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From Symmetric Functions to Partition Identities

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Abstract: In this paper, we show that some classical results from q-analysis and partition theory are specializations of the fundamental relationships between complete and elementary symmetric functions.

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1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. The positive integers in the sequence are called parts [1]. To indicate that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n, we consider the notation $\lambda \vdash n$.

In the following, we recall some essential facts about monomial symmetric functions. Proofs and more details can be found in Macdonald's book [2]. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an integer partition with $k \leqslant n$ then, the monomial symmetric function

$$m_\lambda(x_1, x_2, \ldots, x_n) = m_{(\lambda_1, \lambda_2, \ldots, \lambda_k)}(x_1, x_2, \ldots, x_n)$$

is the sum of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ and all distinct monomials obtained from this by a permutation of variables. For instance, with $\lambda = (2,1,1)$ and $n = 4$, we have:

$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3, x_4) = {} & x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1^2 x_2 x_4 + x_1 x_2^2 x_4 + x_1 x_2 x_4^2 \\ & + x_1^2 x_3 x_4 + x_1 x_3^2 x_4 + x_1 x_3 x_4^2 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2. \end{aligned}$$

If every monomial in a symmetric function has total degree k, then we say that this symmetric function is homogeneous of degree k.

The kth complete homogeneous symmetric function h_k is the sum of all monomials of total degree k in these variables, i.e.,

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} m_\lambda(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

and the kth elementary symmetric function is defined by

$$e_k(x_1, x_2, \ldots, x_n) = m_{(1^k)}(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

where $e_0(x_1, x_2, \ldots, x_n) = h_0(x_1, x_2, \ldots, x_n) = 1$. In particular, the case $\lambda = (k)$ provides the kth power sum symmetric function

$$p_k(x_1, x_2, \ldots, x_n) = m_{(k)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^k,$$

with $p_0(x_1, x_2, \ldots, x_n) = n$.

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In this paper, we aim to show that some results from q-analysis and partition theory can easily be derived as specializations of the fundamental relationships between complete and elementary symmetric functions. To do this, we consider the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & n, k \text{ integers, } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise,} \end{cases}$$

as specializations of symmetric functions, namely

$$h_k(1, q, \ldots, q^n) = \begin{bmatrix} n+k \\ k \end{bmatrix} \tag{1}$$

and

$$e_k(1, q, \ldots, q^{n-1}) = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{2}$$

Here, and in the following, we use the customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$. All identities may be understood in the sense of formal power series in q.

The content of this paper is structured as follows. In the next section, we consider the generating function of the complete homogeneous symmetric functions and derive the q-identities obtained by Cauchy and Euler. In Section 3, we note that Rothe's q-binomial theorem is a specialization of the generating function of the elementary symmetric functions. In Section 4, we consider the derivates of the generating functions of the complete and elementary symmetric functions and obtain Uchimura's identity, which provides connections between partitions and divisors. In Section 5, Newton's identities allow for us to obtain a curios q-identity of Euler. Combinatorial interpretations involving well-known functions in partition theory accompany these results in each section.

2. Complete Homogeneous Symmetric Functions

It is well-known that the complete homogeneous symmetric functions are characterized by the following formal power series identity in t:

$$H(t) = \sum_{k=0}^{\infty} h_k(x_1, x_2, \ldots, x_n)\, t^k = \prod_{i=1}^{n} (1 - x_i t)^{-1}. \tag{3}$$

From (3), with x_j replaced by q^{j-1} for each $j \in \{1, 2, \ldots, n\}$, we obtain a well-known identity, which was proved by Cauchy ([3], Theorem 26).

Theorem 1 (Cauchy)**.** *If n is any nonnegative integer and $|q|$ and $|t|$ are both less than* 1*, then*

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} t^k = \frac{1}{(t;q)_{n+1}}.$$

Some combinatorial interpretations of this theorem can easily be derived if we consider the following partition functions.

