_H \mathcal{R}ic)(F,G)E - (\nabla_H \mathcal{R}ic)(E,G)F] + q[g(F,G)(\nabla_H Q)E - g(E,G)(\nabla_H Q)F]. \tag{34}$$

We know that $(div\mathcal{R})(E,F,G) = (\nabla_E \mathcal{R}ic)(E,G) - (\nabla_F \mathcal{R}ic)(E,G)$ and from (13) we find:

$$(\nabla_E \mathcal{R}ic)(F,G) = \frac{2\pi}{\hbar\eta}(\rho+\varepsilon)[(\nabla_E u)(F)u(G) + u(F)(\nabla_E u)(G)] + \frac{4\pi}{\hbar\eta}[(\nabla_E u)(F)q(G) + u(F)(\nabla_E q)(G)] + \frac{2\pi}{\hbar\eta}(\nabla_E \tau)(F,G), \tag{35}$$

since $\frac{2\pi}{\hbar\eta}(\rho+\varepsilon)$, $\frac{4\pi}{\hbar\eta}$ and $\frac{2\pi}{\hbar\eta}$ are constant. Now after contracting (34) and using (35), we arrive at:

$$\begin{aligned}(divC^\sharp)(E,F,G) = (p+q+\frac{2d}{3})[&\frac{2\pi}{\hbar\eta}(\rho+\varepsilon)((\nabla_E u)(F)q(G) + u(F)(\nabla_E q)(G) \\ &- (\nabla_F u)(E)q(G) - u(E)(\nabla_F q)(G)) \\ &+ \frac{4\pi}{\hbar\eta}((\nabla_E u)(F)q(G) + u(F)(\nabla_E q)(G) \\ &- (\nabla_F u)(E)q(G) - u(E)(\nabla_F q)(G)) \\ &+ \frac{2\pi}{\hbar\eta}((\nabla_E \tau)(F,G) - (\nabla_F \tau)(E,G))].\end{aligned} \tag{36}$$

Applying the condition that the time-like velocity vector field ξ and space-like thermal flux vector field ζ of the relativistic TFS are parallel vector fields that provide $\nabla_E \xi = 0$ and $\nabla_E \zeta = 0$. Thus, we obtain a pair of equations:

$$g(\nabla_E \xi, F) = 0, \quad i.e., \quad (\nabla_E u)(F) = 0. \tag{37}$$

$$g(\nabla_E \zeta, F) = 0, \quad i.e., \quad (\nabla_E q)(F) = 0. \tag{38}$$

Therefore, from (36), (37) and (38), it follows that:

$$(divC^\sharp)(E,F,G) = \frac{2\pi}{\hbar\eta}(p+q+\frac{2d}{3})[(\nabla_E \tau)(F,G) - (\nabla_F \tau)(E,G)]. \tag{39}$$

Consequently, we can articulate the following:

Theorem 10. *If in a relativistic TFS the associated scalars are constant and time-like velocity vector field ξ and space-like thermal flux vector field ζ are parallel, then the divergence-free pseudo-quasi-conformal curvature tensor and the thermal stress tensor τ of Codazzi type are equivalent.*

Moreover, in light of Remark 2 and Equation (39), we gain the following corollary:

Corollary 6. *If in a relativistic TFS the associated scalars are constant and time-like velocity vector field ξ and space-like thermal flux vector field ζ are parallel, then:*

1. *The divergence-free projective curvature tensor and the thermal stress tensor τ of Codazzi type are equivalent;*
2. *The divergence-free quasi-conformal curvature tensor and the thermal stress tensor τ of Codazzi type are equivalent;*
3. *The divergence-free conformal curvature tensor and the thermal stress tensor τ of Codazzi type are equivalent;*

4. *The divergence-free concircular curvature tensor and the thermal stress tensor τ of Codazzi type are equivalent.*

