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Solutions of Time Fractional (1 + 3)-Dimensional Partial Differential Equations by the Natural Transform Decomposition Method (NTDM)

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Abstract: The current study employs the natural transform decomposition method (NTDM) to test fractional-order partial differential equations (FPDEs). The present technique is a mixture of the natural transform method and the Adomian decomposition method. For the purpose of checking the precis of our technique, some examples are offered, and the series solutions of these equations are introduced by using NTDM. The outcome shows that the suggested approach is very active and straightforward for obtaining a series solutions of FPDEs and is more accurate if we compare it with existing methods.

Keywords: natural transform; fractional-order linear and nonlinear; approximate solution; inverse natural transform

MSC: 35A22; 44A30

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1. Introduction

During the last decades, various numerical methods have been improved in the field of fractional calculus. The fractional differentiation equations play a crucial role in several theoretical physical, biological, and applied engineering problems, such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, and biological population models [1–5]. The ADSTM method can be applied to solve the energy balance equations of the porous fin with several temperature dependent properties—see [6]. The authors in [7] discussed the approximate solution of the atmospheric internal waves model by applying FRDTM method. Various approximation and numerical techniques have been utilized to solve fractional differential equations [8,9]. Recently, different new methods for fractional differential equations have been suggested, for example, the fractional differential transform method (FDTM) [10,11], fractional variational iteration method (FVIM) [4], fractional Adomian decomposition method (FROM) [12,13], natural transform decomposition method (NTDM) [14–18], homotopy perturbation method (HPM) [19], and Sumudu transform method (STM) [20,21]. The definition of the natural transform, including its properties, was introduced by Khan in [22], which was later used by Belgacem and his colleagues to obtain the relation between this transform and the Laplace and Sumudu transforms [23]. Some physical problems have been modeled by fractional PDEs and solved by utilizing NTDM, for example, the analytical solution of the system of nonlinear PDEs is proposed in [24]. The main goal of this work is to apply the natural transform decomposition method (NTDM) to solve some types of fractional linear and nonlinear partial differential equations (PDEs). The organization of this work is divided into five sections. In Section 2, definitions and properties of the natural transform method (NTM) are addressed. In Section 3, we discuss the methodology of FNTDM. In Section 4, we offered three examples of fractional PDEs and solved them by NTDM. Finally, Section 5 contains the concluding notions.

2. Basic Definitions and Properties of the Natural Transform Method (NTM)

In this part, some definitions and properties of fractional calculus with natural transform are addressed.

Definition 1. *The natural transform (NT) of a function $g(\nu)$ is defined by the integral [22]*

$$\mathbb{N}^{+}[g(\nu)] = \varphi(s,\mu) = \int_{0}^{\infty} g(\mu\nu)e^{-s\nu}d\nu, \qquad \mu > 0,\ s > 0 \tag{1}$$

where s and μ are transform variables.

Definition 2. *If $n \in \mathbb{N}$, where $n-1 \leq \gamma < n$ and $\varphi(s,\mu)$ is natural transform of a function $g(\nu)$, then the (NT) of Caputo fractional derivative of $\frac{\partial^{\gamma} g(\zeta,\nu)}{\partial \nu^{\gamma}}$ is denoted by [18]*

$$\mathbb{N}^{+}\left[\frac{\partial^{\gamma} g(\zeta,\nu)}{\partial \nu^{\gamma}}\right] = \frac{s^{\gamma}}{\mu^{\gamma}}\varphi(s,\mu) - \sum_{k=0}^{n-1}\frac{s^{\gamma-(k+1)}}{\mu^{\gamma-k}}\left[\frac{\partial^{\gamma} g(\zeta,\nu)}{\partial \nu^{\gamma}}\right]_{\nu=0} \tag{2}$$

Definition 3. *The inverse natural transform (INTM) of $\varphi(s,\mu)$ is defined by*

$$\mathbb{N}^{-1}[\varphi(s,\mu)] = g(\nu) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \varphi(s,\mu)e^{\frac{s\nu}{\mu}}ds, \qquad \mu > 0,\ s > 0 \tag{3}$$