Definition 1. *Let n, i and j be non-negative integers. We define $p(n; i, j)$ as the number of partitions of n into at most i parts, with each being, at most, j.*

For example, $p(5;3,4) = 4$, and the partitions in question are

$$(4,1),\ (3,2),\ (3,1,1),\ (2,2,1).$$

The generating function of the $p(n;i,j)$ is given by the following Cayley's theorem ([3], Theorem 24) often attributed to Sylvester.

Theorem 2 (Cayley)**.** *Let i and j be positive integers. Then,*

$$\sum_{n=0}^{\infty} p(n;i,j)\, q^n = \begin{bmatrix} i+j \\ i \end{bmatrix}.$$

Definition 2. *Let n and i be a non-negative integers. We define:*

(i) $p_e(n;i)$ *as the number of partitions of n divided into an even number of parts, with most being i parts;*
(ii) $p_o(n;i)$ *as the number of partitions of n divided into an odd number of parts, with most being i parts;*
(iii) $p(n;i)$ *as the number of partitions of n divided into, at most, i parts.*

It is clear that $p(n;i) = p_e(n;i) + p_o(n;i)$. For example, the partitions of 5 into at most 3 parts are:

$$(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1).$$

We see that $p(5;3) = 5$, $p_e(5;3) = 2$ and $p_o(5;3) = 3$. On the other hand, it is well-known that

$$\sum_{n=0}^{\infty} \left(p_e(n;i) \pm p_o(n;i)\right) q^n = \frac{1}{(\pm q;q)_i}.$$

We have the following result.

Corollary 1. *Let n and i be nonnegative integers. Then,*

$$p_e(n;i+1) \pm p_o(n;i+1) = \sum_{j=0}^{n} (\pm 1)^j\, p(n-j;i,j).$$

Proof. The case $t = \pm q$ of Theorem 1 reads as follows

$$\sum_{j=0}^{\infty} \begin{bmatrix} i+j \\ j \end{bmatrix} (\pm q)^j = \frac{1}{(\pm q;q)_{i+1}}.$$

We can write

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (\pm 1)^j p(n;i,j)\, q^{n+j} = \sum_{n=0}^{\infty} \left(p_e(n;i+1) \pm p_o(n;i+1)\right) q^n.$$

The identity follows by comparing coefficients of q^n on both sides of this equation. □

Take into account that $p(n;i,j)$ is symmetric in i and j, i.e., $p(n;i,j) = p(n;j,i)$, the identity given by Corollary 1 can be rewritten as

$$p_e(n;i+1) \pm p_o(n;i+1) = \sum_{j=0}^{n} (\pm 1)^j p(n-j;j,i).$$

Definition 3. *Let n, and i be a non-negative integer. We define:*

(i) $p_{\mathcal{O}}(n;i)$ *as the number of partitions of n into odd parts, each, at most, $2i-1$;*
(ii) $p_{\mathcal{O}}(n)$ *as the number of partitions of n into odd parts.*

For example, the partitions of 5 into odd parts are:

$$(5),\ (3,1,1),\ (1,1,1,1,1).$$

We can see that $p_{\mathcal{O}}(5) = 3$, $p_{\mathcal{O}}(5;1) = 1$ and $p_{\mathcal{O}}(5;2) = 2$. It is well-known that

$$\sum_{n=0}^{\infty} p_{\mathcal{O}}(n;i)\, q^n = \frac{1}{(q;q^2)_i}$$

and

$$\sum_{n=0}^{\infty} p_{\mathcal{O}}(n)\, q^n = \frac{1}{(q;q^2)_\infty}.$$

By replacing q by q^2 and t by q in Theorem 1, we deduce the following partition identity.

Corollary 2. *Let n and j be nonnegative integers. Then,*

$$p_{\mathcal{O}}(n;j+1) = \sum_{i=0}^{\lfloor n/2 \rfloor} p\big(\lfloor n/2 \rfloor - i; 2i, j\big).$$

The limiting case $n \to \infty$ of Theorem 1 is given by the following theorem of Euler ([3]: Theorem 25).

