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Inertial Method for Solving Pseudomonotone Variational Inequality and Fixed Point Problems in Banach Spaces

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Abstract: In this paper, we introduce a new iterative method that combines the inertial subgradient extragradient method and the modified Mann method for solving the pseudomonotone variational inequality problem and the fixed point of quasi-Bregman nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces. Under some standard assumptions imposed on cost operators, we prove a strong convergence theorem for our proposed method. Finally, we perform numerical experiments to validate the efficiency of our proposed method.

Keywords: Bregman distance; quasi-Bregman nonexpansive mapping; fixed point problem; subgradient and extragrdient method; inertial term; pseudomonotone operator; variational inequality problem

MSC: 47H09; 47J25

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1. Introduction

Let C be a nonempty subset of a real Banach space E with the norm $||.||$ and the duality space E^*. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. For any nonlinear operator $A : C \to E^*$, According to Stampacchia [1], the variational inequality problem (VIP) is defined as follows:

$$\text{Find } d \in C \text{ such that } \langle A(d), e - d \rangle \geq 0 \ \ \forall\, e \in C. \tag{1}$$

We use $VI(C, A)$ to represent the solution set of (1). The study of VIP originates from solving a minimization problem involving infinite-dimensional functions and variational calculus. As an analytical application of mechanics to the solution of partial differential equations in infinite-dimensional spaces, Hartman and Stampacchia [2] initiated the systematic study of VIP in 1964. In 1966, Stampacchia [1] demonstrated the first VIP existence and uniqueness solution. In 1979, Smith [3] originally used VIP to solve variational inequality problems in finite-dimensional spaces when he formulated the traffic assignment problem. He was unaware that his formulation was an exact variational inequality problem before Dafermos [4] realized it in 1980 while working on traffic and equilibrium problems. Since then, a variety of VIP models have been used in real-world settings. These models have a rich theoretical mathematics, some intriguing crossovers between various fields, and several significant applications in engineering and economics. Furthermore, variational inequalities give us a tool for a wide range of issues in mathematical programming, such as nonlinear systems of equation, issues with optimization, and fixed point theorems. Numerous real-world "equilibrium" problems systematically employ variational inequalities (see [5]).

There are a number of well-known techniques for resolving variational inequalities. The regularized method and the projection method are two prominent and general approaches to solving VIPs. Numerous methods have been considered and put forth to solve the VIP (1) problem based on these directives. The extragradient method, which Korpelevich [6] first proposed and which was later expanded upon due to the strong assumption of his result, uses two projections on the underlying feasible closed and convex set over each iteration. This can have an impact on the computational effectiveness of the method. There are ways to circumvent these problems. The first is the subgradient extragradient technique, Algorithm 1, first proposed by Censor et al. [7]. This method substitutes a projection onto a particular constructible half-space for the second projection onto C. They use the following approach:

Algorithm 1: Subgradient Extragradient Technique

$$\begin{cases} f_n = P_C(e_n - \tau A e_n), \\ T_n = \{d \in H : \langle e_n - \tau A e_n - f_n, d - e_n \rangle \leq 0\}, \\ e_{n+1} = P_{T_n}(e_n - \tau A f_n), \ \forall\, n \geq 0, \end{cases}$$

where $\tau \in (0, \frac{1}{L})$. We are aware that several authors have studied iterative methods for solving variational inequality problems and fixed points of nonexpansive and quasi-nonexpansive mappings, as well as their generalizations, in real Hilbert spaces (see, for instance [7,8] and the references therein). Bregman [9] developed methods using the Bregman distance function D_f in (2) rather than the norm when constructing and investigating feasibility and optimization problems. This approach was used to navigate problems that arise when the useful illustrations of nonexpansive operators in Hilbert spaces H, such as the metric projection P_C onto a nonempty, closed, and convex subset C of H, are no longer nonexpansive in Banach spaces. This led to the development of a growing body of research on approximating solutions to problems involving variational inequality, fixed points, and other issues (see, e.g., [10,11] and the references therein).

Recently, Ma et al. [12] developed the following Algorithm 2, known as the modified subgradient extragradient method, for solving variational inequality and fixed point problems in the context of Banach space:

Algorithm 2: Modified Subgradient Extragradient Method

Let $\lambda_0 > 0$, $\mu \in (0,1)$. For any $e_0 \in C$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$.

(Step1) Calculate $f_n = P_C(Je_n - \lambda A(e_n))$. If $e_n \equiv f_n$ and $Te_n = e_n$, then stop: $e_n \in VI(C, A) \cap F(T)$; otherwise, go to next step.

(Step2) Construct $T_n = \{e \in E : \langle Je_n - \lambda_n A(e_n) - Jf_n, e - f_n \rangle \leq 0\}$ and compute

$$\begin{cases} a_n = P_{T_n}(Je_n - \lambda_n A f_n), \\ b_n = J^{-1}(\alpha_n J e_0 + (1 - \alpha_n) a_n) \\ e_{n+1} = J^{-1}(\beta_n J a_n + (1 - \beta_n) J(T b_n)), \ \forall\, n \geq 0, \end{cases}$$

(Step3) Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu(||e_n - f_n||^2 + ||a_n - f_n||^2)}{2\langle A(e_n) - A(f_n), a_n - f_n \rangle}, \lambda_n + \theta_n \right\}, & \text{if } \langle A(e_n) - A(f_n), a_n - f_n \rangle > 0 \\ \lambda_n + \theta_n, & \text{otherwise.} \end{cases}$$

Let $n := n + 1$ and return to Step 1.

where P_C is the generalized projection on E, J is the duality mapping, $A : E \to E^*$ is the pseudomonotone mapping, and T is the nonexpansive mapping. It was proven that the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to a point $x^* \in VI(C, A) \cap F(T)$, where $x^* = P_{VI(C,A)\cap F(T)}x_0$, under some mild conditions, in 2-uniformly convex real Banach spaces. For more information on the common solution of VIP and fixed point problems in real Banach spaces, which is more general than Hilbert spaces, the reader may refer to any of the following recent papers: [13,14].

Motivated by the above results, this paper investigates the strong convergence of the inertial subgradient extragradient method for solving the pseudomonotone variational inequality problem and the fixed point problem of quasi-Bregman nonexpansive mapping in p-uniformly convex and uniformly smooth real spaces. We demonstrate that, under a number of suitable conditions placed on the parameters, the suggested method strongly converges to a point in $VI(C, A) \cap (F(T))$. Finally, we offer a few numerical experiments that support our main finding in comparison to previous published papers.

2. Preliminaries

Let $1 < q \leq 2 \leq p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Consider E to be a real normed space with dual E^* and $S := \{x \in E : ||x|| = 1\}$. If for any x, y in S with $x \neq y$, $\lambda \in (0, 1)$; then E is (i) *strictly convex* space, if $||\lambda x + (1 - \lambda)y|| < 1$ exists; (ii) *smooth* space if $\lim\limits_{t\to 0} \frac{||x+ty||-||x||}{t}$ exists for each $x, y \in S$.

A function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{||x + y||}{2} : x, y \in S(E), ||x - y|| \geq \epsilon\right\},$$

is known as the modulus of convexity. For any $\epsilon \in (0, 2]$, the space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$; additionally, E is p-uniformly convex ($1 < p < \infty$) if there exists a positive constant c_p such that $\delta_E(\epsilon) \geq c_p\epsilon^p$, for all $\epsilon \in (0, 2]$. As a result, each p-uniformly convex space is also uniformly convex. The function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{||x + \tau y|| + ||x - y||}{2} - 1 : x, y \in S\right\}$$

is the formula for the modulus of smoothness of E. Additionally, E is referred to as uniformly smooth if $\lim\limits_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$; if a positive real integer C_q exits such that $\rho_E(\tau) \leq C_q\tau^q$ for any $\tau > 0$, E is referred to as being q-uniformly smooth. As a result, each and every q-uniformly smooth space is uniformly smooth. If and only if the dual, E^*, is p-uniformly convex, then E is q-uniformly smooth, see [15]. It is widely known that L_p, ℓ_p, and W_p^m are 2-uniformly convex and q-uniformly smooth for $1 \leq q < 2$; 2-uniformly smooth and p-uniformly convex for $2 \leq p < \infty$ (see [16]). The expression,

$$J_E^p(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p; ||x^*|| = ||x||^{p-1}\ \forall x \in E\},$$

defines the *generalized duality* mapping J_E^p from E to 2^{E^*}. The mapping $J_E^2 = J$ is frequently referred to as the *normalized duality* mapping in the case where $p = 2$. It is common knowledge that on bounded subsets of E, J_E^p is norm-to-norm uniformly continuous if E is uniformly smooth. It follows that J_E^p is single-valued if E is smooth. It is well-known that if the duality mapping $J_{E^*}^q$ from E^* to E is injective and sujective, then E is reflexive and strictly convex with a strictly convex dual, and $J_E^p J_{E^*}^q = I_{E^*}$ (identity map in E^*) (see [17]), thus, $J_E^p = (J_{E^*}^q)^{-1}$. For examples of generalized duality mapping, let $a = (a_1, a_2, \cdots) \in \ell_p(1 < p < \infty)$. The generalized duality mapping J_E^p in ℓ_p is therefore defined by

$$J_E^p(a) = (|a_1|^{p-1}\mathrm{sgn}(a_1), |a_2|^{p-1}\mathrm{sgn}(a_2), \cdots).$$

