*Article*

# Innovative Strategy for Constructing Soft Topology

**Amlak I. Alajlan** [1,2,*] and **Ahmad M. Alghamdi** [1]

1   Mathematics Department, Faculty of Sciences, Umm Al-Qura University, P.O. Box 14035, Makkah 21955, Saudi Arabia; amghamdi@uqu.edu.sa
2   Department of Mathematics, College of Sciences, Qassim University, Buraydah 51452, Saudi Arabia
*   Correspondence: s44277172@st.uqu.edu.sa

**Abstract:** To address the complexity of daily problems, soft set theory has emerged as a valuable tool, providing innovative mathematical techniques to manage vast amounts of data and ambiguity. The study of soft topology involves the investigation of various properties of soft sets and functions, as well as the development of new mathematical models and techniques for addressing uncertainty. The main motivation of this paper is to delve deeper into the subject and devise new methodologies to address real-world challenges more effectively and unlock the full potential of soft sets in various applications. In this paper, we present a novel soft topology, which is constructed using soft single points on a nonempty set $V$ in relation to a topology on $V$. We investigate and study the behaviors and properties associated with this particular type of soft topology. Furthermore, we shed light on the soft separation axioms with this type of soft topology and investigate whether these axioms are inherited from the corresponding ordinary topology or not. Our study is concerned with examining the connection between ordinary topologies and the soft topologies generated that arise from them, with the aim of identifying their interdependencies and potential implications. By studying the connection between soft topologies and their corresponding ordinary topologies, researchers are able to gain a deeper understanding of the properties and behaviors of these structures and develop new modeling approaches for dealing with uncertainty and complexity in data.

**Keywords:** soft set; soft single point; soft topology; soft separation axioms; topological space; soft continuity

**MSC:** 03E72; 47H10; 54A05; 54D10; 54D15

## 1. Introduction

Experts across various disciplines, including medical science, economics, systems engineering, artificial intelligence, and other fields, regularly encounter the challenge of molding complex systems that involve uncertainty. While classical probability theory, fuzzy set theory [1], and rough set theory [2] are commonly used mathematical tools to address such problems, they may not always yield satisfactory results due to limitations in their parameterizations. In 1999, Molodtsov [3] explored a fresh strategy for dealing with uncertainty, known as soft set theory, which overcomes the shortcomings of other methods. The technique involves the use of soft sets, which are a parametrized family of subsets of a universal set. Unlike other methods, soft set theory does not impose any specific conditions for object illumination and parameters can be selected in various forms, such as sentences, words, numbers, or mappings. As a result, the theory is highly flexible and convenient to apply in practical settings. In addition, Molodtsov applied soft set theory to various areas including the study of function smoothness, game theory, and more, demonstrating its versatility and broad applicability. In [4], Maji et al. examined the role of soft sets in decision-making problems and presented a technique for aggregating soft set parameters to facilitate decision making. In this context, a soft set can be employed to represent each option or alternative under consideration, with its parameters reflecting the various factors

that are relevant to the decision, such as financial cost, danger, potential benefit, and other quantitative or qualitative measures. By combining and analyzing these parameters, the proposed method enables decision makers to arrive at informed and effective decisions. In 2003, Maji et al. [5] presented and analyzed the concept of soft operations between soft sets. Yang [6] focused on rectifying certain properties in soft set theory that were introduced in [5]. In [7], several researchers have sought to refine some soft operations or develop new ones, in order to enhance their functionality and utility in soft set theory. Recently, in 2019, Hussain et al. [8] defined soft single points and soft real points. They have found intriguing and fundamental characteristics.

The exploration of general topology is a notable and significant area of mathematics that focuses on the application of concepts from set theory and structures employed in the field of topology. In 2011, Shabir and Naz [9] initiated a new area of study within topology called soft topology, which combines the theories of soft sets and topology. They established several fundamental concepts such as soft open (closed), soft neighborhood, soft closure, soft separation axioms, soft regular spaces, and soft normal spaces. In 2012, Hussain and Ahmad [10] furthered the study of soft topology by exploring the properties of soft open (closed), soft neighborhood, and soft closure. In addition, they presented the concepts of the soft interior, soft exterior, and soft boundary. Zarlutana et al. [11] explored and studied the concept of soft mappings and soft continuous mapping between two collections of soft sets. In 2015, Hussain [12] established further characterization of soft continuous mapping.

Numerous research studies have showcased practical applications of soft sets and their topologies across diverse disciplines to address real-world issues. This underscores the importance of studying and investigating these sets and their topologies, providing a strong motivation for us to delve deeper into the subject and devise new methodologies to address real-world challenges more effectively and unlock the full potential of soft sets in various applications. This leads to improved solutions in the face of uncertainty and complexity.

This article aims to establish a cohesive perspective to advance the development of soft topological spaces by building upon the findings of ordinary topology. Our paper presents an innovative methodology to construct a soft topology by utilizing the set V both as an initial universe set and as a set of parameters. Let $\kappa$ be a topology on $V$, and let $\tilde{B} = \{\tilde{L}_{v,Q} : \text{for all } v \in V \text{ and } Q \in \kappa\} \,\tilde{\cup}\, \tilde{\Phi}$ be the collection of all soft single points associated with $\kappa$, where $Q$ is an open neighborhood of an element $v$ in $\kappa$. We proceed to construct a soft topology based on this collection $\tilde{B}$, which forms a base for this soft topology. We will study the relationship between the soft closure (the soft interior) and the closure (the interior) of any soft set, with respect to soft topology and its corresponding ordinary topology. We will prove that two soft topologies that are generated by soft single points are soft homeomorphic if their corresponding ordinary topologies are homeomorphic. We will define a soft relative topology generated by soft single points with respect to an ordinary relative topology. Furthermore, we will study the soft separation axioms that were introduced in [9] of a soft topology generated by soft single points, with respect to an ordinary topology. We will investigate whether these axioms are inherited from an ordinary topology or not. Investigation methods of producing soft topology from classical topology is an important and fruitful area of research in topology and related fields. Researchers are able to gain a deeper understanding of the properties and behaviors of these structures and develop new tools and techniques for studying these spaces, which can then be applied to other areas of mathematics and science.

The organization of the paper is summarized in the following sections. This section comprises an introduction and two subsections. The first subsection provides a comprehensive literature review for constructing soft topology, while the second subsection points out the related topics of soft sets and their applications. Section 2 provides a review of important definitions and results from soft set theory and soft topological spaces that are relevant to the present study. In Section 3, we introduce the generated soft topology by soft single points on a nonempty set $V$ with respect to any ordinary topology on $V$. We prove that two soft topologies are soft homeomorphic if their corresponding ordinary

topologies are homeomorphic. Also, we examine the soft relative topology generated by soft single points with respect to a relative topology. In Section 4, we investigate the soft separation axioms for this type of soft topology. We verify whether the soft separation axioms of this type of soft topology are inherited from its corresponding ordinary topology or not. In Section 5, we present our conclusion which supports our results and explains the future work.

### 1.1. Literature Review of Studies of Constructing Soft Topology

Let us review the literature that examined the topology and different methods of constructing soft topology. In [13], Milan conducted the study on soft topological space, exploring their connection with a topological space defined on the Cartesian product of two sets through a homeomorphism. In 2019, Terepeta [14] discussed a method of constructing a soft topology from any ordinary topology on a nonempty set $V$ while considering any possible set of parameters. In [15], Al Ghour et al. explored a process of constructing a soft topology by utilizing an indexed family of ordinary topologies on a nonempty set $V$ while considering any possible set of parameters. In 2020, Alcantud [16] thoroughly revisited the standard method for deriving soft topology from ordinary topology, which was introduced in [14]. Furthermore, they delved into the concepts of soft separability and soft countability within the framework of this soft topology. In 2022, Azzam et al. [17] conducted an exploration of six soft operators and employed them to construct soft topologies. They inferred that all resulting soft topologies are equivalent and share identical properties with their classical counterparts when considering enriched and extended conditions. Recently, El-Atik and Azzam [18] conducted research on a method of transformation to depict the complex networks of the human brain more simplistically. In addition, they explored a topological model for simplicial complexes and utilized it to represent the brain as a union of simplicial complexes. This approach potentially offers a way to diagnose brain cancer.

### 1.2. Related Topics to Soft Set Theory and Their Applications

In 1965, Zadeh [1] proposed a fuzzy set theory as a way to handle uncertainty in data representation and reasoning. Maji et al. [19] presented the concept of fuzzy soft sets. This concept combines the principles of fuzzy sets and soft sets, providing a more comprehensive framework for handling uncertainty in decision making and data analysis. The application of fuzzy sets and fuzzy soft sets has been studied by many researchers across various domains.

In 1982, Pawlak [2] presented a rough set theory: a mathematical framework that offers a systematic approach to address the challenges of uncertainty and imprecision in data analysis. In 2011, Feng et al. [20] worked to generalize the rough sets model that builds upon soft sets and proposed the concept of soft rough sets. The soft rough set is characterized as a parametrized subset of a set, utilized for determining lower and upper approximations of a subset, diverging from the traditional approach of equivalent classes. This soft rough set has been applied in decision making by many researchers. Several academic experts employed soft rough sets to address medical challenges, like the diagnosis of Chikungunya disease, the diagnosis of Coronavirus disease, and many medical applications. One of our plans is to conduct a comprehensive study and undertake the construction of soft topology on these sets in the future.

## 2. Preliminaries

This section provides a summary of relevant definitions and results with soft sets theory and soft topology that will be referenced later in the paper. In the rest of the paper, we use $V$ to denote an initial universe set and $P(V)$ to denote its power set. The symbol $\varnothing$ is used to denote the empty set.

*2.1. Soft Sets Theory*

**Definition 1.** *[3] Suppose that $D$ is a set of parameters and $L$ is a mapping from $D$ into $P(V)$. A soft set over $V$, symbolized by $\tilde{L}_D$, is defined as:*

$$\tilde{L}_D = \{(d, L(d)) : d \in D \text{ and } L(d) \subseteq V\}.$$

*The collection of all soft sets over $V$ with respect to the set of parameters $D$ will be denoted by $S(V)_D$.*

**Definition 2.** *[5,7]*

- *A soft set $\tilde{L}_D$ over $V$ is said to be a null soft set if $L(d) = \varnothing$ for all $d \in D$ and is symbolized by $\tilde{\Phi}_D$.*
- *A soft set $\tilde{L}_D$ over $V$ is said to be an absolute soft set if $L(d) = V$ for all $d \in D$ and is symbolized by $\tilde{V}_D$.*
- *The relative complement of a soft set $\tilde{L}_D$ is the soft set*

$$\tilde{L}_D^c = \{(d, L(d)^c) : d \in D \text{ and } L^c : D \to P(V) \text{ such that } L^c(d) = L(d)^c\},$$

*where $L(d)^c$ is the complement of a subset $L(d)$.*

**Definition 3.** *Suppose that $\tilde{L}_D$ and $\tilde{Q}_E$ are soft sets over $V$.*

- *Ref. [6] $\tilde{L}_D$ is a soft subset of $\tilde{Q}_E$, symbolized by $\tilde{L}_D \tilde{\subseteq} \tilde{Q}_E$, if $D \subseteq E$ and $L(d) \subseteq Q(d)$ for all $d \in D$. The soft sets $\tilde{L}_D$ and $\tilde{Q}_E$ are soft equal if $\tilde{L}_D \tilde{\subseteq} \tilde{Q}_E$ and $\tilde{Q}_E \tilde{\subseteq} \tilde{L}_D$.*
- *Ref. [7] The restricted intersection of two soft sets $\tilde{L}_D$ and $\tilde{Q}_E$, symbolized by $\tilde{L}_D \tilde{\cap} \tilde{Q}_E$, is the soft set $\tilde{T}_H$, such that $H = D \cap E$, and $T$ represents a map from $H$ into $P(V)$ given by $T(h) = L(h) \cap Q(h)$ for all $h \in H$.*
- *Ref. [5] The union of two soft sets $\tilde{L}_D$ and $\tilde{Q}_E$, symbolized by $\tilde{L}_D \tilde{\cup} \tilde{Q}_E$, is the soft set $\tilde{T}_H$, such that $H = D \cup E$, and $T$ represents a map from $H$ into $P(V)$ defined as:*

$$T(h) = \begin{cases} L(h) & \text{if } h \in D - E, \\ Q(h) & \text{if } h \in E - D, \\ L(h) \cup Q(h) & \text{if } h \in D \cap E. \end{cases}$$

**Definition 4.** *Suppose that $\tilde{L}_D, \tilde{Q}_D \in S(V)_D$. We say that $\tilde{L}_D$ and $\tilde{Q}_D$ are soft disjoint if $L(d) \cap Q(d) = \varnothing$ for all $d \in D$.*

**Definition 5.** *[9] Suppose that $\tilde{L}_D$ is a soft set over $V$ and $v \in V$. We say that $v$ belongs to the soft set $\tilde{L}_D$ if $v \in L(d)$ for all $d \in D$. If $v$ does not belong to $L(d)$ for some $d \in D$, then $v \notin \tilde{L}_D$.*

Now, we recall the definition of soft mappings between two collections of soft sets.