Theorem 3 (Euler). *If $|q| < 1$ and $|t| < 1$, then*

$$\sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} = \frac{1}{(t;q)_\infty}.$$

We consider the following partition functions.

Definition 4. *Let n, i and j be a non-negative integer. We define:*

(i) $p_e(n)$ *as the number of partitions of n into an even number of parts;*
(ii) $p_o(n)$ *as the number of partitions of n into an odd number of parts;*
(iii) $p(n)$ *as the number of partitions of n.*

It is clear that $p(n) = p_e(n) + p_o(n)$. For example, the partitions of 5 are:

$$(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$$

We see that $p_e(5) = 3$, $p_o(5) = 4$ and $p(5) = 7$. In addition, we know that

$$\sum_{n=0}^{\infty} \big(p_e(n) \pm p_o(n)\big)\, q^n = \frac{1}{(\pm q;q)_\infty}.$$

Thus, the case $t = \pm q$ of Theorem 3 allows for us to derive the following partition identities.

Corollary 3. *Let n be a nonnegative integer. Then,*

$$p_e(n) \pm p_o(n) = \sum_{j=0}^{n} (\pm 1)^j\, p(n-j;j).$$

Using Theorem 3, with q replaced by q^2 and t replaced by q, we can derive the limiting case $j \to \infty$ of Corollary 2.

Corollary 4. *Let n be a nonnegative integer. Then,*

$$p_{\mathcal{O}}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} p\big(\lfloor n/2 \rfloor - i; 2i\big).$$

3. Elementary Symmetric Functions

Recall that the elementary symmetric functions are characterized by the following identity of the formal power series in t:

$$E(t) = \sum_{k=0}^{\infty} e_k(x_1, x_2, \ldots, x_n)\, t^k = \prod_{i=1}^{n} (1 + x_i t). \tag{4}$$

The following result is known as Rothe's q-binomial theorem ([3], Theorem 12). This can be obtained by (4), replacing x_j with q^{j-1} for each $j \in \{1, 2, \ldots, n\}$.

Theorem 4 (Rothe's q-binomial theorem)**.** *If n is any nonnegative integer and $|q|$ and $|t|$ are both less than 1, then*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}}\, t^k = (-t; q)_n.$$

In analogy with Definition 2, we consider the following functions, involving partitions into distinct parts.

Definition 5. *Let n, i and j be a non-negative integer. We define:*

(i) $p_{\mathcal{D}e}(n; i)$ *as the number of partitions of n into an even number of distinct parts, with each, at most, i;*

(ii) $p_{\mathcal{D}o}(n; i)$ *as the number of partitions of n into an odd number of distinct parts, with each, at most, i;*

(iii) $p_{\mathcal{D}}(n; i)$ *as the number of partitions of n into distinct parts, each, at most, i.*

It is clear that $p_{\mathcal{D}}(n; i) = p_{\mathcal{D}e}(n; i) + p_{\mathcal{D}o}(n; i)$. For example, the partitions of 9 into distinct parts, with each, at most, 6 are:

$$(6,3),\ (6,2,1),\ (5,4),\ (5,3,1),\ (4,3,2).$$

We see that $p_{\mathcal{D}e}(9; 6) = 2$, $p_{\mathcal{D}o}(9; 6) = 3$ and $p_{\mathcal{D}}(9; 6) = 5$. Moreover, we know that

$$\sum_{n=0}^{\infty} \big(p_{\mathcal{D}e}(n; i) \pm p_{\mathcal{D}o}(n; i)\big) q^n = (\mp q; q)_i.$$

Thus, the case $t = \pm q$ of Theorem 4 allows for us to derive the following partition identities.

