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Regression Estimation with Errors in the Variables via the Laplace Transform

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Abstract: This paper considers nonparametric regression estimation with errors in the variables. It is a standard assumption that the characteristic function of the covariate error does not vanish on the real line. This assumption is rather strong. In this paper, we assume the covariate error distribution is a convolution of uniform distributions, the characteristic function of which contains zeros on the real line. Our regression estimator is constructed via the Laplace transform. We prove its strong consistency and show its convergence rate. It turns out that zeros in the characteristic function have no effect on the convergence rate of our estimator.

Keywords: nonparametric regression; errors in variables; Laplace transform; strong consistency; convergence rate

MSC: 62G08; 62G20

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1. Introduction

This paper considers a regression model with errors in the variables. Suppose observations $(W_1, Y_1), \cdots, (W_n, Y_n)$ are i.i.d. (independent and identically distributed) random variables generated by the model

$$W_j = X_j + \delta_j, \quad Y_j = m(X_j) + \epsilon_j, j = 1, \cdots, n. \tag{1}$$

The i.i.d. random variables δ_j are independent of X_j and Y_j. ϵ_j are independent of X_j, $E\epsilon_j = 0$ and $E\epsilon_j^2 < +\infty$. The functions f_δ (known) and f_X (unknown) stand for the densities of δ_j and X_j, respectively. The goal is to estimate the regression function $m(x)$ from the observations $(W_1, Y_1), \cdots, (W_n, Y_n)$. Errors-in-variables regression problems have been extensively studied in the literature, see, for example, ([1–7]). Regression models with errors in the variables play an important role in many areas of science and social science ([8–10]).

Nadaraya and Watson ([11,12]) propose a kernel regression estimator for the classical regression model ($\delta_j = 0$). Since the Fourier transform can transform a complex convolution to an ordinary product, it is a common method to deal with the deconvolution problem. Fan and Truong [4] generalize the Nadaraya–Watson regression estimator from the classical regression model to the regression model (1) via the Fourier transform. They study the convergence rate by assuming the integer order derivatives of f_X and m to be bounded. Compared to integer-order derivatives, it is more precise to describe the smoothness by the Hölder condition. Meister [6] shows the convergence rate under the local Hölder condition.

The above references on model (1) both assume that the characteristic function of the covariate errors δ_j does not have zeros on the real line. The assumption is rather strong. For example, if f_δ is of uniform density on $[-1, 1]$, it vanishes at $v = k\pi, k = \pm 1, \pm 2, \cdots$ in the Fourier domain. Delaigle and Meister [1] consider the regression model (1) with a Fourier-oscillating noise, which means the Fourier transform of f_δ vanishes periodically.

They show that if f_X and m are compact, then they can be estimated with the standard rate, as in the case where f_δ does not vanish in the Fourier domain. Guo and Liu ([13–15]) extend Delaigle and Meister [1]'s work to multivariate cases.

The compactness is the cost of eliminating the zero points effect in the Fourier domain. Belomestny and Goldenshluger [16] apply the Laplace transform to construct a deconvolution density estimator without assuming the density to be compact. They provide sufficient conditions under which the zeros of the corresponding characteristic function have no effect on the estimation accuracy. Goldenshluger and Kim [17] also construct a deconvolution density estimator via the Laplace transform; they study how zero multiplicity affects the estimation accuracy. Motivated by the above work, we apply the Laplace transform to study the regression model (1) with errors following a convolution of uniform distributions.

The organization of the paper is as follows. In Section 2, we present some knowledge about the covariate error distribution and functional classes. Section 3 introduces the kernel regression estimator via the Laplace transform. The consistency and convergence rate of our estimator are discussed in Section 4 and Section 5, respectively.

2. Preparation

This section will introduce the covariate error distribution and functional classes.

For a integrable function f, the bilateral Laplace transform [18] is defined by

$$\hat{f}(z) := \int_{-\infty}^{+\infty} f(t)e^{-zt}dt.$$

The Laplace transform $\hat{f}(z)$ is an analytic function in the convergence region Σ_f, which is a vertical strip:

$$\Sigma_f := \{z \in \mathbb{C} : \sigma_f^- < \mathrm{Re}(z) < \sigma_f^+\}, \qquad \text{for some } -\infty \le \sigma_f^- < \sigma_f^+ \le +\infty.$$

The inverse Laplace transform is given by the formula

$$f(t) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \hat{f}(z)e^{zt}dz = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(s+iv)e^{(s+iv)t}dv, \quad s \in (\sigma_f^-, \sigma_f^+).$$

Let the covariate error distribution be a γ-fold convolution of the uniform distribution on $[-\theta,\theta], \theta > 0$. This means

$$\delta = Z_1 + \cdots + Z_\gamma,$$

where $Z_i (i = 1, 2, \cdots, \gamma)$ are i.i.d and $Z_i \sim U(-\theta,\theta)$ with density f_Z. Hence,

$$\hat{f}_\delta(z) = \left[\hat{f}_Z(z)\right]^\gamma = \left[\frac{\sinh(\theta z)}{\theta z}\right]^\gamma = \frac{(1-e^{2\theta z})^\gamma}{(-2\theta z)^\gamma e^{\gamma\theta z}}, \qquad z \in \mathbb{C}. \tag{2}$$

Here, $\hat{f}_\delta(z)$ is the product of two functions; the function $(1-e^{2\theta z})^\gamma$ has zeros only on the imaginary axis, the function $\dfrac{1}{(-2\theta z)^\gamma e^{\gamma\theta z}}$ does not have zeros for the analyticity of $(-2\theta z)^\gamma e^{\gamma\theta z}$. The zeros of $\hat{f}_\delta(z)$ are $z_k = \dfrac{ik\pi}{\theta}$, where $k \in \mathbb{Z}\backslash\{0\}$.

Now, we introduce some functional classes.

Definition 1. *For $A > 0$, $\sigma > 0$, and $\beta > 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy the local Hölder condition with smoothness parameter β if f is k times continuously differentiable and*

$$\left|f^{(k)}(y) - f^{(k)}(\tilde{y})\right| \le A\left|y-\tilde{y}\right|^{\beta_0}, \qquad \forall y, \tilde{y} \in [x-\sigma, x+\sigma], \tag{3}$$

where $\beta = k + \beta_0$ and $0 < \beta_0 \le 1$. All these functions are denoted by $\mathscr{H}_{\sigma,\beta;x}(A)$.

If (3) holds for any $y, \tilde{y} \in \mathbb{R}$, f satisfies the Hölder condition with smoothness parameter β. All these functions are denoted by $\mathscr{H}_\beta(A)$.

Clearly, k in Definition 1 equals $\max\{l \in \mathbb{N} : l < \beta\}$. In later discussions, $\lfloor \beta \rfloor := \max\{l \in \mathbb{N} : l < \beta\}$.

Example 1. *Function*

$$f_1(x) := \begin{cases} 1 - |x|, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Then, $f_1 \in \mathscr{H}_1(A)$ *and* $f_1 \in \mathscr{H}_{\sigma,1;x}(A)$*.*

It is easy to see that $f \in \mathscr{H}_\beta(A)$ must be contained in $\mathscr{H}_{\sigma,\beta;x}(A)$ for each $x \in \mathbb{R}$. However, the reverse is not necessarily true.

Example 2 ([19])**.** *Consider the function*

$$f_2(x) := \sum_{l=0}^{\infty} (1 - 2^l |x - 2l|)\chi_l(x),$$

where $\chi_l(x)$ *is the indicator function on the interval* $[2l - 2^{-l}, 2l + 2^{-l}]$ *for a non-negative integer* l*. Then,* $f_2 \in \mathscr{H}_{\sigma,1;x}(A)$ *for each* $x \in \mathbb{R}$*. However,* $f_2 \notin \mathscr{H}_1(A)$*.*

Note that (3) is a local Hölder condition around $x \in \mathbb{R}$. When we consider the pointwise estimation, it is natural to assume the unknown function to satisfy a local smoothness condition.

Definition 2. *Let* $r > 0$ *and* $B > 0$ *be real numbers. We say that a function* f *belongs to the functional class* $\mathscr{M}_r(B)$ *if*

$$\max\{\| f \|_\infty, \max_{0 < r_1 \le r} \int_{-\infty}^{+\infty} |x|^{r_1} |f(x)| dx\} \le B.$$

We denote $\mathscr{F}_{\sigma,\beta,r;x}(A, B) = \mathscr{H}_{\sigma,\beta;x}(A) \cap \mathscr{M}_r(B)$.