6. Generalized Ricci Recurrent Relativistic TFS

Definition 6 ([40]). *A non-flat semi-Riemannian manifold is said to be a generalized Ricci recurrent manifold if its Ricci tensor satisfies the following condition:*

$$(\nabla_E \mathcal{R}ic)(F,G) = a(E)\mathcal{R}ic(F,G) + b(E)g(F,G), \tag{40}$$

where a and b are non-zero 1*-forms. If $b = 0$, then the manifold turn into a Ricci recurrent manifold [41].*

A time-like velocity vector field ξ corresponding to the associated 1-form u is said to be recurrent if [42]:

$$(\nabla_E u)(F) = A(E)u(F), \tag{41}$$

where A is a non-zero 1-form.

In addition, let us assume that the generators ξ and ζ corresponding to the associated 1-form u and q are recurrent. Then we have

$$(\nabla_E u)(F) = \lambda(E)u(F), \quad (\nabla_E q)(F) = \mu(E)q(F), \tag{42}$$

wherein λ and μ are non-zero 1-forms.

Now, adopting (42) and (35), we obtain:

$$\begin{aligned}(\nabla_G \mathcal{R}ic)(E,F) &= \frac{4\pi}{\hbar\eta}(\rho+\varepsilon)\lambda(G)u(E)u(F) \\ &+ \frac{4\pi}{\hbar\eta}[\mu(G)u(E)q(F) + \lambda(G)u(E)q(F)] + d(\nabla_G\tau)(E,F).\end{aligned} \tag{43}$$

Consider that 1-form λ and μ are equal, i.e., $\lambda(G) = \mu(G)$ for all G. Then, we find from (43) that:

$$(\nabla_G \mathcal{R}ic)(E,F) = \frac{4\pi}{\hbar\eta}\lambda(G)u(E)u(F) + \frac{4\pi}{\hbar\eta}(\rho+\varepsilon)\lambda(G)u(E)q(F) + \frac{2\pi}{\hbar\eta}(\nabla_G\tau)(E,F). \tag{44}$$

In view of (13) and (44), we have:

$$(\nabla_G \mathcal{R}ic)(E,F) = \alpha_1(G)\mathcal{R}ic(E,F) + \beta_1(G)g(E,F) + d(\nabla_G\tau)(E,F), \tag{45}$$

where $\alpha_1(G) = 2\lambda(G)$ and $\beta_1(G) = -\frac{4\pi}{\hbar\eta}\lambda(G)$. Then, we can articulate the following.

Theorem 11. *If the generators ξ and ζ of a relativistic TFS corresponding to the associated 1 forms u and q are recurrent with the same vector of recurrence and the associated scalars are constant along the new condition thermal stress tensor τ is covariant, then the relativistic TFS is a generalized Ricci recurrent relativistic TFS.*

7. Ricci Solitons on Relativistic TFS with a ψQ Vector Field

Definition 7 ([43]). *A vector field ψ on a semi-Riemannian manifold M is said to be a ψQ-vector field if it satisfies:*

$$\nabla_E \psi = \sigma Q E, \tag{46}$$

where ∇, σ and Q is the Levi-Civita connection, a constant, and a Ricci operator, respectively. If $\sigma \neq 0$, then vector field ψ is said to be a proper ψQ-vector field, and if $\sigma = 0$ in (46), then vector filed ψ is said to be covariantly constant.

Now, we obtain an interesting finding.

Theorem 12. *If a relativistic TFS M admitting an RS (M, g, ψ, θ), such that the potential vector field ψ is a proper ψQ-vector field, then (M, g, ψ, θ) is a relativistic TFS.*

Proof. In view of (11) and (13), we obtain:

$$\frac{1}{2}\mathfrak{L}_V g(E,F) - \left\{\frac{\mathcal{R}}{2} - \Lambda + \theta\right\} g(E,F) - \frac{2\pi}{\hbar\eta}(\rho+\varepsilon)u(E)u(F) - \frac{4\pi}{\hbar\eta}u(E)q(F) - \frac{2\pi}{\hbar\eta}\tau(E,F) = 0. \tag{47}$$

By the definition of the Lie-derivative and (46), one has:

$$(\mathfrak{L}_\psi g)(E,F) = 2\sigma\mathcal{R}ic(E,F) \tag{48}$$

for any E, F.