Corollary 5. *Let n, j be nonnegative integers. Then*

$$p_{\mathcal{D}e}(n; j) \pm p_{\mathcal{D}o}(n; j) = \sum_{i=0}^{j} (\pm 1)^i p\big(n - i(i+1)/2; i, j - i\big).$$

Proof. The case $t = \pm q$ of Theorem 4 reads as follows

$$\sum_{i=0}^{j} (\pm 1)^i \begin{bmatrix} j \\ i \end{bmatrix} q^{\binom{i+1}{2}} = (\mp q; q)_j.$$

Theorem 2 implies that $\begin{bmatrix} j \\ i \end{bmatrix}$ is the generating function for partitions into at most i parts, with each, at most, $j-i$. Thus, we can write

$$\sum_{i=0}^{j}\sum_{n=0}^{\infty}(\pm 1)^i p(n;i,j-i)q^{n+i(i+1)/2} = \sum_{n=0}^{\infty}\big(p_{\mathcal{D}e}(n;j) \pm p_{\mathcal{D}o}(n;j)\big)q^n.$$

The identity can be derived by comparing coefficients of q^n on both sides of this equation. □

Definition 6. *Let n, and i be a non-negative integer. We define:*

(i) $p_{\mathcal{OD}}(n;i)$ *as the number of partitions of n into distinct odd parts, with each, at most, $2i-1$;*
(ii) $p_{\mathcal{OD}}(n)$ *as the number of partitions of n into distinct odd parts.*

For example, the partitions of 18 into distinct odd parts are:

$$(17,1),\ (15,3),\ (13,5),\ (11,7),\ (9,5,3,1).$$

We see that $p_{\mathcal{OD}}(18) = 5$, $p_{\mathcal{OD}}(18;5) = 1$, $p_{\mathcal{OD}}(18;6) = 2$, $p_{\mathcal{OD}}(18;7) = 3$, $p_{\mathcal{OD}}(18;8) = 4$. It is well-known that

$$\sum_{n=0}^{\infty} p_{\mathcal{OD}}(n;i)q^n = (-q;q^2)_i$$

and

$$\sum_{n=0}^{\infty} p_{\mathcal{OD}}(n)q^n = (-q;q^2)_\infty.$$

By replacing q by q^2 and t with q in Theorem 4, we deduce the following partition identities.

Corollary 6. *Let n and j be nonnegative integers. Then,*

(i) $$p_{\mathcal{OD}}(2n;j) = \sum_{i=0}^{\infty} p\big(n-2i^2;2i,j-2i\big);$$

(ii) $$p_{\mathcal{OD}}(2n+1;j) = \sum_{i=0}^{\infty} p\big(n-2i(i+1);2i+1,j-(2i+1)\big).$$

The limiting case $n \to \infty$ of Theorem 4 offers another theorem of Euler ([3], Theorem 27).

Theorem 5 (Euler)**.** *If $|q| < 1$, then*

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}t^k}{(q;q)_k} = (-t;q)_\infty.$$

We consider the following partition functions.

Definition 7. *Let n be a non-negative integer. We define:*

(i) $p_{\mathcal{D}e}(n)$ *as the number of partitions of n into an even number of distinct parts;*
(ii) $p_{\mathcal{D}o}(n)$ *as the number of partitions of n into an odd number of distinct parts;*
(iii) $p_{\mathcal{D}}(n)$ *as the number of partitions of n into distinct parts.*

It is clear that $p_{\mathcal{D}}(n) = p_{\mathcal{D}e}(n) + p_{\mathcal{D}o}(n)$. For example, the partitions of 9 into distinct parts are:

$$(9),\ (8,1),\ (7,2),\ (6,3),\ (6,2,1),\ (5,4),\ (5,3,1),\ (4,3,2).$$

We see that $p_{\mathcal{D}e}(9) = 4$, $p_{\mathcal{D}o}(9) = 4$ and $p_{\mathcal{D}}(9) = 8$. In addition, we know that

$$\sum_{n=0}^{\infty} \left(p_{\mathcal{D}e}(n) \pm p_{\mathcal{D}o}(n)\right) q^n = (\mp q; q)_\infty.$$

Thus, the case $t = \pm q$ of Theorem 5 allows for us to derive the following partition identities.