3. Kernel Estimator

This section will construct the kernel regression estimator. Two kernels K and $L_{s,h}$ will be used.

Assume that the kernel $K : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) $\int_{-1}^{1} K(x)dx = 1$, $K \in C^\infty(\mathbb{R})$ and $\mathrm{supp}(K) \subseteq [-1, 1]$;

(ii) There exists a fixed positive integer k_0 such that

$$\int_{-1}^{1} x^j K(x)dx = 0, \quad j = 1, \cdots, k_0.$$

Example 3 ([20])**.** *Function*

$$K(x) = a\varphi(x),$$

where

$$\varphi(x) := \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$$

and $a = \left(\int_{-\infty}^{+\infty} \varphi(x)dx\right)^{-1}$*. Then, the kernel* $K(x)$ *satisfies conditions (i) and (ii) with* $k_0 = 1$*.*

Motivated by Belomestny and Goldenshluger [16], we will construct the regression estimator via the Laplace transform. Note that $\hat{f}_\delta(-z)$ does not have zeros out of the imaginary axis. Then, the kernel $L_{s,h}$ is defined by the inverse Laplace transform

$$\begin{aligned} L_{s,h}(t) &:= \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \frac{\hat{K}(zh)}{\hat{f}_\delta(-z)} e^{zt} dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\hat{K}((s+iv)h)}{\hat{f}_\delta(-s-iv)} e^{(s+iv)t} dv, \end{aligned} \tag{4}$$

where $s \neq 0$, $h > 0$ and $\hat{K}(\cdot)$ is the Laplace transform of kernel K with the convergence region $\Sigma_K = \mathbb{C}$. There is a complex-valued improper integral in (4). One can use the property of the Laplace transform to compute it, see [18].

The following lemma provides a infinite series of kernel $L_{s,h}(t)$. It is a specific form of Lemma 2 in [16] . In order to explain the construction of the estimator, we give the details of the proof.

Lemma 1. *Let (2) hold and $\int_{-\infty}^{+\infty} |\hat{K}(ivh)||v|^\gamma dv < \infty$.*
(a) If $s > 0$, then

$$L_{+,h}(t) := L_{s,h}(t) = \frac{(2\theta)^\gamma}{2\pi}\sum_{l_1=0}^{\infty}\cdots\sum_{l_\gamma=0}^{\infty}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma e^{iv[t-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)]}dv.$$

(b) If $s < 0$, then

$$L_{-,h}(t) := L_{s,h}(t) = \frac{(-2\theta)^\gamma}{2\pi}\sum_{l_1=0}^{\infty}\cdots\sum_{l_\gamma=0}^{\infty}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma e^{iv[t+\gamma\theta+2\theta(l_1+\cdots+l_\gamma)]}dv.$$

Proof. (a) If $s > 0$, we have

$$\frac{1}{1-e^{-2\theta(s+iv)}} = \sum_{l=0}^{\infty} e^{-2\theta l(s+iv)}.$$

Therefore,

$$\frac{1}{[1-e^{-2\theta(s+iv)}]^\gamma} = \left[\sum_{l=0}^{\infty} e^{-2\theta l(s+iv)}\right]^\gamma = \sum_{l_1=0}^{\infty}\cdots\sum_{l_\gamma=0}^{\infty} e^{-2\theta(s+iv)(l_1+\cdots+l_\gamma)}.$$

By (2) and (4),

$$\begin{aligned} L_{+,h}(t) := L_{s,h}(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{K}((s+iv)h)[2\theta(s+iv)]^\gamma}{[1-e^{-2\theta(s+iv)}]^\gamma} e^{(s+iv)(t-\gamma\theta)}dv \\ &= \frac{(2\theta)^\gamma}{2\pi}\sum_{l_1=0}^{\infty}\cdots\sum_{l_\gamma=0}^{\infty}\int_{-\infty}^{+\infty}\hat{K}((s+iv)h)(s+iv)^\gamma e^{(s+iv)[t-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)]}dv \\ &= \frac{(2\theta)^\gamma}{2\pi}\sum_{l_1=0}^{\infty}\cdots\sum_{l_\gamma=0}^{\infty}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma e^{iv[t-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)]}dv. \end{aligned}$$

(b) If $s < 0$, then

$$\left[1 - e^{-2\theta(s+iv)}\right]^{-\gamma} = \left[\frac{1 - e^{2\theta(s+iv)}}{-e^{2\theta(s+iv)}}\right]^{-\gamma} = (-1)^{\gamma} e^{2\gamma\theta(s+iv)} \left[1 - e^{2\theta(s+iv)}\right]^{-\gamma}$$
$$= (-1)^{\gamma} e^{2\gamma\theta(s+iv)} \sum_{l_1=0}^{\infty} \cdots \sum_{l_\gamma=0}^{\infty} e^{2\theta(s+iv)(l_1+\cdots+l_\gamma)}.$$

Similarly,

$$L_{-,h}(t) := L_{s,h}(t) = \frac{(-2\theta)^{\gamma}}{2\pi} \sum_{l_1=0}^{\infty} \cdots \sum_{l_\gamma=0}^{\infty} \int_{-\infty}^{+\infty} \hat{K}(ivh)(iv)^{\gamma} e^{iv[t+\gamma\theta+2\theta(l_1+\cdots+l_\gamma)]} dv.$$

This ends the proof. □

The truncation is used to deal with infinite series. Select parameter N so that $\frac{N}{\gamma} \in \mathbb{N}_+$. The cut-off kernels are defined by

$$L_{+,h}^{(N)}(t) := \frac{(2\theta)^{\gamma}}{2\pi} \sum_{l_1=0}^{\frac{N}{\gamma}} \cdots \sum_{l_\gamma=0}^{\frac{N}{\gamma}} \int_{-\infty}^{+\infty} \hat{K}(ivh)(iv)^{\gamma} e^{iv[t-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)]} dv, \tag{5}$$

$$L_{-,h}^{(N)}(t) := \frac{(-2\theta)^{\gamma}}{2\pi} \sum_{l_1=0}^{\frac{N}{\gamma}} \cdots \sum_{l_\gamma=0}^{\frac{N}{\gamma}} \int_{-\infty}^{+\infty} \hat{K}(ivh)(iv)^{\gamma} e^{iv[t+\gamma\theta+2\theta(l_1+\cdots+l_\gamma)]} dv. \tag{6}$$

Denote

$$L_{s,h}^{(N)}(t) := \begin{cases} L_{+,h}^{(N)}(t), & s > 0, \\ L_{-,h}^{(N)}(t), & s < 0. \end{cases}$$

Motivated by the Nadaraya–Watson regression estimator, we define the regression estimator of $m(x)$ as

$$\tilde{m}_{s,h}^{(N)}(x) := \frac{\tilde{p}_{s,h}^{(N)}(x)}{\tilde{f}_{X,s,h}^{(N)}(x)}, \tag{7}$$

where

$$\tilde{f}_{X,s,h}^{(N)}(x) := \frac{1}{n} \sum_{j=1}^{n} L_{s,h}^{(N)}(W_j - x) \quad \text{and} \quad \tilde{p}_{s,h}^{(N)}(x) := \frac{1}{n} \sum_{j=1}^{n} Y_j L_{s,h}^{(N)}(W_j - x). \tag{8}$$

In what follows, we will write $\tilde{m}_{+,h}^{(N)}(x)$ and $\tilde{m}_{-,h}^{(N)}(x)$ for the estimator (7) associated with $s > 0$ and $s < 0$, respectively. Finally, our regression estimator is denoted by

$$\tilde{m}_{h}^{(N)}(x) := \begin{cases} \tilde{m}_{+,h}^{(N)}(x), & x \geq 0, \\ \tilde{m}_{-,h}^{(N)}(x), & x < 0. \end{cases} \tag{9}$$

4. Strong Consistency

In this section, we investigate the consistency of the regression estimator (9). Roughly speaking, consistency means that the estimator $\tilde{m}_{h}^{(N)}(x)$ converges to $m(x)$ as the sample size tends to infinity.