Definition 4. *The Caputo operator of order γ for a fractional derivative [25] is presented by the following mathematical expression for $n \in \mathbb{N}, \zeta > 0, g \in \mathbb{C}_{\nu}$:*

$$D^{\gamma}g(\zeta) = \frac{\partial^{\gamma} g(\zeta)}{\partial \zeta^{\gamma}} = I^{n-\gamma}\left[\frac{\partial^{\gamma} g(\zeta)}{\partial \zeta^{\gamma}}\right], \text{ if } n-1 < \gamma \leq n \in \mathbb{N} \tag{4}$$

Definition 5. *Riemann–Liouville fractional order integral [26]:*

$$I_{\zeta}^{\gamma}g(\zeta) = \left\{\begin{array}{ll} g(\zeta) & \text{if } \gamma = 0 \\ \frac{1}{\Gamma(\gamma)}\int_0^{\zeta}(\zeta-\mu)^{\gamma-1}g(\mu)d\mu & \text{if } \gamma > 0 \end{array}\right\} \tag{5}$$

where Γ describes the concept of the gamma variable by

$$\Gamma(\Phi)\int_0^{\infty} e^{-\zeta}\zeta^{\Phi-1}d\zeta, \qquad \Phi \in \mathbb{C} \tag{6}$$

Important properties: some basic properties of the natural transform method (NTM) are given as follows:

$$\begin{aligned} \mathbb{N}^{+}[v'(\nu)] &= \frac{s^2}{\mu^2}\varphi(s,\mu) - \frac{s}{\mu^2}v(0) - \frac{1}{\mu}v'(0). \\ \mathbb{N}^{+}[v''(\nu)] &= \frac{s^3}{\mu^3}\varphi(s,\mu) - \frac{s^2}{\mu^3}v(0) - \frac{s}{\mu^3}v'(0) - \frac{1}{\mu}v''(0). \end{aligned}$$

3. Natural Transform and Decomposition Method (NTDM)

Here, we demonstrate the pertinence of the (NTDM) to obtain the general solution of FPDEs.

$$D_{\nu}^{\alpha}\Psi(\zeta,\nu) + L\Psi(\zeta,\nu) + N\Psi(\zeta,\nu) = g(\zeta,\nu), \quad 0 < \alpha \leq 2, \zeta,\nu \geq 0 \tag{7}$$

subject to the initial conditions

$$\Psi(\zeta,0) = g_1(\zeta), \qquad \Psi_\nu(\zeta,0) = g_2(\zeta), \tag{8}$$

where symbols L and N indicate the linear and nonlinear operators, respectively, g is the source function, and $D_\nu^\alpha = \frac{\partial^\alpha}{\partial \nu^\alpha}$ is the Caputo operator. By applying the natural transform method (NTM) to both sides of Equation (7), we obtain

$$\mathbb{N}^+[D_\nu^\alpha \Psi(\zeta,\nu)] + \mathbb{N}^+[L\Psi(\zeta,\nu)] + \mathbb{N}^+[N\Psi(\zeta,\nu)] = \mathbb{N}^+[g(\zeta,\nu)], \tag{9}$$

By employing the differentiation property of the natural transform method, one can obtain

$$\frac{s^\alpha}{\mu^\alpha}\mathbb{N}^+[\Psi(\zeta,\nu)] - \frac{s^{\alpha-1}}{\mu^\alpha}\Psi(\zeta,0) - \frac{s^{\alpha-2}}{\mu^{\alpha-1}}\Psi_\nu(\zeta,0) = \mathbb{N}^+[g(\zeta,\nu)] - \mathbb{N}^+[L\Psi(\zeta,\nu) + N\Psi(\zeta,\nu)], \tag{10}$$

and

$$\mathbb{N}^+[\Psi(\zeta,\nu)] = \frac{1}{s}g_1(\zeta) + \frac{\mu}{s^2}g_2(\zeta) + \frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[g(\zeta,\nu)] - \frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[L\Psi(\zeta,\nu) + N\Psi(\zeta,\nu)], \tag{11}$$

taking the inverse NT for both sides of Equation (11), we obtain

$$\Psi(\zeta,\nu) = g_1(\zeta) + \nu g_2(\zeta) + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[g(\zeta,\nu)]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[L\Psi(\zeta,\nu) + N\Psi(\zeta,\nu)]\right] \tag{12}$$