Additionally, if $E = L_p[\alpha, \beta](1 < p < \infty)$, we have the generalized duality mapping J_E^p for any $f \in L_p[\alpha, \beta]$ expressed as

$$J_E^p(g)(s) = |g(s)|^{p-1}\mathrm{sgn}(g(s)), \;\; s \in [\alpha, \beta].$$

We recall the following definitions, which were introduced in [18]. For any closed unit ball B in E with radius $r > 0$, we have $rB = \{u \in E : ||u|| \leq r\}$. If $\rho_r(t) > 0$ for every r, $t > 0$, and $\rho_r : [0, \infty) \to [0, \infty)$ express as

$$\rho_r(t) = \inf_{x,y \in rB, ||x-y||=t, \delta \in (0,1)} \frac{\delta f(x) + (1-\delta) f(y) - f(\delta x + (1-\delta) y)}{(\delta(1-\delta))},$$

for all $t \geq 0$ then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded sets. The ρ_r function is also known as the gauge of uniform convexity of f, and is well known and nondecreasing. The following lemma, which is widely known, if f is uniformly convex, is crucial for the verification of our main result.

Lemma 1 ([19])**.** *Let E be a Banach spance and $f : E \to \mathbb{R}$ a uniformly convex function on bounded subsets of E. If $r > 0$ and $\delta_j \in (0,1)$ for each $i = 0, 1, 2, \cdots, s$ with $\sum_{i=0}^{s} \delta_i = 1$, we have*

$$f\Big(\sum_{i=0}^{s} \delta_i x_i\Big) \leq \sum_{i=0}^{s} \delta_i f(x_i) - \delta_j \delta_k \rho_r(||x_j - x_k||)$$

where ρ_r is its gauge of uniform convexity of f, for each $j, k \in \{0, 1, 2, \cdots, s\}$, $x_i \in rB$.

The Bregman distance in relation to f is given by

$$\Delta_f(x, y) = f(x) - f(y) - \langle f'(y), x - y \rangle, \;\; \text{for every } x, y \in E. \tag{2}$$

Let $f_p(x) := \frac{1}{p}||x||$ in particular. The derivative of the function f_p is the generalized duality mapping J_E^p from E to 2^{E^*}. Consequently, the Bregman distance with regard to f_p is described by

$$\Delta_p(x, y) = \frac{1}{p}||x||^p - \langle J_E^p(y), x \rangle + \frac{1}{q}||y||^p. \tag{3}$$

The three-point identity, a crucial property of the Bregman distance, is defined as:

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle J_E^p(z) - J_E^p(y), x - z \rangle, \;\; \forall\, x, y, z \in E. \tag{4}$$

Due to the lack of symmetry, the Bregman distance is not a metric in the traditional sense, but it does possess some distance-like characteristics. If E is a p-uniformly convex space, then the Bregman distance function Δ_p and the metric function satisfy the relation shown below (see [20]), which proves to be extremely helpful in the demonstration of our result: let $\tau_p > 0$ be any fixed constant.

$$\tau_p ||x - y||^p \leq \Delta_p(x, y) \leq \langle J_E^p(x) - J_E^p(y), x - y \rangle \tag{5}$$

for all $x, y \in E$. Additionally, for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, recall from Young's inequality, that

$$\langle J_E^p(x), y \rangle \leq ||J_E^p(x)||||y|| \leq \frac{1}{q}||x||^p + \frac{1}{p}||y||^p. \tag{6}$$

Let E be a smooth and strictly convex real Banach space and C a nonempty, closed, and convex subset of E. The *Bregman projection operator* in the sense of Bregman [9] is $\Pi_C : E \to C$ defined by

$$\Pi_C x = \arg\min_{y \in C} \Delta_p(y, x), \ \ x \in E. \tag{7}$$

The Bregman projection is described in the following way [21]:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x \rangle \leq 0, \ \ \forall z \in C. \tag{8}$$

With respect to Bregman function Δ_p, we obtain

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \ \ \forall z \in C. \tag{9}$$

The Bregman projection in terms of f_2 and the metric projection are identical in Hilbert spaces, but otherwise they are different. More significantly, in Banach spaces, the metric projection cannot share the same property, (9), as the Bregman projection.

If E is smooth, strictly convex, and reflexive Banach space. We defined the function $V_p : E \times E^{*} \to \mathbb{R}$ in relation to f_p, as follows:

$$V_p(x, \bar{x}) = \frac{1}{p}||x||^p + \frac{1}{q}||\bar{x}||^q - \langle x, \bar{x} \rangle, \ \ \forall x \in E, \ \bar{x} \in E^*, \tag{10}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ (see [22]). It is well known that V_p is nonnegative, and with respect to the Bregman function, we also have

$$V_p(x, \bar{x}) = \Delta_p(x, J_{E^*}^q(\bar{x})), \ \ \forall x \in E, \ \bar{x} \in E^*. \tag{11}$$

Furthermore, V_p satisfies the following inequality:

$$V_p(x, \bar{x}) \leq V_p(x, \bar{x} + \bar{y}) - \langle \bar{y}, J_{E^*}^q(x) - x \rangle, \ \ \forall x \in E \text{ and } \bar{x}, \bar{y} \in E^*. \tag{12}$$

Additionally, in the second variable and for all $z \in E$; V_p is convex, that is

$$\Delta_p\Big(z, J_{E^*}^q\Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big)\Big) \leq V_p\Big(z, \Big(\sum_{i=1}^N t_i J_E^p(x_i)\Big)\Big) = \sum_{i=1}^N \Delta_p(z, x_i), \tag{13}$$

where $\{x_i\}_{i=1}^N \subset E$, $\{t_i\}_{i=1}^N \subset (0,1)$ and $\sum_{i=1}^N t_i = 1$ (see [23–25]).

We also need the nonlinear operators, which are introduced below.

If C is a nonempty subset of E, a Banach space, and $T : C \to E$ is a mapping, then T is nonexpansive, if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$, and T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $||Tx - q|| \leq ||x - q||$ for all $x \in C$ and $q \in F(T)$, where $F(T) := \{x \in C : T(x) = x\}$ denotes the set of fixed point of T. An element q in C is asymptotic fixed point of T, if for any sequence $\{x_n\}$ in C, converges weakly to q such that $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. We describe the set set of asymptotic fixed point of T by $F(\hat{T})$.

Definition 1 ([26]). *Let C be a nonempty subset of a real Banach space E that is uniformly smooth and p-uniformly convex ($(0 < p < \infty)$). Let $T : C \to E$ be a mapping with $F(T) \neq \emptyset$, then T is said to be:*

(n1) quasi-Bregman nonexpansive if

$$\Delta_p(q, Tx) \leq \Delta_p(q, x), \qquad \forall\, x \in C, \ q \in F(T);$$

(n2) *Bregman nonexpansive if*

$$\Delta_p(q, Tx) \leq \Delta_p(q, x), \qquad \forall\, x \in C,\ q \in F(T),\ \hat{F}(T) = F(T);$$

(n3) *Bregman firmly nonexpansive if, for all $x, y \in C$*

$$\langle J_E^p(Tx) - J_E^p(Ty), Tx - Ty\rangle \leq \langle J_E^p(x) - J_E^p(y), Tx - Ty\rangle$$

or equivalently,

$$\Delta_p(Tx, Ty) + \Delta_p(Ty, Tx) + \Delta_p(Tx, x) + \Delta_p(Ty, y) \leq \Delta_p(Tx, y) + \Delta_p(Ty, x).$$

The well known demiclosedness principle plays an important role in our main result.

