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Optimal Sparse Control Formulation for Reconstruction of Noise-Affected Images

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Abstract: We discuss the optimal control formulation for enhancement and denoising of satellite multiband images and propose to take it in the form of an L^1 control problem for a quasi-linear parabolic equation with a nonlocal $p[u]$ Laplacian and with a cost functional of a tracking type. The main characteristic features of the considered parabolic problem is that the variable exponent $p(t,x)$ and the diffusion anisotropic tensor $D(t,x)$ are not predefined well a priori; instead, these characteristics nonlocally depend on the form of the solution of this problem (i.e., $p_u = p(t,x,u)$ and $D_u = D(t,x,u)$). We prove the existence of optimal pairs with sparse L^1 controls used for the indirect approach and a special family of approximation problems.

Keywords: parabolic equation; optimal control; variable order of nonlinearity; noncoercive problem; existence issues

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1. Introduction and Some Preliminaries

The main goal of any image denoising problem is to restore the noise-free gray-scale image $u : \Omega \to \mathbb{R}$ from the observed one $f : \Omega \to \mathbb{R}$. In this paper, we start from the assumption that the observed image can be represented as

$$f = u + v + n,$$

where n is the white Gaussian noise following the Gaussian distribution $\mathcal{N}(0, \sigma^2)$ and v stands for the noise with a probably strong impulsive nature, which the Gaussian model fails to describe. We assume that both noises occur simultaneously and independently in the entire domain.

To eliminate both the Gaussian noise n and impulse noise v, we propose the following optimal control problem:

$$(\mathcal{R}) \quad \text{Minimize } J(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{1}{2}\int_\Omega |u(T) - f_0|^2\,dx \tag{1}$$

subject to the constraints

$$\begin{gathered}\frac{\partial u}{\partial t} = \operatorname{div}\Big(|D_u(t,x)\nabla u|^{p_u(t,x)-2} D_u(t,x)\nabla u\Big) + \kappa(f - u - v)\\ \text{in } Q_T = (0,T)\times\Omega,\end{gathered} \tag{2}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{3}$$

$$u(0,\cdot) = f_0(\cdot) \quad \text{in } \Omega, \tag{4}$$

$$v_a(x) \le v(t,x) \le v_b(x), \quad \text{a.e. in } Q_T. \tag{5}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded, open, simply connected set, its boundary $\partial\Omega$ is assumed to be sufficiently smooth, $T > 0$ is a positive value, $\kappa \in \mathbb{R}$ is a given positive parameter, $f \in L^2(\Omega)$ is the original noise-corrupted image, $f_0 \in L^2(\Omega)$ is the pre-denoised image when applying a median filter to f, $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leq v_b(x)$ a.e. in Ω are given distributions, and

$$\|v\|^2_{L^2(0,T;L^1(\Omega))} = \int_0^T \left(\int_\Omega |v|\, dx \right)^2 dx \tag{6}$$

is the so-called directional sparsity term which, in fact, measures the L^1-norm in the space of the L^2-norm in time. Additionally, $D_u = D(t,x,u)$ is the matrix of anisotropy, and as for the variable exponent $p_u : Q_T \to \mathbb{R}$, we define it with the rule

$$p_u(t,x) := 1 + g\left(\frac{1}{h} \int_{t-h}^{t} |(\nabla G_\sigma * \widetilde{u}(\tau,\cdot))(x)|\, d\tau \right), \quad \forall\, (t,x) \in Q_T, \tag{7}$$

where

$$g(s) = \delta + \frac{a^2(1-\delta)}{a^2+s^2}, \quad \forall\, s \in [0,+\infty), \tag{8}$$

$$G_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} \exp\left(-\frac{|x|^2}{2\sigma^2} \right), \quad \sigma > 0, \tag{9}$$

$$(G_\sigma * \widetilde{u}(t,\cdot))(x) = \int_{\mathbb{R}^2} G_\sigma(x-y)\widetilde{u}(t,y)\, dy, \tag{10}$$

Here, $\widetilde{u}$ stands for the zero extension of u to the entire space $\mathbb{R} \times \mathbb{R}^2$, and $h > 0$ and $0 < \delta \ll 1$ are given small positive values. As for the parameters $\lambda > 0$ and $a > 0$, they act as regularization and smoothing parameters.

It is clear now that, for each function $u \in L^2(0,T;W^{1,1}(\Omega))$, the inclusion $p_u(t,x) \in [p^-, p^+] \subset (1,2]$ holds almost everywhere in Q_T with $p^- = 1+\delta$ and $p^+ = 2$.

The study of optimal control problems for PDEs with variable nonlinearity is motivated by various applications in the image enhancement, where some special cases of Equations (2)–(5) appear as the natural generalization of the classical Perona–Malik model [1–4]. We also refer to [5], where the authors dealt with a special case of the model in Equations (2)–(5) and show the given class of optimal control problems is well posed.

The main benefit of the proposed model in Equations (2)–(5) is the manner in which this model accommodates the local image information. It is easy to see that if the gradient of the noisy image f is sufficiently large (i.e., likely edges) at some places, then only total variation (or shortly TV-based) diffusion will be used there. However, if at some points the gradient is sufficiently close to zero (i.e., it is a homogeneous region), then the model becomes isotropic. At the rest of the locations, the diffusion is somewhere between Gaussian and TV-based. However, as immediately follows from Equation (7) and definition of the matrix D_u, the type of anisotropy is not completely predefined by the structure of the original noisy image f. Moreover, the image u after denoising may have other shapes of homogeneous regions and other structures with other locations for the edges.

In spite of the fact that there are many other different variants for the choice of the diffusivity term in Equation (2) using, for instance, the so-called directional total variation [6] and flexible space-variant anisotropic regularization [7], to the best of our knowledge, the effective choice of this operator for general image denoising problem with different noise distributions remains an open problem.

In recent years, many different techniques have been proposed for the reconstruction of noise-affected digital images. In particular, the following nonlinear hybrid diffusion model, which is a symbiosis of the mean curvature diffusion with the Gaussian heat diffusion, has been proposed for image denoising (see [8]):

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{(|\nabla u|^2+1)^{(2-p(|\nabla u|^2))/2}}\right), \quad \text{in } (0,T)\times\Omega, \tag{11}$$

$$u(0,x) = f, \quad \text{in } \Omega, \tag{12}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{13}$$

where

$$p(|\nabla u|^2) = 1 + \frac{1}{1+k|\nabla u|^2}, \tag{14}$$

Here, f is an input image, $k > 0$ and $T > 0$ are fixed constants, Ω is a bounded open domain of $\mathbb{R}^2$ with a sufficiently smooth boundary, and ν is the unit outwardly normal to the boundary $\partial\Omega$.

The important characteristic of this model is the fact that it has a hybrid diffusion type which combines the mean curvature diffusion with the heat diffusion such that inside those regions, where the gradient of u is small enough, the new model acts like a heat equation and results in isotropic smoothing, whereas near the region's contours where the magnitude of the gradient is large, this model acts like a mean curvature equation. From this point of view, the model in Equations (11)–(13) can be interpreted as some generalization of the well-known ones (in particular, the Perona–Malik model [9] or the models with the $p(x)$-Laplacian operator that was proposed in [4]). However, in general, it would be erroneously to assert that all the above-mentioned models can be obtained as a particular case of Equations (11)–(13).

It is worth emphasizing that because of the variable character of exponent p in Equation (2), we have a gap between the coercivity and monotonicity conditions. In light of this, the problem in Equations (1)–(4) can be specified as an optimal control problem for the quasi-linear parabolic equations with variable growth conditions, and it can be interpreted as a generalization of the parabolic version of the $p(t,x)$-Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p(t,x)-2}\nabla u\right) \tag{15}$$

with a variable exponent that depends only on t and x. We can indicate extensive research which is devoted to Equation (15). A rather complete insight to the theory of parabolic $p(t,x)$-Laplacian equations can be found in [10–15].

However, to the best of our knowledge, the above-mentioned results for the solvability issues for evolutional equations of the type in Equation (2) mainly concern the parabolic IBVPs with exponents depending on (t,x) only, whereas hardly any attention has been paid to the IBVP of the form in Equation (2) with D_u and the exponent p_u given by the rule in Equation (7). Moreover, in contrast to most of the existing results, in this paper, we do not predefine p_u and D_u a priori. Instead, we associate these characteristics with a particular solution for the IBVPs (Equations (2)–(5)). Therefore, the unknown solution u can affect the rate of nonlinearity of p and the tensor D. It is also worth mentioning that in contrast to most existing publications (see, for instance, [10,16]), we do not assume that the dependency of p_u and D_u on u is local. We show that all weak solutions to this problem live in the corresponding 'personal' functional spaces, and, in light of the special assumptions for the structure of D_u and p_u, the problem in Equations (2)–(5) can have the weak solutions that do not possess the standard variational properties of solutions to the parabolic equations. In particular, it is unknown whether a weak solution to the above problem satisfies the standard energy equality and is unique.

In spite of the fact that a number of different regularizations have been suggested in the literature for optimal control problems related to the degenerate elliptic equations (see, for instance, [17–19]), the question about solvability of the proposed optimal control problem is still open, to the best of our knowledge.

In light of this, our primary goal is to study the solvability issues for the OCP (Equations (1)–(5)). In particular, a couple of characteristic features of the proposed problem can be emphasized here. The first one is that the tensor of anisotropy D_u and the exponent p_u depend not only on (t, x) but also on $u(t, x)$. The second feature is that the optimal control problem is formulated with $L^1(\Omega; L^2(0, T))$ as the control cost (together with additional pointwise control constraints). As a result, the optimal control may have directional sparsity (i.e., its support is constant in time, whereas the control v can be identically zero on some parts of the domain Ω).