The NTDM solution $\Psi(\zeta,\nu)$ is described by the following infinite series:

$$\Psi(\zeta,\nu) = \sum_{n=0}^{\infty}\Psi_n(\zeta,\nu), \tag{13}$$

The nonlinear term $N\Psi(\zeta,\nu)$ satisfies the property

$$N\Psi(\zeta,\nu) = \sum_{n=0}^{\infty}B_n, \tag{14}$$

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\mu^n}\left[N\sum_{n=0}^{\infty}(\mu^n\Psi_n)\right]\right]_{\mu=0}, \qquad n = 0,1,2,\ldots \tag{15}$$

Substituting Equations (13) and (14) into Equation (12), we obtain

$$\begin{aligned}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\Psi(\zeta,\nu)\right] \quad &= \quad g_1(\zeta) + \nu g_2(\zeta) + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[g(\zeta,\nu)]\right] \\ &\quad -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[L\sum_{n=0}^{\infty}\Psi_n(\zeta,\nu) + \sum_{n=0}^{\infty}B_n\right]\right]\end{aligned} \tag{16}$$

We define the repetition relation

$$
\begin{aligned}
\Psi_0(\zeta,\nu) &= g_1(\zeta)+\nu g_2(\zeta)+\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[g(\zeta,\nu)]\right] \\
\Psi_1(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_0(\zeta,\nu)+B_0]\right] \\
\Psi_2(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_1(\zeta,\nu)+B_1]\right] \\
\Psi_3(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_2(\zeta,\nu)+B_2]\right] \\
&\vdots \\
\Psi_{n+1}(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_n(\zeta,\nu)+B_n]\right]
\end{aligned} \tag{17}
$$

Therefore, the precise solution is denoted by

$$\Psi(\zeta,\nu)=\Psi_0(\zeta,\nu)+\Psi_1(\zeta,\nu)+\Psi_2(\zeta,\nu)+\Psi_3(\zeta,\nu)+\Psi_4(\zeta,\nu)+\Psi_5(\zeta,\nu)+\cdots \tag{18}$$

4. Illustrative Examples

In this part, we examine the above method using three examples and then compare the approximate solutions with the exact solutions.

Example 1. *Consider the following one-dimensional nonlinear wave-like equation with variable coefficients:*

$$
\begin{aligned}
\frac{\partial^\alpha\Psi(\zeta,\nu)}{\partial\nu^\alpha} &= \zeta^2\frac{\partial}{\partial\zeta}\left[\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right]-\zeta^2\left[\left(\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right)^2-\Psi(\zeta,\nu)\right] \\
0 &< \alpha\le 2,\ \nu>0,
\end{aligned} \tag{19}
$$

subject to the initial condition

$$\Psi(\zeta,0)=0,\qquad \Psi_\nu(\zeta,0)=\zeta^2, \tag{20}$$

By employing the NTM for both sides of Equation (19), we have

$$
\begin{aligned}
&\frac{s^\alpha}{\mu^\alpha}\mathbb{N}^+[\Psi(\zeta,\nu)]-\frac{s^{\alpha-1}}{\mu^\alpha}\Psi(\zeta,0)-\frac{s^{\alpha-2}}{\mu^{\alpha-1}}\Psi_\nu(\zeta,0) \\
=\ &\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}\left[\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right]-\zeta^2\left[\left(\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right)^2-\Psi(\zeta,\nu)\right]\right]
\end{aligned} \tag{21}
$$

Using the initial conditions in Equation (20) and rearranging the terms, we have

$$\Psi(\zeta,\nu)=\frac{\mu}{s^2}\zeta^2+\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}\left[\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right]-\zeta^2\left[\left(\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right)^2-\Psi(\zeta,\nu)\right]\right] \tag{22}$$