**Definition 6.** *[11] Suppose that $D$ and $E$ are sets of parameters and $V$ and $Z$ are initial universe sets. We assume $S(V)_D$ and $S(Z)_E$ are two collections of soft sets and*

$$\lambda : V \to Z \text{ and } \mu : D \to E$$

*are mappings. Then a soft mapping*

$$\Upsilon_{\lambda,\mu} : S(V)_D \to S(Z)_E$$

*is defined by :*

1. *The image of $\tilde{L}_D \in S(V)_D$ is a soft set in $S(Z)_E$ such that for all $e \in E$,*

$$\Upsilon_{\lambda,\mu}(\tilde{L}_D)(e) = \begin{cases} \underset{d \in \mu^{-1}(e)}{\cup} \lambda(L(d)) & \text{if } \mu^{-1}(e) \neq \varnothing, \\ \varnothing & \text{if } \mu^{-1}(e) = \varnothing. \end{cases}$$

2.     The inverse image of $\tilde{Q}_E \in S(Z)_E$, symbolized by $Y^{-1}_{\lambda,\mu}(\tilde{Q}_E)$, is the soft set $\tilde{L}_D$ such that $L(d) = \lambda^{-1}(Q(\mu(d)))$ for all $d \in D$.

**Remark 1.** *The map $Y_{\lambda,\mu}$ is injective (resp. surjective, bijective) if $\lambda$ and $\mu$ in Definition 6 are injective (resp. surjective, bijective) maps.*

The following definitions and results center around the soft single points that were presented in [8].

**Definition 7.** *Suppose that $V$ is a nonempty set. Then for each $v \in V$ and $M \subseteq V$, we have a function called a fixed point function,*

$$L :\{v\} \to P(V)$$
$$v \longmapsto M.$$

The set $\{(v, L(v))\}$ is called a soft single point on $V$. The function $L$ varies uniquely for each $v \in V$ and $M \subseteq V$. The collection of all soft single points on $V$ is symbolized by $S_V$.

**Remark 2.**

- The set $\{(v, \varnothing)\}$ is considered an empty soft single point for each $v \in V$.
- If $V$ is a set containing n elements, then the total count of soft single points on $V$ is equal to $|S_V| = n2^n$.

**Example 1.** *Consider the set $V = \{v_1, v_2, v_3\}$. Then*

$S_V = \{\{(v_1, V)\}, \{(v_2, V)\}, \{(v_3, V)\}, \{(v_1, \{v_1\})\}, \{(v_1, \{v_2\})\}, \{(v_1, \{v_3\})\},$
$\{(v_2, \{v_1\})\}, \{(v_2, \{v_2\})\}, \{(v_2, \{v_3\})\}, \{(v_3, \{v_1\})\}, \{(v_3, \{v_2\})\}, \{(v_3, \{v_3\})\},$
$\{(v_1, \{v_1, v_2\})\}, \{(v_2, \{v_1, v_2\})\}, \{(v_3, \{v_1, v_2\})\}, \{(v_1, \{v_2, v_3\})\}, \{(v_2, \{v_2, v_3\})\},$
$\{(v_3, \{v_2, v_3\})\}, \{(v_1, \{v_1, v_3\})\}, \{(v_2, \{v_1, v_3\})\}, \{(v_3, \{v_1, v_3\})\}, \{(v_1, \varnothing)\},$
$\{(v_2, \varnothing)\}, \{(v_3, \varnothing)\}\}.$

We observe that $|S_V| = 24 = 3 \times 2^3$.

**Definition 8.** *Two soft single points $\{(v_1, M_1)\}, \{(v_2, M_2)\}$ on a nonempty set $V$ are said to be equal if and only if $v_1 = v_2$ and $M_1 = M_2$.*

*2.2. Soft Topology*

**Definition 9.** *[9] Let $\tilde{\kappa}$ be a collection of $S(V)_D$. Then, $(V, \tilde{\kappa})_D$ is called a soft topological space if and only if the following requirements are met:*

1.     $\tilde{V}_D$ and $\tilde{\Phi}_D$ belong to $\tilde{\kappa}$.
2.     The union of any number of soft sets in $\tilde{\kappa}$ belongs to $\tilde{\kappa}$.
3.     The restricted intersection of any two soft sets in $\tilde{\kappa}$ belongs to $\tilde{\kappa}$.

Every element of $\tilde{\kappa}$ is called a soft open set in $V$. The relative complement of the soft open set is called a soft closed set in $V$.

**Proposition 1.** *[9] Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space. Then, for all $d \in D$, we have $\kappa_d = \{L(d) : \tilde{L}_D \in \tilde{\kappa}\}$ is a topology on $V$.*

The validity of the converse of the above proposition is generally not true, as stated in reference [9].

**Definition 10.** *[9,10] Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space and $\tilde{L}_D \in S(V)_D$.*

1. *The soft closure of $\tilde{L}_D$ is the restricted intersection of all soft closed sets which contain $\tilde{L}_D$ and is symbolized by $Cl_{\tilde{\kappa}}(\tilde{L}_D)$.*
2. *The soft interior of $\tilde{L}_D$ is the union of all soft open sets which are contained in $\tilde{L}_D$ and is symbolized by $Int_{\tilde{\kappa}}(\tilde{L}_D)$.*

**Theorem 1.** *[9] Let $(V, \tilde{\kappa})_D$ be a soft topological space and $\tilde{L}_D, \tilde{Q}_D \in S(V)_D$.*

1. $Cl_{\tilde{\kappa}}(\tilde{\Phi}_D) = \tilde{\Phi}_D$ *and* $Cl_{\tilde{\kappa}}(\tilde{V}_D) = \tilde{V}_D$.
2. $\tilde{L}_D \tilde{\subseteq} Cl_{\tilde{\kappa}}(\tilde{L}_D)$.
3. $\tilde{L}_D$ *is a soft closed set if and only if* $\tilde{L}_D = Cl_{\tilde{\kappa}}(\tilde{L}_D)$.
4. $Cl_{\tilde{\kappa}}(Cl_{\tilde{\kappa}}(\tilde{L}_D)) = Cl_{\tilde{\kappa}}(\tilde{L}_D)$.
5. $\tilde{L}_D \tilde{\subseteq} \tilde{Q}_D \Rightarrow Cl_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\subseteq} Cl_{\tilde{\kappa}}(\tilde{Q}_D)$.
6. $Cl_{\tilde{\kappa}}(\tilde{L}_D \tilde{\cup} \tilde{Q}_D) = Cl_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\cup} Cl_{\tilde{\kappa}}(\tilde{Q}_D)$.
7. $Cl_{\tilde{\kappa}}(\tilde{L}_D \tilde{\cap} \tilde{Q}_D) \tilde{\subseteq} Cl_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\cap} Cl_{\tilde{\kappa}}(\tilde{Q}_D)$.

**Theorem 2.** *[10] Let $(V, \tilde{\kappa})_D$ be a soft topological space and $\tilde{L}_D, \tilde{Q}_D \in S(V)_D$.*

1. $Int_{\tilde{\kappa}}(\tilde{\Phi}_D) = \tilde{\Phi}_D$ *and* $Int_{\tilde{\kappa}}(\tilde{V}_D) = \tilde{V}_D$.
2. $Int_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\subseteq} \tilde{L}_D$.
3. $\tilde{L}_D$ *is a soft open set if and only if* $\tilde{L}_D = Int_{\tilde{\kappa}}(\tilde{L}_D)$.
4. $Int_{\tilde{\kappa}}(Int_{\tilde{\kappa}}(\tilde{L}_D)) = Int_{\tilde{\kappa}}(\tilde{L}_D)$.
5. $\tilde{L}_D \tilde{\subseteq} \tilde{Q}_D \Rightarrow Int_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\subseteq} Int_{\tilde{\kappa}}(\tilde{Q}_D)$.
6. $Int_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\cap} Int_{\tilde{\kappa}}(\tilde{Q}_D) = Int_{\tilde{\kappa}}(\tilde{L}_D \tilde{\cap} \tilde{Q}_D)$.
7. $Int_{\tilde{\kappa}}(\tilde{L}_D) \tilde{\cup} Int_{\tilde{\kappa}}(\tilde{Q}_D) \tilde{\subseteq} Int_{\tilde{\kappa}}(\tilde{L}_D \tilde{\cup} \tilde{Q}_D)$.

**Definition 11.** *[9] Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space and $\tilde{L}_D \in S(V)_D$. Then, the closure of $\tilde{L}_D$ with respect to the topology $\kappa_d$ is the soft set $Cl_{\kappa_d}(\tilde{L}_D)$, where $Cl_{\kappa_d}(\tilde{L}_D) = \{(v, Cl_{\kappa_d}(L(d))) : \text{for all } d \in D\}$.*

**Proposition 2.** *[9] For any soft topological space $(V, \tilde{\kappa})_D$ and $\tilde{L}_D \in S(V)_D$, we have $Cl_{\kappa_d}(\tilde{L}_D) \tilde{\subseteq} Cl_{\tilde{\kappa}}(\tilde{L}_D)$.*

**Proposition 3.** *[9] Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space and $\tilde{L}_D \in S(V)_D$. Then, $Cl_{\kappa_d}(\tilde{L}_D) = Cl_{\tilde{\kappa}}(\tilde{L}_D)$ if and only if $Cl_{\kappa_d}(\tilde{L}_D)$ is a soft closed set.*

**Theorem 3.** *[11] Suppose that $(V, \tilde{\kappa}_1)_D$ and $(Z, \tilde{\kappa}_2)_E$ are two soft topological spaces. Suppose that $\lambda : V \rightarrow Z$ and $\mu : D \rightarrow E$ are two mappings. Then, a map $Y_{\lambda, \mu} : S(V)_D \rightarrow S(Z)_E$ is said to be soft continuous if and only if for all $\tilde{Q}_E \in \tilde{\kappa}_2$, $Y_{\lambda, \mu}^{-1}(\tilde{Q}_E) \in \tilde{\kappa}_1$.*

**Definition 12.** *[12] Suppose that $(V, \tilde{\kappa}_1)_D$ and $(Z, \tilde{\kappa}_2)_E$ are two soft topological spaces. Suppose that $\lambda : V \rightarrow Z$ and $\mu : D \rightarrow E$ are two mappings. Then, a map $Y_{\lambda, \mu} : S(V)_D \rightarrow S(Z)_E$ is said to be a soft homeomorphism if*

1. $Y_{\lambda, \mu}$ *is soft continuous;*
2. $Y_{\lambda, \mu}$ *is bijection;*
3. $Y_{\lambda, \mu}^{-1}$ *is soft continuous.*

**Definition 13.** *[9] Let $(V, \tilde{\kappa})_D$ be a soft topological space and $M$ be a nonempty subset of $V$. Then, the soft relative topology on $M$ is defined as:*

$$\tilde{\kappa}_M = \{\tilde{N}_D : N(d) = L(d) \cap M \text{ for all } d \in D \text{ and } \tilde{L}_D \in \tilde{\kappa}\}.$$

Now, we state the main definitions of soft separation axioms with respect to ordinary points in $V$, which were introduced in [9].