This paper is structured as follows. The main assumptions for the structure of the anisotropic diffusion tensor $D_u(t, x)$ and variable exponent $p_u(t, x)$ and some preliminaries are given in Section 2. We also discuss in this section the basic auxiliary results concerning the Sobolev–Orlicz spaces with a variable exponent. In Section 3, we focus mainly on the solvability issues for the IBVPs (Equations (2)–(5)). To this end, we follow the indirect approach using a special technique of passing to the limit in the sequences of variational problems. Precise statement of the optimal control problems for a quasi-linear elliptic equation with the sparse control is discussed in Section 4. We also discuss in this section the main topological properties of feasible solutions to the given OCP, and as a consequence, we derive the conditions where the set of optimal solutions is nonempty. As for the optimality conditions, their substantiation, and the results of numerical simulations, these issues are the subject of a forthcoming paper. With that in mind, we will realize the principle of variational convergence of constrained minimization problems and utilize some key ideas from [20–23].

2. Main Assumptions and Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, simply connected set and its boundary $\partial\Omega$ be sufficiently smooth. For simplicity, we assume that the unit's outward normal $\nu = \nu(x)$ is well defined for a.e., $x \in \partial\Omega$. Let $T > 0$ be a given value. We also set $Q_T = (0, T) \times \Omega$. For any measurable subset $D \subset \Omega$, we denote by $|D|$ its two-dimensional Lebesgue measure $\mathcal{L}^2(D)$. Let $\overline{D}$ be its closure and ∂D stand for the boundary of D. We also make use of the following notation: $\operatorname{diam} \Omega = \sup_{x,y \in \Omega} |x - y|$.

For two vectors $\xi \in \mathbb{R}^2$ and $\eta \in \mathbb{R}^2$, the notation $(\xi, \eta) = \xi^t \eta$ stands for the standard vector inner product in $\mathbb{R}^2$, where ^{t} stands for the transpose operator. As for the norm $|\xi|$, we take this as the Euclidean norm given by the rule $|\xi| = \sqrt{(\xi, \xi)}$.

2.1. Functional Spaces

Let Y be a real Banach space endowed with the norm $\|\cdot\|_Y$, and let Y' be its dual. With $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, we denote the weak and weak * convergence in the spaces Y and Y', respectively. Let $\langle \cdot, \cdot \rangle_{Y';Y}$ be the duality form on $Y' \times Y$.

For a given exponent $1 \leq p \leq +\infty$, the Lebesgue space $L^p(\Omega; \mathbb{R}^2)$ is defined by the standard rule

$$L^p(\Omega; \mathbb{R}^2) = \left\{ g : \Omega \to \mathbb{R}^2 \ : \ \|g\|_{L^p(\Omega;\mathbb{R}^2)} < +\infty \right\}.$$

Here, $\|g\|_{L^p(\Omega;\mathbb{R}^2)} = \left(\int_\Omega |g(x)|^p \, dx \right)^{1/p}$ for $1 \leq p < +\infty$. The inner product of two functions g and f in $L^p(\Omega; \mathbb{R}^2)$ with $p \in [1, \infty)$ is given by

$$(g, f)_{L^p(\Omega;\mathbb{R}^2)} = \int_\Omega (g(x), f(x)) \, dx = \int_\Omega \sum_{k=1}^{2} g_k(x) f_k(x) \, dx.$$

Let $C_c^\infty(\mathbb{R}^2)$ be the locally convex space of all infinitely differentiable functions with compact support in $\mathbb{R}^2$. We also define the Banach space $W^{1,p^-}(\Omega)$ with $p^- > 1$ as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|y\|_{W^{1,p^-}(\Omega)} = \left(\int_\Omega \left(|y|^{p^-} + |\nabla y|^{p^-}\right) dx\right)^{1/p^-}.$$

We denote by $\left(W^{1,p^-}(\Omega)\right)'$ the dual space of $W^{1,p^-}(\Omega)$. Let us remark that in this case, the embedding $L^2(\Omega) \hookrightarrow \left(W^{1,p^-}(\Omega)\right)'$ is continuous.

Given a real separable Banach space Y, we denote by $C([0,T];Y)$ the space of all continuous functions from $[0,T]$ into Y.

We recall that a function $u : [0,T] \to Y$ is said to be Lebesgue-measurable if there exists a sequence of step functions $\{u_k\}_{k\in\mathbb{N}}$ (i.e., $u_k = \sum_{j=1}^{n_k} a_j^k \chi_{A_j^k}$ for a finite number n_k of Borel subsets $A_j^k \subset [0,T]$ and with $a_j^k \in X$) such that this sequence converges to u almost everywhere with respect to the Lebesgue measure in $[0,T]$. Then, $L^p(0,T;Y)$, for $1 \le p < \infty$, is the space of all measurable functions $u : [0,T] \to Y$ such that

$$\|u\|_{L^p(0,T;Y)} = \left(\int_0^T \|u(t)\|_Y^p \, dt\right)^{\frac{1}{p}} < \infty.$$

As for $L^\infty(0,T;X)$, it is the space of measurable functions such that

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X < \infty.$$

This choice makes $L^p(0,T;X)$ a Banach space and guarantees that its dual can be identified with $L^{p'}(0,T;X')$, where $p' = p/(p-1)$ and X' is the dual space to X. In particular, for functions $f \in L^2(0,T;L^1(\Omega))$, the continuous Minkowski inequality yields $f \in L^1(0,T;L^2(\Omega))$ and moreover

$$\|f\|_{L^2(0,T;L^1(\Omega))} := \left(\int_0^T \left(\int_\Omega |f| \, dx\right)^2 dx\right)^{1/2} \le \int_\Omega \left(\int_0^T |f|^2 \, dt\right)^{1/2} dx =: \|f\|_{L^1(0,T;L^2(\Omega))}.$$

Hence, we have $L^2(0,T;L^1(\Omega)) \hookrightarrow L^1(0,T;L^2(\Omega))$. The full presentation of this topic can be found in [24].

2.2. Variable Exponent

Let $v \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given function. Let $p_v : Q_T \to \mathbb{R}$ be the exponent that can be associated with $v : Q_T \mapsto \mathbb{R}$ using the rule in Equation (7).

Since $G_\sigma \in C^\infty(\mathbb{R}^2)$, it follows from Equation (7) and from the absolute continuity of the Lebesgue integral that $1 < p_v(t,x) \le 2$ in Q_T and $p_v \in C^1([0,T];C^\infty(\mathbb{R}^2))$, even if v is just an absolutely integrable function in Q_T. Then, we observe that for each $t \in [0,T]$, $p_v(t,x) \approx 2$ if $v(t,x)$ contains homogeneous features or is smooth enough, and $p_v(t,x) \approx 1$ in those places of Ω where some discontinuities are present in $v(t,x)$. Thus, the sparse texture of the function v can be characterized by the exponent $p_v(t,x)$.

For our further analysis, we make use of the following result. (For comparison, we refer to Lemma 2.1 in [25]).

Lemma 1. *Let $\{v_k\}_{k\in\mathbb{N}} \subset L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given sequence of measurable functions. We assume that all elements of this sequence are extended by zero outside of Q_T and*

$$\begin{gathered}\sup_{k\in\mathbb{N}} \|v_k\|_{L^\infty(0,T;L^2(\Omega))} < +\infty, \\ v_k \to v \ \text{ weakly in } L^1(0,T;L^1(\Omega)) \text{ for some } \ v \in L^1(0,T;L^1(\Omega)).\end{gathered} \tag{16}$$

Let

$$\left\{ p_{v_k} = 1 + g\left(\frac{1}{h} \int_{t-h}^{t} |(\nabla G_\sigma * v_k(\tau,\cdot))|\, d\tau \right) \right\}_{k\in\mathbb{N}}$$

be the corresponding sequence of exponents. Then, there exists a constant $C > 0$ depending on G, Ω, and $\sup_{k\in\mathbb{N}} \|v_k\|_{L^1(0,T;L^1(\Omega))}$ such that

$$p^- := 1+\delta \le p_{v_k}(t,x) \le p^+ := 2, \quad \forall\,(t,x) \in Q_T,\ \forall\, k \in \mathbb{N}, \tag{17}$$

$$\left\{ p_{v_k}(\cdot) \right\} \subset \mathfrak{S} = \left\{ q \in C^{0,1}(Q_T) \;\middle|\; \begin{array}{c} |q(t,x) - q(s,y)| \le C(|x-y| + |t-s|), \\ \forall\,(t,x),(s,y) \in \overline{Q_T}, \\ 1 < p^- \le q(\cdot,\cdot) \le p^+ \text{ in } \overline{Q_T}. \end{array} \right\} \tag{18}$$

$$\begin{gathered} p_{v_k} \to p_v = 1 + g\left(\frac{1}{h} \int_{t-h}^{t} |(\nabla G_\sigma * v(\tau,\cdot))(\cdot)|\, d\tau \right) \\ \text{uniformly in } \overline{Q_T} \text{ as } k \to \infty. \end{gathered} \tag{19}$$

Proof. Since the sequence $\{v_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;L^1(\Omega))$, and the Gaussian kernel G_σ is smooth, it follows that

$$\begin{aligned} \frac{1}{h} \int_{t-h}^{t} \left| (\nabla G_\sigma * \widetilde{v}_k(\tau,\cdot))(x) \right| d\tau &\le h^{-1} \int_{t-h}^{t} \left(\int_\Omega |\nabla G_\sigma(x-y)| |\widetilde{v}_k(\tau,y)|\, dy \right) d\tau \\ &\le \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} h^{-1} \|v_k\|_{L^1(0,T;L^1(\Omega))}, \\ 2 \ge p_{v_k}(t,x) = 1 + \delta + {}& \frac{a^2(1-\delta)h^2}{a^2h^2 + \left(\int_{t-h}^{t} |(\nabla G_\sigma * \widetilde{v}_k(\tau,\cdot))(x)|\, d\tau \right)^2} \\ &\ge 1 + \delta + \frac{a^2h^2(1-\delta)}{a^2h^2 + \|v_k\|^2_{L^1(0,T;L^1(\Omega))} \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}}, \\ &\qquad \forall\,(t,x) \in Q_T, \end{aligned}$$

where

$$\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} = \max_{\substack{z=x-y \\ x\in\overline{\Omega}, y\in\overline{\Omega}}} \Big[|\nabla G_\sigma(z)| + |G_\sigma(z)| \Big] = \frac{e^{-1}}{\left(\sqrt{2\pi}\sigma\right)^2} \left[1 + \frac{1}{\sigma^2} \operatorname{diam} \Omega \right]. \tag{20}$$

Then, the L^1-boundedness of $\{v_k\}_{k\in\mathbb{N}}$ guarantees the existence of a value $\widehat{\delta} \in (0,1)$ such that $\widehat{\delta} > \delta$ and $p_{v_k}(t,x) \ge 1 + \widehat{\delta}$. Hence, the estimate in Equation (17) holds true for all $k \in \mathbb{N}$.