Definition 2. *Assume that C is a nonempty, closed, convex subset of a uniformly convex Banach space E and that $T : C \to C$ is a nonlinear mapping. Then, T is called demiclosed at 0; if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = 0$, then $x = Tx$.*

Next, we outline a few ideas about the monotonicity of an operator.

Definition 3. *Let E be a Banach space that has E^* as its dual. The operator $A : E \to E^*$ is referred to as:*

(m1) *$p - L$-Lipschitz, if*

$$||Ax - Ay|| \leq L||x - y||^p \ \ \forall\, x,\ y \in E,$$

where $L \geq 0$ and $p \in [1, \infty)$ are two constants.

(m2) *monotone, if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in E$;*

(m3) *pseudomonotone, if for all $x, y \in E$, $\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0$;*

(m4) *weakly sequentially continuous if for any $\{x_n\}$ in E such that $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.*

It is clear that $(m2) \Rightarrow (m3)$; the example that follows demonstrates that the implication's converse is not generally true. Let $A(x) = 1 - x$ for all $x \in E := [0, 1]$. Then, A is pseudomonotone but not monotone.

When demonstrating the strong convergence of our sequence, the following result is helpful:

Lemma 2 ([27])**.** *Let $\{a_n\}$ be a nonnegative sequence of real numbers, and $\{\alpha_n\}$ a real sequence of numbers in $(0, 1)$, with*

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and $\{b_n\}$ is a real sequence of numbers. Suppose that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \forall n \geq 1.$$

If $\limsup\limits_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim\limits_{n\to\infty} a_n = 0$.

3. Main Results

For the purpose of solving pseudomonotone variational inequality and fixed point problems, in this section, we formulate Algorithm 3, combining a modified inertial Mann-type method with a subgradient extragradient algorithm. For the convergence of the method, we require the following conditions:

Assumption 1. *(C1) E is a $p-$uniformly convex real Banach space which is also uniformly smooth and C is a nonempty, closed, and convex subset of E.*

(C2) $A : C \to E^*$ *is pseudomonotone and* $(p-1) - L-$*Lipschitz continuous on E.*

(C3) A is weakly sequentially continuous; that is, for any $\{x_n\} \subset E$, *we have* $x_n \rightharpoonup x^*$, *which implies* $Ax_n \rightharpoonup Ax^*$.

(C4) $\{\delta_n\}$ *be a sequence in* (a, b) *for some* $0 < a < b$; $\{\mu_n\}$ *is a positive sequence in* $\left(0, \frac{p\tau_p}{2^{p-1}}\right)$, *where* τ_p *is defined in (5),* $\mu_n = \circ(\alpha_n)$, *where* α_n *is a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(C5) $T : E \to E$ *is a quasi-Bregman nonexpansive mapping with* $F(T) \neq \emptyset$.

(C6) Denote the set of solutions by $\Gamma := VI(A, C) \cap F(T)$ *and is assumed to be nonempty. Then* Γ *is closed and convex.*

Now, we describe the modified inertial Mann-type subgradient extragradient methods for finding a common solution for the fixed point problem and the pseudomonotone variational inequality problem:

Algorithm 3: Modified Inertial Mann-type Subgradient Extragradient Method

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Initialization: Choose $x_0, x_1 \in E$ to be arbitrary, $\theta \in (0, \tau_p)$, $\mu \in (0, \tau_p)$ and $\lambda_1 > 0$.

Iterative Steps: Calculate x_{n+1} as follows:

(Step1) Given the iterates x_{n-1} and x_n for each $n \geq 1$, $\theta > 0$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_E^p(x_n) - J_E^p(x_{n-1})\|}\}, & if \;\; x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise} \end{cases} \tag{14}$$

(Step2) Compute

$$\begin{cases} y_n = J_{E^*}^q[(1-\alpha_n)[J_E^p(x_n) + \theta_n(J_E^p(x_n) - J_E^p(x_{n-1}))]], \\ w_n = P_C(J_{E^*}^q[J_E^p(y_n) - \lambda_n A(y_n)]), \end{cases} \tag{15}$$

If $x_n = w_n = y_n$ for some $n \geq 1$, then stop. Otherwise

(Step3) Construct

$$T_n = \{y \in E : \langle J_E^p(y_n) - \lambda_n A(y_n)) - w_n, y - w_n \rangle \leq 0\}$$

and Compute

$$\begin{cases} v_n = P_{T_n}(J_{E^*}^q[J_E^p(y_n) - \lambda_n A(w_n)]), \\ x_{n+1} = J_{E^*}^q((1-\delta_n)J_E^p(v_n) + \delta_n J_E^p(Tv_n)). \end{cases} \tag{16}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||y_n - w_n||^p + ||v_n - w_n||^p)}{\min\{p,q\}\langle A(y_n) - A(w_n), v_n - w_n\rangle}, \lambda_n\}, \text{ if } \langle A(y_n) - A(w_n), v_n - w_n\rangle > 0, \\ \\ \lambda_n, \quad \text{otherwise} \end{cases} \tag{17}$$

Set $n := n+1$ and return to **Step 1**.

Lemma 3. *The sequence $\{\lambda_n\}$ generated by (17) is monotonically decreasing and bounded from below by* $\min\{\lambda_1, \frac{\mu}{L}\}$.

Proof. Let $x^* \in \Gamma$ and $u_n := J_{E^*}^q(J_E^p x_n + \theta_n(J_E^p x_n - J_E^p x_{n-1}))$, then it follows from (5), (6) and (14) that

$$\begin{aligned}\langle J_E^p u_n - J_E^p x_n, u_n - x^*\rangle &\leq \|u_n - x^*\|\|J_E^p u_n - J_E^p x_n\| \\ &= \theta_n\|J_E^p x_n - J_E^p x_{n-1}\|\ \|u_n - x^*\| \\ &\leq \theta_n\|J_E^p x_n - J_E^p x_{n-1}\|[\frac{1}{p}\|u_n - x^*\|^p + \frac{1}{q}] \\ &\leq \frac{\theta_n}{p}\|J_E^p x_n - J_E^p x_{n-1}\|[2^{p-1}(\|x_n - u_n\|^p + \|x_n - x^*\|^p)] \\ &+ \frac{\theta_n}{q}\|J_E^p x_n - J_E^p x_{n-1}\| \\ &\leq \frac{2^{p-1}\mu_n}{p\tau_p}(\Delta_p(x_n, u_n) + \Delta_p(x_n, x^*)) + \frac{\mu_n}{q}.\end{aligned}$$

Using (4), we obtain

$$\begin{aligned}\Delta_p(u_n, x^*)) &= \Delta_p(x_n, x^*) - \Delta_p(x_n, u_n) + \langle J_E^p u_n - J_E^p x_n, u_n - x^*\rangle \\ &\leq \Big(1 + \frac{2^{p-1}\mu_n}{p\tau_p}\Big)\Delta_p(x_n, x^*) - \Big(1 - \frac{2^{p-1}\mu_n}{p\tau_p}\Big)\Delta_p(x_n, u_n) + \frac{\mu_n}{q}. \quad (18)\end{aligned}$$

Observe from (C5) that for any $\epsilon \in \left(0, \frac{p\tau_p}{2^{p-1}}\right)$, there exists a natural number N such that for all $n \geq N$

$$\frac{\mu_n}{\alpha_n} < \epsilon^2 \quad \text{which implies } \frac{\mu_n 2^{p-1}}{p\tau_p} < \alpha_n\epsilon,$$

then for some $M > 0$, by letting σ denotes the zero vector in E, then from (13), (15) and (18), we obtain

$$\begin{aligned}\Delta_p(y_n, x^*) &= \Delta_p(J_{E^*}^q[(1-\alpha_n)J_E^p(u_n)], x^*) \\ &\leq (1-\alpha_n)\Delta_p(u_n, x^*) + \alpha_n\Delta_p(\sigma, x^*) \\ &\leq (1-\alpha_n[1-\epsilon])\Delta_p(x_n, x^*) - (1-\alpha_n\epsilon)\Delta_p(x_n, u_n) \\ &+\alpha_n[\Delta_p(\sigma, x^*) + M]. \quad (19)\end{aligned}$$