Theorem 1 (Strong consistency)**.** *Consider the model (1) with (2). Suppose $f_X, p := mf_X \in \mathscr{M}_r(B)(r > \frac{1}{2})$, $E|Y_1|^{8(\gamma+1)} < +\infty$ and kernel function K satisfies condition (i). If x is the Lebesgue point of both f_X and p ($f_X(x) \neq 0$), then $\tilde{m}_h^{(N)}(x)$ satisfies*

$$\lim_{n\to\infty} \tilde{m}_h^{(N)}(x) \overset{\text{a.s.}}{=} m(x)$$

with $h = n^{-\frac{1}{6(\gamma+1)}}$ and $n^{\frac{1}{3(\gamma+1)}} \leq N \leq 2n^{\frac{1}{3(\gamma+1)}}$.

Proof. (1) We consider the estimator $\tilde{m}_{+,h}^{(N)}(x)$ for $x \geq 0$.

Note that $\tilde{m}_{+,h}^{(N)}(x) = \dfrac{\tilde{p}_{+,h}^{(N)}(x)}{\tilde{f}_{X,+,h}^{(N)}(x)}$, $m(x) = \dfrac{p(x)}{f_X(x)}$ and $f_X(x) \neq 0$. Then, it is sufficient to prove $\lim\limits_{n\to\infty} \tilde{p}_{+,h}^{(N)}(x) \overset{\text{a.s.}}{=} p(x)$ and $\lim\limits_{n\to\infty} \tilde{f}_{X,+,h}^{(N)}(x) \overset{\text{a.s.}}{=} f_X(x)$.

Now, we prove $\lim\limits_{n\to\infty} \tilde{p}_{+,h}^{(N)}(x) \overset{\text{a.s.}}{=} p(x)$. For any $\epsilon > 0$,

$$P\big[|\tilde{p}_{+,h}^{(N)}(x) - p(x)| > \sqrt{\epsilon}\big] \leq P\left[|\tilde{p}_{+,h}^{(N)}(x) - E\tilde{p}_{+,h}^{(N)}(x)| > \frac{\sqrt{\epsilon}}{2}\right] + \chi_{(\frac{\sqrt{\epsilon}}{2},\infty)}(|E\tilde{p}_{+,h}^{(N)}(x) - p(x)|).$$

By Markov's inequality, we obtain

$$P\big[|\tilde{p}_{+,h}^{(N)}(x) - p(x)| > \sqrt{\epsilon}\big] \leq c_1\epsilon^{-s}E|\tilde{p}_{+,h}^{(N)}(x) - E\tilde{p}_{+,h}^{(N)}(x)|^{2s} + \chi_{(\frac{\sqrt{\epsilon}}{2},\infty)}(|E\tilde{p}_{+,h}^{(N)}(x) - p(x)|) \quad (10)$$

for $s := 4(\gamma+1)$. This motivates us to derive an upper bound on $E|\tilde{p}_{+,h}^{(N)}(x) - E\tilde{p}_{+,h}^{(N)}(x)|^{2s}$. Combining (5) with (8), we have

$$\begin{aligned}\tilde{p}_{+,h}^{(N)}(x) &= \frac{1}{n}\sum_{j=1}^{n} Y_j L_{+,h}^{(N)}(W_j - x)\\ &= \frac{(2\theta)^\gamma}{2\pi n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma(Y_je^{ivW_j})e^{-iv(x+\gamma\theta)}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{-i2\theta v(l_1+\cdots+l_\gamma)}\right]dv,\end{aligned}$$

and

$$E\tilde{p}_{+,h}^{(N)}(x) = \frac{(2\theta)^\gamma}{2\pi n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma E(Y_je^{ivW_j})e^{-iv(x+\gamma\theta)}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{-i2\theta v(l_1+\cdots+l_\gamma)}\right]dv. \quad (11)$$

We obtain

$$E|\tilde{p}_{+,h}^{(N)}(x) - E\tilde{p}_{+,h}^{(N)}(x)|^{2s} = \left[\frac{(2\theta)^\gamma}{2\pi n}\right]^{2s} E\left|\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma\Psi_j(v)\Phi_{+,N}(x,v)dv\right|^{2s},$$

where $\Psi_j(v) := Y_je^{ivW_j} - E(Y_je^{ivW_j})$, $\Phi_{+,N}(x,v) := e^{-iv(x+\gamma\theta)}\left[\sum\limits_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum\limits_{l_\gamma=0}^{\frac{N}{\gamma}} e^{-i2\theta v(l_1+\cdots+l_\gamma)}\right]$.

Thus,

$$
\begin{aligned}
&E|\tilde{p}_{+,h}^{(N)}(x)-E\tilde{p}_{+,h}^{(N)}(x)|^{2s}\\
&=\left[\frac{(2\theta)^{\gamma}}{2\pi n}\right]^{2s}\sum_{j_1=1}^{n}\cdots\sum_{j_{2s}=1}^{n}E\Bigg[\Bigg(\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{k=1}^{s}\hat{K}(iv_{2k-1}h)\hat{K}(-iv_{2k}h)(iv_{2k-1})^{\gamma}(-iv_{2k})^{\gamma}\\
&\qquad\times\Phi_{+,N}(x,v_{2k-1})\Phi_{+,N}(x,-v_{2k})\Psi_{j_{2k-1}}(v_{2k-1})\Psi_{j_{2k}}(-v_{2k})dv_1\cdots dv_{2s}\Bigg)\Bigg]\\
&=\left[\frac{(2\theta)^{\gamma}}{2\pi n}\right]^{2s}\sum_{j_1=1}^{n}\cdots\sum_{j_{2s}=1}^{n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\prod_{k=1}^{s}\hat{K}(iv_{2k-1}h)\hat{K}(-iv_{2k}h)(iv_{2k-1})^{\gamma}(-iv_{2k})^{\gamma}\right.\\
&\qquad\left.\times\Phi_{+,N}(x,v_{2k-1})\Phi_{+,N}(x,-v_{2k})\right]E\left[\prod_{k=1}^{s}\Psi_{j_{2k-1}}(v_{2k-1})\Psi_{j_{2k}}(-v_{2k})\right]dv_1\cdots dv_{2s}\\
&\le\left[\frac{(2\theta)^{\gamma}}{2\pi n}\right]^{2s}\sum_{j_1=1}^{n}\cdots\sum_{j_{2s}=1}^{n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\prod_{k=1}^{s}|\hat{K}(iv_{2k-1}h)||\hat{K}(-iv_{2k}h)||v_{2k-1}|^{\gamma}|v_{2k}|^{\gamma}\right]\\
&\qquad\times(\frac{N}{\gamma}+1)^{2\gamma s}\left|E\left[\prod_{k=1}^{s}\Psi_{j_{2k-1}}(v_{2k-1})\Psi_{j_{2k}}(-v_{2k})\right]\right|dv_1\cdots dv_{2s}.
\end{aligned}
\tag{12}
$$

Let $\#A$ denote the number of elements contained in the set A. If $\#\{j_1,\cdots,j_{2s}\}>s$, at least one of Ψ_{j_l} is independent of all other $\Psi_{j_{l'}}$, $l'\neq l$. Hence,

$$E\left[\prod_{k=1}^{s}\Psi_{j_{2k-1}}(v_{2k-1})\Psi_{j_{2k}}(-v_{2k})\right]=0.$$

On the other hand, if $\#\{j_1,\cdots,j_{2s}\}=s_1$ for $s_1\le s$, by Jensen's inequality, we obtain

$$
\begin{aligned}
\left|E\left[\prod_{k=1}^{s}\Psi_{j_{2k-1}}(v_{2k-1})\Psi_{j_{2k}}(-v_{2k})\right]\right|&=\left|E\left[\Psi_{j_1'}(t_1')\right]^{\lambda_1}\cdots\left[\Psi_{j_{s_1}'}(t_{s_1}')\right]^{\lambda_{s_1}}\right|\le E\Big|\Psi_1(t_1)\Big|^{2s}\\
&\le E(|Y_1|+E|Y_1|)^{2s}\le 4^sE|Y_1|^{2s},
\end{aligned}
$$

where $\lambda_1+\cdots+\lambda_{s_1}=2s$. Let $J_n=\{(j_1,\cdots,j_{2s}):\#\{j_1,\cdots,j_{2s}\}\le s, j_i\in\{1,\cdots,n\}, i=1,\cdots,2s\}$. Then,

$$E|\tilde{p}_{+,h}^{(N)}(x)-E\tilde{p}_{+,h}^{(N)}(x)|^{2s}\le\left[\frac{(2\theta)^{\gamma}}{2\pi n}\right]^{2s}(c_2N^{2\gamma s})(4^sE|Y_1|^{2s})\sum_{j\in J_n}\left(\int_{-\infty}^{+\infty}|\hat{K}(ivh)||v|^{\gamma}dv\right)^{2s}.$$