Corollary 7. *Let n be a nonnegative integer. Then,*

$$p_{\mathcal{D}e}(n) \pm p_{\mathcal{D}o}(n) = \sum_{j=0}^{n} (\pm 1)^j p\big(n - j(j+1)/2; j\big).$$

Using Theorem 5, with q replaced by q^2 and t replaced by q, we can derive the limiting case $j \to \infty$ of Corollary 6.

Corollary 8. *Let n be a nonnegative integer. Then,*

(i) $$p_{\mathcal{OD}}(2n) = \sum_{i=0}^{\infty} p\big(n - 2i^2; 2i\big);$$

(ii) $$p_{\mathcal{OD}}(2n+1) = \sum_{i=0}^{\infty} p\big(n - 2i(i+1); 2i+1\big).$$

4. Partitions and Divisors

Some interesting connections between partitions and divisors can easily be derived if we consider the derivatives of the generating functions of the complete and elementary symmetric functions.

Theorem 6. *Let n be a non-negative integer. Then,*

(i) $$\sum_{k=1}^{n+1} \frac{q^{k-1}t}{1 - q^{k-1}t} = \frac{1}{(t;q)_{n+1}} \sum_{k=1}^{n+1} (-1)^{k-1} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\binom{k}{2}} k\, t^k;$$

(ii) $$\sum_{k=1}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} k\, t^k = \frac{1}{(t;q)_{n+1}^2} \sum_{k=1}^{n+1} (-1)^{k-1} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\binom{k}{2}} k\, t^k.$$

Proof. (i) We have

$$\frac{d}{dt} \ln\big(E(t)\big) = \sum_{k=1}^{n+1} \frac{d}{dt} \ln(1 + x_k t) = \sum_{k=1}^{n+1} \frac{x_k}{1 + x_k t}$$

On the other hand, we can write

$$\frac{d}{dt} \ln\big(E(t)\big) = \left(\prod_{k=1}^{n+1} \frac{1}{1 + x_k t}\right) \left(\sum_{k=1}^{n+1} k\, e_k(x_1, x_2, \ldots, x_{n+1}) t^{k-1}\right).$$

Thus, we deduce that

$$\sum_{k=1}^{n+1} \frac{x_k\, t}{1 + x_k\, t} = \left(\prod_{k=1}^{n+1} \frac{1}{1 + x_k t}\right) \left(\sum_{k=1}^{n+1} k\, e_k(x_1, x_2, \ldots, x_{n+1}) t^{k}\right).$$

The first identity easily follows by replacing t by $-t$ and x_k by q^{k-1} for each $k \in \{1, 2, \ldots, n+1\}$.

(ii) We can write

$$\sum_{k=1}^{\infty} k\, h_k(x_1, x_2, \ldots, x_{n+1}) t^{k-1} = \frac{d}{dt} \prod_{i=1}^{n+1} (1 - x_i t)^{-1}$$
$$= \left(\prod_{i=1}^{n+1} (1 - x_i t)^{-2} \right) \left(\sum_{k=1}^{n+1} (-1)^k\, k\, e_k(x_1, x_2, \ldots, x_{n+1}) t^{k-1} \right).$$

The proof easily follows by replacing x_k by q^{k-1} for each $k \in \{1, 2, \ldots, n+1\}$. □

The first identity of Theorem 6 is known and can be seen in ([4], Equation (7)). The following identity can be derived as a consequence of Theorem 6.

Corollary 9. *Let n be a non-negative integer. Then,*

$$\sum_{k=1}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} k\, t^k = \frac{1}{(t;q)_{n+1}} \sum_{k=1}^{n+1} \frac{q^{k-1} t}{1 - q^{k-1} t}.$$

We consider the following divisor functions.