On using the inverse NTM of Equation (22), we have

$$\Psi(\zeta,\nu) = \zeta^2\nu + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}\left[\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right] - \zeta^2\left[\left(\frac{\partial^2\Psi(\zeta,\nu)}{\partial\zeta^2}\right)^2 - \Psi(\zeta,\nu)\right]\right]\right] \tag{23}$$

Now, we assume an infinite series solution for the Equation (23) which is defined by Equation (13); then, Equation (23) becomes

$$\begin{aligned}\sum_{n=0}^{\infty}\Psi_n(\zeta,\nu) &= \zeta^2\nu + \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial}{\partial\zeta}[H(\Psi_\zeta\Psi_{\zeta\zeta})]\right]\right] \\ &\quad -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}F(\Psi^2_{\zeta\zeta})\right]\right] \\ &\quad -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\Psi_n(\zeta,\nu)\right]\right]\end{aligned} \tag{24}$$

where $H(\Psi_\zeta\Psi_{\zeta\zeta})$ and $F(\Psi^2_{\zeta\zeta})$ are the Adomian polynomials, which represent the nonlinear terms. The few nonlinear terms are as follows:

$$\begin{aligned}H_0(\Psi_\zeta\Psi_{\zeta\zeta}) &= \Psi_{0\zeta}\Psi_{0\zeta\zeta}, \\ H_1(\Psi_\zeta\Psi_{\zeta\zeta}) &= \Psi_{0\zeta}\Psi_{1\zeta\zeta} + \Psi_{1\zeta}\Psi_{0\zeta\zeta}, \\ H_2(\Psi_\zeta\Psi_{\zeta\zeta}) &= \Psi_{0\zeta}\Psi_{2\zeta\zeta} + \Psi_{1\zeta}\Psi_{1\zeta\zeta} + \Psi_{2\zeta}\Psi_{0\zeta\zeta}, \\ H_3(\Psi_\zeta\Psi_{\zeta\zeta}) &= \Psi_{0\zeta}\Psi_{3\zeta\zeta} + \Psi_{1\zeta}\Psi_{2\zeta\zeta} + \Psi_{2\zeta}\Psi_{1\zeta\zeta} + \Psi_{3\zeta}\Psi_{0\zeta\zeta}, \\ &\vdots\end{aligned} \tag{25}$$

and

$$\begin{aligned}F_0(\Psi^2_{\zeta\zeta}) &= \Psi^2_{0\zeta\zeta}, \\ F_1(\Psi^2_{\zeta\zeta}) &= 2\Psi_{0\zeta\zeta}\Psi_{1\zeta\zeta}, \\ F_2(\Psi^2_{\zeta\zeta}) &= 2\Psi_{0\zeta\zeta}\Psi_{2\zeta\zeta} + \Psi^2_{1\zeta\zeta}, \\ F_3(\Psi^2_{\zeta\zeta}) &= 2\Psi_{0\zeta\zeta}\Psi_{3\zeta\zeta} + 2\Psi_{1\zeta\zeta}\Psi_{2\zeta\zeta}, \\ &\vdots\end{aligned} \tag{26}$$