Moreover, as immediately follows from the relations

$$
\begin{aligned}
&\left|p_{v_k}(t,x) - p_{v_k}(t,y)\right| \\
&\leq \frac{a^2h^2(1-\delta)}{a^4h^4}\left|\left(\int_{t-h}^{t}|(\nabla G_\sigma * v_k(\tau,\cdot))(x)|\,d\tau\right)^2 - \left(\int_{t-h}^{t}|(\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau\right)^2\right| \\
&\leq \frac{1-\delta}{a^2h^2}\int_0^T (|(\nabla G_\sigma * v_k(\tau,\cdot))(x)| + |(\nabla G_\sigma * v_k(\tau,\cdot))(y)|)\,d\tau \\
&\quad \times \int_0^T |(\nabla G_\sigma * v_k(\tau,\cdot))(x) - (\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau \\
&\leq \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\|v_k\|_{L^1(0,T;L^1(\Omega))}}{a^2h^2} \\
&\quad \times \int_0^T\int_\Omega |v(\tau,z)|\,dz\,d\tau \max_{z\in\Omega}|\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)| \\
&= \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1^2}{a^2h^2}\max_{z\in\Omega}|\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)|, \quad \forall\, x,y\in\overline{\Omega}
\end{aligned} \tag{21}
$$

with $\gamma_1^2 = \left(\sup_{k\in\mathbb{N}}\|v_k\|_{L^1(0,T;L^1(\Omega))}\right)^2$, and from the smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a positive constant $C_G > 0$ such that $C_G > 0$ does not depend on k and, for each $t \in [0,T]$, the following estimate

$$
\left|p_{v_k}(t,x) - p_{v_k}(t,y)\right| \leq \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1^2 C_G}{a^2h^2}|x-y|,\ \forall\, x,y\in\overline{\Omega}
$$

holds true. Arguing in a similar manner, we see that

$$
\begin{aligned}
&\left|p_{v_k}(s,y) - p_{v_k}(t,y)\right| \\
&\leq \frac{1-\delta}{a^2h^2}\left|\left(\int_{t-h}^{t}|(\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau\right)^2 - \left(\int_{s-h}^{s}|(\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau\right)^2\right| \\
&\leq \frac{2(1-\delta)\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1}{a^2h^2} \\
&\quad \times \left|\int_s^t |(\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau - \int_{s-h}^{t-h}|(\nabla G_\sigma * v_k(\tau,\cdot))(y)|\,d\tau\right| \\
&\leq \frac{4(1-\delta)\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\gamma_1\gamma_2}{a^2h^2}|s-t|, \quad \forall\, t,s\in[0,T]
\end{aligned} \tag{22}
$$

with $\gamma_2 = \sup_{k\in\mathbb{N}}\|v_k\|_{L^\infty(0,T;L^2(\Omega))}$.

Taking into account the estimates in Equation (21)–(22), and by setting

$$
C := \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1}{a^2h^2}\left(\gamma_1 C_G + 2\gamma_2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\right), \tag{23}
$$

we obtain

$$
\begin{aligned}
\left|p_{v_k}(s,x) - p_{v_k}(t,y)\right| &\leq \left|p_{v_k}(t,x) - p_{v_k}(t,y)\right| + \left|p_{v_k}(t,y) - p_{v_k}(s,y)\right| \\
&\leq C[|y-x| + |t-s|], \\
&\forall\,(t,x),(s,y)\in\overline{Q_T} := [0,T]\times\overline{\Omega}.
\end{aligned} \tag{24}
$$

Thus, we see that $\{p_{v_k}\} \subset \mathfrak{S}$. Since each element of the sequence $\{p_{v_k}\}_{k\in\mathbb{N}}$ has the same modulus of continuity, and $\max_{(t,x)\in\overline{Q_T}}|p_{v_k}(t,x)| \leq p^+$, it follows that this sequence is equicontinuous and uniformly bounded. Hence, under the Arzelà–Ascoli theorem, the sequence $\{p_{v_k}\}_{k\in\mathbb{N}}$ is relatively compact with respect to the norm topology of $C(\overline{Q_T})$. Then, in light of the estimate in Equation (24), the fact that the set $\mathfrak{S}$ is closed with respect to the uniform convergence, and

$$\frac{1}{h}\int_{t-h}^{t}|(\nabla G_\sigma * v_k(\tau,\cdot))(x)|\,d\tau \to \frac{1}{h}\int_{t-h}^{t}|(\nabla G_\sigma * v(\tau,\cdot))(x)|\,d\tau$$
$$\text{as } k\to\infty,\ \forall\,(t,x)\in Q_T$$

by definition of the weak convergence in $L^1(0,T;L^1(\Omega))$, we finally deduce that $p_{v_k}\to p_v$ uniformly in $\overline{Q_T}$ as $k\to\infty$, where

$$p_v(t,x) = 1+g\left(\frac{1}{h}\int_{t-h}^{t}|(\nabla G_\sigma * v(\tau,\cdot))(\cdot)|\,d\tau\right)$$

in Q_T. □

2.3. Anisotropic Diffusion Tensor

Let $\mathbb{S}^2$ be the set of symmetric quadratic matrices $B=[b_{ij}]_{i,j=1}^2$, ($b_{ij}=b_{ji}\in\mathbb{R}$). We endow $\mathbb{S}^2$ with the Euclidian scalar product $B\cdot A=\mathrm{tr}(B\,A):=\sum_{i,j=1}^2 a_{ij}b_{ij}$ and with the corresponding Euclidian norm $\|B\|_{\mathbb{S}^2}=(B\cdot B)^{1/2}=\sqrt{\mathrm{tr}(B^2)}$. We also introduce the spectral norm $\|B\|_2:=\sup\left\{|B\zeta|\ :\ \xi\in\mathbb{R}^2 \text{ with } |\zeta|=1\right\}$ of the matrices $B\in\mathbb{S}^2$. Note that the following relation $\|B\|_2\le\|B\|_{\mathbb{S}^2}\le\sqrt{2}\|B\|_2$ holds true for all $B\in\mathbb{S}^2$.

Let $v\in L^1(0,T;L^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ be a given function. Wee suppose that v is zero-extended outside of Q_T. With $v_\sigma(t,x)$, we denote its convolution with a Gaussian kernel $G_\sigma(x)$ (see Equations (9)–(10)).

By analogy with [26,27], we associate with the function $v:Q_T\mapsto\mathbb{R}$ the structure tensor $J_\rho(v_\sigma)$, using for that the following representation:

$$J_\rho(v_\sigma) := \frac{1}{h}\int_{t-h}^{t} G_\rho * (\nabla v_\sigma\otimes\nabla v_\sigma)\,d\tau = \frac{1}{h}\int_{t-h}^{t} G_\rho * \left(\nabla v_\sigma(\nabla v_\sigma)^t\right)d\tau, \tag{25}$$

where G_ρ is defined in Equation (9) and

$$\nabla v_\sigma(t,x) = (\nabla G_\sigma * \widetilde{v}(t,\cdot))(x).$$

It is easy to check that $J_\rho(v_\sigma) = \begin{bmatrix} j_{11} & j_{12}\\ j_{12} & j_{22}\end{bmatrix}$ is the positively semi-definite matrix. Moreover, this matrix is uniformly bounded in Ω. To check, it is enough to notice that

$$\begin{aligned}\zeta^t J_\rho(v_\sigma)\zeta &\le 2\frac{1}{h}\int_{t-h}^{t}\int_\Omega G_\rho(x-z)|\nabla v_\sigma(s,\cdot)|^2|\zeta|^2\,dzds\\ &\le \frac{2\mu e^{-1}h^{-1}}{\left(\sqrt{2\pi}\rho\right)^2}|\Omega|\int_{t-h}^{t}\|v(s,\cdot)\|^2_{L^1(\Omega)}\,ds|\zeta|^2,\\ &\le \frac{2e^{-1}\mu}{\left(\sqrt{2\pi}\rho\right)^2}\|v\|^2_{L^\infty(0,T;L^1(\Omega))}|\Omega||\zeta|^2,\quad\forall\,(t,x)\in Q_T,\end{aligned} \tag{26}$$

$$\zeta^t J_\rho(u_\sigma)\zeta = \frac{1}{h}\int_{t-h}^{t}\int_\Omega G_\rho(x-z)(\nabla v_\sigma(s,z),\xi)^2_{\mathbb{R}^2}\,dzds\ge 0,\quad\forall\,(t,x)\in Q_T, \tag{27}$$

for any $\zeta\in\mathbb{R}^2$, where $\mu=\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}$.

Having this in mind, we define the following diffusion tensor $D_v(t,x)$:

$$D_v := \gamma I + J_\rho(v_\sigma), \tag{28}$$

where $0 < \gamma \ll 1$ is a small positive value and $I \in (\mathbb{R}^2, \mathbb{R}^2)$ stands for the unit matrix. In fact, D_v can be interpreted as some relaxation of the anisotropic tensor J_ρ (we refer to [1,2] for comparison).