**Definition 14.** *Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space.*

1. $(V, \tilde{\kappa})_D$ is said to be a soft $T_0$-space if for any two distinct points $v, w \in V$, there exists a soft open set $\tilde{L}_D$ such that either $v \in \tilde{L}_D$ and $w \notin \tilde{L}_D$ or $v \notin \tilde{L}_D$ and $w \in \tilde{L}_D$.
2. $(V, \tilde{\kappa})_D$ is said to be a soft $T_1$-space if for any two distinct points $v, w \in V$, there exist two soft open sets $\tilde{L}_D$ and $\tilde{Q}_D$ such that $v \in \tilde{L}_D, w \notin \tilde{L}_D$ and $v \notin \tilde{Q}_D, w \in \tilde{Q}_D$.
3. $(V, \tilde{\kappa})_D$ is said to be a soft $T_2$-space if for any two distinct points $v, w \in V$, there exist two soft open sets $\tilde{L}_D$ and $\tilde{Q}_D$ such that $v \in \tilde{L}_D, w \in \tilde{Q}_D$, and $\tilde{L}_D \tilde{\cap} \tilde{Q}_D = \tilde{\Phi}_D$.
4. $(V, \tilde{\kappa})_D$ is said to be a soft regular space if for any soft closed set $\tilde{T}_D$ and $v \notin \tilde{T}_D$, there exist two soft open sets $\tilde{L}_D$ and $\tilde{Q}_D$ such that $v \in \tilde{L}_D, \tilde{T}_D \tilde{\subseteq} \tilde{Q}_D$, and $\tilde{L}_D \tilde{\cap} \tilde{Q}_D = \tilde{\Phi}_D$.
5. $(V, \tilde{\kappa})_D$ is said to be a soft normal space if for any two soft disjoint closed sets $\tilde{R}_D$ and $\tilde{T}_D$ in $\tilde{\kappa}$, there exist two soft open sets $\tilde{L}_D$ and $\tilde{Q}_D$ such that $\tilde{R}_D \tilde{\subseteq} \tilde{L}_D, \tilde{T}_D \tilde{\subseteq} \tilde{Q}_D$, and $\tilde{L}_D \tilde{\cap} \tilde{Q}_D = \tilde{\Phi}_D$.

**Definition 15.** *Suppose that $(V, \tilde{\kappa})_D$ is a soft topological space.*

1. $(V, \tilde{\kappa})_D$ *is said to be a soft $T_3$-space if it is both soft $T_1$-space and soft regular space.*
2. $(V, \tilde{\kappa})_D$ *is said to be a soft $T_4$-space if it is both soft $T_1$-space and soft normal space.*

### 3. A New Approach for Constructing a Soft Topology from an Ordinary Topology

This section is focused on introducing a soft topology that is generated by soft single points on $V$ in relation to any ordinary topology on $V$. Additionally, we examine several properties pertaining to this particular type of soft topology.

**Definition 16.** *Suppose that $(V, \kappa)$ is a topological space. We define a fixed point function for each $v \in V$ and $Q \in \kappa$,*

$$L : V \to P(V)$$
$$v \longmapsto Q$$

*such that $Q$ is an open neighborhood of $v$ in $\kappa$ and $L(w) = \varnothing$ for all $w \neq v$. The soft set*

$$\tilde{L}_{v,Q} = \{(v, L(v)) : v \in V\} \in S(V)_V$$
$$= \{(v, Q), (w, \varnothing), \text{ for all } w \neq v\}$$

*is called a soft single point on $V$ with respect to $\kappa$. The set of all soft single points on $V$ with respect to $\kappa$ is symbolized by $B$.*

**Definition 17.** *Suppose that $(V, \kappa)$ is a topological space. Let*

$$\tilde{B} = B \; \tilde{\cup} \; \tilde{\Phi} = \{\tilde{L}_{v,Q} : \text{for all } v \in V \text{ and } Q \in \kappa\} \; \tilde{\cup} \; \tilde{\Phi},$$

*where $\tilde{\Phi} = \{(v, \varnothing) : \text{for all } v \in V\}$. Then, the soft topology generated by soft single points on $V$ with respect to $\kappa$ is as follows:*

$$\tilde{\kappa} = \{\tilde{\cup} \, \tilde{L}_{v,Q} : \tilde{L}_{v,Q} \in \tilde{B}\}$$

*and is symbolized by $(STGP)_{(V, \kappa)}$.*

To clarify our terminology, V will henceforth indicate both the initial universe set and the set of parameters. We will use the notation $S(V)$ to represent the collection of all soft sets over $V$ and any soft set over V will be symbolized by $\tilde{L}$ instead of $\tilde{L}_V$.

The first thing we do is to prove that $\tilde{\kappa}$ in the above definition meets the soft topology conditions.

**Theorem 4.** *Suppose that $(V, \kappa)$ is a topological space. Then, $\tilde{\kappa}$ in Definition 17 is a soft topology on $V$.*

**Proof.**

1.  Since $V \in \kappa$ and $V$ is an open neighborhood for each $v \in V$, $\tilde{L}_{v,V} \in \tilde{B}$ for all $v \in V$. Then, their union belongs to $\tilde{\kappa}$. Thus, $\tilde{V} = \{\tilde{\cup}\tilde{L}_{v,V} : \text{for all } v \in V\} \in \tilde{\kappa}$. Also, $\tilde{\Phi} \in \tilde{\kappa}$ by Definition 17.

2.  Let $\tilde{L}, \tilde{Q} \in \tilde{\kappa}$. Then,
    $\tilde{L} = \{(v, L(v)) : v \in V, L(v) \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } L(v) = \varnothing\}$,
    $\tilde{Q} = \{(v, Q(v)) : v \in V, Q(v) \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } Q(v) = \varnothing\}$.
    If $L(v)$ and $Q(v)$ are two open neighborhoods of $v \in V$, then $L(v) \cap Q(v)$ is an open neighborhood of $v$ and $L(v) \cap Q(x) \in \kappa$. Thus,
    $\tilde{L} \tilde{\cap} \tilde{Q} = \{(v, L(v) \cap Q(v)) : v \in V, L(v) \cap Q(v) \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } L(v) \cap Q(v) = \varnothing\} \in \tilde{\kappa}$.

3.  Let $\tilde{Q}_i \in \tilde{\kappa}$. Then, we have for all $i$, $\tilde{Q}_i = \{\tilde{\cup} \tilde{L}_{v,G} : \tilde{L}_{v,G} \in \tilde{B}\}$. Since $\tilde{\kappa}$ is the union of an arbitrary number of elements in $\tilde{B}$, then $\underset{i}{\tilde{\cup}} \tilde{Q}_i \in \tilde{\kappa}$.

Hence, $\tilde{\kappa}$ is a soft topology on $V$. $\square$

**Example 2.** *Let $V = \{v, w, x\}$ and $\kappa = \{V, \varnothing, \{w\}\}$ be a topology on $V$. Then,*

$$\tilde{B} = \{\tilde{L}_{v,V}, \tilde{L}_{w,V}, \tilde{L}_{x,V}, \tilde{L}_{w,\{w\}}\} \tilde{\cup} \tilde{\Phi},$$

*where $\tilde{L}_{v,V} = \{(v, V), (w, \varnothing), (x, \varnothing)\}$ and similarly for $\tilde{L}_{w,V}, \tilde{L}_{x,V}$, and $\tilde{L}_{w,\{w\}}$. Thus,*

$$\tilde{\kappa} = \left\{ \begin{array}{c} \tilde{L}_1 = \{(v, V), (w, \varnothing), (x, \varnothing)\}, \tilde{L}_2 = \{(v, \varnothing), (w, V), (x, \varnothing)\}, \\ \tilde{L}_3 = \{(v, \varnothing), (w, \varnothing), (x, V)\}, \tilde{L}_4 = \{(v, \varnothing), (w, \{w\}), (x, \varnothing)\}, \\ \tilde{L}_5 = \{(v, V), (w, V), (x, \varnothing)\}, \tilde{L}_6 = \{(v, V), (w, \varnothing), (x, V)\}, \\ \tilde{L}_7 = \{(v, V), (w, \{w\}), (x, \varnothing)\}, \tilde{L}_8 = \{(v, \varnothing), (w, V), (x, V)\}, \\ \tilde{L}_9 = \{(v, \varnothing), (w, \{w\}), (x, V)\}, \tilde{L}_{10} = \{(v, V), (w, \{w\}), (x, V)\}, \\ \tilde{L}_{11} = \{(v, \varnothing), (w, \varnothing), (x, \varnothing)\}, \tilde{L}_{12} = \{(v, V), (w, V), (x, V)\} \end{array} \right\}$$

*is a soft topology on $V$.*

**Definition 18.** *Suppose that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$. Then, each element of $\tilde{\kappa}$ is called a soft single point open set.*

**Remark 3.** *The set $\tilde{B}$ in Definition 17 is a soft base of $\tilde{\kappa}$ since every soft single open set is a union of elements from $\tilde{B}$.*

**Example 3.**

-   *Let $V$ be an infinite set and $\kappa$ be a co-finite topology on $V$. Then,*

    $$\tilde{B} = \{\tilde{L}_{v,Q} : v \in V, Q \text{ is an open neighborhood of } v \text{ in } \kappa\} \tilde{\cup} \tilde{\Phi}$$

    *is a soft base of $\tilde{\kappa}$. Thus, $\tilde{\kappa}$ is called a soft co-finite topology generated by soft single points on $V$ with respect to $\kappa$.*

-   *Let $\mathbb{R}$ be the set of real numbers and $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Then,*

    $$\tilde{B} = \{\tilde{L}_{r,Q} : r \in \mathbb{R}, Q \text{ is an open neighborhood of } r \text{ in } \mathcal{U}\} \tilde{\cup} \tilde{\Phi}$$

    *is a soft base of $\tilde{\mathcal{U}}$. Thus, $\tilde{\mathcal{U}}$ is called a soft usual topology generated by soft single points on $\mathbb{R}$ with respect to $\mathcal{U}$.*

**Theorem 5.** *Suppose that $(V, \kappa)$ is a topological space and $Q \subseteq V$. We assume that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$. Then, $Q$ is an open set in $\kappa$ if and only if $\tilde{L}_{v,Q}$ is a soft single point open set in $\tilde{\kappa}$ for all $v \in Q$.*

**Proof.** *Necessity.* Suppose that $Q$ is an open set in $\kappa$. Then, $Q$ is an open neighborhood for all $v \in Q$. Thus, $\tilde{L}_{v,Q} \in \tilde{B}$ for all $v \in Q$. Hence, $\tilde{L}_{v,Q} \in \tilde{\kappa}$ for all $v \in Q$.