Using (8), (10) and (16), we obtain

$$\begin{aligned}\Delta_p(v_n, x^*) &= \Delta_p(P_{T_n}[J_{E^*}^q(J_E^p(y_n) - \lambda_n A(w_n))], x^*) \\ &\leq \Delta_p(J_{E^*}^q(J_E^p(y_n) - \lambda_n Aw_n), x^*) - \Delta_p(J_{E^*}^q(J_E^p(y_n) - \lambda_n A(w_n)), v_n) \\ &= V_p(J_E^p(y_n) - \lambda_n Aw_n), x^*) - V_p(J_E^p(y_n) - \lambda_n A(w_n)), v_n) \\ &= \frac{1}{p}||x^*||^p - \langle J_E^p(y_n), x^*\rangle + \lambda_n\langle A(w_n), x^*\rangle + \langle J_E^p(y_n), v_n\rangle \\ &-\lambda_n\langle A(w_n), v_n\rangle - \frac{1}{p}||v_n||^p \\ &+\frac{1}{p}||v_n||^p] + \lambda_n\langle J_E^p(y_n), x^* - v_n\rangle \\ &= \Delta_p(y_n, x^*) - \Delta_p(y_n, v_n) + \lambda_n\langle A(w_n), x^* - v_n\rangle\end{aligned}$$

Since $w_n = P_C(J^q_{E^*}[J^p_E(y_n) - \lambda_n A(w_n)])$ is in C and A is pseudomonotone, then $\langle A(w_n), w_n - x^*\rangle \geq 0$. Thus

$$\langle A(w_n), x^* - v_n\rangle \leq \langle A(w_n), w_n - v_n\rangle.$$

By using definition of T_n, we have

$$\langle J^p_E(y_n) - \lambda_n A(y_n) - J^p_E(w_n), v_n - w_n\rangle \leq 0$$

hence

$$\begin{aligned} &\langle J^p_E(y_n) - \lambda_n A(w_n) - J^p_E(w_n), v_n - w_n\rangle \\ &\leq \quad \lambda_n\langle A(y_n) - A(w_n), v_n - w_n\rangle. \end{aligned}$$

Using (4), (5), (10) and (17), we obtain

$$\begin{aligned} \Delta_p(v_n, x^*) &\leq \Delta_p(y_n, x^*) - \Delta_p(y_n, v_n) + \lambda_n\langle A(w_n), x^* - v_n\rangle \\ &\leq \Delta_p(y_n, x^*) - \Delta_p(y_n, w_n) - \Delta_p(w_n, v_n) \\ &\quad +\lambda_n\langle A(y_n) - A(w_n), v_n - w_n\rangle \\ &\leq \Delta_p(y_n, x^*) - \Delta_p(y_n, w_n) - \Delta_p(w_n, v_n) \\ &\quad +\frac{\mu\lambda_n}{\min\{p,q\}\lambda_{n+1}}\Big(||y_n - w_n||^p + ||v_n - w_n||^p\Big) \\ &\leq \Delta_p(y_n, x^*) - \Delta_p(y_n, w_n) - \Delta_p(w_n, v_n) \\ &\quad +\frac{\mu\lambda_n}{\tau_p\min\{p,q\}\lambda_{n+1}}\Big(\Delta_p(y_n, w_n) + \Delta_p(w_n, v_n)\Big) \\ &= \Delta_p(y_n, x^*) - \Big(1 - \frac{\mu\lambda_n}{\tau_p\min\{p,q\}\lambda_{n+1}}\Big)\Delta_p(y_n, w_n) \\ &\quad -\Big(1 - \frac{\mu\lambda_n}{\tau_p\min\{p,q\}\lambda_{n+1}}\Big)\Delta_p(w_n, v_n) \end{aligned} \tag{20}$$

Since $\lim\limits_{n\to\infty}\lambda_n$ exists and $\mu \in (0, \tau_p)$, then $\lim\limits_{n\to\infty}\Big(1 - \frac{\mu\lambda_n}{\tau_p\min\{p,q\}\lambda_{n+1}}\Big) = 1 - \frac{\mu}{\tau_p\min\{p,q\}} > 0$, then for all $n \geq N$, using Lemma 1 and (10), it then follows from the definition of (x_{n+1}) in (16), (19) and (20) that

$$\begin{aligned} \Delta_p(x_{n+1}, x^*) &= V_p((1-\delta_n)J^p_E(v_n) + \delta_n J^p_E(Tv_n), x^*) \\ &\leq \frac{1}{p}||x^*||^p - (1-\delta_n)\langle J^p_E(v_n), x^*\rangle - \delta_n\langle J^p_E(Tv_n), x^*\rangle + \frac{(1-\delta_n)}{q}||J^p_E(v_n)||^q \\ &\quad +\frac{\delta_n}{q}||J^p_E(Tv_n)||^q - (1-\delta_n)\delta_n\rho_r(||J^p_E(v_n) - J^p_E(Tv_n)||) \\ &\leq \Delta_p(v_n, x^*) - (1-\delta_n)\delta_n\rho_r(||J^p_E(v_n) - J^p_E(Tv_n)||) \qquad (21) \\ &\leq (1 - \alpha_n[1-\epsilon])\Delta_p(x_n, x^*) + \alpha_n[\Delta_p(\sigma, x^*) + M] \\ &\quad -\Big(1 - \frac{\mu\lambda_n}{\tau_p\min\{p,q\}\lambda_{n+1}}\Big)[\Delta_p(y_n, w_n) + \Delta_p(w_n, v_n)] \\ &\quad -(1-\alpha_n\epsilon)\Delta_p(x_n, u_n) - (1-\delta_n)\delta_n\rho_r(||J^p_E(v_n) - J^p_E(Tv_n)||) \\ &\leq (1 - \alpha_n[1-\epsilon])\Delta_p(x_n, x^*) + \alpha_n[\Delta_p(\sigma, x^*) + M] \\ &\leq \max\Big\{\Delta_p(x_n, x^*), \frac{[\Delta_p(\sigma, x^*) + M]}{1-\epsilon}\Big\} \\ &\vdots \quad \vdots \\ &\leq \max\Big\{\Delta_p(x_N, x^*), \frac{[\Delta_p(\sigma, x^*) + M]}{1-\epsilon}\Big\} \end{aligned}$$

By induction

$$\Delta_p(x_n, x^*) \le \max\left\{\Delta_p(x_N, x^*), \frac{[\Delta_p(\sigma, x^*) + M]}{1-\epsilon}\right\} \quad n \ge N.$$

Thus, $\{\Delta_p(x_n, x^*)\}$ is bounded and from (5), we know that $\tau_p||x_n - x^*||^p \le \Delta_p(x_n, x^*)$ then we conclude that $\{x_n\}$ is bounded. This means that $\{v_n\}$, $\{w_n\}$, and $\{y_n\}$ are also bounded. □

We know the following lemma, which was essentially proved in [13], is important and crucial in the proof of our main result.

Lemma 4 ([13], Lemma 3.4)**.** *Let $\{y_n\}$ and $\{w_n\}$ be two sequences formulated in Algorithm 3. If there exists a subsequence $\{y_{n_s}\}$ of $\{y_n\}$ that converges weakly to a point $z \in E$ and $\lim_{s\to\infty}||y_{n_s} - w_{n_s}|| = 0$, then $z \in VI(C, A)$.*

We demonstrate that the Algorithm 3 converges strongly under the assumptions (C1)–(C6) based on the analysis described above and Lemma 4.

Theorem 1. *Suppose that Assumption 1 holds. Then, the sequence $\{x_n\}$ defined by Algorithm 3 converges strongly to the unique solution of the Γ.*

Proof. Let $x^* \in \Gamma$, letting $u_n := J^q_{E^*}(J^p_E x_n + \theta_n(J^p_E x_n - J^p_E x_{n-1}))$, then using (11), (12), (15) and (18), we obtain

$$\begin{aligned}
\Delta_p(y_n, x^*) &= V_p((1-\alpha_n)J^p_E(u_n), x^*) \\
&\le V_p(\alpha_n J^p_E(x^*) + (1-\alpha_n)J^p_E(u_n), x^*) + \alpha_n\langle y_n - x^*, J^p_E(x^*)\rangle \\
&\le (1-\alpha_n)\Delta_p(u_n, x^*) + \alpha_n\langle y_n - x^*, J^p_E(x^*)\rangle \\
&\le (1-\alpha_n)\Big(1 + \frac{2^{p-1}\mu_n}{p\tau_p}\Big)\Delta_p(x_n, x^*) + \alpha_n\langle y_n - x^*, J^p_E(x^*)\rangle + \frac{\mu_n}{q} \\
&\quad -(1-\alpha_n)\Big(1 - \frac{2^{p-1}\mu_n}{p\tau_p}\Big)\Delta_p(x_n, u_n).
\end{aligned}$$