Then, it easily follows from Equations (26), (27), and (28) that for any distribution $v \in C([0,T]; L^1(\Omega))$, the estimate

$$d_1^2|\zeta|^2 \leq \zeta^t [D_v(t,x)]^2 \zeta \leq d_2^2 |\zeta|^2, \quad \forall \zeta \in \mathbb{R}^2, \quad \forall (t,x) \in Q_T. \tag{29}$$

holds true with

$$d_1 = \gamma, \quad d_2 = d_1 + \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^2} \|v\|^2_{L^\infty(0,T;L^1(\Omega))} |\Omega| \|G_\sigma\|^2_{C^1(\overline{\Omega - \Omega})}.$$

For simplicity, we suppose that $d_2 \geq 1$.

Following in many aspects the proof of Lemma 1, it is easy to establish the following result:

Lemma 2. *Let $\{u_k\}_{k\in\mathbb{N}}, u \subset L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be measurable functions with the properties in Equation (16). We assume that each of these functions is extended by zero outside of Q_T. Let $\{D_{u_k}(t,x)\}_{k\in\mathbb{N}}$ be a collection of the associated diffusion tensors. Then, we have*

$$d_1^2|\zeta|^2 \leq \zeta^t \left[D_{u_k}(t,y)\right]^2 \zeta \leq d_2^2 |\zeta|^2, \quad \forall (t,y) \in Q_T, \forall \zeta \in \mathbb{R}^2, \quad \forall k \in \mathbb{N}, \tag{30}$$

$$D_{u_k}(t,y) \to D_u(t,y) \text{ uniformly in } \overline{Q_T} \text{ as } k \to \infty, \tag{31}$$

$$\left\{D_{u_k}\right\} \subset \mathfrak{D}, \tag{32}$$

where

$$d_1 = \gamma, \quad , d_2 = d_1 + \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^2} \|G_\sigma\|^2_{C^1(\overline{\Omega - \Omega})} \sup_{k \in \not\in} \|u_k\|^2_{L^\infty(0,T;L^1(\Omega))} |\Omega|,$$

$$\mathfrak{D} = \left\{ B \in C^{0,1}(Q_T; \mathbb{R}^{2\times 2}) \;\middle|\; \begin{array}{c} \|B(s,x) - B(t,y)\|_2 \leq C(|x-y| + |t-s|), \\ \forall (t,x), (s,y) \in \overline{Q_T}. \end{array} \right\}$$

2.4. On Orlicz Spaces

Let $z \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given distribution. Let $p_z : Q_T \to \mathbb{R}$ be the corresponding exponent defined by Equation (7). Then, we have

$$1 < p^- \leq p_z(t,x) \leq p^+ < \infty \text{ a.e. in } Q_T \tag{33}$$

where p^- and p^+ are the constants given by Equation (17) (see Lemma 1). Let $p_z'(t,x) = \frac{p_z(t,x)}{p_z(t,x)-1}$ be the conjugate exponent. Then, we have

$$2 = \underbrace{\frac{p^+}{p^+ - 1}}_{(p^+)'} \leq p_z'(t,x) \leq \underbrace{\frac{p^-}{p^- - 1}}_{(p^-)'} = \frac{p^-}{\delta} \text{ a.e. in } Q_T. \tag{34}$$

Let $L^{p_z(\cdot)}(Q_T)$ be the set of measurable functions $f : Q_T \to \mathbb{R}$ such that their modular is finite; in other words, let

$$\rho_{p_z(t,x)}(f) := \int_{Q_T} |f(t,s)|^{p_z(t,s)} \, dsdt < \infty. \tag{35}$$

which is equipped with the Luxembourg norm

$$\|f\|_{L^{p_z(\cdot)}(Q_T)} = \inf\left\{\lambda > 0 \,:\, \int_{Q_T} |\lambda^{-1} f(t,x)|^{p_z(t,x)}\, dxdt \le 1\right\}. \tag{36}$$

Here, $L^{p_z(\cdot)}(Q_T)$ becomes a Banach space (see [28,29] for details). The space $L^{p_z(\cdot)}(Q_T)$ is a sort of Musielak–Orlicz space. In fact, it can be denoted by a generalized Lebesgue space because its main properties are inherited from the classical Lebesgue spaces. In particular, the two-sides of the inequality in Equation (33) implies that $L^{p_z(\cdot)}(Q_T)$ is reflexive and separable, and the set $C_0^\infty(Q_T)$ is dense in $L^{p_z(\cdot)}(Q_T)$. Moreover, under the condition in Equation (33), $L^\infty(Q_T) \cap L^{p_z(\cdot)}(Q_T)$ is also dense in $L^{p_z(\cdot)}(Q_T)$.

Its dual can be identified with $L^{p'_z(\cdot)}(Q_T)$, and therefore, each continuous functional $F = F(f)$ on $L^{p_z(\cdot)}(Q_T)$ has the following representation (see Lemma 13.2 in [15]):

$$F(f) = \int_{Q_T} fw\, dxdt, \quad \text{with } w \in L^{p'_z(\cdot)}(Q_T).$$

Since the relation between the modular in Equation (35) and the norm in Equation (36) is not so direct as in the classical Lebesgue spaces, it can be proven from its definitions in Equations (35) and (36) that

$$\begin{gathered} \min\left\{\|f\|^{p^-}_{L^{p_z(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_z(\cdot)}(Q_T)}\right\} \le \rho_{p_z(t,x)}(f) \le \max\left\{\|f\|^{p^-}_{L^{p_z(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_z(\cdot)}(Q_T)}\right\}, \\ \min\left\{\rho^{\frac{1}{p^-}}_{p_z(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_z(t,x)}(f)\right\} \le \|f\|_{L^{p_z(\cdot)}(Q_T)} \le \max\left\{\rho^{\frac{1}{p^-}}_{p_z(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_z(t,x)}(f)\right\}. \end{gathered} \tag{37}$$

The following consequence of Equation (37) is very useful:

$$\begin{gathered} \|g\|^{p^-}_{L^{p_z(\cdot)}(Q_T)} - 1 \le \int_{Q_T} |g(t,s)|^{p_z(t,s)}\, dsdt \le \|g\|^{p^+}_{L^{p_z(\cdot)}(Q_T)} + 1, \\ \forall\, g \in L^{p_z(\cdot)}(Q_T), \end{gathered} \tag{38}$$

$$\begin{gathered} \|g_k - g\|_{L^{p_z(\cdot)}(Q_T)} \to 0 \quad \Longleftrightarrow \quad \int_{Q_T} |g_k(t,s) - g(t,s)|^{p_z(t,s)}\, dsdt \to 0 \\ \text{as } k \to \infty. \end{gathered} \tag{39}$$

Moreover, if $g \in L^{p_z(\cdot)}(Q_T)$, then

$$\|g\|_{L^{p^-}(Q_T)} \le \|g\|_{L^{p_z(\cdot)}(Q_T)} (1 + T|\Omega|)^{1/p^-}, \tag{40}$$

$$\|g\|_{L^{p_z(\cdot)}(Q_T)} \le \|g\|_{L^{p^+}(Q_T)} (1 + T|\Omega|)^{1/(p^+)'}, \quad (p^+)' = \frac{p^+}{p^+ - 1}, \ \forall\, f \in L^{p^+}(Q_T), \tag{41}$$

See, for instance, [28–30] for more information.

In generalized Lebesgue spaces, there holds a version of Young's inequality

$$|fg| \le \varepsilon \frac{|f|^{p_z(\cdot)}}{p_z(\cdot)} + C(\varepsilon) \frac{|g|^{p'_w(\cdot)}}{p'_w(\cdot)},$$

with some positive constant $C(\varepsilon)$ and arbitrary $\varepsilon > 0$.

The next assertion can be interpreted as an analogue of the Hölder inequality in variable Lebesgue spaces (we refer to [28,29] for the details).

Proposition 1. *If $f_1 \in L^{p_z(\cdot)}(Q_T;\mathbb{R}^2)$ and $f_2 \in L^{p'_z(\cdot)}(Q_T;\mathbb{R}^2)$, then $(f_1, f_2) \in L^1(Q_T)$ and*

$$\int_{Q_T} (f_1, f_2)\, dxdt \le 2\|f_1\|_{L^{p_z(\cdot)}(Q_T;\mathbb{R}^2)} \|f_2\|_{L^{p'_z(\cdot)}(Q_T;\mathbb{R}^2)}. \tag{42}$$

As a biproduct of Equation (42), we have that, for a bounded domain $Q_T = (0,T) \times \Omega$ and for $p_z(\cdot)$ satisfying Equation (33), the following imbedding

$$L^{p_z(\cdot)}(Q_T) \hookrightarrow L^{r(\cdot)}(Q_T) \ \text{ whenever } \ p_z(t,s) \geq r(t,s) \ \text{ for a.e. } (t,s) \in Q_T \tag{43}$$

is continuous.

Let $\widehat{\delta} \in (0,1]$, and let $\{q_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ be a given sequence of exponents. Assume that

$$\begin{aligned} q, q_k \in C^{0,\widehat{\delta}}(\overline{Q_T}) \ \text{ for } k = 1,2,\dots, \text{and} \\ q_k(\cdot) \to q(\cdot) \ \text{ uniformly in } \overline{Q_T} \text{ as } k \to \infty. \end{aligned} \tag{44}$$

With these sequences of exponents, we associate another sequence $\left\{f_k \in L^{q_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. As a result, we see that each element f_k lives in the individual Orlicz space $L^{q_k(\cdot)}(Q_T)$. In fact, we deal with a sequence in the scale of spaces $\left\{L^{q_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. The sequence $\left\{f_k \in L^{q_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty} \int_{Q_T} |f_k(t,x)|^{q_k(t,x)}\, dxdt < +\infty. \tag{45}$$

Definition 1. *Let $\left\{f_k \in L^{q_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ be a bounded sequence. Then, we say that this sequence weakly convergences in the variable space $L^{q_k(\cdot)}(Q_T)$ to a function $f \in L^{q(\cdot)}(Q_T)$, where $q \in C^{0,\delta}(\overline{Q_T})$ is the limit of $\{q_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ in the norm topology of $C(\overline{Q_T})$, if*

$$\lim_{k\to\infty} \int_{Q_T} f_k \varphi\, dxdt = \int_{Q_T} f\varphi\, dxdt, \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^2). \tag{46}$$

In order to proceed further, we recall some results concerning the lower semicontinuity property of the norm in the variable $L^{q_k(\cdot)}$ space with respect to the weak convergence in $L^{q_k(\cdot)}(Q_T)$. (For the detailed proof, we refer to Lemma 3.1 in [31]). For comparison, see also Lemma 13.3 in [15] and Lemma 2.1 in [25].