Comparing both sides of Equation (24), we can obtain

$$
\begin{aligned}
\Psi_0(\zeta,\nu) &= \zeta^2\nu, \\
\Psi_1(\zeta,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial}{\partial\zeta}(\Psi_{0\zeta}\Psi_{0\zeta\zeta}) - \zeta^2\sum_{n=0}^{\infty}\Psi_{0\zeta\zeta}^2 - \sum_{n=0}^{\infty}\Psi_0\right]\right] \\
&= -\zeta^2\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)}, \\
\Psi_2(\zeta,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}(\Psi_{0\zeta}\Psi_{1\zeta\zeta} + \Psi_{1\zeta}\Psi_{0\zeta\zeta}) - \zeta^2(2\Psi_{0\zeta\zeta}\Psi_{1\zeta\zeta}) - \Psi_1\right]\right] \\
&= \zeta^2\frac{\nu^{2\alpha+1}}{\Gamma(2\alpha+2)}, \\
\Psi_3(\zeta,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}(\Psi_{0\zeta}\Psi_{2\zeta\zeta} + \Psi_{2\zeta}\Psi_{0\zeta\zeta} + \Psi_{1\zeta}\Psi_{1\zeta\zeta})\right]\right] \\
&\quad \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[-\zeta^2\left(2\Psi_{0\zeta\zeta}\Psi_{2\zeta\zeta} + \Psi_{1\zeta\zeta}^2\right) - \Psi_2\right]\right] \\
&= -\zeta^2\frac{\nu^{3\alpha+1}}{\Gamma(3\alpha+2)}, \\
\Psi_4(\zeta,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2\frac{\partial}{\partial\zeta}(\Psi_{0\zeta}\Psi_{3\zeta\zeta} + \Psi_{1\zeta}\Psi_{2\zeta\zeta} + \Psi_{2\zeta}\Psi_{1\zeta\zeta} + \Psi_{3\zeta}\Psi_{0\zeta\zeta})\right]\right] \\
&\quad -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\zeta^2(2\Psi_{0\zeta\zeta}\Psi_{3\zeta\zeta} + 2\Psi_{1\zeta\zeta}\Psi_{2\zeta\zeta})\right]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_3]\right] \\
&= \zeta^2\frac{\nu^{4\alpha+1}}{\Gamma(4\alpha+2)}, \\
&\;\;\vdots
\end{aligned}
$$

The NTDM solution for the above equation is

$$\Psi(\zeta,\nu) = \zeta^2\left[\nu - \frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\nu^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{\nu^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{\nu^{4\alpha+1}}{\Gamma(4\alpha+2)} - \frac{\nu^{5\alpha+1}}{\Gamma(5\alpha+2)} + \cdots\right]. \tag{27}$$

If we substitute $\alpha = 2$ in Equation (27), the approximate solution of Equation (19) becomes

$$\Psi(\zeta,\nu) = \zeta^2\left[\nu - \frac{\nu^3}{3!} + \frac{\nu^5}{5!} - \frac{\nu^7}{7!} + \frac{\nu^9}{9!} - \cdots\right]. \tag{28}$$

Therefore, the solution of Equation (19) in a closed form is

$$\Psi(\zeta,\nu) = \zeta^2\sin(\nu).$$

Figure 1: The solution $\Psi(\zeta,\nu)$ for Example 1 when $\alpha = 2$.

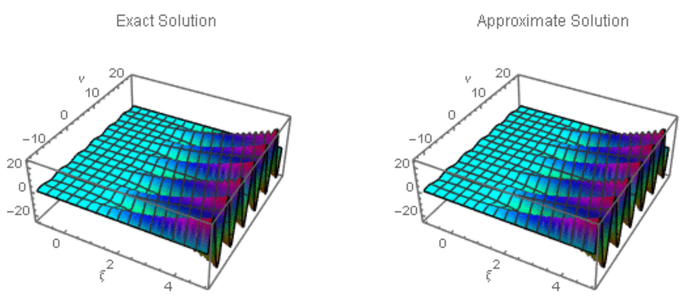

Figure 1. $\psi(\zeta,\nu) = \zeta^2\sin\nu$.

Figure 2: The solution $\Psi(\zeta, \nu)$ for Example 1 when $\alpha = 2, 1.80, 1.50$.

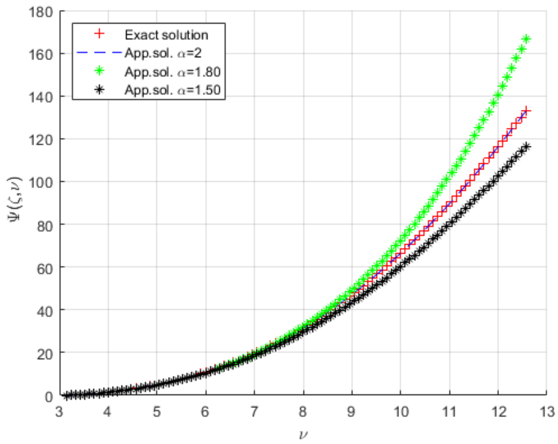

Figure 2. $\psi(\zeta, \nu) = \zeta^2 \sin \nu$.