Since $|v|^k|\hat{K}(iv)|\le c(k)$ for all k, we obtain that $\int_{-\infty}^{+\infty}|\hat{K}(ivh)||v|^{\gamma}dv\le c_3h^{-(\gamma+2)}$ for $k=\gamma+2$. This, with $\#J_n\le c_4n^s$, leads to

$$E|\tilde{p}_{+,h}^{(N)}(x)-E\tilde{p}_{+,h}^{(N)}(x)|^{2s}\le c_5n^{-s}N^{2\gamma s}h^{-2s(\gamma+2)}\le c_6n^{\frac{-s}{3(\gamma+1)}}.\tag{13}$$

Inserting this into (10), we obtain

$$P\big[|\tilde{p}_{+,h}^{(N)}(x)-p(x)|>\sqrt{\epsilon}\big]\le c_7n^{\frac{-s}{3(\gamma+1)}}+\chi_{(\frac{\sqrt{\epsilon}}{2},\infty)}(|E\tilde{p}_{+,h}^{(N)}(x)-p(x)|).\tag{14}$$

Note that (W_j, Y_j) are identically distributed. Then, it follows from (11) and $E(Y_j e^{ivW_j}) = E(Y_j e^{ivX_j})E(e^{iv\delta_j})$ that

$$\begin{aligned} &E\tilde{p}_{+,h}^{(N)}(x) \\ &= \frac{(2\theta)^\gamma}{2\pi}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma E(Y_1e^{ivX_1})E(e^{iv\delta_1})e^{-iv(x+\gamma\theta)}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}e^{-i2\theta v(l_1+\cdots+l_\gamma)}\right]dv, \end{aligned} \tag{15}$$

where

$$\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}\left[e^{-i2\theta v}\right]^{l_1+\cdots+l_\gamma} = \left[\sum_{l=0}^{\frac{N}{\gamma}}e^{-iv(2\theta l)}\right]^\gamma = \left[\frac{1-(e^{-i2\theta v})^{\frac{N}{\gamma}+1}}{1-e^{-i2\theta v}}\right]^\gamma. \tag{16}$$

By (2), we have

$$\begin{aligned} E\tilde{p}_{+,h}^{(N)}(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}E(Y_1e^{ivX_1})\hat{K}(ivh)e^{-ivx}\left[1-(e^{-i2\theta v})^{\frac{N}{\gamma}+1}\right]^\gamma dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}E\left[e^{ivX_1}\cdot E(Y_1|X_1)\right]\hat{K}(ivh)e^{-ivx}\sum_{l=0}^{\gamma}\binom{\gamma}{l}(-1)^l(e^{-i2\theta v})^{l(\frac{N}{\gamma}+1)}dv \\ &= \sum_{l=0}^{\gamma}\binom{\gamma}{l}(-1)^l\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}p(t)e^{ivt}dt\right]\hat{K}(ivh)e^{-iv[x+2\theta l(\frac{N}{\gamma}+1)]}dv \\ &= \sum_{l=0}^{\gamma}\binom{\gamma}{l}(-1)^l\frac{1}{h}\int_{-\infty}^{+\infty}p(t)K\left(\frac{t-x-2\theta l(\frac{N}{\gamma}+1)}{h}\right)dt \\ &= \int_{-\infty}^{+\infty}\frac{1}{h}K(\frac{t-x}{h})p(t)dt + T_{+,\frac{N}{\gamma}}(p;x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} T_{+,\frac{N}{\gamma}}(p;x) &:= \sum_{l=1}^{\gamma}\binom{\gamma}{l}(-1)^l\frac{1}{h}\int_{-\infty}^{+\infty}p(t)K\left(\frac{t-x-2\theta l(\frac{N}{\gamma}+1)}{h}\right)dt \\ &= \sum_{l=1}^{\gamma}\binom{\gamma}{l}(-1)^l\int_{-1}^{1}K(y)p\left(yh+x+2\theta l(\frac{N}{\gamma}+1)\right)dy. \end{aligned}$$

Hence,

$$\begin{aligned} |E\tilde{p}_{+,h}^{(N)}(x)-p(x)| &\le \left|\int_{-\infty}^{+\infty}\frac{1}{h}K(\frac{t-x}{h})p(t)dt-p(x)\right| + \left|T_{+,\frac{N}{\gamma}}(p;x)\right| \\ &= \left|\int_{-1}^{1}K(y)[p(x+yh)-p(x)]dy\right| + \left|T_{+,\frac{N}{\gamma}}(p;x)\right|. \end{aligned} \tag{18}$$

Since $p \in \mathscr{M}_r(B)$ and considering the boundedness of K,

$$\left|T_{+,\frac{N}{\gamma}}(p;x)\right| \le c_8\sum_{l=1}^{\gamma}\binom{\gamma}{l}\int_{-1}^{1}\left|p\left(yh+x+2\theta l(\frac{N}{\gamma}+1)\right)\right|dy \le \frac{c_9B}{h(x+2\theta N\cdot\frac{1}{\gamma})^r} \le \frac{c_{10}B}{hN^r} \tag{19}$$

holds for an h that is small enough. It follows from $r > \frac{1}{2}$, $h = n^{-\frac{1}{6(\gamma+1)}}$ and $N \ge n^{\frac{1}{3(\gamma+1)}}$ that

$$\left|T_{+,\frac{N}{\gamma}}(p;x)\right| \xrightarrow{n\to\infty} 0.$$

Note that the kernel function K satisfies condition (i) and $p \in L(\mathbb{R})$, then

$$\Big|\int_{-1}^{1} K(y)[p(x+yh)-p(x)]dy\Big| \xrightarrow{h\to 0} 0$$

holds for each Lebesgue point x of p. Hence, for an n that is sufficiently large, the term $\chi_{(\frac{\sqrt{\epsilon}}{2},\infty)}(|E\tilde{p}_{+,h}^{(N)}(x)-p(x)|)$ vanishes. This, with (14), shows

$$P\big[|\tilde{p}_{+,h}^{(N)}(x)-p(x)|>\sqrt{\epsilon}\big] \le c_7 n^{\frac{-s}{3(\gamma+1)}} \tag{20}$$

for an n that is large enough. Since $s=4(\gamma+1)$, we have

$$\sum_{n=1}^{\infty} P\big[|\tilde{p}_{+,h}^{(N)}(x)-p(x)|>\sqrt{\epsilon}\big] < \infty.$$

For any $\epsilon>0$, it follows from the Borel–Cantelli lemma that

$$P\Big\{\overline{\lim_{n\to\infty}}\big[|\tilde{p}_{+,h}^{(N)}(x)-p(x)|>\sqrt{\epsilon}\big]\Big\}=0.$$

Thus,

$$\lim_{n\to\infty} \tilde{p}_{+,h}^{(N)}(x) \overset{\text{a.s.}}{=} p(x).$$

When putting $Y_j \equiv 1$ almost surely, we have

$$\lim_{n\to\infty} \tilde{f}_{X,+,h}^{(N)}(x) \overset{\text{a.s.}}{=} f_X(x).$$

Hence,

$$\lim_{n\to\infty} \tilde{m}_{+,h}^{(N)}(x) \overset{\text{a.s.}}{=} m(x).$$

(2) We consider the estimator $\tilde{m}_{-,h}^{(N)}(x)$ for $x<0$. Inserting (6) into (8), we obtain

$$\begin{aligned}\tilde{p}_{-,h}^{(N)}(x) &= \frac{1}{n}\sum_{j=1}^{n} Y_j L_{-,h}^{(N)}(W_j-x)\\ &= \frac{(-2\theta)^\gamma}{2\pi n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma(Y_j e^{ivW_j})e^{-iv(x-\gamma\theta)}\Bigg[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{i2\theta v(l_1+\cdots+l_\gamma)}\Bigg]dv,\end{aligned}$$

and

$$E\tilde{p}_{-,h}^{(N)}(x) = \frac{(-2\theta)^\gamma}{2\pi n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma E(Y_j e^{ivW_j})e^{-iv(x-\gamma\theta)}\Bigg[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{i2\theta v(l_1+\cdots+l_\gamma)}\Bigg]dv. \tag{21}$$