For any $\epsilon > 0$ such that $\epsilon \in \Big(0, \frac{p\tau_p}{2^{p-1}}\Big)$, there exists a natural number N such that for all $n \ge N$, we obtain

$$\begin{aligned}
\Delta_p(y_n, x^*) &\le (1-\alpha_n(1-\epsilon))\Delta_p(x_n, x^*) + \alpha_n[\langle y_n - x^*, J^p_E\rangle + \frac{\mu_n}{\alpha_n q}] \\
&\quad -(1-\alpha_n\epsilon)\Delta_p(x_n, u_n).
\end{aligned}$$

Using (20) and (21), it follows that

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\le (1-\alpha_n[1-\epsilon])\Delta_p(x_n, x^*) + \alpha_n[\langle y_n - x^*, J^p_E(x^*)\rangle + \frac{\mu_n}{\alpha_n q}] \\
&\quad -\Big(1 - \frac{\mu\lambda_n}{\tau_p \min\{p,q\}\lambda_{n+1}}\Big)[\Delta_p(y_n, w_n) + \Delta_p(w_n, v_n)] \\
&\quad -(1-\alpha_n\epsilon)\Delta_p(x_n, u_n) - (1-\delta_n)\delta_n\rho_r(||J^p_E(v_n) - J^p_E(Tv_n)||) \qquad (22) \\
&\le (1-\alpha_n[1-\epsilon])\Delta_p(x_n, x^*) + \alpha_n[\langle y_n - x^*, J^p_E(x^*)\rangle + \frac{\mu_n}{\alpha_n q}]. \qquad (23)
\end{aligned}$$

Next, using Lemma 2 and (23), it remains to show that

$$\limsup_{s\to\infty}\langle y_{n_s} - x^*, J^p_E(x^*)\rangle \le 0$$

for every subsequence $\{\Delta_p(x_{n_s}, x^*)\}$ of $\{\Delta_p(x_n, x^*)\}$ satisfying

$$\liminf_{s\to\infty}(\Delta_n(x_{n_s+1}, x^*) - \Delta_p(x_{n_s}, x^*)) \geq 0.$$

Now, let $\{\Delta_p(x_{n_s}, x^*)\}$ be a subsequence of $\{\Delta_p(x_n, x^*)\}$ such that

$$\liminf_{s\to\infty}(\Delta_p(x_{n_s+1}, x^*) - \Delta_p(x_{n_s}, x^*)) \geq 0$$

holds and, from (22), we denotes $\{\Upsilon_{n_s}\}$ as follows:

$$\begin{aligned} \Upsilon_{n_s} \quad := \quad & (1-\alpha_{n_s}\epsilon)\Delta_p(x_{n_s}, u_{n_s}) + (1-\delta_{n_s})\delta_{n_s}\rho_r(||J_E^p(v_{n_s}) - J_E^p(Tv_{n_s})||) \\ & + \Big(1 - \frac{\mu\lambda_{n_s}}{\tau_p \min\{p,q\}\lambda_{n_s+1}}\Big)[\Delta_p(y_{n_s}, w_{n_s}) + \Delta_p(w_{n_s}, v_{n_s})] \end{aligned} \tag{24}$$

thus, from (22), we obtain

$$\begin{aligned} \limsup_{s\to\infty}\Upsilon_{n_s} \quad & \leq \quad \limsup_{s\to\infty}\Big(\Delta_p(x_{n_s}, x^*) - \Delta_p(x_{n_s+1}, x^*)\Big) \\ & \qquad + \limsup_{s\to\infty}\alpha_{n_s}\Big(||y_{n_s} - x^*||||J_E^p(x^*)|| + \frac{\mu_{n_s}}{\alpha_{n_s}q} - (1-\epsilon)\Delta_p(x_{n_s}, x^*)\Big) \\ & \leq \quad \limsup_{s\to\infty}\Big(\Delta_p(x_{n_s}, x^*) - \Delta_p(x_{n_s+1}, x^*)\Big) \\ & = \quad -\liminf_{s\to\infty}\Big(\Delta_p(x_{n_s+1}, x^*) - \Delta_p(x_{n_s}, x^*)\Big) \\ & \leq \quad 0. \end{aligned}$$

Hence, $\limsup_{s\to\infty}\Upsilon_{n_s} \leq 0$, which implies that $\lim_{s\to\infty}\Upsilon_{n_s} = 0$. It follows from (24) that

$$\lim_{s\to\infty}\Delta_p(x_{n_s}, u_{n_s}) = 0 = \lim_{s\to\infty}\Delta_p(y_{n_s}, w_{n_s}) = \lim_{s\to\infty}\Delta_p(w_{n_s}, v_{n_s}) \tag{25}$$

and

$$\lim_{s\to\infty}\rho_r(||J_E^p(v_{n_s}) - J_E^p(Tv_{n_s})||) = 0.$$

By the property of ρ_r, we obtain

$$\lim_{s\to\infty}||J_E^p(v_{n_s}) - J_E^p(Tv_{n_s})|| = 0 \tag{26}$$

and, since $J_{E^*}^q$ is uniformly continuous on a bounded subset of E^*, we obtain

$$\lim_{s\to\infty}||v_{n_s} - Tv_{n_s}|| = 0. \tag{27}$$

Additionally, using (5) and (25), we obtain

$$\lim_{s\to\infty}||x_{n_s} - u_{n_s}|| = 0 = \lim_{s\to\infty}||y_{n_s} - w_{n_s}|| = \lim_{s\to\infty}||w_{n_s} - v_{n_s}|| = 0. \tag{28}$$

With J_E^p being uniformly norm-to-norm continuous on bounded sets, we also have

$$\lim_{s\to\infty}||J_E^p x_{n_s} - J_E^p u_{n_s}|| = 0 = \lim_{s\to\infty}||J_E^p y_{n_s} - J_E^p w_{n_s}|| = \lim_{s\to\infty}||J_E^p w_{n_s} - J_E^p v_{n_s}|| = 0. \tag{29}$$

However, we understand from the definition that $y_n := J_{E^*}^q(1-\alpha_n)J_E^p u_n$, where $u_n = J_{E^*}^q[J_E^p x_n - (J_E^p x_n - J_E^p x_{n-1})]$, then

$$||J_E^p y_n - J_E^p u_n|| = \alpha_n||J_E^p u_n||$$

which implies from the fact $\lim_{n\to\infty}\alpha_n = 0$ and the boundedness of $\{J_E^p u_n\}$ that

$$\lim_{s\to\infty}||J_E^p y_{n_s} - J_E^p u_{n_s}|| = 0 \tag{30}$$

with

$$||J_E^p v_{n_s} - J_E^p x_{n_s}|| \le ||J_E^p v_{n_s} - J_E^p w_{n_s}|| + ||J_E^p w_{n_s} - J_E^p y_{n_s}|| + ||J_E^p y_{n_s} - J_E^p u_{n_s}|| + ||J_E^p u_{n_s} - J_E^p x_{n_s}||$$

it follows from (29) and (30) that

$$\lim_{s\to\infty}||J_E^p v_{n_s} - J_E^p x_{n_s}|| = 0 \tag{31}$$

Moreover, from (28) and (30), since $J_{E^*}^q$ is also uniformly continuous, we obtain from (30) that

$$\lim_{s\to\infty}||y_{n_s} - x_{n_s}|| = 0 \tag{32}$$

and from (16), we obtain $||J_E^p x_{n+1} - J_E^p v_n|| = \delta_n ||J_E^p T v_n - J_E^p v_n||$ and with (26), since δ_n in $(0,1)$ for all $n \ge 1$, we obtain

$$\lim_{s\to\infty}||J_E^p x_{n_s+1} - J_E^p v_{n_s}|| = 0.$$

Thus, from (31), we obtain

$$\lim_{s\to\infty}||J_E^p x_{n_s+1} - J_E^p x_{n_s}|| = 0.$$

By uniform continuity of $J_{E^*}^q$ on a bounded subset of E^*, we conclude, respectively, from (31),we obtain

$$\lim_{s\to\infty}||v_{n_s} - x_{n_s}|| = 0 \tag{33}$$

and

$$\lim_{s\to\infty}||x_{n_s+1} - x_{n_s}|| = 0.$$

Since $\{x_{n_s}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{s_k}}\}$ of $\{x_{n_s}\}$ that converges weakly to some point z in E. By using (33), we obtain $v_{n_s} \rightharpoonup z$; from (27) and Definition 2, we conclude that $z \in F(T)$. Furthermore, from (32), we obtain that $y_{n_s} \rightharpoonup z$. This together with $\lim_{s\to\infty}||y_{n_s} - w_{n_s}|| = 0$ in (28) and Lemma 4, we conclude that $z \in VI(C,A)$, therefore $z \in \Gamma$. Finally, using σ as a zero point in C, it follows from the definition of the Bregman projection that

$$\begin{aligned}
\limsup_{s\to\infty}\langle y_{n_s} - x^*, J_E^p(x^*)\rangle &= \lim_{k\to\infty}\langle y_{n_{s_k}} - x^*, J_E^p(x^*)\rangle \\
&= \langle z - x^*, J_E^p(x^*)\rangle \\
&= \langle x^* - z, J_E^p(\sigma) - J_E^p(x^*)\rangle \\
&\le 0
\end{aligned} \tag{34}$$