*Sufficiency.* We assume that $\tilde{L}_{v,Q}$ is a soft single point open set in $\tilde{\kappa}$ for all $v \in Q$. Then, $\tilde{L}_{v,Q} \in \tilde{B}$ for all $v \in Q$. By the definition of $\tilde{B}$, $Q$ is an open set in $\kappa$. $\square$

**Theorem 6.** *Let $\tilde{\kappa}_1$ be an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ be an $(STGP)_{(V,\kappa_2)}$. Then, their intersection is an $(STGP)_{(V,\kappa_1 \cap \kappa_2)}$.*

**Proof.** Suppose that $\tilde{B}_1 = \{\tilde{L}_{v,Q_1} : \text{for all } v \in V \text{ and } Q_1 \in \kappa_1\} \tilde{\cup} \tilde{\Phi}$ and $\tilde{B}_2 = \{\tilde{L}_{v,Q_2} : \text{for all } v \in V \text{ and } Q_2 \in \kappa_2\} \tilde{\cup} \tilde{\Phi}$ are soft bases of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, respectively. Then,

$$
\begin{aligned}
\tilde{\kappa}_1 \cap \tilde{\kappa}_2 &= \{\tilde{R} : \tilde{R} \in \tilde{\kappa}_1 \cap \tilde{\kappa}_2\} \\
&= \{\tilde{R} : \tilde{R} = \{\tilde{\cup}\tilde{L}_{v,Q} : \tilde{L}_{v,Q} \in \tilde{B}_1 \cap \tilde{B}_2\}\} \\
&= \{\tilde{R} : \tilde{R} = \{\tilde{\cup}\tilde{L}_{v,Q} : v \in V, Q \in \kappa_1 \cap \kappa_2\}\}.
\end{aligned}
$$

Hence, $\tilde{\kappa}_1 \cap \tilde{\kappa}_2$ is a soft topology generated by soft single points on $V$ with respect to $\kappa_1 \cap \kappa_2$. $\square$

**Remark 4.** *If $\tilde{\kappa}_1$ is an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ is an $(STGP)_{(V,\kappa_2)}$, then $\tilde{\kappa}_1 \cup \tilde{\kappa}_2$ is an $(STGP)_{(V,\kappa_1 \cup \kappa_2)}$ in a case that $\kappa_1 \cup \kappa_2$ is a topology on $V$.*

**Theorem 7.** *Let $\tilde{\kappa}_1$ be an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ be an $(STGP)_{(V,\kappa_2)}$. If $\kappa_1$ is finer or strictly finer than $\kappa_2$, then $\tilde{\kappa}_1$ is finer or strictly finer than $\tilde{\kappa}_2$. We say that $\tilde{\kappa}_2$ is coarser or strictly coarser than $\tilde{\kappa}_1$.*

**Proof.** Suppose that $\kappa_1$ is finer or strictly finer than $\kappa_2$. We assume that $\tilde{B}_1$ and $\tilde{B}_2$ are soft bases of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, respectively. Let $\tilde{R} \in \tilde{\kappa}_2$. Then,

$$
\begin{aligned}
\tilde{R} &= \{\tilde{\cup}\tilde{L}_{v,Q} : \tilde{L}_{v,Q} \in \tilde{B}_2\} \\
&= \{\tilde{\cup}\tilde{L}_{v,Q} : v \in V, Q \in \kappa_2\} \\
&= \{\tilde{\cup}\tilde{L}_{v,Q} : v \in V, Q \in \kappa_1\}, \text{ since } \kappa_2 \subseteq \kappa_1.
\end{aligned}
$$

Hence, $\tilde{R} \in \tilde{\kappa}_1$. This proves the theorem. $\square$

**Remark 5.** *If $\kappa_1$ and $\kappa_2$ are incomparable, then $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are incomparable too.*

**Definition 19.** *The relative complement of a soft single point open set is called a soft single point closed set.*

The following lemma provides evidence that De-Morgan's laws apply to all soft sets over $V$ with respect to the set of parameters. The proof can be found in ([9]).

**Lemma 1.** *Let $\tilde{L}, \tilde{Q} \in S(V)$. Then,*
1. $(\tilde{L} \tilde{\cup} \tilde{Q})^c = \tilde{L}^c \tilde{\cap} \tilde{Q}^c$,
2. $(\tilde{L} \tilde{\cap} \tilde{Q})^c = \tilde{L}^c \tilde{\cup} \tilde{Q}^c$.

**Theorem 8.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$, where $\kappa$ is a topology on $V$. Then,*
1. *$\tilde{\Phi}$ and $\tilde{V}$ are soft single point closed sets;*
2. *The restricted intersection of any numbers of soft single point closed sets is a soft single point closed set;*
3. *The union of any two soft single point closed sets is a soft single point closed set.*

**Proof.** It is clear by applying Theorem 4 and Lemma 1. $\square$

While the converse of Proposition 1 may not hold (as noted in [9]), our subsequent result confirms that the converse does indeed hold in the particular situation we are considering.

**Theorem 9.** *Let $(V, \kappa)$ be a topological space. Then, $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$ if and only if $\kappa_v = \{L(v) : \check{L} \in \tilde{\kappa}\}$ is a topology on $V$ for all $v \in V$.*

**Proof.** *Necessity.* Suppose that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$. Therefore, by Proposition 1, $\kappa_v$ is a topology on $V$ for all $v \in V$.

*Sufficiency.* Suppose that $\kappa_v = \{L(v) : \check{L} \in \tilde{\kappa}\}$ is a topology on $V$ for all $v \in V$. By the definition of $\tilde{\kappa}$, $\kappa_v = \{\emptyset$ and $Q$, where $Q$ is an open neighborhood of $v$ in $\kappa\}$ for all $v \in V$. For all $v \in V$, we define the soft sets $\check{L}_{v,Q} = \{(v, Q): v \in Q$ and $(w, \emptyset)$ for all $w \neq v\}$. Therefore, we have the set $\tilde{B} = \{\check{L}_{v,Q} :$ for all $v \in V$ and $Q \in \kappa\} \,\tilde{\cup}\, \tilde{\Phi}$. Thus, by Theorem 4, $\tilde{\kappa} = \{\tilde{\cup}\, \check{L}_{v,Q} : \check{L}_{v,Q} \in \tilde{B}\}$ is an $(STGP)_{(V,\kappa)}$. $\square$

As Propositions 2 and 3 state, $Cl_{\kappa_d}(\check{L}) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(\check{L})$ for any $\check{L} \in S(V)_D$, and the equality holds if and only if $Cl_{\kappa_d}(\check{L})$ is a soft closed set. However, the converse holds in general in our specific case, as demonstrated by the following theorem.

**Theorem 10.** *Suppose that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $Cl_{\kappa_v}(\check{L}) = Cl_{\tilde{\kappa}}(\check{L})$.*

**Proof.** Suppose that $Cl_{\tilde{\kappa}}(\check{L}) = \tilde{R}$. Then, $\tilde{R}$ is the smallest soft single point closed set which contains $\check{L}$. For all $v \in V$, $L(v) \subseteq R(v)$, and $R(v)$ is a closed set which contains $L(v)$ in $\kappa_v$. By the definition of $\tilde{\kappa}$, we have for all closed set $R(v)$ in $\kappa_v$,

$$\{(v, R(v)) : R(v)^c \text{ is an open neighborhood of } v \text{ in } \kappa \text{ and } (w, \emptyset) \text{ for all } w \neq v\}$$

is a soft single point closed set in $\tilde{\kappa}$. Since $\tilde{R}$ is the smallest soft single point closed set which contains $\check{L}$, then $R(v)$ is the smallest closed set which contains $L(v)$ in $\kappa_v$ for all $v \in V$. Therefore, $Cl_{\kappa_v}(\check{L}) = \tilde{R}$. $\square$

Now, we introduce the definition of the closure of any $\check{L} \in S(V)$ with respect to any ordinary topology on $V$.

**Definition 20.** *Let $(V, \kappa)$ be a topological space. Let $\check{L} \in S(V)$. Then, the closure of $\check{L}$ with respect to $\kappa$ is defined as:*

$$Cl_\kappa(\check{L}) = \{(v, Cl_\kappa(L(v)) : v \in V\},$$

*where $Cl_\kappa(L(v))$ is the closure of $L(v)$ in $\kappa$.*

In the theorem that follows, we describe the relationship between $Cl_\kappa(\check{L})$ and $Cl_{\tilde{\kappa}}(\check{L})$ for any $\check{L} \in S(V)$.

**Theorem 11.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $Cl_\kappa(\check{L}) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(\check{L})$.*

**Proof.** We assume that $Cl_\kappa(\check{L}) = \tilde{R}$. Then, $Cl_\kappa(L(v)) = R(v)$ for all $v \in V$. Therefore, for all $v \in V$, $R(v)$ is the smallest closed set which contains $L(v)$ in $\kappa$. Now, if $Cl_{\kappa_v}(\check{L}) = \tilde{Q}$, then $L(v) \subseteq Q(v)$ for all $v \in V$, and $Q(v)$ is a closed set which contains $L(v)$ in $\kappa$. This implies that $R(v) \subseteq Q(v)$ since $R(v)$ is the smallest closed set which contains $L(v)$ in $\kappa$ for all $v \in V$. Thus, $Cl_\kappa(\check{L}) \,\tilde{\subseteq}\, Cl_{\kappa_v}(\check{L})$. Hence, by Theorem 10, $Cl_\kappa(\check{L}) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(\check{L})$. $\square$

**Corollary 1.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then $Cl_{\tilde{\kappa}}(Cl_\kappa(\check{L})) = Cl_{\tilde{\kappa}}(\check{L})$.*

**Proof.** We have by Theorem 11, $\check{L} \,\tilde{\subseteq}\, Cl_\kappa(\check{L}) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(\check{L})$. Then,

$$Cl_{\tilde{\kappa}}(\check{L}) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(Cl_\kappa(\check{L})) \,\tilde{\subseteq}\, Cl_{\tilde{\kappa}}(Cl_{\tilde{\kappa}}(\check{L})) = Cl_{\tilde{\kappa}}(\check{L}).$$

Hence, $Cl_{\tilde{\kappa}}(Cl_\kappa(\check{L})) = Cl_{\tilde{\kappa}}(\check{L})$. $\square$

In general, Theorem 11 cannot be reversed, as shown by the following example.