From (47) and (48), we obtain:

$$\mathcal{R}ic(E,F) = -\frac{(\alpha+\theta)}{\sigma}g(E,F) + \frac{\beta}{\sigma}u(E)u(F) + \frac{\gamma}{\sigma}u(E)q(F) + \frac{\omega}{\sigma}\tau(E,F). \tag{49}$$

□

Adopting $E = F = \xi$ in (49), we find:

$$\theta = \sigma(\alpha-\beta) - (\alpha+\beta). \tag{50}$$

Hence, we state the following results.

Theorem 13. *Let M be a relativistic TFS admitting an RS (M, g, ξ, θ) with a proper ξQ- time-like velocity vector field ξ; then, RS is growing, stable or decreasing according to $\sigma(\alpha-\beta) > (\alpha+\beta)$, $\sigma(\alpha-\beta) = (\alpha+\beta)$ and $\sigma(\alpha-\beta) < (\alpha+\beta)$, respectively.*

Corollary 7. *Let M be a relativistic TFS admitting an RS (M, g, ξ, θ), such that the time-like velocity vector field ξ is ξQ, which is covariantly constant; then, RS decreases.*

Again, putting $E = \xi$ in (49) yields:

$$\left[(\alpha-\beta) + \frac{(\alpha+\theta)}{\sigma} - \frac{\beta}{\sigma}\right]u(F) - \frac{\gamma}{\sigma}q(F) = 0, \tag{51}$$

which signifies:

$$q(F) = \frac{1}{\gamma}[\alpha(\sigma-1) - \beta(\sigma+1) - \theta]u(F), \tag{52}$$

where

$$\alpha = \frac{R}{2} - \Lambda \tag{53}$$

and $\beta = \frac{4\pi}{\hbar\eta}(\rho+\varepsilon)$. As a result, the following can be said.

Theorem 14. *A relativistic TFS admitting an RS (M, g, ξ, θ) with a proper ξQ- time-like velocity vector field ξ, after which the relativistic TFS admits thermal flux, provided:*

$$\alpha(\sigma-1) - \beta(\sigma+1) - \theta \neq 0.$$

Corollary 8. *A relativistic TFS admitting an RS (M, g, ξ, θ) with a covariantly constant ξQ time-like velocity vector field ξ, after which the relativistic TFS admits thermal flux, provided:*

$$-(\alpha+\beta) - \theta \neq 0.$$

In view of Remark 1 and Equations (53), (4)–(6), we obtain the following results:

Theorem 15. *With a relativistic TFS admitting an RS (M, g, ξ, θ) with a proper ξQ- time-like velocity vector field ξ and relativistic TFS admitting thermal flux, the value of the cosmological constant is positive if and only if $\frac{\mathcal{R}}{2} > \alpha$.*

Theorem 16. *With a relativistic TFS admitting an RS (M, g, ξ, θ) with a proper ξQ- time-like velocity vector field ξ and relativistic TFS admitting thermal flux, the TFS is an accelerating universe if and only if $\frac{\mathcal{R}}{2} > \alpha$.*

Theorem 17. *With a relativistic TFS admitting an RS (M, g, ξ, θ) with a proper ξQ- time-like velocity vector field ξ and relativistic TFS admitting thermal flux, $\Lambda > 0$, the dark energy is $\epsilon_\Lambda = \frac{c^4}{8\pi G}[\frac{R}{2} - \alpha]$, pressure is $p_\Lambda = -\frac{c^4}{8\pi G}[\frac{R}{2} - \alpha]$ and vacuum energy density is $\rho_\Lambda = \frac{c^2}{8\pi G}[\frac{R}{2} - \alpha]$.*