We know from (23), that

$$\Delta_p(x_{n_s+1}, x^*) \le (1 - \alpha_{n_s}[1-\epsilon])\Delta_p(x_{n_s}, x^*) + \alpha_{n_s}[\langle y_{n_s} - x^*, J_E^p(x^*)\rangle + \frac{\mu_{n_s}}{\alpha_{n_s} q}]. \tag{35}$$

Hence, combining (34), (35), and together with Lemma 2, we conclude that $\lim_{n\to\infty} \Delta_p(x_n, x^*) = 0$, and together with the fact that $\tau_p||x_n - x^*||^p \le \Delta_p(x_n, x^*)$, we obtain $x_n \to x^*$ as $n \to \infty$. This complete the proof. $\square$

We obtain the following corollary from Theorem 1 by setting $T = 0$ in Algorithm 3.

Corollary 1. *Let (C1)-(C3) of Assumption 1 hold. Choose $x_0, x_1 \in E$ to be arbitrary, $\theta \in (0, \tau_p)$, $\mu \in (0, \tau_p)$, and $\lambda_1 > 0$. Calculate x_{n+1} as follows:*

(Step1) *Given the iterates x_{n-1} and x_n for each $n \ge 1$, $\theta > 0$, choose θ_n such that $0 \le \theta_n \le \bar{\theta_n}$, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_E^p(x_n) - J_E^p(x_{n-1})\|}\}, & if \;\; x_n \ne x_{n-1}, \\ \\ \theta, & \text{otherwise} \end{cases}$$

(Step2) *Compute*

$$\begin{cases} y_n = J_{E^*}^q[(1-\alpha_n)[J_E^p(x_n) + \theta_n(J_E^p(x_n) - J_E^p(x_{n-1}))]], \\ w_n = P_C(J_{E^*}^q[J_E^p(y_n) - \lambda_n A(y_n)]), \end{cases}$$

If $x_x = w_n = y_n$ for some $n \ge 1$, then stop. Otherwise

(Step3) *Construct*

$$T_n = \{y \in E : \langle J_E^p(y_n) - \lambda_n A(y_n)) - w_n, y - w_n\rangle \le 0\}$$

and Compute

$$x_{n+1} = P_{T_n}(J_{E^*}^q[J_E^p(y_n) - \lambda_n A(w_n)]),$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||y_n - w_n||^p + ||v_n - w_n||^p)}{\min\{p,q\}\langle A(y_n) - A(w_n), v_n - w_n\rangle}, \lambda_n\}, & if \;\; \langle A(y_n) - A(w_n), v_n - w_n\rangle > 0, \\ \\ \lambda_n, & \text{otherwise} \end{cases}$$

Set $n := n+1$ and return to **Step 1**.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a point $p \in VI(C, A)$.

Next, if, in Algorithm 3, we assume that $A = 0$, we obtain the following corollary:

Corollary 2. *Let E be a p-uniformly convex and uniformly smooth real Banach space with sequentially continuous duality mapping $J_{E^*}^p$. Let $T : E \to E$ be a quasi-Bregman nonexpansive mapping such that $F(T) \ne \emptyset$. Suppose $\{\delta_n\}$ is a sequence in (a, b) for some $0 < a < b$ and $\{\mu_n\}$ is a positive sequence in $\left(0, \frac{p\tau_p}{2^{p-1}}\right)$, where τ_p is defined in (5), $\mu_n = \circ(\alpha_n)$, where α_n is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated in Algorithm 4 as follows:*

Algorithm 4: First Modified Inertial Mann-type Method

Initialization: *Choose* $x_0, x_1 \in E$ *to be arbitrary,* $\theta \in (0, \tau_p)$*,* $\mu \in (0, \tau_p)$ *and* $\lambda_1 > 0$*.*
Iterative Steps: *Calculate* x_{n+1} *as follows:*
(Step1) *Given the iterates* x_{n-1} *and* x_n *for each* $n \geq 1$*,* $\theta > 0$*, choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta_n}$*, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_E^p(x_n) - J_E^p(x_{n-1})\|}\}, & if \;\; x_n \neq x_{n-1}, \\ \theta, & \text{otherwise} \end{cases}$$

(Step2) *Compute*

$$\begin{cases} y_n = J_{E^*}^q[(1-\alpha_n)[J_E^p(x_n) + \theta_n(J_E^p(x_n) - J_E^p(x_{n-1}))]], \\ x_{n+1} = J_{E^*}^q((1-\delta_n)J_E^p(v_n) + \delta_n J_{E_1}^p(Ty_n)). \end{cases}$$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a point* $p \in F(T)$*.*

Proof. We observe that the necessary assertion is provided by the method of proof of Theorem 1. □

Let $B : E \to E^*$ be a set-valued mapping with domain $D(B) = \{x \in E : B(x) \neq \emptyset\}$ and range $R(B) = \{x^* \in E^* : x^* \in B(x)\}$, and the graph of B is given as $Gra(B) := \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. Then B is said to be monotone if $\langle x^* - y^*, x - y\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in Gra(B)$, and B is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on E. Let $B : E \to 2^{E^*}$ be a mapping. Additionally, B is called a monotone mapping if, for any $x, y \in \mathrm{dom}B$, we have

$$u \in Bx \;\text{ and }\; v \in By \Rightarrow \langle u - v, x - y\rangle \geq 0.$$

B is called maximal if B is monotone and the graph of B is not properly contained in the graph of any other monotone operator. It is known that if B is maximal monotone, then the set $B^{-1}(0) := \{u \in E : 0 \in B(u)\}$ is closed, and convex. The resolvent of B is the operator $\mathrm{Res}_\sigma^B : E \to 2^E$ defined by

$$\mathrm{Res}_\sigma^B = (J_E^p + \sigma B)^{-1} \circ J_E^p.$$

It is known that Res_σ^B is single-valued, Bregman firmly nonexpansive, and $\hat{F}(\mathrm{Res}_\sigma^B) = F(\mathrm{Res}_\sigma^B) = B^{-1}(0)$ (see [28,29]). Since every Bregman firmly nonexpansive is quasi-Bregman nonexpansive, from Corollary 2, we obtain the following result as a special case:

Corollary 3. *Let* E *be a* p*-uniformly convex and uniformly smooth real Banach space with sequentially continuous duality mapping* J_E^p*. Let* $B : E \to 2^{E^*}$ *be a maximal monotone with* $B^{-1}(0) \neq 0$*. Suppose* $\{\delta_n\}$ *be a sequence in* (a, b) *for some* $0 < a < b$ *and* $\{\mu_n\}$ *is a positive sequence in* $\left(0, \frac{p\tau_p}{2^{p-1}}\right)$*, where* τ_p *is defined in* (5)*,* $\mu_n = \circ(\alpha_n)$*, where* α_n *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence generated in Algorithm 5 as follows:*

Algorithm 5: Second Modified Inertial Mann-type Method

Initialization: *Choose* $x_0, x_1 \in E$ *to be arbitrary,* $\theta \in (0, \tau_p)$, $\mu \in (0, \tau_p)$ *and* $\lambda_1 > 0$.

Iterative Steps: *Calculate* x_{n+1} *as follows:*

(Step1) *Given the iterates* x_{n-1} *and* x_n *for each* $n \geq 1$, $\theta > 0$, *choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_E^p(x_n) - J_E^p(x_{n-1})\|}\}, & if \;\; x_n \neq x_{n-1}, \\ \theta, & \text{otherwise} \end{cases}$$

(Step2) *Compute*

$$\begin{cases} y_n = J_{E^*}^q[(1-\alpha_n)[J_E^p(x_n) + \theta_n(J_E^p(x_n) - J_E^p(x_{n-1}))]], \\ x_{n+1} = J_{E^*}^q((1-\delta_n)J_E^p(v_n) + \delta_n J_{E_1}^p(Res_\sigma^B y_n)). \end{cases}$$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a point* $p \in B^{-1}(0)$.