Definition 8. *Let n and k be positive integers. We define:*

(i) $\tau(n;k)$ *as the number of divisors of n less than or equal to k;*
(ii) $\tau(n)$ *as the number of divisors of n.*

We known that

$$\sum_{n=0}^{\infty} \tau(n;k)\, q^n = \sum_{n=1}^{k} \frac{q^n}{1 - q^n}$$

and

$$\sum_{n=0}^{\infty} \tau(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}.$$

By replacing t with q in Corollary 9, we easily deduce the following identity involving partitions and divisors.

Corollary 10. *Let n and j be positive integers. Then,*

$$\sum_{i=1}^{n} i\, p(n-i; i, j) = \sum_{i=0}^{n} p(n-i; j+1)\, \tau(i; j+1).$$

The limiting case $n \to \infty$ of Theorem 6 and Corollary 9 reads as follows.

Theorem 7. *For $|q| < 1$, we have*

(i) $$\sum_{k=1}^{\infty} \frac{q^{k-1}\, t}{1 - q^{k-1} t} = \frac{1}{(t;q)_\infty} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k\, t^k}{(q;q)_k} q^{\binom{k}{2}};$$

(ii) $$\sum_{k=1}^{\infty} \frac{k\, t^k}{(q;q)_k} = \frac{1}{(t;q)_\infty^2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k\, t^k}{(q;q)_k} q^{\binom{k}{2}};$$

(iii) $$\sum_{k=1}^{\infty} \frac{k\, t^k}{(q;q)_k} = \frac{1}{(t;q)_\infty} \sum_{k=1}^{\infty} \frac{q^{k-1}\, t}{1 - q^{k-1}\, t}.$$

We note that the case $t = q$ of the first identity of Theorem 7 can be seen in ([4], Theorem 1):

$$\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} = \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k\, q^{\binom{k+1}{2}}}{(q;q)_k}.$$

This identity was stated without proof by Eisenstein ([3], Theorem 39). We have the following combinatorial interpretation of this identity.

Corollary 11. *Let n be a positive integer. Then,*

$$\sum_{k=-\infty}^{\infty} (-1)^k \tau\big(n-k(3k-1)/2\big) = \sum_{k=1}^{\infty} (-1)^{k-1} k\, p\big(n-k(k+1)/2;k\big).$$

The case $t=q$ of the third identity of Theorem 7 is known as Uchimura's theorem ([3], Theorem 38).

Theorem 8 (Uchimura)**.** *For $|q|<1$,*

$$\sum_{n=1}^{\infty} \tau(n)\, q^n = (q;q)_\infty \sum_{n=1}^{\infty} \frac{n\, q^n}{(q;q)_n}.$$

We consider the following counting function.

Definition 9. *Let n be a non-negative integer. We define $s(n)$ as the number of parts in all the partitions of n.*

The partitions of 4 are:

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

We have $s(4)=1+2+2+3+4=12$. It is known that

$$\sum_{n=1}^{\infty} s(n)\, q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$

This generating function allows for us to derive two identities:

$$\tau(n) = \sum_{k=-\infty}^{\infty} (-1)^k\, s\big(n-k(3k-1)/2\big)$$

and

$$s(n) = \sum_{k=1}^{n} \tau(k)\, p(n-k).$$

The case $t=q$ of the third identity of Theorem 7 allows for us to deduce a new decomposition for $s(n)$.

Corollary 12. *Let n be a non-negative integer. Then,*

$$s(n) = \sum_{k=1}^{n} k\, p(n-k;k).$$

5. Newton's Identities

There is a fundamental relationship between the elementary symmetric functions and the complete homogeneous ones:

$$\sum_{j=0}^{k} (-1)^k\, e_k(x_1,x_2,\ldots,x_n)\, h_{n-k}(x_1,x_2,\ldots,x_n) = 0,$$

which is valid for all $k>0$, and any number of variables n. By replacing x_i with q^{i-1}, we derive the following identity.