**Example 4.** *Let $\kappa$ and $\tilde{\kappa}$ be the same as in Example 2.*
*Let $\check{L} = \{(v, \{v\}), (w, \{w\}), (x, \{w\})\} \in S(V)$. Then,*

$$Cl_{\tilde{\kappa}}(\check{L}) = \{(v, V), (w, V), (x, V)\} \text{ and } Cl_{\kappa}(\check{L}) = \{(v, \{v, x\}), (w, V), (x, V)\}.$$

*Hence, $Cl_{\kappa}(\check{L}) \tilde{\subset} Cl_{\tilde{\kappa}}(\check{L})$ and $Cl_{\kappa}(\check{L}) \neq Cl_{\tilde{\kappa}}(\check{L})$.*

**Theorem 12.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $\check{L}$ is a soft single point closed set if and only if $Cl_{\kappa}(\check{L}) = Cl_{\tilde{\kappa}}(\check{L}) = \check{L}$.*

**Proof.** *Necessity.* Suppose that $\check{L}$ is a soft single point closed set. Then, $Cl_{\tilde{\kappa}}(\check{L}) = \check{L}$. Since $\check{L}$ is soft single point closed set, $L(v)$ is a closed set in $\kappa$ for all $v \in V$ by the definition of $\tilde{\kappa}$. Thus, $Cl_{\kappa}(\check{L}) = \check{L} = Cl_{\tilde{\kappa}}(\check{L})$.
  *Sufficiency.* It is clear.   □

**Corollary 2.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $Cl_{\kappa}(Cl_{\tilde{\kappa}}(\check{L})) = Cl_{\tilde{\kappa}}(\check{L})$.*

**Proof.** Since $Cl_{\tilde{\kappa}}(\check{L})$ is a soft single point closed set, by Theorem 12, $Cl_{\kappa}(Cl_{\tilde{\kappa}}(\check{L})) = Cl_{\tilde{\kappa}}(\check{L})$.
□

**Theorem 13.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $Cl_{\kappa}(\check{L})$ is a soft single point closed set if and only if $Cl_{\kappa}(\check{L}) = Cl_{\tilde{\kappa}}(\check{L})$.*

**Proof.** *Necessity.* Suppose that $Cl_{\kappa}(\check{L}) = \tilde{R}$ and $\tilde{R}$ is a soft single point closed set. Then, $\tilde{R}$ is a soft single point closed set which contains $\check{L}$. If $Cl_{\tilde{\kappa}}(\check{L}) = \tilde{Q}$, then $\tilde{Q}$ is the smallest soft single point closed set which contains $\check{L}$. Thus, $\tilde{Q} \subseteq \tilde{R}$ since $\tilde{Q}$ is the smallest soft single point closed set which contains $\check{L}$. Therefore, $Cl_{\tilde{\kappa}}(\check{L}) \tilde{\subseteq} Cl_{\kappa}(\check{L})$. By Theorem 11, we obtain $Cl_{\kappa}(\check{L}) = Cl_{\tilde{\kappa}}(\check{L})$.
  *Sufficiency.* It is clear.   □

  Now, we proceed to define the interior of any $\check{L} \in S(V)$ with respect to any ordinary topology on $V$.

**Definition 21.** *Suppose that $(V, \kappa)$ is a topological space and $\check{L} \in S(V)$. Then, the interior of $\check{L}$ with respect to $\kappa$ is defined as:*

$$Int_{\kappa}(\check{L}) = \{(v, Int_{\kappa}(L(v)) : v \in V\},$$

*where $Int_{\kappa}(L(v))$ is the interior of $L(v)$ in $\kappa$.*

**Theorem 14.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. The interior of $\check{L}$ with respect to the topology $\kappa_v$ is the soft set $Int_{\kappa_v}(\check{L})$, where $Int_{\kappa_v}(\check{L})(v) = Int_{\kappa}(L(v))$ for all $v \in V$. Then, $Int_{\kappa_v}(\check{L}) = Int_{\tilde{\kappa}}(\check{L})$.*

**Proof.** Suppose that $Int_{\tilde{\kappa}}(\check{L}) = \tilde{R}$. Then, $\tilde{R}$ is the largest soft single point open set which contained in $\check{L}$. Therefore, $R(v) \subseteq L(v)$ for all $v \in V$, and $R(v)$ is an open set which contained in $L(v)$ in $\kappa_v$. By the definition of $\tilde{\kappa}$, we have for all open set $R(v)$ in $\kappa_v$,

  $$\{(v, R(v)) : R(v) \text{ is an open neighborhood of } v \text{ in } \kappa \text{ and } (w, \varnothing) \text{ for all } w \neq v\}$$

is a soft single point open set. Since $\tilde{R}$ is the largest soft single point open set which contained in $\check{L}$, then $R(v)$ is the largest open set which contained in $L(v)$ in $\kappa_v$ for all $v \in V$. Therefore, $Int_{\kappa_v}(\check{L}) = \tilde{R}$.   □

  The relationship between $Int_{\kappa}(\check{L})$ and $Int_{\tilde{\kappa}}(\check{L})$ for any $\check{L} \in S(V)$ is explained in the following theorem.

**Theorem 15.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\check{L} \in S(V)$. Then, $Int_{\tilde{\kappa}}(\check{L}) \tilde{\subseteq} Int_{\kappa}(\check{L})$.*

**Proof.** Suppose that $Int_{\tilde{\kappa}}(\tilde{L}) = \tilde{R}$. Then $\tilde{R}$ is the largest soft single point open set which contained in $\tilde{L}$. Therefore, $R(v) \subseteq L(v)$ for all $v \in V$. By the definition of $\tilde{\kappa}$, $R(v)$ is an open neighborhood of $v$ in $\kappa$ which contained in $L(v)$ for all $v \in V$. Now, if $Int_{\kappa}(\tilde{L}) = \tilde{Q}$, then $Q(v)$ is the the largest open neighborhood of $v$ in $\kappa$ which is contained in $L(v)$ for all $v \in V$. Therefore, $R(v) \subseteq Q(v)$ since $Q(v)$ is the the largest open neighborhood of $v$ in $\kappa$ which contained in $L(v)$ for all $v \in V$. Hence, $Int_{\tilde{\kappa}}(\tilde{L}) \tilde{\subseteq} Int_{\kappa}(\tilde{L})$. □

**Corollary 3.** *Suppose that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$ and $\tilde{L} \in S(V)$. Then, $Int_{\tilde{\kappa}}(Int_{\kappa}(\tilde{L})) = Int_{\tilde{\kappa}}(\tilde{L})$.*

**Proof.** By Theorem 15, we have $Int_{\tilde{\kappa}}(\tilde{L}) \tilde{\subseteq} Int_{\kappa}(\tilde{L}) \tilde{\subseteq} \tilde{L}$. Then,

$$Int_{\tilde{\kappa}}(Int_{\tilde{\kappa}}(\tilde{L})) = Int_{\tilde{\kappa}}(\tilde{L}) \tilde{\subseteq} Int_{\tilde{\kappa}}(Int_{\kappa}(\tilde{L})) \tilde{\subseteq} Int_{\tilde{\kappa}}(\tilde{L}).$$

Thus, $Int_{\tilde{\kappa}}(Int_{\kappa}(\tilde{L})) = Int_{\tilde{\kappa}}(\tilde{L})$. □

From the following example, it is evident that the converse of Theorem 15 is not true in general.

**Example 5.** *Let $\kappa$ and $\tilde{\kappa}$ be the same as in Example 2.*

*Let $\tilde{L} = \{(v, \{v, w\}), (w, \{w\}), (x, \{w\})\} \in S(V)$. Then,*

*$Int_{\tilde{\kappa}}(\tilde{L}) = \{(v, \varnothing), (w, \{w\}), (x, \varnothing)\}$ and $Int_{\kappa}(\tilde{L}) = \{(v, \{w\}), (w, \{w\}), (x, \{w\})\}$.*

*Hence, $Int_{\tilde{\kappa}}(\tilde{L}) \tilde{\subset} Int_{\kappa}(\tilde{L})$ and $Int_{\kappa}(\tilde{L}) \neq Int_{\tilde{\kappa}}(\tilde{L})$.*

**Theorem 16.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\tilde{L} \in S(V)$. Then, $\tilde{L}$ is a soft single point open set if and only if $Int_{\tilde{\kappa}}(\tilde{L}) = Int_{\kappa}(\tilde{L}) = \tilde{L}$.*

**Proof.** *Necessity.* Suppose that $\tilde{L}$ is a soft single point open set. By the definition of $\tilde{\kappa}$, $L(v)$ is an open set in $\kappa$ for all $v \in V$. Then, $Int_{\kappa}(\tilde{L}) = \tilde{L} = Int_{\tilde{\kappa}}(\tilde{L})$.
　*Sufficiency.* It is clear. □

**Corollary 4.** *Suppose that $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$ and $\tilde{L} \in S(V)$. Then $Int_{\kappa}(Int_{\tilde{\kappa}}(\tilde{L})) = Int_{\tilde{\kappa}}(\tilde{L})$.*

**Proof.** Since $Int_{\tilde{\kappa}}(\tilde{L})$ is a soft single point open set, by Theorem 16, $Int_{\kappa}(Int_{\tilde{\kappa}}(\tilde{L})) = Int_{\tilde{\kappa}}(\tilde{L})$. □

**Theorem 17.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, $Int_{\kappa}(\tilde{L})$ is a soft single point open set if and only if $Int_{\kappa}(\tilde{L}) = Int_{\tilde{\kappa}}(\tilde{L})$.*

**Proof.** *Necessity.* Suppose that $Int_{\kappa}(\tilde{L}) = \tilde{R}$ and $\tilde{R}$ is a soft single point open set. Then, $\tilde{R}$ is a soft single point open set which is contained in $\tilde{L}$. If $Int_{\tilde{\kappa}}(\tilde{L}) = \tilde{Q}$, then $\tilde{Q}$ is the largest soft single point open set which is contained in $\tilde{L}$. Therefore, $\tilde{R} \tilde{\subseteq} \tilde{Q}$ since $\tilde{Q}$ is the largest soft single point open set which is contained in $\tilde{L}$. It follows that $Int_{\kappa}(\tilde{L}) \tilde{\subseteq} Int_{\tilde{\kappa}}(\tilde{L})$. Thus, by Theorem 15, we obtain $Int_{\kappa}(\tilde{L}) = Int_{\tilde{\kappa}}(\tilde{L})$. *Sufficiency.* It is clear. □

Suppose that $\kappa_1$ is a topology on $V$ and $\kappa_2$ is a topology on $Z$. We assume that $\tilde{\kappa}_1$ is an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ is an $(STGP)_{(Z,\kappa_2)}$. We study whether $\kappa_1$ and $\kappa_2$ being homeomorphic as topological spaces implies that $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are homeomorphic as soft topological spaces. In our situation, where $V$ and $Z$ are treated as initial universe sets and the sets of parameters, Definition 6 specifies that we should only consider the mapping $\lambda =: \mu : V \to Z$.

**Theorem 18.** *Suppose that $(V, \kappa_1)$ and $(Z, \kappa_2)$ are two topological spaces. Let $\tilde{\kappa}_1$ be an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ be an $(STGP)_{(Z,\kappa_2)}$. Let $\lambda : V \to Z$ be a map and $Y_{\lambda} : S(V) \to S(Z)$ be an associated map. Then, $Y_{\lambda}$ is a soft continuous map if and only if the map $\lambda$ is continuous.*

**Proof.** *Necessity.* Suppose that $Y_\lambda$ is a soft continuous map. Let $Q$ be a nonempty open set in $\kappa_2$, $\tilde{L}_{z,Q} = \{(z, Q) : Q$ is an open neighborhood of $z$ in $\kappa_2$ and $L(w) = \varnothing$ for all $w \neq z\} \in \tilde{\kappa}_2$. Since $Y_\lambda$ is a soft continuous map, $Y_\lambda^{-1}(\tilde{L}_{z,Q}) \in \tilde{\kappa}_1$. By Remark 1, we have for all $v \in V$,

$$Y_\lambda^{-1}(\tilde{L}_{z,Q})(v) = \lambda^{-1}(L(\lambda(v))) \in \kappa_1, \text{ where } \lambda(v) \in Z.$$

Then, we have two cases:

- If $\lambda(v) = z$, then $\lambda^{-1}(L(\lambda(v))) = \lambda^{-1}(Q) \in \kappa_1$.
- If $\lambda(v) \neq z$, then $\lambda^{-1}(L(\lambda(v))) = \lambda^{-1}(\varnothing) \in \kappa_1$.

Since $Q$ is an open neighborhood of $z$ in $\kappa_2$, then $\lambda^{-1}(L(\lambda(v))) = \lambda^{-1}(Q) \in \kappa_1$. Hence, $\lambda$ is a continuous map.