Remark 1. *The following are considered:*

(a) *Theorem 1 improves, extends, and generalizes the corresponding results [12,13,30–33] in the sense that either our method requires an inertial term to improve the convergence rate and/or the space considered is more general.*

(b) *We observe that the result in Corollary 1 improves, and extends the results in [7,34–36] from Hilbert space to a p-uniformly convex and uniformly smooth real Banach space as well as from solving the monotone variational inequality problem to the pseudomonotone variational inequality problem.*

(c) *Corollary 3 improves, and extends the corresponding results of Wei et al. [37], Ibaraki [38], and Tianchai [39] in the sense that our iterative method does not require computation of* C_{n+1} *for each* $n \geq 1$ *or the class of mappings considered in our corollary is more general and inertial in our method, which aids in increasing the convergence rate of the sequence generated by the method.*

4. Numerical Example

In this section, we intend to demonstrate the efficiency of our Algorithm 3 with the aid of numerical experiments. Furthermore, we compare our iterative method with the methods of Censor et al. [7] (Algorithm 1) and Ma et al. [12] (Algorithm 2).

Example 1. *Let* $E = L_2[0,1]$ *and* $C = \{x \in L_2[0,1] : \langle a, x \rangle \leq b\}$, *where* $a = t^2 + 1$ *and* $b = 1$, *with norm* $||x|| = \sqrt{\int_0^1 |x(t)|^2 dt}$ *and inner product* $\langle x, y \rangle = \int_0^t x(t)y(t)dt$, *for all* $x, y \in L_2([0,1])$, $t \in [0,1]$. *Define metric projection* P_C *as follows:*

$$P_C(x) = \begin{cases} x, & if\, x \in C \\ \frac{b - \langle a,x \rangle}{||a||_{L_2}} a + x, & otherwise. \end{cases} \tag{36}$$

Let $A : L_2[0,1] \to L_2[0,1]$ *be defined by* $A(x(t)) = e^{-||x||} \int_0^t x(s)ds$, *for all* $x \in L_2[0,1]$, $t, s \in [0,1]$, *then,* A *is pseudomonotone and uniformly continuous mapping (see [40]) and let* $T(x(t)) = \int_0^t x(s)ds$, *for all* $x \in L_2[0,1]$, $t \in [0,1]$, *then* T *is nonexpansive mapping. For the control parameters, we use* $\alpha_n = \frac{1}{5n+2}$, $\delta_n = \frac{1}{2} - \alpha_n$, $\mu_n = \frac{\alpha_n}{n^{0.01}}$ *and* $\theta_n = \bar{\theta}_n$. *We define the sequence* $TOL_n := ||x_{n+1} - x_n||^2$ *and apply the stopping criterion* $TOL_n < \varepsilon$ *for the iterative processes because the solution to the problem is unknown.* ε *is the predetermined error. Here, the* *terminating condition is set to $\varepsilon = 10^{-5}$. For the numerical experiments illustrated in Figure 1 and Table 1 below, we take into consideration the resulting cases.*

Case 1: $x_0 = t^3$ *and* $x_1 = t^2 + t$.
Case 2: $x_0 = t^3$ *and* $x_1 = t$.
Case 3: $x_0 = (t^2/2) + t$ *and* $x_1 = 2t^3 + t$.
Case 4: $x_0 = t^2$ *and* $x_1 = (t/5)^3 + t$.

Table 1. Comparison of Algorithm 3, Algorithm 2, and Algorithm 1.

Cases		Algorithm 3	Algorithm 2	Algorithm 1
1	Iter.	16	38	89
	CPU (time)	4.2781	5.7812	7.7115
2	Iter.	13	45	104
	CPU (time)	3.3712	6.7367	10.3962
3	Iter.	15	46	116
	CPU (time)	3.8396	7.0305	10.5921
4	Iter.	15	40	94
	CPU (time)	3.7721	6.2475	7.9981

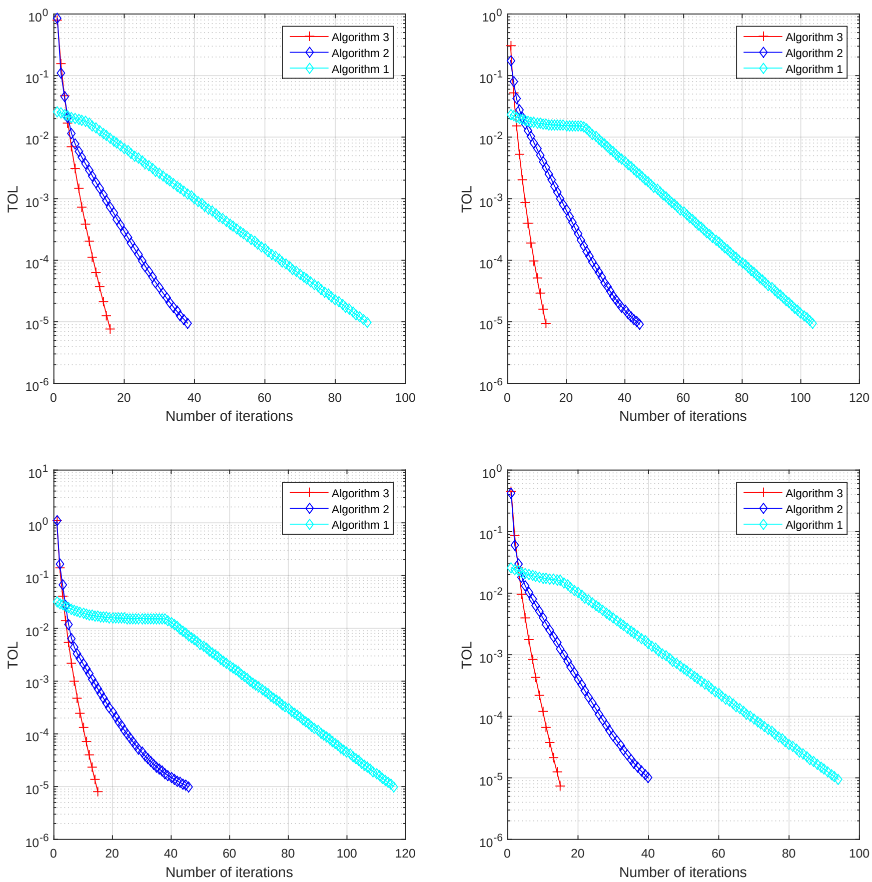

Figure 1. (**Top Left**): Case 1; (**Top Right**): Case 2; (**Bottom Left**): Case 3; (**Bottom Right**): Case 4, the error plotting of comparison of Algorithm 3, Algorithm 2, and Algorithm 1 for Example 1.

Example 2. *Let $E = \mathbb{R}^N$. Define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(x) = Mx + q$, where the matrix M is formed as: $M = V \sum V'$, where $V = I - \frac{2vv'}{\|v\|^2}$ and $\sum = diag(\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1N})$ are the householder and the diagonal matrix, and*

$$\sigma_{1j} = \cos\frac{j\pi}{N+1} + 1 + \frac{\cos\frac{\pi}{N+1} + 1 - \widehat{C}(\cos\frac{N\pi}{N+1} + 1)}{\widehat{C} - 1}, \quad j = 1, 2, \cdots, N,$$

with $\widehat{C}$ been the present condition number of M ([41], Example 5.2). In the numerical computation, we choose $\widehat{C} = 10^4$, $q = 0$ and uniformly take the vector $v \in \mathbb{R}^N$ in $(-1, 1)$. Thus, A is pseudomonotone and Lipschitz continuous with $K = ||M||$ (see [41]). By setting $C = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$, Matlab is used to efficiently compute the projection onto C. Moreover, we examine various instances of the problem's dimension. That is, $N = 20, 30, 40, 60$, with starting points $x_1 = (1, 1, \ldots, 1)'$ and $x_0 = (0, 0, \ldots, 0)'$. In this example, we take the stopping criterion to be $\varepsilon = 10^{-5}$ and obtain the numerical results shown in Table 2 and Figure 2.

Table 2. Numerical results for Example 2 with $\varepsilon = 10^{-5}$.