Proposition 2. *Assume that a sequence of exponents $\{q_k\}_{k\in\mathbb{N}}$ satisfies the condition in Equation (33), $q_k \to q$ as $k \to \infty$ a.e.in Q_T, and $\left\{g_k \in L^{q_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is a sequence that is bounded and weakly convergent in $L^{q^-}(Q_T)$ to g. Then, $g \in L^{q(\cdot)}(Q_T)$, $g_k \rightharpoonup g$ in variable $L^{q_k(\cdot)}(Q_T)$, and*

$$\liminf_{k\to\infty} \int_{Q_T} |g_k(t,s)|^{q_k(t,s)}\, dsdt \geq \int_{Q_T} |g(t,s)|^{q(t,s)}\, dsdt. \tag{47}$$

We also recall the inequality which is well known in the theory of p-Laplace equations. If $1 < p \leq 2$, then for all $\xi, \zeta \in \mathbb{R}^N$, the following estimate holds true:

$$(p-1)|\xi-\zeta|^2 \leq \left(\left[|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right], \xi-\zeta\right)\left(|\xi|^p + |\zeta|^p\right)^{\frac{2-p}{p}}.$$

2.5. On a Weighted Sobolev Space with a Variable Exponent

Let $z \in C([0,T]; L^2(\Omega))$ be a given distribution. Let $D_z(t,x)$ be a diffusion tensor associated with z by the rule in Equation (28). We define the weighted Sobolev space $W_z(Q_T)$ as the set of functions $v(t,x)$ such that

$$\begin{aligned} v \in L^2(Q_T), \quad v(t,\cdot) \in W^{1,1}(\Omega) \ \text{ for almost all } t \in [0,T], \\ \int_{Q_T} |D_z(t,x)\nabla v|^{p_z(t,x)}\, dxdt < +\infty. \end{aligned} \tag{48}$$

We define the norm on the space $W_z(Q_T)$ as follows:

$$\|v\|_{W_z(Q_T)} = \|v\|_{L^2(Q_T)} + \|D_z \nabla v\|_{L^{p_z(\cdot)}(Q_T;\mathbb{R}^2)}, \tag{49}$$

where the second term in Equation (49) is the norm of the function $D_z(t,x)\nabla v(t,x)$ in $L^{p_z(\cdot)}(Q_T;\mathbb{R}^2)$. Since

$$d_1^2|\zeta|^2 \leq \zeta^t [D_z(t,s)]^2 \zeta \leq d_2^2|\zeta|^2, \quad \forall\, \zeta \in \mathbb{R}^2,\ \forall\,(t,s) \in Q_T, \tag{50}$$

it follows that $W_w(Q_T)$ is a reflexive Banach space. Since $p_z : Q_T \to \mathbb{R}$ is the Lipschitz continuous exponent, it follows that the smooth and compactly supported functions are dense in the weighted Sobolev space $W_z(Q_T)$ (see [32]). Thus, $W_z(Q_T)$ can be represented as the closure of the set $\{\varphi \in C^\infty(\overline{Q}_T)\}$ with respect to the norm $\|\cdot\|_{W_z(Q_T)}$.

2.6. On Passage to the Limit in Fluxes

A standard situation in the study of many variational problems can be described as follows. Let, for a given $k \in \mathbb{N}$, $v_k \in L^2(0,T;W^{1,p^-}(\Omega))$ be a solution in the sense of distributions of the following parabolic equation of a monotone type:

$$\frac{\partial v_k}{\partial t} - \operatorname{div} B_k(t,x,\nabla v_k) = f, \quad (t,x) \in Q_T, \tag{51}$$

with $f \in L^2(\Omega)$. It is assumed that for all $\xi \in \mathbb{R}^2$, we have the following property: $B_k(\cdot,\cdot,\xi) \to B(\cdot,\cdot,\xi)$ as $k \to \infty$ pointwise a.e. with respect to the first two arguments. We also assume that the corresponding flow $w_k = B_k(\cdot,\cdot,\nabla v_k) \in L^{(p^+)'}(Q_T;\mathbb{R}^N)$ converges weakly such that

$$v_k \rightharpoonup v \ \text{ in } \ L^2(0,T;W^{1,p^-}(\Omega)), \quad w_k \rightharpoonup w \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^N),$$
$$1 < p^- < p^+,\ (p^+)' = \frac{p^+}{p^+ - 1}.$$

The main question is to find out whether a flux converges to a flux (i.e., check whether the following equality $B(t,x,\nabla v) = w$ holds true). Since the weak convergence $u_k \rightharpoonup u$ and the nonlinearity of the function $B(\cdot,\cdot,u)$ with respect to u do not guarantee that the limit passage $B_k(\cdot,\cdot,\nabla v_k) \rightharpoonup B(\cdot,\cdot,\nabla v)$ is valid, it makes this situation rather difficult and nontrivial. Therefore, the important problem is to show that $w = B(\cdot,\cdot,\nabla v)$. The following result gives the answer to the above question (for the proof, we refer to a celebrated paper [33]):

Theorem 1. *Let us assume that the following assumptions are satisfied:*

(C1) $B_k(t,s,\xi)$ *and* $B(t,s,\xi)$ *are continuous in* $\xi \in \mathbb{R}^N$ *for a.e.* $(t,s) \in Q_T$ *and measurable with respect to* $(t,s) \in Q_T$ *for each* $\xi \in \mathbb{R}^N$ *(i.e., they are* $\mathbb{R}^N$*-valued Carathéodory functions);*

(C2) $\Big(B_k(t,s,\xi) - B_k(t,s,\zeta), \xi - \zeta\Big) \geq 0$, $B_k(t,s,0) = 0 \ \forall\, \xi,\zeta \in \mathbb{R}^N$ *and for a.e.* $(t,s) \in Q_T$*;*

(C3) $|B_k(t,s,\xi)| \leq c(|\xi|) < \infty$ *and* $\lim_{k\to\infty} B_k(t,s,\xi) = B(t,s,\xi)$ *for all* $\xi \in \mathbb{R}^N$ *and for a.e.* $(t,s) \in Q_T$*;*

(C4) $v_k \rightharpoonup v$ *in* $L^{p^-}(0,T;W^{1,p^-}(\Omega))$, $p^- > 1$*, and* $\{v_k\}_{k\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;L^2(\Omega))$*;*

(C5) $w_k = B_k(t,s,\nabla v_k) \rightharpoonup w$ *in* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$, $p^+ > 1$*;*

(C6) $v_k \in L^{p^+}(0,T;W^{1,p^+}(\Omega))$ *for all* $k \in \mathbb{N}$*, and* $\sup_{k\in\mathbb{N}} \|(w_k,\nabla v_k)\|_{L^1(Q_T)} < \infty$*;*

(C7) $1 < p^- < p^+ < 2p^-$.

Then, $B_k(t,s,\nabla v_k) \rightharpoonup B(t,s,\nabla v)$ *weakly in the Lebesgue space* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$ *as* $k \to \infty$*.*

The next results reveal some other properties of the weak convergence in $L^1(\Omega)$.

Lemma 3 ([30]). *Let Ψ be a set of functions $F(t,s,\zeta)$ such that each of them is convex with respect to $\zeta \in \mathbb{R}^N$. These functions are measurable with respect to $(t,s) \in Q_T$ and satisfy the estimate*

$$c_1|\zeta|^{p^-} \le F(t,s,\zeta) \le c_2|\zeta|^{p^+}, \quad 1 < p^- \le p^+ < \infty,\ c_1, c_2 > 0.$$

If F_k and F belong to the set Ψ, and

$$\lim_{k\to\infty} F_k(t,s,\zeta) = F(t,s,\zeta) \quad \text{for a.e. } (t,s) \in Q_T \text{ and any } \zeta \in \mathbb{R}^N,$$

then the following lower semicontinuity property is valid:

$$\liminf_{k\to\infty} \int_{Q_T} F_k(t,s,u_k)\,dxdt \ge \int_{Q_T} F(t,s,u)\,dxdt \tag{52}$$

provided that $u_k \rightharpoonup u$ in $L^1(Q_T;\mathbb{R}^N)$.