*Sufficiency.* Suppose that $\lambda$ is a continuous map. Then, for all $Q \in \kappa_2$, $\lambda^{-1}(Q) \in \kappa_1$. Let $\tilde{Q}$ be a soft single point open set in $\tilde{\kappa}_2$. Then, we can write $\tilde{Q}$ as follows:

$$\tilde{Q} = \{(w, Q(w)) : Q(w) \text{ is either an open neighborhood of } w \text{ in } \kappa_1 \text{ or } Q(w) = \varnothing\}.$$

By Remark 1, for all $v \in V$,

$$Y_\lambda^{-1}(\tilde{Q})(v) = \lambda^{-1}(Q(\lambda(v))), \lambda(v) \in Z.$$

Since $\lambda$ is a continuous map, then $\lambda^{-1}(Q(\lambda(v))) \in \kappa_1$ for all $v \in V$. Then, we have two cases:

- If $\lambda(v) \in Q(\lambda(v))$, then $\lambda^{-1}(Q(\lambda(v)))$ is an open neighborhood of $v$ in $\kappa_1$.
- If $Q(\lambda(v)) = \varnothing$, then $\lambda^{-1}(Q(\lambda(v))) = \varnothing$.

Thus, $Y_\lambda^{-1}(\tilde{Q}) \in \tilde{\kappa}_1$. Hence, $Y_\lambda$ is a soft continuous map. $\square$

**Theorem 19.** *Suppose that $\tilde{\kappa}_1$ is an $(STGP)_{(V,\kappa_1)}$ and $\tilde{\kappa}_2$ is an $(STGP)_{(Z,\kappa_2)}$. If $\kappa_1$ and $\kappa_2$ are homeomorphic, then $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are soft homeomorphic.*

**Proof.** Suppose that $\kappa_1$ and $\kappa_2$ are homeomorphic. Then, we have a homeomorphism map $\lambda : V \to Z$. By Remark 1, $Y_\lambda : S(V) \to S(Z)$ is a bijection map since $\lambda$ is a bijection map. By Theorem 18, $Y_\lambda : S(V) \to S(Z)$ and $Y_\lambda^{-1} : S(Z) \to S(V)$ are soft continuous maps since $\lambda : V \to Z$ and $\lambda^{-1} : Z \to V$ are continuous maps. Hence, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are soft homeomorphic. $\square$

Suppose that $M$ is a nonempty subset of $V$ and $\tilde{\kappa}$ is an $(STGP)_{(V,\kappa)}$. By using Definition 13, we obtain a soft relative topology on $M$ that has soft open sets $\tilde{L}$, and $L(m)$ may not be an open neighborhood of $m$ in $\kappa$. So, we recast the definition of the soft relative topology to obtain a soft relative topology generated by soft single points, with respect to an ordinary relative topology, as follows:

**Definition 22.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Let $M \subseteq V$ and $\kappa_M$ be a relative topology on $M$. Then,*

$$\tilde{\kappa}_M = \{\tilde{\cup}\tilde{L}_{v,Q} : \tilde{L}_{v,Q} \in \tilde{B}_M\}$$

*is called a soft relative topology generated by soft single points on $M$ with respect to $\kappa_M$, where $\tilde{B}_M = \{\tilde{L}_{v,Q} : v \in M, Q \text{ is an open neighborhood of } v \text{ in } \kappa_M\} \tilde{\cup} \tilde{\Phi}_M$.*

**Theorem 20.** *Suppose that $M \subseteq V$. Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\tilde{\kappa}_M$ be a soft relative topology generated by soft single points on $M$ with respect to $\kappa_M$. Then, $\tilde{Q}$ is a soft single point open set in $\tilde{\kappa}_M$ if and only if there exists a soft single point open set $\tilde{L}$ in $\tilde{\kappa}$ such that*

$$\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}, \tilde{M} = \{(v, M) : \text{for all } v \in M\}.$$

**Proof.** *Necessity.* Suppose that $\tilde{Q}$ is a soft single point open set in $\tilde{\kappa}_M$. Then, we can write $\tilde{Q}$ as follows:

$$\tilde{Q} = \{(v, Q(v)) : Q(v) \text{ is either an open neighborhood of } v \text{ in } \kappa_M \text{ or } Q(v) = \varnothing, v \in M\}.$$

Since $Q(v)$ is an open set in $\kappa_M$, there exists an open set $L \in \kappa$ such that $Q(v) = M \cap L$. Therefore, $\tilde{Q} = \{(v, M \cap L) : v \in M \cap L$ or $M \cap L = \varnothing, L \in \kappa\}$. Thus, $\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}$, where $\tilde{L} = \{(v, L) : v \in V, L$ is either an open neighborhood of $v$ in $\kappa$ or $L = \varnothing\}$. Hence, $\tilde{L}$ is a soft single point open set in $\tilde{\kappa}$.

*Sufficiency.* Suppose that $\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}$, where $\tilde{L}$ is a soft single point open set in $\tilde{\kappa}$. Since $\tilde{L} \in \tilde{\kappa}$, we can write $\tilde{L}$ as follows:

$\tilde{L} = \{(v, L(v)) : v \in V, L(v)$ is either an open neighborhood of $v$ in $\kappa$ or $L(v) = \varnothing\}$.

Then, $\tilde{Q} = \{(v, L(v)) : v \in V, L(v)$ is either an open neighborhood of $v$ in $\kappa$ or $L(v) = \varnothing\} \tilde{\cap} \{(v, M) : $ for all $v \in M\}$. Since $L(v) \in \kappa$, then $Q(v) = L(v) \cap M \in \kappa_M$. Thus,

$\tilde{Q} = \{(v, Q(v)) : Q(v)$ is either an open neighborhood of $v$ in $\kappa_M$ or $Q(v) = \varnothing\}$.

Hence, $\tilde{Q}$ is a soft single point open set in $\tilde{\kappa}_M$. $\quad\square$

**Theorem 21.** *Suppose that $M \subseteq V$. Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\tilde{\kappa}_M$ be a soft relative topology generated by soft single points on $M$ with respect to $\kappa_M$. Then, $\tilde{Q}$ is a soft single point closed set in $\tilde{\kappa}_M$ if and only if there exists a soft single point closed set $\tilde{L}$ in $\tilde{\kappa}$ such that*

$$\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}, \tilde{M} = \{(v, M) \text{ for all } v \in M\}.$$

**Proof.** *Necessity.* Suppose that $\tilde{Q}$ is a soft single point closed set in $\tilde{\kappa}_M$. Then, we can write $\tilde{Q}$ as follows:

$\tilde{Q} = \{(v, Q(v)) : Q(v)^c$ is either an open neighborhood of $v$ in $\kappa_M$ or $Q(v)^c = \varnothing, v \in M\}$.

Since $Q(v)$ is a closed set in $\kappa_M$, then there exists a closed set $L \in \kappa$ such that $Q(v) = M \cap L$. Also, since $Q(v)^c = M - Q(v)$, $Q(v)^c = M \cap (L \cap M)^c = M \cap L^c$. Then,

$$\tilde{Q} = \{(v, M \cap L) : v \in M \cap L^c \text{ or } M \cap L^c = \varnothing, L \in \kappa\}.$$

Thus, $\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}$, where $\tilde{L} = \{(v, L) : v \in V, L^c$ is either an open neighborhood of $v$ in $\kappa$ or $L^c = \varnothing\}$. Hence, $\tilde{L}$ is a soft single point closed set in $\tilde{\kappa}$.

*Sufficiency.* Suppose that $\tilde{Q} = \tilde{L} \tilde{\cap} \tilde{M}$, where $\tilde{L}$ is a soft single point closed set in $\tilde{\kappa}$. So, we can write $\tilde{L}$ as follows:

$\tilde{L} = \{(v, L(v)) : v \in V, L(v)^c$ is either an open neighborhood of $v$ in $\kappa$ or $L(v)^c = \varnothing\}$.

Therefore, $\tilde{Q} = \{(v, L(v)) : v \in V, L(v)^c$ is either an open neighborhood of $v$ in $\kappa$ or $L(v^c) = \varnothing\} \tilde{\cap} \{(v, M) : $ for all $v \in M\}$. Then $Q(v) = L(v) \cap M$ for all $v \in M$. Since $L(v)$ is a closed set in $\kappa$, then $Q(v)$ is a closed set in $\kappa_M$ for all $v \in M$. Thus,

$\tilde{Q} = \{(v, Q(v)) : Q(v)^c$ is either an open neighborhood of $v$ in $\kappa_M$ or $Q(v)^c = \varnothing\}$.

Hence, $\tilde{Q}$ is a soft single point closed set in $\tilde{\kappa}_M$. $\quad\square$

## 4. Soft Separation Axioms

The focus of this section is on the soft separation axioms of a soft topology $\tilde{\kappa}$, which is generated by soft single points on $V$ with respect to a topology $\kappa$ on $V$. We aim to analyze whether the soft topology $\tilde{\kappa}$ inherits the soft separation axioms from $\kappa$ and to examine the conditions that must be met for $\tilde{\kappa}$ to satisfy these axioms.

**Theorem 22.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, $\tilde{\kappa}$ is a soft $T_0$-space if and only if $\kappa$ is a $T_0$-space.*

**Proof.** *Necessity.* Suppose that $\tilde{\kappa}$ is a soft $T_0$-space. Then, for all $w, x \in V$ such that $w \neq x$, there exists a soft single point open set $\tilde{L}$ such that $w \in \tilde{L}$ and $x \notin \tilde{L}$ or $w \notin \tilde{L}$ and $x \in \tilde{L}$. We assume that $w \in \tilde{L}$ and $x \notin \tilde{L}$. Then, $w \in L(v)$ for all $v \in V$ and $x \notin L(v)$ for some $v \in V$, say $L(v) = Q$. By the definition of $\tilde{\kappa}$, $Q$ is an open set in $\kappa$. Therefore, we have the open set $Q$ in $\kappa$ such that $w \in Q$ and $x \notin Q$. Hence, $\kappa$ is a $T_0$-space.

*Sufficiency.* Suppose that $\kappa$ is a $T_0$-space. Then, for all $w, x \in V$ such that $w \neq x$, there exists an open set $Q$ in $\kappa$ such that $w \in Q$ and $x \notin Q$ or $w \notin Q$ and $x \in Q$. We assume that $w \in Q$ and $x \notin Q$. Therefore, $\tilde{L}_{w,Q} = \{(w, Q)$ and $(v, \varnothing)$ for all $v \neq w\} \in \tilde{\kappa}$. Also, we have $\tilde{R} = \{(w, \varnothing)$ and $(v, V)$ for all $v \neq w\} \in \tilde{\kappa}$. Then, $\tilde{T} = \tilde{L}_{w,Q} \,\tilde{\cup}\, \tilde{R} = \{(w, Q)$ and $(v, V)$ for all $v \neq w\} \in \tilde{\kappa}$. It follows that $w \in \tilde{T}$ since $w \in Q$ and $w \in V$ and $x \notin \tilde{T}$ since $x \notin Q$. Hence, $\tilde{\kappa}$ is a soft $T_0$-space.　□

**Theorem 23.** *Let M be a nonempty subset of V. Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\tilde{\kappa}_M$ be an $(STGP)_{(M,\kappa_M)}$. If $\tilde{\kappa}$ is a soft $T_0$-space, then $\tilde{\kappa}_M$ is a soft $T_0$-space.*

**Proof.** It follows from Theorem 22 and the fact that a $T_0$-separation axiom has a hereditary property in the ordinary topology.　□

The subsequent theorem provides clarity that a $T_1$-separation axiom yields identical results.