In the next example, we apply the natural transform decomposition method to solve a non-constant coefficient two-dimensional partial differential equation.

Example 2. *Consider the following two-dimensional fractional wave-like equation [19]:*

$$\frac{\partial^\alpha \Psi(\zeta, \eta, \nu)}{\partial \nu^\alpha} = \frac{1}{12}\left[\zeta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \eta^2}\right], \; 0 < \alpha \leq 2, \nu > 0 \tag{29}$$

subject to conditions

$$\Psi(\zeta, \eta, 0) = \zeta^4, \qquad \Psi_\nu(\zeta, \eta, 0) = \eta^4 \tag{30}$$

Utilizing the NTM for both sides of Equation (29), we can obtain

$$\frac{s^\alpha}{\mu^\alpha}\mathbb{N}^+[\Psi(\zeta, \eta, \nu)] - \frac{s^{\alpha-1}}{\mu^\alpha}\Psi(\zeta, \eta, 0) - \frac{s^{\alpha-2}}{\mu^{\alpha-1}}\Psi_\nu(\zeta, \eta, 0) = \frac{1}{12}\mathbb{N}^+\left[\zeta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \eta^2}\right], \tag{31}$$

By placing conditions Equation (30) into Equation (31), we obtain

$$\Psi(\zeta, \eta, \nu) = \frac{1}{s}\zeta^4 + \frac{\mu}{s^2}\eta^4 + \frac{1}{12}\mathbb{N}^+\left[\zeta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \eta^2}\right], \tag{32}$$

Employing the inverse natural transform method of Equation (32), we have

$$\Psi(\zeta, \eta, \nu) = \zeta^4 + \eta^4 \nu + \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta, \eta, \nu)}{\partial \eta^2}\right]\right]. \tag{33}$$

Now, we suppose an infinite series solution for the Equation (13); then, Equation (33) becomes

$$\sum_{n=0}^{\infty} \Psi_n(\zeta, \eta, \nu) = \zeta^4 + \eta^4 \nu + \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2 \sum_{n=0}^{\infty} \frac{\partial^2 \Psi_n(\zeta, \eta, \nu)}{\partial \zeta^2} + \eta^2 \sum_{n=0}^{\infty} \frac{\partial^2 \Psi_n(\zeta, \eta, \nu)}{\partial \eta^2}\right]\right] \tag{34}$$

Making both sides of Equation (34) equivalent, we have

$$\begin{aligned}
\Psi_0(\zeta,\eta,\nu) &= \zeta^4+\eta^4\nu,\\
\Psi_1(\zeta,\eta,\nu) &= \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_0(\zeta,\eta,\nu)}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_0(\zeta,\eta,\nu)}{\partial\eta^2}\right]\right]\\
&= \zeta^4\frac{\nu^{\alpha}}{\Gamma(\alpha+1)}+\eta^4\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)},\\
\Psi_2(\zeta,\eta,\nu) &= \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_1(\zeta,\eta,\nu)}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_1(\zeta,\eta,\nu)}{\partial\eta^2}\right]\right]\\
&= \zeta^4\frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)}+\eta^4\frac{\nu^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
\Psi_3(\zeta,\eta,\nu) &= \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_2(\zeta,\eta,\nu)}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_2(\zeta,\eta,\nu)}{\partial\eta^2}\right]\right]\\
&= \zeta^4\frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)}+\eta^4\frac{\nu^{3\alpha+1}}{\Gamma(3\alpha+2)},\\
\Psi_4(\zeta,\eta,\nu) &= \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_3(\zeta,\eta,\nu)}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_3(\zeta,\eta,\nu)}{\partial\eta^2}\right]\right]\\
&= \zeta^4\frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)}+\eta^4\frac{\nu^{4\alpha+1}}{\Gamma(4\alpha+2)},\\
\Psi_5(\zeta,\eta,\nu) &= \mathbb{N}^{-1}\left[\frac{1}{12}\mathbb{N}^+\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_4(\zeta,\eta,\nu)}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_4(\zeta,\eta,\nu)}{\partial\eta^2}\right]\right]\\
&= \zeta^4\frac{\nu^{5\alpha}}{\Gamma(5\alpha+1)}+\eta^4\frac{\nu^{5\alpha+1}}{\Gamma(5\alpha+2)},\\
&\ \ \vdots
\end{aligned}$$