We obtain

$$E|\tilde{p}_{-,h}^{(N)}(x)-E\tilde{p}_{-,h}^{(N)}(x)|^{2s} = \left[\frac{(2\theta)^\gamma}{2\pi n}\right]^{2s} E\Big|\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma \Psi_j(v)\Phi_{-,N}(x,v)dv\Big|^{2s},$$

where $\Phi_{-,N}(x,v) := e^{-iv(x-\gamma\theta)}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{i2\theta v(l_1+\cdots+l_\gamma)}\right]$. Similar to (12) and (13), we obtain

$$E|\tilde{p}_{-,h}^{(N)}(x)-E\tilde{p}_{-,h}^{(N)}(x)|^{2s}\le c_{11}n^{\frac{-s}{3(\gamma+1)}}.$$

By (21), we have

$$\begin{aligned}E\tilde{p}_{-,h}^{(N)}(x)=&\frac{(-2\theta)^\gamma}{2\pi}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^\gamma E(Y_1e^{ivX_1})E(e^{iv\delta_1})e^{-iv(x-\gamma\theta)}\\&\times\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{i2\theta v(l_1+\cdots+l_\gamma)}\right]dv,\end{aligned}$$

where

$$\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} e^{i2\theta v(l_1+\cdots+l_\gamma)}=\left[\frac{1-(e^{i2\theta v})^{\frac{N}{\gamma}+1}}{1-e^{i2\theta v}}\right]^\gamma.$$

By $\dfrac{1}{(1-e^{i2\theta v})^\gamma}=\dfrac{1}{(-e^{i2\theta v})^\gamma(1-e^{-i2\theta v})^\gamma}$ and (2), we have that $\dfrac{(-i2\theta v)^\gamma e^{iv(\gamma\theta)}}{(1-e^{i2\theta v})^\gamma}=\dfrac{1}{\hat{f}_\delta(-iv)}$. So,

$$E\tilde{p}_{-,h}^{(N)}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{K}(ivh)E(Y_1e^{ivX_1})e^{-ivx}\left[1-(e^{i2\theta v})^{\frac{N}{\gamma}+1}\right]^\gamma dv.$$

Similar to (17), we obtain

$$E\tilde{p}_{-,h}^{(N)}(x)=\int_{-\infty}^{+\infty}\frac{1}{h}K(\frac{t-x}{h})p(t)dt+T_{-,\frac{N}{\gamma}}(p;x),$$

where

$$\begin{aligned}T_{-,\frac{N}{\gamma}}(p;x):=&\sum_{l=1}^{\gamma}\binom{\gamma}{l}(-1)^l\frac{1}{h}\int_{-\infty}^{+\infty}p(t)K\left(\frac{t-x+2\theta l(\frac{N}{\gamma}+1)}{h}\right)dt\\=&\sum_{l=1}^{\gamma}\binom{\gamma}{l}(-1)^l\int_{-1}^{1}K(y)p\left(yh+x-2\theta l(\frac{N}{\gamma}+1)\right)dy.\end{aligned}$$

Thus, we have

$$|E\tilde{p}_{-,h}^{(N)}(x)-p(x)|\le\left|\int_{-1}^{1}K(y)[p(x+yh)-p(x)]dy\right|+\left|T_{-,\frac{N}{\gamma}}(p;x)\right|. \tag{22}$$

Since $p\in\mathscr{M}_r(B)$ and considering the boundedness of K,

$$\left|T_{-,\frac{N}{\gamma}}(p;x)\right|\le c_{12}\sum_{l=1}^{\gamma}\binom{\gamma}{l}\int_{-1}^{1}\left|p\left(yh+x-2\theta l(\frac{N}{\gamma}+1)\right)\right|dy\le\frac{c_{13}B}{h(-x+2\theta N\cdot\frac{1}{\gamma})^r}\le\frac{c_{14}B}{hN^r} \tag{23}$$

holds for an h that is small enough.

Similar to $x\ge 0$, we get

$$\lim_{n\to\infty}\tilde{m}_{-,h}^{(N)}(x)\overset{\text{a.s.}}{=}m(x).$$

This completes the proof. □

Remark 1. *Theorem 1 shows the strong consistency of kernel estimator $\tilde{m}_h^{(N)}(x)$. It is different from the work of Meister [6] in that the density function of our covariate error δ contains zeros in the Fourier domain. Our covariate error belongs to the Fourier oscillating noise considered by Delaigle and Meister [1]. Compared to their work, we construct a regression estimator via the Laplace transform without assuming f_X and m to be compact.*

5. Convergence Rate

In this section, we focus on the convergence rate in the weak sense. Meister [6] introduces the weak convergence rate by modifying the concept of weak consistency. A regression estimator $\hat{m}_n(x)$ is said to attain the weak convergence rate ε_n if

$$\lim_{C\to\infty}\left(\limsup_{n\to\infty}\sup_{(m,f_X)\in\mathscr{P}} P\left[|\hat{m}_n(x)-m(x)|^2\geq C\cdot\varepsilon_n\right]\right)=0.$$

The set $\mathscr{P}$ is the collection of all pairs (m, f_X) that satisfy some conditions. The order of limits is first $n\to\infty$, and then $C\to\infty$. Here, C is independent of n.

Define the set

$$\mathscr{P}_{\beta,r;x} := \{(m,f_X) : f_X, mf_X \in \mathscr{F}_{\sigma,\beta,r;x}(A,B), |m(x)|\leq C_1, f_X(x)\geq C_2, \| m(\cdot)\|_\infty \leq C_3\},$$

where $C_1, C_2, C_3 > 0$.

The following Lemma is used to prove the theorem in this section.

Lemma 2 ([6])**.** *If $p := mf_X$, $\hat{m}_n(x) = \dfrac{p_n(x)}{f_{X,n}(x)}$, $|m(x)| < +\infty$ and $f_X(x)\neq 0$. Then, for a small enough $\epsilon > 0$,*

$$P\left[|\hat{m}_n(x)-m(x)|^2>\epsilon\right]\leq P\left[|p_n(x)-p(x)|^2>c_1\epsilon\right]+P\left[|f_{X,n}(x)-f_X(x)|^2>c_2\epsilon\right]$$

with two positive constants, c_1 and c_2.

Theorem 2. *Consider the model (1) with (2). Assume that $(m, f_X)\in\mathscr{P}_{\beta,r;x}$ with $r = 2\gamma-2$ if $\gamma > 1$, and $r > 0$ if $\gamma = 1$. Suppose kernel K satisfies conditions (i), (ii) with $k_0\geq\beta$. Let $h = n^{\frac{-1}{2\beta+2\gamma+1}}$, $N\geq n^{\frac{\beta+1}{r(2\beta+2\gamma+1)}}$. Then,*

$$\lim_{C\to\infty}\left(\limsup_{n\to\infty}\sup_{(m,f_X)\in\mathscr{P}_{\beta,r;x}} P\left[|\tilde{m}_h^{(N)}(x)-m(x)|^2\geq C\cdot\varepsilon_n\right]\right)=0,$$

where $\varepsilon_n = n^{\frac{-2\beta}{2\beta+2\gamma+1}}$.