Again, using $E = F = \zeta$, we arrive at:

$$\theta = -\{\alpha(\sigma + 1) + \omega I(\sigma - 1)\}. \tag{54}$$

Theorem 18. *Let M be a relativistic TFS admitting an RS (M, g, ζ, θ) with a proper ζQ-space-like velocity vector field ζ; then, RS decreases.*

Corollary 9. *If M is a relativistic TFS admitting an RS (M, g, ζ, θ) with a covariantly constant ζQ-space-like velocity vector field ζ, then RS is growing, stable or decreasing according to $\omega I > \alpha$, $\omega I = \alpha$ and $\omega I < \alpha$, respectively.*

Example 1. *Let $M = \{(x, y, z, t) \in \mathbb{R}^4 : t \neq 0\}$, where (x, y, z, t) are the standard coordinates of $\mathbb{R}^4$.*

Let (e_1, e_2, e_3, e_4) be the set of linearly independent vector fields of M, and is defined as:

$$e_1 = t\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad e_2 = t\frac{\partial}{\partial y}, \qquad e_3 = t\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \qquad e_4 = (t)^3\frac{\partial}{\partial t}.$$

Let g be the Riemannian metric M, defined by:

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_4, e_4) = -1 \qquad g(e_i, e_j) = 0, \text{for} \quad i \neq j, i, j = 1, 2, 3, 4.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_4)$ for any $Z \in \chi(M)$.

Furthermore, let φ be the $(1,1)$ tensor field, defined by:

$$\varphi(e_1) = e_1, \quad \varphi(e_2) = e_2, \qquad \varphi(e_3) = e_3, \quad \varphi(e_4) = 0, \quad \xi = (t)^3\frac{\partial}{\partial t}$$

and let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Thus, using the linearity of φ and g, we obtain:

$$[e_1, e_2] = -(t)e_2, \quad [e_1, e_4] = -(t)^2 e_1, \quad [e_2, e_4] = -(t)^2 e_2, \quad [e_3, e_4] = -(t)^2 e_3.$$

Then, for $e_4 = \xi$ and using Koszul's formula for the Lorentzian metric g, we obtain:

$$\nabla_{e_1} e_1 = -(t)^2 e_4, \quad \nabla_{e_2} e_1 = te_2, \quad \nabla_{e_1} e_4 = -(t)^2 e_1, \quad \nabla_{e_2} e_4 = -(t)^2 e_2$$

$$\nabla_{e_3} e_4 = -(t)^2 e_3, \quad \nabla_{e_3} e_3 = -(t)^2 e_4, \quad \nabla_{e_2} e_2 = -(t)^2 e_4 - te_1. \tag{55}$$

$$\nabla_{e_3} e_1 = \frac{1}{2}e^{2z} e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{2}e^{2z} e_1, \quad \nabla_{e_3} e_0 = 0. \tag{56}$$

From (55), we find that the structure (φ, ξ, η, g) is a Lorentzian structure on M. Consequently, $M^4(\varphi, \xi, \eta, g)$ is an Lorentzian manifold (four-dimensional spacetime model).

The non-vanishing components of Riemannian curvature and the Ricci tensors are given by:

$$R(e_1,e_4)e_1 = (t)^4 e_4, \quad R(e_2,e_4)e_2 = (t)^4 e_4, \quad R(e_3,e_4)e_3 = (t)^4 e_4,$$

$$R(e_1,e_3)e_3 = (t)^4 e_1, \quad R(e_1,e_3)e_1 = -(t)^4 e_3, \quad R(e_2,e_3)e_2 = -(t)^4 e_3,$$

$$R(e_1,e_4)e_4 = (t)^4 e_1, \quad R(e_2,e_4)e_4 = (t)^4 e_2, \quad R(e_1,e_2)e_2 = [(t)^4 - (t)^2]e_1,$$

$$R(e_2,e_3)e_3 = (t)^4 e_2, \quad R(e_3,e_4)e_4 = (t)^4 e_3, \quad R(e_1,e_2)e_1 = -[(t)^4 - (t)^2]e_2,$$