Lemma 4 ([34]). *Let $B_k(t,s,\zeta)$ and $B(t,s,\zeta)$ be $\mathbb{R}^N$-valued Carathéodory functions satisfying properties (C1–C3) such that*

$$u_k \rightharpoonup v,\ \ w_k = B_k(t,s,u_k) \rightharpoonup w \ \text{ in } L^1(Q_T;\mathbb{R}^N) \text{ as } k \to \infty,$$

and $(w,u) \in L^1(Q_T)$. Then, we have

$$\liminf_{k\to\infty} \int_{Q_T} (B_k(t,s,u_k),u_k)\,dxdt \ge \int_{Q_T} (w,u)\,dxdt. \tag{53}$$

3. Existence Theorem for the Weak Solutions of Parabolic Equations with a Variable Order of Nonlinearity

In this section, we focus on the solvability issues for the following problem:

$$\frac{\partial u}{\partial t} - \operatorname{div} A_u(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad \text{in } Q_T, \tag{54}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{55}$$

$$u(0,\cdot) = f_0 \quad \text{in } \Omega. \tag{56}$$

Here, $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ are given distributions, we have

$$A_z(t,x,\nabla u) := |D_z(t,x)\nabla u|^{p_z(t,x)-2} D_z(t,x)\nabla u, \tag{57}$$

the exponent $p_z : Q_T \to (1,2]$ is defined in Equation (7), the matrix $D_z(t,x)$ is given by Equation (28), ∂_ν is the outward normal derivative, and $v \in \mathcal{V}_{ad}$ stands for the control with the following class of admissible controls $\mathcal{V}_{ad}$:

$$\mathcal{V}_{ad} = \Big\{ v \in L^2(Q_T) \ :\ v_a(x) \le v(t,x) \le v_b(x), \text{ a.e. in } Q_T \Big\}. \tag{58}$$

As follows from Equations (57) and (28) and Lemma 1, the mapping $(t,s,\xi) \mapsto A_z(t,s,\xi)$ is a Carathéodory function for each fixed $z \in C([0,T];L^2(\Omega))$ (i.e., $A_z(t,s,\zeta)$ is measurable with respect to (t,s) for each $\zeta \in \mathbb{R}^2$), and this function is continuous with respect to the third argument $\zeta \in \mathbb{R}^2$. Moreover, the following conditions (monotonicity, coerciveness, and boundedness) hold for a.e. $(t,s) \in Q_T$ [15]:

$$\Big(A_z(t,s,\xi) - A_z(t,s,\zeta), \xi - \zeta\Big) \geq 0, \quad \forall\, \xi, \zeta \in \mathbb{R}^2, \tag{59}$$

$$\begin{aligned}(A_z(t,s,\xi), \xi) &= |D_z(t,s)\xi|^{p_z(t,s)-2}\Big(D_z(t,s)\xi, D_z^{-1}(t,s)D_z(t,s)\xi\Big)\\ &\overset{\text{by (50)}}{\geq} d_2^{-1}d_1^{p_z(t,s)}|\xi|^{p_z(t,s)} \geq d_2^{-1}d_1^2|\xi|^{p_z(t,s)}, \quad \forall\, \xi \in \mathbb{R}^2,\end{aligned} \tag{60}$$

$$|A_z(t,s,\xi)|^{p_z'(t,s)} \leq d_2^{p_z(t,s)}|\xi|^{p_z(t,s)} \leq d_2^2|\xi|^{p_z(t,s)}, \quad \forall\, \xi \in \mathbb{R}^2. \tag{61}$$

However, if we have $z = u$, then $-\operatorname{div}\, A_u(t,x,\nabla u) + \kappa u$ provides an example of a nonmonotone, strongly nonlinear, and noncoercive operator in divergence form. In contrast to [35], where there existence of strong solutions to the similar class of the initial boundary value problems (IBVPs) was proven, we make use of the concept of weak solutions to the above problem. However, the issue of their uniqueness is, apparently, still open [36] (Chapter III).

Definition 2. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$ be given distributions. We say that a function u is a weak solution to the IBVPs in Equations (54)–(56) if $u \in W_u(Q_T)$; in other words, we have*

$$\begin{gathered} u \in L^2(Q_T),\ u(t,\cdot) \in W^{1,1}(\Omega) \ \text{for a.e. } t \in [0,T],\\ \int_{Q_T} |D_u(t,x)\nabla u|^{p_u(t,x)}\, dxdt < +\infty, \end{gathered} \tag{62}$$

and the integral identity

$$\begin{aligned}\int_{Q_T}\Big(-u\frac{\partial\varphi}{\partial t} + (A_u(t,x,\nabla u), \nabla\varphi) + \kappa u\varphi\Big)\, dx\, dt&\\ = \kappa\int_{Q_T}(f - v)\varphi\, dxdt + \int_\Omega f_0\varphi|_{t=0}\, dx&\end{aligned} \tag{63}$$

holds for any $\varphi \in \Phi$, where $\Phi = \big\{\varphi \in C^\infty(\overline{Q}_T)\ :\ \varphi|_{t=T} = 0\big\}$.

To clarify the sense in which the initial value $u(0,\cdot) = f_0$ is assumed for the weak solutions, we give the following assertion (for the proof, we refer to Proposition 2.2 in [25]):

Proposition 3. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$ and $v \in \mathcal{V}_{ad}$ be given distributions. Let $u \in W_u(Q_T)$ be a weak solution to the problem in Equations (54)–(56) in the sense of Definition 2. Then, for any $\eta \in C^\infty(\overline{\Omega})$, the scalar function $h(t) = \int_\Omega u(t,x)\eta(x)\, dx$ belongs to $W^{1,1}(0,T)$, and $h(0) = \int_\Omega f_0(x)\eta(x)\, dx$.*

We next recall some known results that have recently been proven based on the Schauder fixed-point theorem and using the perturbation technique (see Theorem 3.2 in [25]).

Theorem 2. *Given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, the problem in Equations (54)–(56) admits at least one weak solution $u \in W_u(Q_T)$ for which the following energy inequality*

$$\begin{aligned}\frac{1}{2}\int_\Omega u^2\, dx + \int_0^t\int_\Omega\Big((A_u(s,x,\nabla u), \nabla u) + \kappa u^2\Big)\, dx\, ds&\\ \leq \kappa\int_0^t\int_\Omega (f+v)u\, dxds + \int_\Omega f_0^2\, dx&\end{aligned} \tag{64}$$

holds for all $t \in [0,T]$.

Using Equation (64), we can derive the following estimates:

$$\|u\|^2_{L^2(Q_T)} \le C_1^2 = \kappa T\Big(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)}\Big) + \kappa^{-1}\|f_0\|^2_{L^2(\Omega)}, \tag{65}$$

$$\begin{aligned}\|\nabla u\|_{L^{p_u(\cdot)}(Q_T;\mathbb{R}^2)} &\overset{\text{by (38)}}{\le} \left(\int_{Q_T} |\nabla u|^{p_u(t,x)}\,dxdt + 1\right)^{1/p^-}\\ &\overset{\text{by (60)}}{\le} \left(\frac{d_2}{d_1^2}\left(\frac{3}{2}\|f_0\|^2_{L^2(\Omega)} + \frac{2\kappa+\kappa^2 T}{2}\Big(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)}\Big)\right) + 1\right)^{1/p^-}\\ &=: C_2,\end{aligned} \tag{66}$$

$$\|\nabla u\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \le (1 + T|\Omega|)^{1/p^-} C_2, \tag{67}$$

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}\sqrt{\kappa\Big(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)}\Big) + \|f_0\|^2_{L^2(\Omega)}}. \tag{68}$$

Since the uniqueness issue for the weak solutions of the initial boundary value problem in Equations (54)–(56) seems to be an open question, we adopt the following concept:

Definition 3. *We say that a weak solution $u \in W_u(Q_T)$ to the problem in Equations (54)–(56) for given distributions $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$ is W_0-attainable if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero as $n \to \infty$ such that*

$$\left.\begin{aligned} u_n &\rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)),\\ A_{u_{n-1}}(t,x,\nabla u_n) &\rightharpoonup A_u(t,x,\nabla u) \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^2)\end{aligned}\right\} \text{ as } n \to \infty, \tag{69}$$

where

$$\begin{aligned} u_n \in W(0,T) = \left\{w \in L^2(0,T;W^{1,2}(\Omega)),\ \frac{dw}{dt} \in L^2\Big(0,T;\Big[W^{1,2}(\Omega)\Big]'\Big)\right\}, \quad \forall\, n \in \mathbb{N},\\ A_w(t,x,\nabla u) := |D_z(t,x)\nabla u|^{p_z(t,x)-2} D_z(t,x)\nabla u,\end{aligned} \tag{70}$$

and, for each $n \in \mathbb{N}$, u_n is the weak solutions to the following perturbed problem:

$$\frac{\partial u}{\partial t} - \varepsilon_n \Delta u - \operatorname{div} A_{u_{n-1}}(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad \text{in } Q_T, \tag{71}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{72}$$

$$u(0,\cdot) = f_0 \quad \text{in } \Omega. \tag{73}$$

Remark 1. *It is worth emphasizing that (see the recent results in [25]) can now be specified as follows. Given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, the initial-boundary value problem in Equations (54)–(56) admits at least one W_0-attainable weak solution $u \in W_u(Q_T)$ for which the energy inequality in Equation (64) holds true for all $t \in [0,T]$. Moreover, as follows from the estimates in Equations (65)–(68), this solution is bounded in $L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.*

4. Setting of the Optimal Control Problem and Existence Result

As was pointed out in the previous sections, the operator $-\operatorname{div} A_u(t,x,\nabla u) + \kappa u$ provides an example of a nonlinear operator in divergence form which is neither monotone nor coercive. In this case (see Theorem 2), a weak solution to the initial boundary value problem (Equations (54)–(56)) under some admissible control $v \in \mathcal{V}_{ad}$ may not be unique. Moreover, it is unknown whether all weak solutions to Equations (54)–(56) satisfy the energy inequality in Equation (64), which plays a crucial role in the derivation of a priori estimates (Equations (65)–(68)).

Our prime interest in this section is to consider the following optimal control problem of the tracking type:

$$\begin{gathered}\text{Minimize } J(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{\mu}{2}\int_\Omega |u(T) - f_0|^2\,dx \\ \text{subject to the constraints (2)–(4), (58),}\end{gathered} \tag{74}$$

where $f \in L^2(\Omega)$ is the original noise-corrupted image, $f_0 \in L^2(\Omega)$ is the pre-denoised image when applying a median filter to f, and $v_a, v_b \in L^2(\Omega)$ and $v_a(x) \le v_b(x)$ a.e. in Ω are given distributions.