Proof. (1) We assume that $x\geq 0$ and consider the estimator $\tilde{m}_{+,h}^{(N)}(x)$. Applying Lemma 2 and Markov's inequality, we obtain

$$P\left[|\tilde{m}_{+,h}^{(N)}(x)-m(x)|^2\geq C\cdot\varepsilon_n\right]\leq\frac{c_3}{C\varepsilon_n}\left(E|\tilde{p}_{+,h}^{(N)}(x)-p(x)|^2+E|\tilde{f}_{X,+,h}^{(N)}(x)-f_X(x)|^2\right),\tag{24}$$

where c_3 is the larger of $\frac{1}{c_1}$ and $\frac{1}{c_2}$, and c_1, c_2 appear in Lemma 2. Then,

$$E|\tilde{p}_{+,h}^{(N)}(x)-p(x)|^2=\mathrm{var}[\tilde{p}_{+,h}^{(N)}(x)]+|E\tilde{p}_{+,h}^{(N)}(x)-p(x)|^2,\tag{25}$$

and

$$E|\tilde{f}_{X,+,h}^{(N)}(x)-f_X(x)|^2=\mathrm{var}[\tilde{f}_{X,+,h}^{(N)}(x)]+|E\tilde{f}_{X,+,h}^{(N)}(x)-f_X(x)|^2.\tag{26}$$

First, we estimate $|E\tilde{p}^{(N)}_{+,h}(x)-p(x)|^2$ and $|E\tilde{f}^{(N)}_{X,+,h}(x)-f_X(x)|^2$. By (18), we have

$$|E\tilde{p}^{(N)}_{+,h}(x)-p(x)|^2 \le 2\Big(\Big|\int_{-1}^{1}K(y)[p(x+yh)-p(x)]dy\Big|^2+\Big|T_{+,\frac{N}{\gamma}}(p;x)\Big|^2\Big). \tag{27}$$

By Taylor expansion of p with the degree $\lfloor\beta\rfloor-1$, there exists $0<\eta<1$ such that

$$\begin{aligned}
&\left|\int_{-1}^{1}K(y)\Big[p(x+yh)-p(x)\Big]dy\right|\\
&=\left|\int_{-1}^{1}K(y)\left[\sum_{j=1}^{\lfloor\beta\rfloor}\frac{(yh)^j}{j!}p^{(j)}(x)+\frac{(yh)^{\lfloor\beta\rfloor}}{\lfloor\beta\rfloor!}\Big(p^{(\lfloor\beta\rfloor)}(x+\eta yh)-p^{(\lfloor\beta\rfloor)}(x)\Big)\right]dy\right|\\
&\le\left|\int_{-1}^{1}K(y)\sum_{j=1}^{\lfloor\beta\rfloor}\frac{(yh)^j}{j!}p^{(j)}(x)dy\right|+\left|\int_{-1}^{1}K(y)\frac{(yh)^{\lfloor\beta\rfloor}}{\lfloor\beta\rfloor!}\Big(p^{(\lfloor\beta\rfloor)}(x+\eta yh)-p^{(\lfloor\beta\rfloor)}(x)\Big)dy\right|.
\end{aligned}$$

Since kernel K satisfies condition (ii) and $\beta\le k_0$, we have

$$\int_{-1}^{1}K(y)\sum_{j=1}^{\lfloor\beta\rfloor}\frac{(yh)^j}{j!}p^{(j)}(x)dy=0.$$

By $p\in\mathscr{H}_{\sigma,\beta;x}(A)$, we find that

$$\begin{aligned}
&\left|\int_{-1}^{1}K(y)\frac{(yh)^{\lfloor\beta\rfloor}}{\lfloor\beta\rfloor!}\Big(p^{(\lfloor\beta\rfloor)}(x+\eta yh)-p^{(\lfloor\beta\rfloor)}(x)\Big)dy\right|\\
&\le\int_{-1}^{1}|K(y)|\frac{|yh|^{\lfloor\beta\rfloor}}{\lfloor\beta\rfloor!}\Big|p^{(\lfloor\beta\rfloor)}(x+\eta yh)-p^{(\lfloor\beta\rfloor)}(x)\Big|dy\\
&\le\int_{-1}^{1}|K(y)|\frac{A|yh|^{\beta}\|\eta|^{\beta-\lfloor\beta\rfloor}}{\lfloor\beta\rfloor!}dy\\
&\le c_4Ah^{\beta}
\end{aligned} \tag{28}$$

holds for an h that is small enough. Equations (19) and (28) imply the following upper bound:

$$|E\tilde{p}^{(N)}_{+,h}(x)-p(x)|^2\le c_5\Big(A^2h^{2\beta}+\frac{B^2}{h^2N^{2r}}\Big). \tag{29}$$

Now, we estimate the term $|E\tilde{f}^{(N)}_{X,+,h}(x)-f_X(x)|^2$. By (8) and (5),

$$\begin{aligned}
&E\tilde{f}^{(N)}_{X,+,h}(x)=E\Big[L^{(N)}_{+,h}(W_1-x)\Big]\\
&=\frac{(2\theta)^{\gamma}}{2\pi}\int_{-\infty}^{+\infty}\hat{K}(ivh)(iv)^{\gamma}E(e^{ivX_1})E(e^{iv\delta_1})e^{-iv(x+\gamma\theta)}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}e^{-i2\theta v(l_1+\cdots+l_\gamma)}\right]dv.
\end{aligned} \tag{30}$$

Note that $E(e^{ivX_1})=\int_{-\infty}^{+\infty}f_X(t)e^{ivt}dt$. Then, similar arguments to (15)–(17) show

$$E\tilde{f}^{(N)}_{X,+,h}(x)=\int_{-\infty}^{+\infty}\frac{1}{h}K(\frac{t-x}{h})f_X(t)dt+T_{+,\frac{N}{\gamma}}(f_X;x). \tag{31}$$

Similar to (27)–(29), we have

$$|E\tilde{f}^{(N)}_{X,+,h}(x)-f_X(x)|^2\le c_6\Big(A^2h^{2\beta}+\frac{B^2}{h^2N^{2r}}\Big). \tag{32}$$

Now, we estimate $\mathrm{var}[\tilde{p}^{(N)}_{+,h}(x)]$ and $\mathrm{var}[\tilde{f}^{(N)}_{X,+,h}(x)]$. By (8), we have

$$\begin{aligned}\mathrm{var}[\tilde{p}^{(N)}_{+,h}(x)] &\le \frac{1}{n}E\Big[|Y_1|^2|L^{(N)}_{+,h}(W_1-x)|^2\Big]\\ &= \frac{1}{n}E\Big[E\Big(|Y_1|^2|L^{(N)}_{+,h}(W_1-x)|^2|X_1\Big)\Big]\\ &= \frac{1}{n}\int_{-\infty}^{+\infty}E(|Y_1|^2|X_1=t)E|L^{(N)}_{+,h}(t+\delta_1-x)|^2 f_X(t)dt.\end{aligned}$$

Note that $\mathrm{var}(Y_1|X_1=t)=E(|Y_1|^2|X_1=t)-m^2(t)$. It follows from $\|\,\mathrm{var}(Y_1|X_1=\cdot)\,\|_\infty = E\epsilon_j^2$ and $\|\,m(\cdot)\,\|_\infty \le C_3$ that $\|\,E(|Y_1|^2|X_1=\cdot)\,\|_\infty \le \|\,\mathrm{var}(Y_1|X_1=\cdot)\,\|_\infty + \|\,m^2(\cdot)\,\|_\infty \le c_7$. Then,

$$\begin{aligned}\mathrm{var}[\tilde{p}^{(N)}_{+,h}(x)] &\le \frac{c_7}{n}\int_{-\infty}^{+\infty}E|L^{(N)}_{+,h}(t+\delta_1-x)|^2 f_X(t)dt\\ &= \frac{c_7}{n}E\Big[E(|L^{(N)}_{+,h}(W_1-x)|^2|X_1)\Big]\\ &= \frac{c_7}{n}\int_{-\infty}^{+\infty}\big|L^{(N)}_{+,h}(\omega-x)\big|^2 f_W(\omega)d\omega.\end{aligned} \tag{33}$$

It follows from (5) that

$$L^{(N)}_{+,h}(t)=\frac{(2\theta)^\gamma}{h^{\gamma+1}}\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}K^{(\gamma)}\Big(\frac{t-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)}{h}\Big).$$

Therefore,

$$\begin{aligned}\mathrm{var}[\tilde{p}^{(N)}_{+,h}(x)] &\le \frac{c_7}{n}\int_{-\infty}^{+\infty}\left|\frac{(2\theta)^\gamma}{h^{\gamma+1}}\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}K^{(\gamma)}\Big(\frac{\omega-x-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)}{h}\Big)\right|^2\\ &\quad\times f_W(\omega)d\omega\\ &\le \frac{c_7(2\theta)^{2\gamma}}{nh^{2\gamma+2}}\int_{-\infty}^{+\infty}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}\left|K^{(\gamma)}\Big(\frac{\omega-x-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)}{h}\Big)\right|\right]^2\\ &\quad\times f_W(\omega)d\omega.\end{aligned} \tag{34}$$