The NTDM solution is

$$\begin{aligned}
\Psi(\zeta,\eta,\nu) &= \Psi_0(\zeta,\eta,\nu)+\Psi_1(\zeta,\eta,\nu)+\Psi_2(\zeta,\eta,\nu)+\Psi_3(\zeta,\eta,\nu)+\Psi_4(\zeta,\eta,\nu)+\cdots\\
\Psi(\zeta,\eta,\nu) &= \zeta^4+\eta^4\nu+\zeta^4\frac{\nu^{\alpha}}{\Gamma(\alpha+1)}+\eta^4\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)}\\
&+\zeta^4\frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)}+\eta^4\frac{\nu^{2\alpha+1}}{\Gamma(2\alpha+2)}+\zeta^4\frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)}\\
&+\eta^4\frac{\nu^{3\alpha+1}}{\Gamma(3\alpha+2)}+\zeta^4\frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)}+\eta^4\frac{\nu^{4\alpha+1}}{\Gamma(4\alpha+2)}\\
&+\zeta^4\frac{\nu^{5\alpha}}{\Gamma(5\alpha+1)}+\eta^4\frac{\nu^{5\alpha+1}}{\Gamma(5\alpha+2)}+\cdots
\end{aligned}\tag{35}$$

If we substitute $\alpha=2$, in Equation (35), the approximate solution of Equation (29) becomes

$$\begin{aligned}
\Psi(\zeta,\eta,\nu) &= \zeta^4\left[1+\frac{\nu^2}{2!}+\frac{\nu^4}{4!}+\frac{\nu^6}{6!}+\frac{\nu^8}{8!}+\cdots\right]\\
&+\eta^4\left[\nu+\frac{\nu^3}{3!}+\frac{\nu^5}{5!}+\frac{\nu^7}{7!}+\frac{\nu^9}{9!}+\cdots\right],
\end{aligned}$$

Hence, the exact solution of Equation (29) in a closed form is

$$\Psi(\zeta,\eta,\nu)=\zeta^4\cosh(\nu)+\eta^4\sinh(\nu)$$

Figure 3: The exact and approximate solutions $\Psi(\zeta, \eta, z, \nu)$ for Example 2 when $\alpha = 2$.

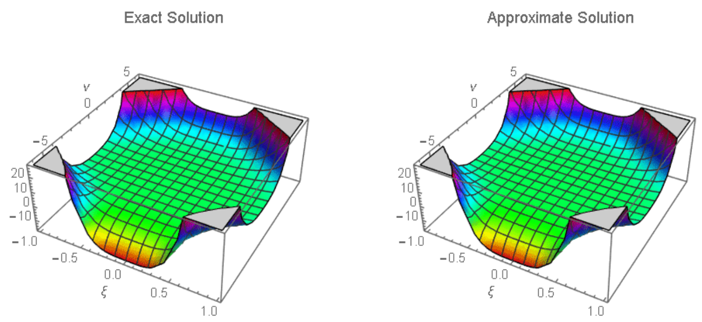

Figure 3. $\Psi(\zeta, \eta, \nu) = \zeta^4 \cosh(\nu) + \eta^4 \sinh(\nu)$

Figure 4: The exact and approximate solutions $\Psi(\zeta, \eta, z, \nu)$ for Example 2 when $\alpha = 2$, 1.5, 1.