From the above expression of the curvature tensor, we can easily calculate the non-vanishing components of the Ricci tensor $\mathcal{R}ic$

$$\mathcal{R}ic(e_1,e_1) = 3(t)^4 - (t)^2, \quad \mathcal{R}ic(e_2,e_2) = 3(t)^4 - (t)^2$$

similarly, we have:

$$\mathcal{R}ic(e_3,e_3) = 3(t)^4, \quad \mathcal{R}ic(e_4,e_4) = 3(t)^4. \tag{57}$$

Therefore:

$$\mathcal{R} = \sum_{i=j=1}^{4} \mathcal{R}ic(e_i,e_j) = 2[6(t)^4 - (t)^2].$$

Now, in light of (11) and (49), we obtain:

$$\mathcal{R}ic(e_i,e_i) + \frac{(\alpha+\theta)}{\sigma}g(e_i,e_i) - \frac{\beta}{\sigma}u(e_i)u(e_i) - \frac{\gamma}{\sigma}u(e_i)q(e_i) - \frac{\omega}{\sigma}\tau(e_i,e_i) = 0$$

for all $i \in \{1,2,3,4\}$, and we have:

$$\mathcal{R}ic(e_i,e_i) + \frac{(\alpha+\theta)}{\sigma}g(e_i,e_i) - \frac{\beta}{\sigma}\delta_{i4} = 0$$

for all $i \in \{1,2,3,4\}$, we gain $\theta = 3\sigma t^4 - (\alpha+\beta)$. Thus, the data $(g, \xi = e_4, \theta)$ are Ricci solitons on TFS (M^4, g), with a proper ξQ- time-like velocity vector field ξ, which is expanding if $3\sigma t^4 < (\alpha+\beta)$, shrinking if $3\sigma t^4 > (\alpha+\beta)$ or steady if $3\sigma t^4 = (\alpha+\beta)$, as illustrated in Theorem 13.

In addition, the data $(g, \xi = e_4, \theta)$ are expanding Ricci solitons on TFS (M^4, g); additionally, a time-like velocity vector field ξ is ξQ covariantly constant if $\theta = -(\alpha+\beta)$ and verified Corollary 7.

Finally, in light of Equation (53), we find the value of $\Lambda = [6(t^4) - (t^2)] - \alpha$, and it will be positive if $[6(t^4) - (t^2)] > \alpha$, which also fulfills the Theorems 16 and 17.

8. Conclusions

In this article, we attempted to analyze the thermodynamical development of the spacetime manifold under the influence of some specific thermal characteristics. We described the TFS as a manifold with three particular objects (thermal density energy, thermal flux and thermal stress). Furthermore, we analyzed a series of geometric properties of these spaces linked to the existence of symmetries and demonstrated that the total density of space is not zero. Certain types of curvature tensors were used in order to characterize the structural configuration of the abovementioned spacetime. On the other hand, the Codazzi type and Ricci cyclic type curvature introduced the geometric or unconventional matter content in Einstein's equitation, depending on the point of the view, in a different way as compared to other extended theories of gravity. With the help of the Codazzi and Cyclic Parallel Ricci conditions, we confined the curvature of TFS and examine new and interesting results. Lastly, we studied the behavior of the RS on TFS with a new sort of vector field and also discovered that an expanding universe with a positive value of the cosmological constant is the condition for a TFS with an RS.

Author Contributions: Conceptualization, M.D.S., S.A.S.; formal analysis, M.D.S., F.M. and A.N.S.; investigation, M.D.S. and A.N.S.; methodology, M.D.S.; project administration and funding F.M.; validation, M.D.S., F.M., A.N.S. and S.A.S.; writing—original draft M.D.S. All authors have read and agreed to the published version of the manuscript.