**Theorem 24.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, $\kappa$ is a $T_1$-space if and only if $\tilde{\kappa}$ is a soft $T_1$-space.*

**Proof.** It is similar to the proof of Theorem 22.　□

**Theorem 25.** *Let M be a nonempty subset of V. Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$ and $\tilde{\kappa}_M$ be an $(STGP)_{(M,\kappa_M)}$. If $\tilde{\kappa}$ is a soft $T_1$-space, then $\tilde{\kappa}_M$ is a soft $T_1$-space.*

**Proof.** It follows from Theorem 24 and the fact that a $T_1$-separation axiom has a hereditary property in the ordinary topology.　□

If $V$ has more than one element, then $\tilde{\kappa}$ cannot be a soft $T_2$-space even if $\kappa$ is a $T_2$-space, as stated by the following theorem.

**Theorem 26.** *Let V be a set with at least two elements and $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, $\tilde{\kappa}$ does not meet the requirements of a soft axiom $T_2$.*

**Proof.** Let $w, x$ be any two distinct elements in $V$. We assume that there exist two soft single point open sets $\tilde{L}$ and $\tilde{Q}$ such that $w \in \tilde{L}$, $x \in \tilde{Q}$, and $\tilde{L} \,\tilde{\cap}\, \tilde{Q} = \tilde{\Phi}$. Since $\tilde{L}, \tilde{Q} \in \tilde{\kappa}$, we can write them as follows:

$$\tilde{L} = \{(v, L(v)) : v \in V, L(v) \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } L(v) = \varnothing\},$$

$$\tilde{Q} = \{(v, Q(v)) : v \in V, Q(v) \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } Q(v) = \varnothing\}.$$

Since $w \in \tilde{L}$ and $x \in \tilde{Q}$, $L(v)$ and $Q(v)$ cannot be empty sets for all $v \in V$. By the definition of $\tilde{\kappa}$, we have $Q(w)$ and $L(w)$ are open neighborhoods of $w$ in $\kappa$. Thus, $L(w) \cap Q(w) \neq \varnothing$ since it is the intersection of two open neighborhoods of $w$. It follows that $\tilde{L} \,\tilde{\cap}\, \tilde{Q} \neq \tilde{\Phi}$. This is a contradiction. Hence, $\tilde{\kappa}$ is not a soft $T_2$-space.　□

The following theorem presents the condition that establishes $\tilde{\kappa}$ as a soft regular space.

**Theorem 27.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, $\tilde{\kappa}$ is a soft regular space if and only if V is a singleton.*

**Proof.** *Necessity.* Suppose that $\tilde{\kappa}$ is a soft regular space. We assume that $V$ is not a singleton. Let $v \in V$ and $v \notin \tilde{Q}$, where $\tilde{Q}$ is a nonempty soft single point closed set. Then, there exist two soft single point open sets $\tilde{L}$ and $\tilde{R}$ such that $v \in \tilde{L}$, $\tilde{Q} \tilde{\subseteq} \tilde{R}$, and $\tilde{L} \,\tilde{\cap}\, \tilde{R} = \tilde{\Phi}$. Since $v \in \tilde{L}$, $L(v) \neq \varnothing$ for all $v \in V$. Also, since $\tilde{Q} \tilde{\subseteq} \tilde{R}$, $R(v) \neq \varnothing$ for some $v \in V$. Then, there exists at least one element $v \in V$ such that $R(v) \neq \varnothing$ and $L(v) \neq \varnothing$. It follows that $L(v) \cap R(v) \neq \varnothing$ since it is intersection of two open neighborhoods of $v$ in $\kappa$. Thus, $\tilde{L} \,\tilde{\cap}\, \tilde{R} \neq \tilde{\Phi}$. This is a contradiction. Hence, $V$ must be a singleton.

*Sufficiency.* Suppose that $V$ is a singleton. Then, $\kappa = \{V, \varnothing\}$ is the only topology on $V$. Thus, $\tilde{\kappa} = \{\tilde{V}, \tilde{\Phi}\}$ is an $(STGP)_{(V,\kappa)}$, and it is a soft regular space.　□

In the case $V$ has more than one element, the example that follows demonstrates that $\kappa$ is a regular space, while $\tilde{\kappa}$ fails to be considered a soft regular space.

**Example 6.** *Let* $V = \{v, w\}$ *and* $\kappa = \{V, \phi, \{v\}, \{w\}\}$ *be a topology on* $V$. *Then,*

$$\tilde{B} = \{\tilde{L}_{v,V}, \tilde{L}_{w,V}, \tilde{L}_{v,\{v\}}, \tilde{L}_{w,\{w\}}\} \,\tilde{\cup}\, \tilde{\Phi},$$

*is a soft base of* $\tilde{\kappa}$. *It is clear that* $\kappa$ *is a regular space. We have* $\tilde{Q} = \{(v, \{w\}), (w, V)\}$ *is a soft single point closed set in* $\tilde{\kappa}$ *and* $v \notin \tilde{Q}$. *Then, there exist only two soft single point open sets* $\tilde{L} = \{(v, \{v\}), (w, V)\}$ *and* $\tilde{V}$ *such that* $v \in \tilde{L}$ *and* $v \in \tilde{V}$. *However,* $\tilde{V}$ *is the only soft single point open set which contains* $\tilde{Q}$. *We note that* $\tilde{L} \,\tilde{\cap}\, \tilde{V} \neq \tilde{\Phi}$. *Thus,* $\tilde{\kappa}$ *is not a soft regular space.*

**Remark 6.** *We note that* $\tilde{\kappa}$ *is a soft* $T_3$-*space if and only if* $V$ *is a singleton.*

Our investigation currently focuses on exploring the specific conditions or criteria that determine whether $\tilde{\kappa}$ can be classified as a soft normal space.

**Theorem 28.** *Let* $\tilde{\kappa}$ *be an* $(STGP)_{(V,\kappa)}$. *If* $\tilde{\kappa}$ *is a soft normal space, then* $\kappa$ *is a normal space.*

**Proof.** We assume that $\tilde{\kappa}$ is a soft normal space. Let $Q_1$ and $Q_2$ be two nonempty disjoint closed sets in $\kappa$. Then,

$$\tilde{Q}_1 = \{(v, Q_1) : Q_1^c \text{ is an open neighborhood of } v \text{ in } \kappa \text{ and } (w, \varnothing) \text{ for all } w \neq v\} \text{ and}$$
$$\tilde{Q}_2 = \{(v, Q_2) : Q_2^c \text{ is an open neighborhood of } v \text{ in } \kappa \text{ and } (w, \varnothing) \text{ for all } w \neq v\}$$

are two nonempty soft single point closed sets. Therefore, $\tilde{Q}_1 \,\tilde{\cap}\, \tilde{Q}_2 = \tilde{\Phi}$ since $Q_1$ and $Q_2$ are nonempty disjoint closed sets. Since $\tilde{\kappa}$ is soft normal space, there exist two soft single point open sets $\tilde{L}_1$ and $\tilde{L}_2$ such that $\tilde{Q}_1 \,\tilde{\subseteq}\, \tilde{L}_1$, $\tilde{Q}_2 \,\tilde{\subseteq}\, \tilde{L}_2$, and $\tilde{L}_1 \,\tilde{\cap}\, \tilde{L}_2 = \tilde{\Phi}$. Thus,

$$Q_1 \subseteq L_1(v), Q_2 \subseteq L_2(v), \text{ and } L_1(v) \cap L_2(v) = \varnothing.$$

By the definition of $\tilde{\kappa}$, $L_1(v)$ and $L_2(v)$ are open sets in $\kappa$. Hence, $\kappa$ is a normal space. □

The reversal of Theorem 28 is not generally valid, as evidenced by the following instance.

**Example 7.** *Let* $V = \{v, w, x\}$ *and* $\kappa = \{V, \phi, \{v\}, \{x\}, \{v, x\}, \{w, x\}\}$ *is a topology on* $V$. *It is clear that* $\kappa$ *is a normal space. Then,*

$$\tilde{B} = \{\tilde{L}_{v,V}, \tilde{L}_{w,V}, \tilde{L}_{x,V}, \tilde{L}_{v,\{v\}}, \tilde{L}_{v,\{v,x\}}, \tilde{L}_{x,\{x\}}, \tilde{L}_{x,\{v,x\}}, \tilde{L}_{x,\{w,x\}}, \tilde{L}_{x,\{w,x\}}\} \,\tilde{\cup}\, \tilde{\Phi},$$

*is a soft base of* $\tilde{\kappa}$. *Therefore,*

$$\tilde{Q}_1 = \{(v, \varnothing), (w, \{v\}), (x, \{w\})\} \text{ and } \tilde{Q}_2 = \{(v, \{w\}), (w, \varnothing), (x, \{v\})\}$$

*are soft single point closed sets and* $\tilde{Q}_1 \,\tilde{\cap}\, \tilde{Q}_2 = \tilde{\Phi}$. *Thus,*

$$\tilde{L}_1 = \{(v, \varnothing), (w, V), (x, \{w, x\})\} \text{ and } \tilde{L}_2 = \{(v, V), (w, \varnothing), (x, \{v, x\})\},$$

*are the smallest soft single point open sets which contain* $\tilde{Q}_1$ *and* $\tilde{Q}_2$, *respectively. Therefore,*

$$\tilde{Q}_1 \,\tilde{\subseteq}\, \tilde{L}_1, \tilde{Q}_2 \,\tilde{\subseteq}\, \tilde{L}_2, \text{ and } \tilde{L}_1 \,\tilde{\cap}\, \tilde{L}_2 \neq \tilde{\Phi}.$$

*Hence,* $\tilde{\kappa}$ *is not a soft normal space.*

Definition 4 illustrates the concept of soft disjointness between two soft sets. Specifically, let $\tilde{L} = \{(v, L(v)) : v \in V \text{ and } L(v) \subseteq V\}$ and $\tilde{Q} = \{(v, G(v)) : v \in V \text{ and } G(v) \subseteq V\}$ be two soft sets over $V$ such that for all $v \in V$, either $L(v) = \varnothing$ while $Q(v) \neq \varnothing$ or the other way around. In this case, we say that $\tilde{L}$ and $\tilde{Q}$ are soft absolutely disjoint sets.

Our attention shifts to a case where $\tilde{\kappa}$ is a soft normal space, as we explore in the following theorem.

**Theorem 29.** *Let $\tilde{\kappa}$ be an $(STGP)_{(V,\kappa)}$. Then, all soft disjoint single point closed sets are soft absolutely disjoint sets in $\tilde{\kappa}$ if and only if $\tilde{\kappa}$ is a soft normal space.*

**Proof.** *Necessity.* Suppose that $\tilde{Q}_1$ and $\tilde{Q}_2$ are two soft disjoint single point closed sets. We assume that they are soft absolutely disjoint sets. Then, we can write them as follows:

$$\tilde{Q}_1 = \{(v, Q_1(v)) : v \in V \text{ and } Q_1(v) \subseteq V\} \text{ and}$$

$$\tilde{Q}_2 = \{(v, Q_2(v)) : v \in V \text{ and if } Q_1(v) \neq \varnothing, \text{ then } Q_2(v) = \varnothing\}.$$

Therefore, there exist two soft single point open sets $\tilde{L}_1$ and $\tilde{L}_2$ defined as:

$$L_1(x) = \begin{cases} V & \text{if } Q_1(v) \neq \varnothing, \\ \varnothing & \text{if } Q_1(v) = \varnothing. \end{cases}, L_2(x) = \begin{cases} V & \text{if } Q_2(v) \neq \varnothing, \\ \varnothing & \text{if } Q_2(v) = \varnothing. \end{cases}$$

It is clear that $\tilde{Q}_1 \tilde{\subseteq} \tilde{L}_1$, $\tilde{Q}_2 \tilde{\subseteq} \tilde{L}_2$, and $\tilde{L}_1 \tilde{\cap} \tilde{L}_2 = \tilde{\Phi}$. Hence, $\tilde{\kappa}$ is a soft normal space.