Theorem 9. *Let n and k be positive integers. Then,*

$$\sum_{j=0}^{k}(-1)^j\, q^{\binom{j}{2}} \begin{bmatrix} n+1 \\ j \end{bmatrix} \begin{bmatrix} n+k-j \\ k-j \end{bmatrix} = 0.$$

The limiting case $n \to \infty$ of this theorem reads as follows

$$\sum_{j=0}^{k}(-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_j (q;q)_{k-j}} = 0.$$

By multiplying this identity by $(q;q)_k$, we obtain the following result which, is the case $t = 1$ of Theorem 4:

$$\sum_{j=0}^{k}(-1)^j\, q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} = 0.$$

The limiting case $k \to \infty$ of this identity is the case $t = 1$ of Theorem 5:

$$\sum_{j=0}^{\infty}(-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_j} = 0.$$

We have the following combinatorial interpretations of the last two identities.

Corollary 13. *Let n and k be positive integers. Then,*

(i) $\displaystyle\sum_{j=0}^{k}(-1)^j\, p\big(n - j(j-1)/2; j, k-j\big) = 0;$

(ii) $\displaystyle\sum_{j=0}^{\infty}(-1)^j\, p\big(n - j(j-1)/2; j\big) = 0.$

The problem of expressing power sum symmetric polynomials in terms of elementary symmetric polynomials and vice versa was solved a long time ago. This was also the case for the problem of expressing power sum symmetric polynomials in terms of complete symmetric polynomials and vice versa. The relations are given as Newton's identities

$$k\, h_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} h_{k-j}(x_1, x_2, \ldots, x_n)\, p_j(x_1, x_2, \ldots, x_n)$$

and

$$k\, e_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k}(-1)^{j-1}\, e_{k-j}(x_1, x_2, \ldots, x_n)\, p_j(x_1, x_2, \ldots, x_n)$$

and are well known. Using these identities, with x_i replaced by q^{i-1}, we can obtain the following identities.

Theorem 10. *Let n and k be positive integers. Then,*

(i) $\displaystyle k \begin{bmatrix} n+k \\ k \end{bmatrix} = \sum_{j=1}^{k} \begin{bmatrix} n+k-j \\ k-j \end{bmatrix} \frac{1-q^{j(n+1)}}{1-q^j};$

(ii) $\displaystyle k\, q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=1}^{k}(-1)^{j-1}\, q^{\binom{k-j}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \frac{1-q^{j(n+1)}}{1-q^j}.$

The limiting case $n \to \infty$ of this theorem reads as follows.

Theorem 11. *Let n and k be positive integers. Then,*

$$(i) \quad \frac{k}{(q;q)_k} = \sum_{j=1}^{k} \frac{1}{(1-q^j)(q;q)_{k-j}};$$

$$(ii) \quad \frac{k\, q^{\binom{k}{2}}}{(q;q)_k} = \sum_{j=1}^{k} (-1)^{j-1} \frac{q^{\binom{k-j}{2}}}{(1-q^j)(q;q)_{k-j}}.$$

We have the following combinatorial interpretations of these identities.

Corollary 14. *Let n and k be positive integers. Then,*

$$(i) \quad k\, p(n;k) = \sum_{i=0}^{n} \sum_{\substack{j=1 \\ j|i}}^{k} p(n-i;k-j);$$

$$(ii) \quad k\, p(n;k) = \sum_{i=0}^{n} \sum_{\substack{j=1 \\ j|i}}^{k} (-1)^{j-1} p\left(n-i+\binom{k}{2}-\binom{k-j}{2};k-j\right).$$

By multiplying both sides of the firs identity of Theorem 11 by $(q;q)_k$, we obtain a curious q-identity of Euler ([3], Theorem 17).