Funding: The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions. The author, Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Eisenhart, L.P. *A Treatise on the Differential Geometry of Curves and Surfaces*; Ginn: Boston, MA, USA, 1909.
2. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Spacetime*; Cambridge Monographs on Mathematical Physics, 1; Cambridge University Press: London, UK, 1973.
3. Novello, M.; Reboucas, M.J. The stability of a rotating universe. *Astrophys. J.* **1978**, *225*, 719–724. [CrossRef]
4. Stephani, H. *General Relativity-An Introduction to the Theory of Gravitational Field*; Cambridge University Press: Cambridge, UK, 1982.
5. O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
6. Pimental, L.O. Energy-momentum tensor in the general scalar tensor theory. *Class. Quantum Grav.* **1989**, *6*, L263. [CrossRef]
7. Bekestein, J.D. Generalized second law of thermodynamics in blackhole physics. *Phy. Rev. D* **1974**, *9*, 3292. [CrossRef]
8. Jacobson, T. Thermodynamic of Spacetime: The Einstein equation of state. *Phys. Rev. Lett.* **1995**, *75*, 1260–1263. [CrossRef] [PubMed]
9. Earman, J. Lambda: The constant that refuses to die. *Arch. Hist. Exact Sci.* **2001**, *55*, 189–220. [CrossRef]
10. Hubble, E.P. A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae. *Proc. Natl. Acad. Sci. USA* **1929**, *15*, 168–173. [CrossRef] [PubMed]
11. Bousso, R. The cosmological constant. *Gen. Rel. Grav.* **2008**, *40*, 607–637. [CrossRef]
12. Bhatti, M.Z.; Yousaf, Z.; Tariq, Z. Structure scalars and their evolution for massive objects in $f(R)$ gravity. *Eur. Phys. J. C* **2021**, *81*, 16. [CrossRef]
13. Yousaf, Z.; Bhatti, M.Z.; Hassan, K. Complexity for self-gravitating fluid distributions in $f(G,T)$ gravity. *Eur. Phys. J. Plus* **2020**, *135*, 397. [CrossRef]
14. Abdel-Rahman, A.-M.M. A Critical density cosmological model with varying gravitational and cosmological constants. *Gen. Rel. Grav.* **1990**, *22*, 655–663. [CrossRef]
15. Peebles, P.J.; Ratra, B. The cosmological constant and dark energy. *Rev. Modern Phys.* **2003**, *75*, 559. [CrossRef]
16. Hipolito-Ricaldi, W.S.; Velten, H.E.S.; Zimdahl, W. Viscous dark fluid cosmology. *Phys. Rev.* **2010**, *82*, 063507.
17. Havas, P.; Swenson, R.J. Relativistic thermodynamics of fluids. I. *Ann. Phys.* **1979**, *118*, 259–306. [CrossRef]
18. Eckart, C. The Thermodynamics of Irreversible Processes III. Relativistic Theory of the simple fluid. *Phys. Rev.* **1940**, *58*, 919–924. [CrossRef]
19. Chaki, M.C. Spacetimes with covariant constant energy momentum tensor. *Int. J. Theoret. Phys.* **1996**, *35*, 1027–1032. [CrossRef]
20. Alias, L.; Romero, A.; Sanchez, M. Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetime. *Gen Rel. Grav.* **1995**, *30*, 915–932. [CrossRef]
21. Mantica, C.A.; Molinari, L.G. A condition for a perfect fluid spacetime to be a generalized Robertson-Walker spacetimes, a survey. *J. Math. Phys.* **2016**, *57*, 049901. [CrossRef]
22. Arslan, K.; Ezentas, R.; Murathan, C.; Ozgur, C. On pseudo Ricci-symmetric manifolds. *Balkan J. Geom. Appl.* **2001**, *6*, 526–531.
23. Duggal, K.L.; Sharma, R. Conformal Killing vector field on spacetime solutions of Einstein's equation and initial data. *Nonlinear Anal.* **2005**, *63*, 447–454. [CrossRef]
24. Guler, S.; Altay, S. A study of generalized quasi-Einstein spacetimes with applications in general relativity. *Int. J. Theor. Phys.* **2016**, *55*, 548–562. [CrossRef]
25. Hamilton, R.S. The Ricci flow on surfaces. *Contemp. Math.* **1988**, *17*, 237–261.
26. Ali, M.; Ahsan, Z. Ricci Solitons and Symmetries of Space time manifold of general relativity. *J. Adv. Res. Class. Mod. Geom.* **2014**, *1*, 75–84.