*Sufficiency.* Suppose that $\tilde{\kappa}$ is a soft normal space. Then, for any two soft disjoint single point closed sets $\tilde{Q}_1$ and $\tilde{Q}_2$, there exist two soft single point open sets $\tilde{L}_1$ and $\tilde{L}_2$ such that $\tilde{Q}_1 \tilde{\subseteq} \tilde{L}_1$, $\tilde{Q}_2 \tilde{\subseteq} \tilde{L}_2$, and $\tilde{L}_1 \tilde{\cap} \tilde{L}_2 = \tilde{\Phi}$. Since $\tilde{Q}_1$ and $\tilde{Q}_2$ are soft single point closed sets, and we can write them as follows:

$$\tilde{Q}_1 = \{(v, Q_1(v)), Q_1(v)^c \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } Q_1(v)^c = \varnothing\},$$

$$\tilde{Q}_2 = \{(v, Q_2(v)), Q_2(v)^c \text{ is either an open neighborhood of } v \text{ in } \kappa \text{ or } Q_2(v)^c = \varnothing\}.$$

We assume that $\tilde{Q}_1$ and $\tilde{Q}_2$ are not soft absolutely disjoint sets. Then, for some $v \in V$, we have $Q_1(v) \neq \varnothing$ and $Q_2(v) \neq \varnothing$. Since $\tilde{Q}_1 \tilde{\subseteq} \tilde{L}_1$ and $\tilde{Q}_2 \tilde{\subseteq} \tilde{L}_2$, then $Q_1(v) \subseteq L_1(v) \neq \varnothing$ and $Q_2(v) \subseteq L_2(v) \neq \varnothing$ for some $v \in V$. Thus, $L_1(v) \cap L_2(v) \neq \varnothing$ for some $v \in V$ since it is intersection of two open neighborhoods of $v$ in $\kappa$. Therefore, $\tilde{L}_1 \tilde{\cap} \tilde{L}_2 \neq \tilde{\Phi}$. This is a contradiction. Hence, $\tilde{Q}_1$ and $\tilde{Q}_2$ must be soft absolutely disjoint sets. $\square$

**Remark 7.** *We note that $\tilde{\kappa}$ is a soft $T_4$-space if and if $\tilde{\kappa}$ is a soft $T_1$-space and all soft disjoint single point closed sets are soft absolutely disjoint sets in $\tilde{\kappa}$.*

The soft normality of $\tilde{\kappa}$ does not imply that $\tilde{\kappa}_M$ is also a soft normal space, as illustrated in the following instance.

**Example 8.** *Let $V = \{v, w, x, y\}$ and $\kappa = \{V, \phi, \{y\}, \{x, y\}, \{w, x, y\}, \{w, y\}\}$ be a topology on $V$. Then, $\tilde{B} = \{\tilde{L}_{v,V}, \tilde{L}_{w,V}, \tilde{L}_{x,V}, \tilde{L}_{y,V}, \tilde{L}_{y,\{y\}}, \tilde{L}_{y,\{w,y\}}, \tilde{L}_{y,\{x,y\}}, \tilde{L}_{y,\{w,x,y\}}, \tilde{L}_{w,\{w,y\}}, \tilde{L}_{w,\{w,x,y\}}, \tilde{L}_{x,\{x,y\}}, \tilde{L}_{x,\{w,x,y\}}\} \tilde{\cup} \tilde{\Phi}$ is a soft base of $\tilde{\kappa}$. It is clear that all soft disjoint single point closed sets in $\tilde{\kappa}$ are soft absolutely disjoint sets. By Theorem 29, $\tilde{\kappa}$ is a soft normal space. Let $M = \{w, x, y\} \subset V$. Therefore, $\kappa_M = \{M, \phi, \{y\}, \{x, y\}, \{w, y\}\}$ is a relative topology on $M$. Thus,*

$$\tilde{B}_M = \{\tilde{L}_{w,M}, \tilde{L}_{x,M}, \tilde{L}_{y,M}, \tilde{L}_{y,\{y\}}, \tilde{L}_{y,\{w,y\}}, \tilde{L}_{y,\{x,y\}}, \tilde{L}_{w,\{w,y\}}, \tilde{L}_{x,\{x,y\}}\} \tilde{\cup} \tilde{\Phi}_M$$

*is a soft base of $\tilde{\kappa}_M$. We have*

$$\tilde{Q}_1 = \{(w, \{x\}), (x, \{w\}), (y, \{x\})\} \text{ and } \tilde{Q}_2 = \{(w, \varnothing), (x, \varnothing), (y, \{w\})\}$$

*are soft disjoint single point closed sets in $\tilde{\kappa}_M$. Thus,*

$$\tilde{L}_1 = \{(w, M), (x, M), (y, \{x, y\})\} \text{ and } \tilde{L}_2 = \{(w, \varnothing), (x, \varnothing), (y, \{w, y\})\},$$

*are the smallest soft single point open sets which contain $\tilde{Q}_1$ and $\tilde{Q}_2$, respectively. Then,*

$$\tilde{Q}_1 \tilde{\subseteq} \tilde{L}_1, \tilde{Q}_2 \tilde{\subseteq} \tilde{L}_2, \text{ and } \tilde{L}_1 \tilde{\cap} \tilde{L}_2 \neq \tilde{\Phi}_M.$$

*Hence, $\tilde{\kappa}_M$ is not a soft normal space.*

## 5. Conclusions

In this paper, we have introduced a new technique for constructing a soft topology on a nonempty set V by using soft single points with respect to an ordinary topology on $V$. The relationship between the soft closure (the soft interior) and the closure (the interior) of a soft set, under both the soft topology and the corresponding ordinary topology, has been investigated. We have demonstrated that the soft topologies that are generated by soft single points exhibit soft homeomorphism if their corresponding ordinary topologies are homeomorphic. In addition, a soft relative topology that arises using soft single points with respect to an ordinary relative topology has been studied. We have examined the soft separation axioms of this type of soft topology, specifically determining whether these axioms are inherited from the ordinary topology or not. We have observed that this soft topology inherits the $T_0$ and $T_1$ axioms from its corresponding ordinary topology, while also being recognized as non-hausdorff space ($T_2 space$) when the set V has more than one element. Furthermore, the conditions that are necessary for this soft topology to satisfy the requirements of a soft regular space and a soft normal space have been examined. The construction of a soft topology from an ordinary topology provides a means of applying the principles of the soft topology to real-world problems. It allows for the use of soft sets in a way that is consistent with the existing framework of an ordinary topology, making it easier to apply soft topology concepts in practical applications. Additionally, the relationship between the soft topology and the corresponding ordinary topology can be explored, leading to a better understanding of the connections between the two and the properties that are shared or differ between them. This method of constructing a soft topology from an ordinary topology is an important step in making these tools more widely applicable and accessible in a variety of fields. To advance this area of research, we plan to investigate and analyze further properties of this soft topology, like soft compactness, soft connectedness, and others. In addition, we plan to extend the study into other directions, such as geometry and algebraic topology, to gain fresh perspectives and expand the scope of knowledge in this field.

**Author Contributions:** Conceptualization, A.I.A.; Methodology, A.I.A.; Validation, A.I.A.; Investigation, A.I.A.; Writing—original draft preparation, A.I.A.; Writing—review and editing, A.I.A. and A.M.A.; Supervision, A.M.A. All authors have read and agreed to the published version of the manuscript.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

## List of Symbols

| Symbols | Descriptions | Symbols | Descriptions |
|---|---|---|---|
| $P(V)$ | The set of all subsets of $V$ | $L(d)^c$ | The complement of a subset $L(d)$ |
| $V, Z$ | Initial universe sets | $\Phi_D$ | A null soft set |
| $D, E, H$ | Sets of parameters | $\tilde{V}_D$ | An absolute soft set |
| $\tilde{L}_D$ | A soft set over $V$ relative to $D$ | $\tilde{L}$ | A soft set over $V$ relative to $V$ |
| $S(V)_D$ | The set of all soft sets over $V$ relative to $D$ | $S(V)$ | The set of all soft sets over $V$ relative to $V$ |
| $\kappa$ | An ordinary topology | $\tilde{\kappa}$ | A soft topology |
| $\kappa_M$ | A relative topology | $\tilde{\kappa}_M$ | A soft relative topology |
| $\Upsilon_{\lambda,\mu}$ | A soft mappings | $\tilde{L}_{v,Q}$ | A soft single point on $V$ relative to $\kappa$. |
| $Cl_{\tilde{\kappa}}(\tilde{L}_D)$ | A soft closure of $\tilde{L}_D$ relative to $\tilde{\kappa}$ | $Int_{\tilde{\kappa}}(\tilde{L}_D)$ | A soft interior of $\tilde{L}_D$ relative to $\tilde{\kappa}$ |
| $Cl_\kappa(\tilde{L}_D)$ | The closure of $\tilde{L}_D$ relative to $\kappa$ | $Int_\kappa(\tilde{L}_D)$ | The interior of $\tilde{L}_D$ relative to $\kappa$ |

# References

1. Zadeh, L. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Pawlak, Z. Rough sets. *Int. J. Inform. Comput. Sci.* **1982**, *11*, 341–356. [CrossRef]
3. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
4. Maji, P.K.; Roy, A.R; Biswas, R. An application of soft sets in a decision making problem. *Comput. Math. Appl.* **2002**, *44*, 1077–1083. [CrossRef]
5. Maji, P.K.; Biswas, R.; Roy, A.R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
6. Min, W.K. A note on soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 3524–3528. [CrossRef]
7. Ali, M.I.; Feng, F.; Liu, X.; Min, W.K.; Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
8. Hussain, S.; Akiz, H.F.; Alajlan, A.I. On algebraic properties of soft real points. *Fixed Point Theory Appl.* **2019**, *2019*, 9. [CrossRef]
9. Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799. [CrossRef]
10. Hussain, S.; Ahmad, B. Some properties of soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 4058–4067. [CrossRef]
11. Zorlutuna, I.; Akdag, M.; Min, W.K.; Atmaca, S. Remarks on soft topological spaces. *Ann. Fuzzy Math. Inf.* **2012**, *3*, 171–185.
12. Hussain, S. On some soft functions. *Math. Sci. Lett.* **2015**, *4*, 55–61.
13. Milan, M. Soft topological space and topology on the Cartesian product. *Hacet. J. Math. Statis.* **2016**, *45*, 1091–1100. [CrossRef]
14. Terepeta, M. On separating axioms and similarity of soft topological spaces. *Soft Comput.* **2019**, *23*, 1049–1057. [CrossRef]
15. Al Ghour, S.; Bin-Saadon, A. On some generated soft topological spaces and soft homogeneity. *Heliyon* **2019**, *5*, e02061. [CrossRef] [PubMed]
16. Alcantud, J.C.R. Soft open bases and a novel construction of soft topologies from bases for topologies. *Mathematics* **2020**, *8*, 672. [CrossRef]
17. Azzam, A.A.; Ameen, Z.A.; Al-shami, T.M.; El-Shafei, M.E. Generating soft topologies via soft set operators. *Symmetry* **2022**, *14*, 914. [CrossRef]
18. El-Atik, A.A.; Azzam, A.A. Topological representations of simplicial complexes and their applications. *Afr. Matem.* **2023**, *34*, 22. [CrossRef]
19. Maji, P.K.; Biswas, R.; Roy, A.R. Fuzzy soft sets. *J. Fuzzy Math.* **2001**, *9*, 677–692.
20. Feng, F.; Liu, X.; Leoreanu-Fotea, V.; Jun, Y.B. Soft sets and soft rough sets. *Inform. Sci.* **2011**, *181*, 1125–1137. [CrossRef]