N		**Algorithm 3**	**Algorithm 2**	**Algorithm 1**
20	Iter. CPU	57	698	2802
		0.0293	0.1414	0.1593
30	Iter. CPU	57	698	2802
		0.0256	0.1405	0.1472
40	Iter. CPU	84	530	2651
		0.0171	0.1081	0.1259
60	Iter. CPU	104	692	2810
		0.0358	0.1379	0.1537

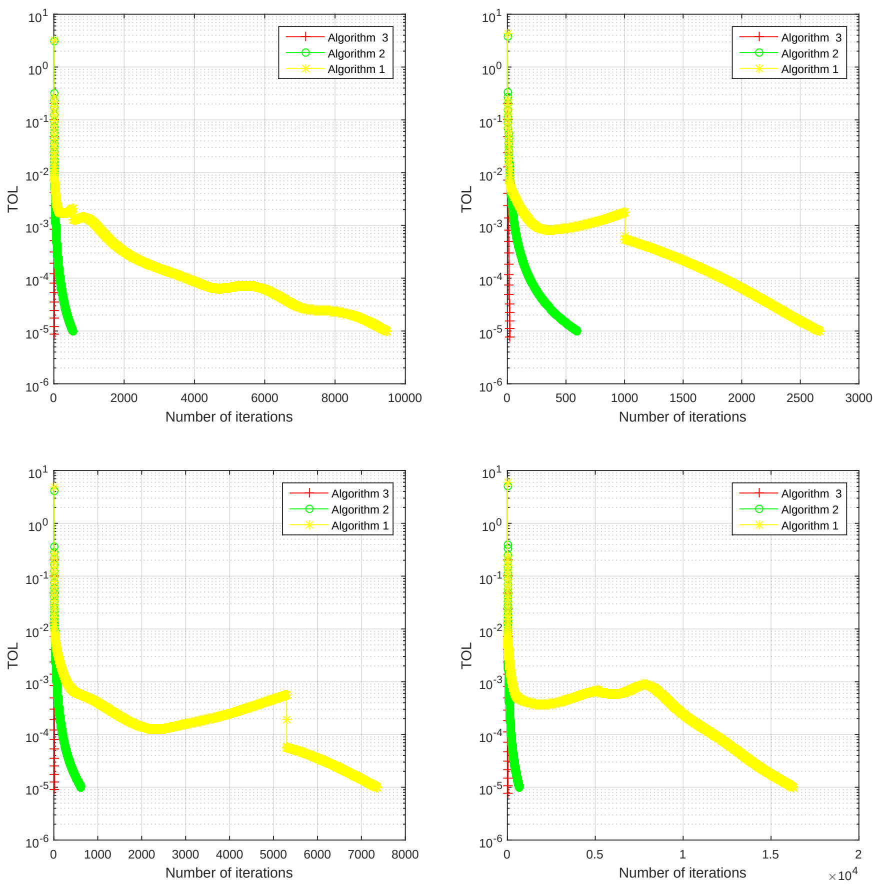

Figure 2. The behavior of TOL_n with $\varepsilon = 10^{-5}$ for Example 2: (**Top Left**): $N = 20$; (**Top Right**): $N = 30$; (**Bottom Left**): $N = 40$; (**Bottom Right**): $N = 60$.

Example 3. *Let* $E = (l_2(\mathbb{R}),\ ||.||_{l_2})$*, where* $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ *and* $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}},\ \ \forall x \in l_2(\mathbb{R})$*. Let* $C = \{x \in l_2(\mathbb{R}) : |x_i| \leq \frac{1}{i},\ i = 1, 2, 3, \dots\}$*. Thus, we have an explicit formula for* P_C*. Now, define the operator* $A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *by*

$$Ax = \left(||x|| + \frac{1}{||x|| + \alpha}\right)\alpha,$$

for some $\alpha > 0$*. Then, A is pseudomonotone on* $l_2(\mathbb{R})$ *(see [42]). In this experiment, the stopping criterion is* $\varepsilon = 10^{-8}$*, and the starting points are selected as follows:*
Case 1: *Take* $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ *and* $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$*.*
Case 2: *Take* $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ *and* $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$*.*
Case 3: *Take* $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ *and* $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$*.*
Case 4: *Take* $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ *and* $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$*.*

The numerical results are reported in Table 3 and Figure 3.

Table 3. Numerical results for Example 3 with $\varepsilon = 10^{-8}$.

Cases		Algorithm 3	Algorithm 2	Algorithm 1
1	Iter. CPU	21	597	8017
		0.0907	0.1926	0.8535
2	Iter. CPU	22	539	2644
		0.0286	0.0506	0.1519
3	Iter. CPU	21	639	10221
		0.0671	0.2695	1.1225
4	Iter. CPU	21	707	19870
		0.0197	0.0609	3.9870

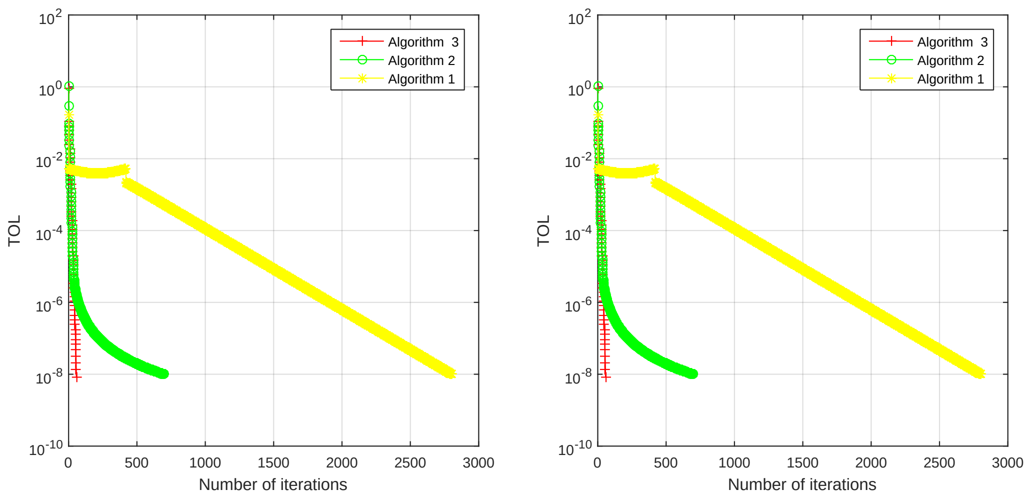

Figure 3. *Cont.*

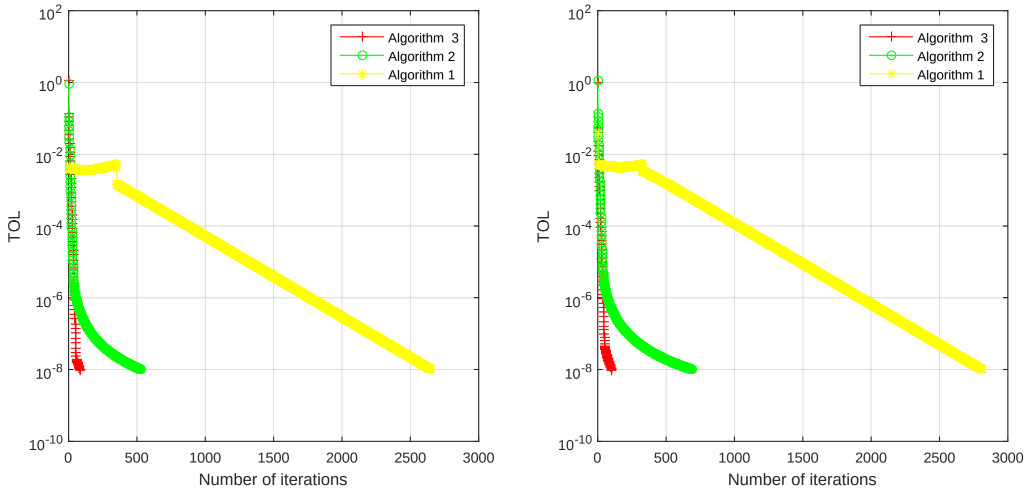

Figure 3. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 3: (**Top Left**): **Case 1**; (**Top Right**): **Case 2**; (**Bottom Left**): **Case 3**; (**Bottom Right**): **Case 4**.

5. Conclusions

The paper has proposed a new inertial subgradient extragradient method with the modified Mann algorithm for solving the Lipschitz pseudomonotone variational inequality problem and the fixed point of quasi-Bregman nonexpansive mapping in p-uniformly convex and uniformly smooth real Banach spaces. Under some suitable conditions imposed on parameters, we have proved the strong convergence of the algorithms. The efficiency of the proposed algorithm has also been illustrated by numerical experiments in comparison with other existing methods.

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