27. Blaga, A.M. Solitons and geometrical structures in a perfect fluid spacetime. *Rocky Mountain J. Math.* **2020**, *50*, 41–53. [CrossRef]
28. Venkatesha; Kumara, H.A. Ricci solitons and Geometrical structure in a Perfect fluid spacetime with Torse-forming vector filed. *Afr. Math.* **2019**, *30*, 725–736. [CrossRef]
29. Chow, B.; Chu, S.C.; Glickenstein, D.; Guenther, C.; Isenberg, J.; Ivey, T.; Knopf, D.; Lu, P.; Luo, F.; Ni, L. *The Ricci Flow: Techniques and Applications*; Part I: Geometric Aspects; AMS: Knoxville, TN, USA, 2007; Volume 135.
30. Deshmukh, S.; Alsodais, H. A Note on Ricci Solitons. *Symmetry* **2020**, *12*, 289. [CrossRef]
31. Siddiqi, M.D. Ricci ρ-soliton and geometrical structure in a dust fluid and viscous fluid sapcetime. *Bulg. J. Phys.* **2019**, *46* 163–173.
32. Siddiqi, M.D.; Siddqui, S.A. Conformal Ricci soliton and Geometrical structure in a perfect fluid spacetime. *Int. J. Geom. Methods Mod. Phys.* **2020**, *17*, 2050083. [CrossRef]
33. Siddiqui, A.N.; Siddiqi, M.D. Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime. *Balkan J. Geom. Appl.* **2021**, *26*, 126–138.
34. Siddiqi, M.D.; De, U.C., Relativistic magneto-fluid spacetimes. *J. Geom. Phys.* **2021**, *170*, 104370. [CrossRef]
35. Siddiqi, M.D.; Chaubey, S.K.; Khan, M.N.I. $f(R,T)$-Gravity Model with Perfect Fluid Admitting Einstein Solitons. *Mathematics* **2021**, *10*, 82. [CrossRef]
36. Siddiqi, M.D.; De, U.C. Relativistic perfect fluid spacetimes and Ricci–Yamabe solitons. *Lett. Math. Phys.* **2022**, *112*, 1. [CrossRef]
37. Merton, G. Codazzi tensor with two eigenvalues functions. *Proc. Amer. Math. Soc.* **2013**, *141*, 3265–3273. [CrossRef]
38. Derdzinski, A.; Shen, C.L. Codazzi tensor fileds, curvature and Pontryagir forms. *Proc. Lond. Math. Soc.* **1983**, *47*, 15–26. [CrossRef]
39. Shaikh A.A.; Jana, S.K. A pseudo-quasi-conformal curvature tensor on a Riemannian manifold. *South East Asian J. Math. Math. Sci.* **2005**, *4*, 15–20.
40. Arslan, K.; De, U.C.; Murathan, C.; Yildiz, A. On gebralized recurrent Riemannian manifolds. *Acta Math. Hungar.* **2009**, *123*, 27–39. [CrossRef]
41. Patterson, E.M. Some theorems on Ricci-recurrent spaces. *J. Lond. Math. Soc.* **1952**, *27*, 287–295. [CrossRef]
42. Schoutten, J.A. *Ricci Calculus*; Springer: Berlin/Heidelberg, Germany, 1954.
43. Hinterleitner, I.; Kiosak, V.A. $\varphi(Ric)$-vector fields in Riemannian spaces. *Arch. Math.* **2008**, *44*, 385–390.