We say that $(v.u)$ is a feasible pair to this problem if

$$\left.\begin{gathered} v \in \mathcal{V}_{ad}, \quad u \in W_u(Q_T), \quad J(v,u) < +\infty, \\ (v,u) \text{ are related by integral identity (63) and inequality (64),} \\ \text{and } u \text{ is a } W_0\text{-attainable weak solution to (54)–(56) for the given } v. \end{gathered}\right\} \tag{75}$$

Let $\Xi \subset L^2(Q_T) \times W_u(Q_T)$ be the set of feasible solutions to the problem in Equation (74). Then, Theorem 2 implies that $\Xi \ne \emptyset$. Since the main topological properties of the set Ξ are unknown, in general, we begin with the following observation:

Theorem 3. *Given $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$, the set Ξ is sequentially closed with respect to the weak topology of $L^2(Q_T) \times L^{p^-}(0,T;W^{1,p^-}(\Omega))$.*

Proof. Let $\{(v_k, u_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a sequence such that

$$v_k \rightharpoonup v \ \text{ in } \ L^2(Q_T), \quad u_k \rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)). \tag{76}$$

Since the set $\mathcal{V}_{ad}$ is convex and closed, it follows from Mazur's theorem that $\mathcal{V}_{ad}$ is sequentially closed with respect to the weak topology of $L^2(Q_T)$. Therefore, $v \in \mathcal{V}_{ad}$. Let us show that $(v,u) \in \Xi$. We will accomplish this in several steps.

Step 1. Under the initial assumptions, for each $k \in \mathbb{N}$, the pair (v_k, u_k) satisfies the energy inequality in Equation (64), and u_k is a W_0-attainable weak solution for Equations (54)–(56). Hence, in light of Definition 3, we may always suppose that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset W(0,T)$ such that $\{u_{k,n}\}_{n\in\mathbb{N}}$ are the weak solutions (in the sense of distributions) to Equations (71)–(73) with $\varepsilon_n = 1/n$ and $v = v_k$, and

$$u_{k,n} \rightharpoonup u_k \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } n \to \infty, \tag{77}$$

$$A_{u_{k,n-1}}(t,x,\nabla u_{k,n}) \rightharpoonup A_{u_k}(t,x,\nabla u_k) \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^2) \quad \text{as } n \to \infty, \tag{78}$$

Moreover, the fact that the energy equality

$$\begin{aligned}\frac{1}{2}\int_\Omega u_{k,n}^2\,dx + \int_0^t\int_\Omega \left(\frac{1}{n}|\nabla u_{k,n}|^2 + \left(A_{u_{k,n-1}}(s,x,\nabla u_{k,n}), \nabla u_{k,n}\right) + \kappa u_{k,n}^2\right)dx\,ds \\ = \kappa\int_0^t\int_\Omega (f - v_k)u_{k,n}\,dx\,ds + \int_\Omega f_0^2\,dx, \quad \forall\, t \in [0,T]\end{aligned} \tag{79}$$

is valid for all $n,k \in \mathbb{N}$ implies the boundedness of the sequence $\{u_{k,k}\}_{k\in\mathbb{N}}$ in the space $L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Hence, by combining this fact with Equations (77) and (76), we deduce that

$$u_{k,k} \rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } k \to \infty, \tag{80}$$

$$u_{k,k} \rightharpoonup u \ \text{ in } \ L^2(0,T;L^2(\Omega)), \quad \text{as } k \to \infty. \tag{81}$$

Step 2. By utilizing the energy equality in Equation (79) and arguing as we did in Equations (65)–(68), we can derive the following a priori estimates:

$$\|u_{k,k}\|^2_{L^2(Q_T)} \le \kappa T\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right) + \kappa^{-1}\|f_0\|^2_{L^2(\Omega)} =: S_1^2, \tag{82}$$

$$\begin{aligned}&\|\nabla u_{k,k}\|^{p^-}_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)}\\ &\qquad\le \frac{d_2}{d_1^2}\left(\frac{3}{2}\|f_0\|^2_{L^2(\Omega)} + \frac{2\kappa+\kappa^2 T}{2}\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right)\right) + 1 =: S_2,\end{aligned} \tag{83}$$

$$\|\nabla u_{k,k}\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \le (1+T|\Omega|)^{1/p^-} S_2, \tag{84}$$

$$\|u_{k,k}\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}\sqrt{\kappa\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right) + \|f_0\|^2_{L^2(\Omega)}}, \tag{85}$$

$$\|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^N)} \le \sqrt{k}\left(\|f_0\|^2_{L^2(\Omega)} + \kappa\|f+v_k\|_{L^2(Q_T)}\|u_{k,k}\|_{L^2(Q_T)}\right) \overset{\text{by (82)}}{\le} \sqrt{k}S_3. \tag{86}$$

for all $k\in\mathbb{N}$, where

$$\sup_{k\in\mathbb{N}}\|v_k\|_{L^2(Q_T)} \le \sqrt{T}\|v_b\|_{L^2(\Omega)} < +\infty. \tag{87}$$

Our main intention in this step is to establish the following asymptotic property:

$$\frac{1}{k}\nabla u_{k,k} \rightharpoonup 0 \quad \text{in } L^2(Q_T;\mathbb{R}^2). \tag{88}$$

Indeed, for any vector-valued test function $\varphi\in C_0^\infty(Q_T)$, we have

$$\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right)dxdt\right| \le \frac{1}{\sqrt{k}}\left(\int_{Q_T}\frac{1}{k}|\nabla u_{k,k}|^2\,dxdt\right)^{1/2}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2}.$$

Hence, the sequence $\left\{\frac{1}{k}\nabla u_{k,k}\right\}_{k\in\mathbb{N}}$ is bounded in $L^2(Q_T;\mathbb{R}^2)$. As a result, we obtain

$$\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right)dxdt\right| \overset{\text{by (86)}}{\le} S_3\frac{1}{\sqrt{k}}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2} \to 0 \quad \text{as } k\to\infty.$$

Step 3. Let us show that in this case, the flux $\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k})$ weakly converges in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ to the flux $A_u(t,x,\nabla u)$ as $k\to\infty$. To accomplish this, it is enough to show that all preconditions (C1–C7) of Theorem 1 are fulfilled.

To begin with, we notice that the conclusion, similar to Equation (80), can also be made with respect to the sequence $\{u_{k,k-1}\}_{k\in\mathbb{N}}$. Then, Lemmas 1 and 2 imply that

$$\begin{gathered}D_{u_{k,k-1}}(t,x) \to D_u(t,x) \text{ and } p_{u_{k,k-1}}(t,x) \to p_u(t,x)\\ \text{uniformly in } \overline{Q_T} \text{ as } j\to\infty.\end{gathered} \tag{89}$$

Moreover, we deduce from Equations (61) and (83) that the sequence

$$\left\{\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k})\right\}_{k\in\mathbb{R}}$$

is bounded in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$. Hence, there exists an element $z\in L^{(p^+)'}(Q_T;\mathbb{R}^2)$ such that

$$\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup z \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \tag{90}$$

We also make use of the following observation: the sequence

$$\left\{ \frac{1}{k}|\nabla u_{k,k}|^2 + \Big(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}), \nabla u_{k,k} \Big) \right\}_{k\in\mathbb{N}} \tag{91}$$

is uniformly bounded in $L^1(Q_T)$. Indeed, this inference is a direct consequence of the estimates in Equations (86) and (83), and the following one:

$$\begin{aligned} |A_{u_{k,k-1}}(t,x,\nabla u_{k,k})||\nabla\varphi| &\le \frac{1}{p'_{u_{k,k-1}}(t,x)}|A_{u_{k,k-1}}(t,x,\nabla u_{k,k})|^{p'_{u_{k,k-1}}(t,x)} \\ &\quad + \frac{1}{p_{u_{k,k-1}}(t,x)}|\nabla\varphi|^{p_{u_{k,k-1}}(t,x)} \\ &\le \frac{d_2^2}{2}|\nabla u_{k,k}|^{p_{u_{k,k-1}}(t,x)} + \frac{1}{p^-}|\nabla\varphi|^{p_{u_{k,k-1}}(t,x)}. \end{aligned} \tag{92}$$

Utilizing this fact together with the properties in Equations (89), (90), (59), (80), and

$$u_{k,k} \in L^{p^+}(0,T;W^{1,p^+}(\Omega))\ \forall\, k \in \mathbb{N} \text{ by (82) and (86),}$$

and taking into account that $1 < 1+\delta = p^- < p^+ = 2 < 2p^-$, we see that all preconditions of Theorem 1 hold true. Hence, in light of the property in Equation (88), the assertion in Equation (90) can be rewritten as follows:

$$\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup A_u(t,x,\nabla u) \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \tag{93}$$

Step 4. At this stage, we show that the limit pair (v,u) is related with the integral identity in Equation (63). First, we notice that $u_{k,k}$ is a weak solution (in the sense of distributions) of Equations (71)–(73) with $n = k$, $\varepsilon_n = 1/k$, and $v = v_k$. Hence, $u_{k,k}$ satisfies the integral identity

$$\begin{aligned} \int_{Q_T} \left(-u_{k,k}\frac{\partial\varphi}{\partial t} + \frac{1}{k}(\nabla u_{k,k},\nabla\varphi) + \Big(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla\varphi\Big) + \kappa u_{k,k}\varphi \right) dx\,dt \\ = \kappa\int_{Q_T}(f-v_k)\varphi\,dxdt + \int_\Omega f_0\varphi|_{t=0}\,dx \quad \forall\,\varphi\in\Phi. \end{aligned} \tag{94}$$

Then, utilizing the properties in Equations (93), (80), and (76) and passing to the limit in Equation (94) as $k\to\infty$, we immediately arrive at the announced identity in Equation (63).