Theorem 12 (Euler)**.** *Let k be a positive integer. Then,*

$$k = \sum_{j=1}^{k} (q;q)_{j-1} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Recently, Merca [4] proved that the complete, elementary and power sum symmetric functions are related by

$$p_k(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{k} (-1)^{j-1} j\, e_j(x_1, x_2, \ldots, x_n)\, h_{k-j}(x_1, x_2, \ldots, x_n).$$

Using this relation, with x_i replaced by q^{i-1}, we can derive the following identity.

Theorem 13. *Let n and k be positive integers. Then,*

$$\sum_{j=1}^{k} (-1)^{j-1}\, j\, q^{\binom{j}{2}} \begin{bmatrix} n+1 \\ j \end{bmatrix} \begin{bmatrix} n+k-j \\ k-j \end{bmatrix} = \frac{1-q^{k(n+1)}}{1-q^k}.$$

The limiting case $n \to \infty$ of this theorem reads as follows:

$$\sum_{j=1}^{k} (-1)^{j-1}\, j\, q^{\binom{j}{2}} \frac{1}{(q;q)_j (q;q)_{k-j}} = \frac{1}{1-q^k}.$$

By multiplying both sides of this identity by $(q;q)_k$, we can obtain the following result.

Theorem 14. *Let k be a positive integer. Then,*

$$\sum_{j=1}^{k} (-1)^{j-1}\, j\, q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} = (q;q)_{k-1}.$$

We note the following combinatorial interpretation of this theorem.

Corollary 15. *Let n and k be positive integers. Then,*

$$p_{\mathcal{D}e}(n;k-1) - p_{\mathcal{D}o}(n;k-1) = \sum_{j=1}^{k}(-1)^{j-1} j\, p\big(n-j(j-1)/2; j, k-j\big).$$

The case $t=-q$ of Theorem 5 is given by

$$(q;q)_\infty = \sum_{j=0}^{\infty}(-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_j}.$$

The limiting case $k\to\infty$ of Theorem 14 provides another representation of Euler's function $(q;q)_\infty$.

Theorem 15. *For $|q|<1$,*

$$(q;q)_\infty = \sum_{j=1}^{\infty}(-1)^{j-1} \frac{j\, q^{\binom{j}{2}}}{(q;q)_j}.$$

As a consequences of this result, we can derive the following recurrence relation for $p(n;k)$.

Corollary 16. *Let n be a positive integer. Then,*

$$\sum_{j=1}^{\infty}(-1)^{j-1} j\, p\big(n-j(j-1)/2; j\big) = \begin{cases} (-1)^n, & \text{if } n = m(3m-1)/2,\ m\in\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

6. Concluding Remarks

The partition identities obtained in this paper are specializations of the fundamental relations between complete and elementary symmetric functions. There are other relations between complete and elementary symmetric functions, which can be used to derive partitions identities. For example, the following relations between complete and elementary symmetric functions

$$\sum_{k=0}^{\lfloor n/2\rfloor} h_k(x_1^2, x_2^2, \ldots, x_m^2)\, e_{n-2k}(x_1, x_2, \ldots, x_m) = h_n(x_1, x_2, \ldots, x_m)$$

and

$$\sum_{k=0}^{\lfloor n/4\rfloor} h_k(x_1^4, \ldots, x_m^4)\, e_{n-4k}(x_1, \ldots, x_m) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k\, h_k(x_1^2, \ldots, x_m^2)\, h_{n-2k}(x_1, \ldots, x_m)$$

was introduced by Merca [5] to obtain generalizations of two identities of Guo and Yang for the q-binomial coefficients. The partition identities that can be derived by considering these relations can be seen in [5,6].

Motivated by the relations obtained in [5], Merca introduced an infinite family of relations between complete and elementary symmetric functions ([7], Theorem 1.1). It would be interesting to see what partition identities can be obtained as combinatorial interpretations of ([7], Theorem 1.1).

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