Let

$$C_{l,\gamma}:=\binom{l+\gamma-1}{\gamma-1},$$

where $C_{l,\gamma}$ is the number of weak compositions of l in γ parts [21]. Note that

$$\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}\left|K^{(\gamma)}\Big(\frac{\omega-x-\gamma\theta-2\theta(l_1+\cdots+l_\gamma)}{h}\Big)\right|\le\sum_{l=0}^{N}C_{l,\gamma}\left|K^{(\gamma)}\Big(\frac{\omega-x-\gamma\theta-2\theta l}{h}\Big)\right|. \tag{35}$$

Then,

$$\begin{aligned}\mathrm{var}[\tilde{p}^{(N)}_{+,h}(x)] &\le \frac{c_7(2\theta)^{2\gamma}}{nh^{2\gamma+2}}\sum_{l=0}^{N}\sum_{j=0}^{N}C_{l,\gamma}C_{j,\gamma}\int_{-\infty}^{+\infty}\left|K^{(\gamma)}\Big(\frac{\omega-x-\theta(\gamma+2l)}{h}\Big)\right|\\ &\quad\times\left|K^{(\gamma)}\Big(\frac{\omega-x-\theta(\gamma+2j)}{h}\Big)\right|f_W(\omega)d\omega.\end{aligned} \tag{36}$$

By $\text{supp}(K) \subseteq [-1,1]$, we have $\text{supp}\left[K^{(\gamma)}\left(\frac{\omega - x - \theta(\gamma+2l)}{h}\right)\right] \subseteq [x+\theta(\gamma+2l)-h, x+\theta(\gamma+2l)+h]$. Denote $I_{+,l}(x) := [x+\theta(\gamma+2l)-h, x+\theta(\gamma+2l)+h]$. If $h < \theta$, the intervals $I_{+,l}(x)$ and $I_{+,j}(x)$ are disjointed for $l \neq j$. For an h that is small enough, we obtain

$$\begin{aligned}\text{var}[\tilde{p}^{(N)}_{+,h}(x)] &\leq \frac{c_7(2\theta)^{2\gamma}}{nh^{2\gamma+2}} \sum_{l=0}^{N} C^2_{l,\gamma} \int_{-\infty}^{+\infty} \left|K^{(\gamma)}\left(\frac{\omega - x - \theta(\gamma+2l)}{h}\right)\right|^2 f_W(\omega)d\omega \\ &\leq \frac{c_8(2\theta)^{2\gamma}}{nh^{2\gamma+1}} \sum_{l=0}^{N} \frac{C^2_{l,\gamma}}{h} \int_{I_{+,l}(x)} f_W(\omega)d\omega.\end{aligned} \tag{37}$$

Denote $\xi_{+,l} := x + \theta(\gamma+2l)$. By $\text{supp}(f_\delta) \subseteq [-\gamma\theta, \gamma\theta]$ and $f_\delta \leq \frac{c_9}{\theta}$,

$$\begin{aligned}\frac{1}{h}\int_{I_{+,l}(x)} f_W(\omega)d\omega &\leq \frac{c_9}{\theta h}\int_{I_{+,l}(x)}\left[\int_{-\gamma\theta}^{\gamma\theta} f_X(\omega - t)dt\right]d\omega \\ &= \frac{c_9}{\theta h}\int_{-\infty}^{+\infty} f_X(u)\left[\int_{-\infty}^{+\infty} \chi_{(u-\gamma\theta, u+\gamma\theta)}(\omega)\chi_{(\xi_{+,l}-h, \xi_{+,l}+h)}(\omega)d\omega\right]du.\end{aligned}$$

Since $h < \gamma\theta$, we have

$$\begin{aligned}\frac{1}{h}\int_{I_{+,l}(x)} f_W(\omega)d\omega &\leq \frac{c_9}{\theta}\left[\int_{-h}^{h}(1+\frac{t}{h})f_X(t+\xi_{+,l}-\gamma\theta)dt + 2\int_{h-\gamma\theta}^{-h+\gamma\theta} f_X(t+\xi_{+,l})dt \right. \\ &\quad \left. + \int_{-h}^{h}(1-\frac{t}{h})f_X(t+\xi_{+,l}+\gamma\theta)dt\right] \\ &\leq \frac{c_{10}}{\theta}\left[\int_{-h}^{h} f_X(t+x+2\theta l)dt + \int_{-\gamma\theta}^{\gamma\theta} f_X(t+x+\theta(\gamma+2l))dt \right. \\ &\quad \left. + \int_{-h}^{h} f_X(t+x+2\theta(\gamma+l))dt\right].\end{aligned} \tag{38}$$

This, with (37), leads to

$$\begin{aligned}\text{var}[\tilde{p}^{(N)}_{+,h}(x)] \leq \frac{c_8 c_{10}(2\theta)^{2\gamma}}{nh^{2\gamma+1}} \sum_{l=0}^{N} \frac{C^2_{l,\gamma}}{\theta}&\left[\int_{-h}^{h} f_X(t+x+2\theta l)dt + \int_{-\gamma\theta}^{\gamma\theta} f_X(t+x+\theta(\gamma+2l))dt \right. \\ &\left. + \int_{-h}^{h} f_X(t+x+2\theta(\gamma+l))dt\right].\end{aligned} \tag{39}$$

When $\gamma > 1$, we obtain

$$\begin{aligned}\sum_{l=0}^{N} \frac{C^2_{l,\gamma}}{\theta}\int_{-h}^{h} f_X(t+x+2\theta l)dt &= \frac{1}{\theta}\int_{x-h}^{x+h} f_X(t)dt + \sum_{l=1}^{N}\frac{C^2_{l,\gamma}}{\theta}\int_{x+2\theta l-h}^{x+2\theta l+h} f_X(t)dt \\ &\leq \frac{1}{\theta} + c_{11}B\theta^{-2\gamma+1}\end{aligned} \tag{40}$$

by $f_X \in \mathscr{M}_{2\gamma-2}(B)$ and similar arguments to [16]. Similarly,

$$\sum_{l=0}^{N} \frac{C^2_{l,\gamma}}{\theta}\int_{-\gamma\theta}^{\gamma\theta} f_X(t+x+\theta(\gamma+2l))dt \leq \frac{1}{\theta} + c_{12}B\theta^{-2\gamma+1},$$

and

$$\sum_{l=0}^{N} \frac{C_{l,\gamma}^2}{\theta} \int_{-h}^{h} f_X(t + x + 2\theta(\gamma + l))dt \leq \frac{1}{\theta} + c_{13} B\theta^{-2\gamma+1}.$$

When $\gamma = 1$, we have that

$$\sum_{l=0}^{N} \frac{1}{\theta} \int_{-h}^{h} f_X(t + x + 2\theta l)dt = \sum_{l=0}^{N} \frac{1}{\theta} \int_{x+2\theta l-h}^{x+2\theta l+h} f_X(t)dt \leq \frac{1}{\theta} \int_{x-h}^{x+2\theta N+h} f_X(t)dt \leq \frac{1}{\theta} \quad (41)$$

holds for $h < \theta$. Similar to (41), for $h < \theta$, we have

$$\sum_{l=0}^{N} \frac{1}{\theta} \int_{-\theta}^{\theta} f_X(t + x + \theta(2l + 1))dt \leq \frac{1}{\theta} \quad \text{and} \quad \sum_{l=0}^{N} \frac{1}{\theta} \int_{-h}^{h} f_X(t + x + 2\theta(l + 1))dt \leq \frac{1}{\theta}.$$

Hence,

$$\mathrm{var}[\tilde{p}_{+,h}^{(N)}(x)] \leq \frac{c_{14}(2\theta)^{2\gamma}}{nh^{2\gamma+1}} \left(B\theta^{-2\gamma+1} + \frac{3}{\theta} \right) \leq c_{15}(B\theta + \theta^{2\gamma-1})(nh^{2\gamma+1})^{-1}. \quad (42)$$

Similar to estimate $\mathrm{var}[\tilde{p}_{+,h}^{(N)}(x)]$, we have

$$\mathrm{var}[\tilde{f}_{X,+,h}^{(N)}(x)] \leq \frac{1}{n} E\left[|L_{+,h}^{(N)}(W_1 - x)|^2\right] \leq c_{16}(B\theta + \theta^{2\gamma-1})(nh^{2\gamma+1})^{-1}. \quad (43)$$