Step 5. In order to show that the limit pair (v,u) satisfies the energy inequality in Equation (64), we have to realize the limit passage as $k\to\infty$ in the following relation (see [25]):

$$\begin{aligned} \frac{1}{2}\int_\Omega u_{k,k}^2\,dx + \int_0^t\int_\Omega \left(\frac{1}{k}|\nabla u_{k,k}|^2 + \Big(A_{u_{k,k-1}}(s,x,\nabla u_{k,k}),\nabla u_{k,k}\Big) + \kappa u_{k,k}^2 \right) dx\,ds \\ = \kappa\int_0^t\int_\Omega (f-v_k)u_{k,k}\,dx\,ds + \int_\Omega f_0^2\,dx \quad \forall\, t\in[0,T]. \end{aligned} \tag{95}$$

This can be viewed as the energy equality for the weak solutions to the problem in Equations (71)–(73) with $n = k$, $\varepsilon_n = 1/k$, and $v = v_k$. With that in mind, we notice that the weak convergence in Equation (80), under the Sobolev embedding theorem, implies the pointwise convergence

$$u_{k,k}^2(t,\cdot) \to u^2(t,\cdot) \quad \text{a.e. in } \Omega \text{ for a.a. } t\in(0,T).$$

Then, in light of the estimate in Equation (85), we have the strong convergence $u_{k,k}^2(t,\cdot) \to u^2(t,\cdot)$ in $L^1(\Omega)$ for a.e. $t \in (0,T)$ (under Lebesgue's dominated convergence theorem), and therefore

$$\frac{1}{2}\lim_{k\to\infty}\int_\Omega u_{k,k}^2(t,x)\,dx = \frac{1}{2}\int_\Omega u^2(t,x)\,dx \quad \text{for a.a. } t \in (0,T). \tag{96}$$

Moreover, taking into account that the $L^2(Q_T)$ norm is less semi-continuous with respect to the weak convergence in Equation (81), we see that

$$\lim_{k\to\infty}\int_0^t\int_\Omega u_{k,k}^2\,dxdt \geq \int_0^t\int_\Omega u^2\,dxdt. \tag{97}$$

We also notice that due to the properties in Equations (88), (93), and (80), we have

$$\nabla u_{k,k} \rightharpoonup \nabla u \text{ and } A_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup A_u(t,x,\nabla u) \text{ in } L^1(Q_T;\mathbb{R}^2) \text{ as } k\to\infty.$$

Since $(A_u(t,x,\nabla u),\nabla u) \in L^1(Q_T)$ (see Equation (92)), it follows from Lemma 4 (see also Proposition 2) that

$$\begin{aligned}\lim_{k\to\infty}\int_0^t\int_\Omega &\left[\frac{1}{k}|\nabla u_{k,k}|^2 + \Big(A_{u_{k,k-1}}(s,x,\nabla u_{k,k}),\nabla u_{k,k}\Big)\right]dxds\\ &\geq \lim_{k\to\infty}\int_0^t\int_\Omega\left[\frac{1}{k}|\nabla u_{k,k}|^2\right]dxds\\ &\quad + \liminf_{k\to\infty}\int_0^t\int_\Omega\Big(A_{u_{k,k-1}}(s,x,\nabla u_{k,k}),\nabla u_{k,k}\Big)\,dxds\\ &\overset{\text{by (86)}}{\geq} \int_0^t\int_\Omega (A_u(s,x,\nabla u),\nabla u)\,dxds.\end{aligned} \tag{98}$$

Therefore, in order to pass to the limit in Equation (95), the asymptotic behavior of the term $\int_{Q_T}(f-v_k)u_{k,k}\,dxdt$ as $k\to\infty$ remains to be found. We prove this in the next step using the well-known Aubin–Lions lemma.

Step 6. We recall that the Aubin–Lions lemma states the criteria for when a set of functions is relatively compact in $L^p(0,T;B)$, where $p\in[1,\infty)$, $T>0$, and B is a Banach space. The standard formulation of the Aubin–Lions lemma states that if U is a bounded set in $L^p(0,T;X)$, and $\partial U/\partial t = \{\partial u/\partial t \,:\, u\in U\}$ is bounded in $L^r(0,T;Y)$, $r\geq 1$, then U is relatively compact in $L^p(0,T;B)$ under the conditions that

$$X \hookrightarrow B \text{ compactly}, \quad B \hookrightarrow Y \text{ continuously}.$$

By setting $U = \{u_{k,k}\}_{k\in\mathbb{N}}$, we deduce from Equations (82)–(85) that

$$\{u_{k,k}\}_{k\in\mathbb{N}} \text{ is bounded in } L^{p^-}(0,T;W^{1,p^-}(\Omega)\cap L^2(\Omega)). \tag{99}$$

Since, under the Sobolev embedding theorem, $W^{1,p^-}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ compactly, it follows from Lebesgue's dominated convergence theorem that the following embeddings are compact as well:

$$W^{1,p^-}(\Omega)\cap L^2(\Omega) \hookrightarrow L^2(\Omega), \quad L^2(\Omega) \hookrightarrow \Big(W^{1,2}(\Omega)\Big)' \text{ (by the duality arguments)}. \tag{100}$$

Furthermore, we have in mind the fact that for each $k \in \mathbb{N}$, the functions $u_{k,k}$ are the solutions in $W(0,T)$ for the variational problem

$$\left\langle \frac{\partial u_{k,k}(t)}{\partial t}, \varphi \right\rangle_{(W^{1,2}(\Omega))';W^{1,2}(\Omega)} + \int_\Omega \left[\frac{1}{k} (\nabla u_{k,k}(t), \nabla \varphi) \right] dx$$

$$+ \int_\Omega \left[\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}(t)), \nabla \varphi \right) + \kappa u_{k,k}(t) \varphi \right] dx \tag{101}$$

$$= \kappa \int_\Omega (f(t) - v_k(t)) \varphi \, dx, \quad \forall \varphi \in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], \tag{102}$$

$$u_{k,k}(0) = f_0. \tag{103}$$

We derive from this the following estimate:

$$\begin{aligned}
\left| \left\langle \frac{\partial u_{k,k}}{\partial t}, \varphi \right\rangle \right| &\leq \frac{1}{\sqrt{k}} \|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^2)} \|\nabla \varphi\|_{L^2(Q_T;\mathbb{R}^2)} \\
&+ 2 \|A_{u_{k,k-1}}(t,x,\nabla u_{k,k})\|_{L^{p'_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)} \|\nabla \varphi\|_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)} \\
&+ \kappa \|u_{k,k}\|_{L^2(Q_T)} \|\varphi\|_{L^2(Q_T)} + \kappa \|f - v_k\|_{L^2(Q_T)} \|\varphi\|_{L^2(Q_T)} \\
&\leq (\text{by } (82)\text{–}(86)) \\
&\leq \left[S_3 + \kappa S_1 + \kappa \|f\|_{L^2(Q_T)} + \kappa \sup_{k \in \mathbb{N}} \|v_k\|_{L^2(Q_T)} \right] \|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))} \\
&+ \left(1 + \int_{Q_T} |A_{u_{k,k-1}}(t,x,\nabla u_{k,k})|^{p'_{u_{k,k-1}}(t,x)} \, dxdt \right)^{1/2} (1 + T|\Omega|)^{1/2} \|\varphi\|_{L^2(Q_T)} \\
&\overset{\text{by } (83),(61)}{\leq} \text{const} \|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall v \in L^2(0,T;W^{1,2}(\Omega)).
\end{aligned}$$

Hence, we have

$$\left\| \frac{\partial u_{k,k}}{\partial t} \right\|_{L^2(0,T;(W^{1,2}(\Omega))')} < +\infty. \tag{104}$$

Utilizing this fact together with Equations (99) and (100), we deduce from the the Aubin–Lions lemma that the set $U = \{u_{k,k}\}_{k \in \mathbb{N}}$ is relatively compact in $L^{p^-}(0,T;L^2(\Omega))$. Hence, we can complement properties with the following one: $u_{k,k} \to u$ strongly in $L^{p^-}(0,T;L^2(\Omega))$ as $k \to \infty$. Since U is bounded in $L^\infty(0,T;L^2(\Omega))$, it leads to the conclusion that

$$u_{k,k} \to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \quad \text{as } k \to \infty. \tag{105}$$

Hence, the term $\int_{Q_T} (f - v_k) u_{k,k} \, dxdt$ is the product of weakly and strongly convergent sequences in $L^2(0,T;L^2(\Omega))$. As a result, we have

$$\lim_{k \to \infty} \int_{Q_T} (f - v_k) u_{k,k} \, dxdt = \int_{Q_T} (f - v) u \, dxdt. \tag{106}$$

Thus, in light of the obtained collection of properties (see Equations (96)–(98) and (106)), the limit passage in Equation (95) as $k \to \infty$ finally leads us to the energy inequality in Equation (64).

Step 7. To end the proof, it remains to notice that due to the properties in Equation (62), which were established in the previous steps, we have $J(v,u) < +\infty$ and $u \in W_u(Q_T)$. Moreover, it has been proven that in this case, the sequence $\{u_{k,k}\}_{k \in \mathbb{N}}$ satisfies all requirements that were mentioned in Definition 3. Hence, $u \in W_u(Q_T)$ is a W_0-attainable weak solution to the problem in Equations (54)–(56). The proof is complete. □

Taking this result into account, let us show that the original optimal control problem (Equation (74)) has a solution. In fact, this issue immediately follows from Theorem 3 and the facts that the set of feasible solutions Ξ is bounded in $L^2(Q_T) \times L^{p^-}(0,T;W^{1,p^-}(\Omega))$ (see the estimates in Equations (65)–(68) and (87)) and the objective functional $J(v,u)$ is less semi-continuous with respect to the weak topology of $L^2(Q_T) \times \left(L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))\right)$. Thus, as a direct consequence, we can finalize this inference as follows:

Corollary 1. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leq v_b(x)$ a.e. in Ω be given distributions, and let $\kappa > 0$, $\sigma > 0$, $\varepsilon > 0$, and $\mu > 0$ be some constants. Then, the optimal control problem (Equation (74)) admits at least one solution $(v^0, u^0) \in \Xi$.*

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