By (29), (32), (42) and (43) with (24)–(26), we obtain

$$P\left[|\tilde{m}_{+,h}^{(N)}(x) - m(x)|^2 \geq C \cdot \varepsilon_n\right] \leq \frac{c_{17}}{C\varepsilon_n} \left[A^2 h^{2\beta} + \frac{B^2}{h^2 N^{2r}} + (B\theta + \theta^{2\gamma-1})(nh^{2\gamma+1})^{-1} \right]. \quad (44)$$

Since $h = n^{\frac{-1}{2\beta+2\gamma+1}}$ and $N \geq n^{\frac{\beta+1}{r(2\beta+2\gamma+1)}}$,

$$P\left[|\tilde{m}_{+,h}^{(N)}(x) - m(x)|^2 \geq C \cdot \varepsilon_n\right] \leq \frac{c_{17}(A^2 + B^2 + B\theta + \theta^{2\gamma-1})}{C\varepsilon_n} \cdot n^{\frac{-2\beta}{2\beta+2\gamma+1}}.$$

Note that $\varepsilon_n = n^{\frac{-2\beta}{2\beta+2\gamma+1}}$. Then,

$$\sup_{(m,f_X) \in \mathscr{P}_{\beta,r;x}} P\left[|\tilde{m}_{+,h}^{(N)}(x) - m(x)|^2 \geq C \cdot \varepsilon_n\right] \leq c_{17}(A^2 + B^2 + B\theta + \theta^{2\gamma-1})C^{-1}. \quad (45)$$

This leads to the result of Theorem 2 for $x \geq 0$.
(2) We consider the estimator $\tilde{m}_{-,h}^{(N)}(x)$ for $x < 0$. By (22), (23) and (28), we have

$$\begin{aligned} |E\tilde{p}_{-,h}^{(N)}(x) - p(x)|^2 &\leq 2\left(\left| \int_{-1}^{1} K(y)[p(x + yh) - p(x)]dy \right|^2 + \left| T_{-,\frac{N}{\gamma}}(p; x) \right|^2 \right) \\ &\leq c_{18} \left(A^2 h^{2\beta} + \frac{B^2}{h^2 N^{2r}} \right). \end{aligned}$$

Similar arguments to (30)–(32) show

$$|E\tilde{f}_{X,-,h}^{(N)}(x) - f_X(x)|^2 \leq c_{19} \left(A^2 h^{2\beta} + \frac{B^2}{h^2 N^{2r}} \right).$$

Similar to (33),

$$\operatorname{var}[\tilde{p}^{(N)}_{-,h}(x)] \leq \frac{c_{20}}{n}\int_{-\infty}^{+\infty}\left|L^{(N)}_{-,h}(\omega-x)\right|^2 f_W(\omega)d\omega,$$

and from (6),

$$L^{(N)}_{-,h}(t) = \frac{(-2\theta)^{\gamma}}{h^{\gamma+1}}\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}} K^{(\gamma)}\Big(\frac{t+\gamma\theta+2\theta(l_1+\cdots+l_\gamma)}{h}\Big).$$

Similar arguments to (34)–(37) show

$$\begin{aligned}\operatorname{var}[\tilde{p}^{(N)}_{-,h}(x)] \leq{}& \frac{c_{20}(2\theta)^{2\gamma}}{nh^{2\gamma+2}}\int_{-\infty}^{+\infty}\left[\sum_{l_1=0}^{\frac{N}{\gamma}}\cdots\sum_{l_\gamma=0}^{\frac{N}{\gamma}}\left|K^{(\gamma)}\Big(\frac{\omega-x+\gamma\theta+2\theta(l_1+\cdots+l_\gamma)}{h}\Big)\right|\right]^2 \\ &\times f_W(\omega)d\omega \\ \leq{}& \frac{c_{21}(2\theta)^{2\gamma}}{nh^{2\gamma+1}}\sum_{l=0}^{N}\frac{C^2_{l,\gamma}}{h}\int_{I_{-,l}(x)} f_W(\omega)d\omega\end{aligned}$$

holds for an h that is small enough, where $I_{-,l}(x) := [x-\theta(\gamma+2l)-h, x-\theta(\gamma+2l)+h]$. Denote $\xi_{-,l} := x-\theta(\gamma+2l)$. Similar to (38),

$$\begin{aligned}\frac{1}{h}\int_{I_{-,l}(x)} f_W(\omega)d\omega \leq{}& \frac{c_{22}}{\theta}\Big[\int_{-h}^{h}(1+\frac{t}{h})f_X(t+\xi_{-,l}-\gamma\theta)dt + 2\int_{h-\gamma\theta}^{-h+\gamma\theta} f_X(t+\xi_{-,l})dt \\ &+ \int_{-h}^{h}(1-\frac{t}{h})f_X(t+\xi_{-,l}+\gamma\theta)dt\Big] \\ \leq{}& \frac{c_{23}}{\theta}\Big[\int_{-h}^{h} f_X(t+x-2\theta(\gamma+l))dt + \int_{-\gamma\theta}^{\gamma\theta} f_X(t+x-\theta(\gamma+2l))dt \\ &+ \int_{-h}^{h} f_X(t+x-2\theta l))dt\Big].\end{aligned}$$

By similar arguments to (39)–(42), we have

$$\operatorname{var}[\tilde{p}^{(N)}_{-,h}(x)] \leq c_{24}(B\theta+\theta^{2\gamma-1})(nh^{2\gamma+1})^{-1},$$

and

$$\operatorname{var}[\tilde{f}^{(N)}_{X,-,h}(x)] \leq \frac{1}{n}E\Big[|L^{(N)}_{-,h}(W_1-x)|^2\Big] \leq c_{25}(B\theta+\theta^{2\gamma-1})(nh^{2\gamma+1})^{-1}.$$

Similar to (45),

$$\sup_{(m,f_X)\in\mathscr{P}_{\beta,r;x}} P\Big[|\tilde{m}^{(N)}_{-,h}(x)-m(x)|^2 \geq C\cdot\varepsilon_n\Big] \leq c_{26}(A^2+B^2+B\theta+\theta^{2\gamma-1})C^{-1}.$$

This leads to the result of Theorem 2 for $x<0$.
This completes the proof. □

Remark 2. *Our convergence rate is the same as that in the ordinary smoothness case of Meister [6], where the density function of the covariate error does not vanish in the Fourier domain. Compared to Delaigle and Meister [1], we do not assume f_X and m to be compact.*

Remark 3. *Belomestny and Goldenshluger [16] consider the density deconvolution problem with non-standard error distributions. They assume the density function to be estimated satisfies the Hölder condition. It is natural to assume a local smooth condition in point estimation. Hence, f_X and mf_X are assumed to satisfy the local Hölder condition in our discussion.*

Remark 4. *Theorem 1 shows the strong consistency of the regression estimator without the smoothness assumption. The main tool used is the Borel–Cantelli lemma which requires a convergent series. It is easy to see from (13) and (20) that the choice of h is not unique. Theorem 2 gives a weak convergence rate, which is defined by modifying the weak consistency. It is natural to assume the smoothness condition when discussing the convergence rate. In Theorem 2, the choice of h is related to the smoothness index β. It follows from our proof (44) that the choice of h is unique in the sense of a constant difference.*

Remark 5. *In our discussion, $\hat{f}_\delta(iv) = \left[\frac{\sinh(i\theta v)}{i\theta v}\right]^\gamma = \left[\frac{\sin(\theta v)}{\theta v}\right]^\gamma$. Substituting this into the proof of Theorem 3.5 in [6], one can obtain the optimality of convergence rate in our Theorem 2. This means that there does not exist an estimator $\tilde{m}(x)$ of the regression function $m(x)$ based on i.i.d data $(W_1, Y_1), \cdots, (W_n, Y_n)$ generated by model (1) with (2), which satisfies*

$$\lim_{C\to\infty}\left(\limsup_{n\to\infty} \sup_{(m,f_X)\in\mathscr{P}_{\beta,r;x}} P\left[|\tilde{m}(x) - m(x)|^2 \geq C\cdot\circ\left(n^{\frac{-2\beta}{2\beta+2\gamma+1}}\right)\right]\right) = 0.$$

It would be interesting to study the numerical illustration of our estimation. We shall investigate this in the future.

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