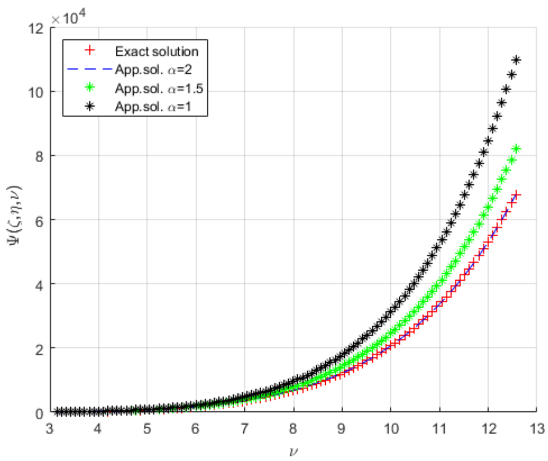

Figure 4. $\Psi(\zeta, \eta, \nu) = \zeta^4 \cosh(\nu) + \eta^4 \sinh(\nu)$.

Example 3. *Consider the 3D fractional heat equation [19]:*

$$\begin{aligned} \frac{\partial^\alpha \Psi(\zeta,\eta,z,\nu)}{\partial \nu^\alpha} &= \frac{1}{36}\left[\zeta^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial \eta^2} + z^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial z^2}\right] + \zeta^4 \eta^4 z^4 \\ 0 &< \alpha \leq 1, \end{aligned} \tag{36}$$

with conditions

$$\Psi(\zeta,\eta,z,0) = 0. \tag{37}$$

Applying the NTM to both sides of Equation (36), we obtain

$$\begin{aligned} &\frac{s^\alpha}{\mu^\alpha}\mathbb{N}^+[\Psi(\zeta,\eta,z,\nu)] - \frac{s^{\alpha-1}}{\mu^\alpha}\Psi(\zeta,\eta,z,0) = \\ &\mathbb{N}^+\left[\frac{1}{36}\left[\zeta^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial \zeta^2} + \eta^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial \eta^2} + z^2 \frac{\partial^2 \Psi(\zeta,\eta,z,\nu)}{\partial z^2}\right]\right] \\ &\qquad + \mathbb{N}^+\left[\zeta^4 \eta^4 z^4\right] \end{aligned} \tag{38}$$

Taking the inverse NTM of Equation (38), we have

$$\begin{aligned}\Psi(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\Psi(\zeta,\eta,z,0)}{s}+\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\zeta^4\eta^4z^4\right]\right]\\ &+\mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi(\zeta,\eta,z,\nu)}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi(\zeta,\eta,z,\nu)}{\partial\eta^2}\right]\right]\\ &+\mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[z^2\frac{\partial^2\Psi(\zeta,\eta,z,\nu)}{\partial z^2}\right]\right]\end{aligned} \tag{39}$$

Then, Equation (39) becomes

$$\begin{aligned}\Psi_0(\zeta,\eta,z,\nu) &= \zeta^4\eta^4z^4\frac{\nu^{\alpha}}{\Gamma(\alpha+1)},\\ \sum_{n=0}^{\infty}\Psi_{n+1}(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_n}{\partial\zeta^2}+\eta^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_n}{\partial\eta^2}\right]\right]\\ &+\mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[z^2\sum_{n=0}^{\infty}\frac{\partial^2\Psi_n}{\partial z^2}\right]\right]\end{aligned} \tag{40}$$

For $n=0$,

$$\begin{aligned}\Psi_1(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi_0}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi_0}{\partial\eta^2}+z^2\frac{\partial^2\Psi_0}{\partial z^2}\right]\right]\\ &= \mathbb{N}^{-1}\left[\zeta^4\eta^4z^4\frac{\mu^{2\alpha}}{s^{2\alpha+1}}\right]\\ &= \zeta^4\eta^4z^4\frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{41}$$

The subsequent terms are given as follows:

$$\begin{aligned}\Psi_2(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi_1}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi_1}{\partial\eta^2}+z^2\frac{\partial^2\Psi_1}{\partial z^2}\right]\right]\\ &= \zeta^4\eta^4z^4\frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)},\\ \Psi_3(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi_2}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi_2}{\partial\eta^2}+z^2\frac{\partial^2\Psi_2}{\partial z^2}\right]\right]\\ &= \zeta^4\eta^4z^4\frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)},\\ \Psi_4(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi_3}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi_3}{\partial\eta^2}+z^2\frac{\partial^2\Psi_3}{\partial z^2}\right]\right]\\ &= \zeta^4\eta^4z^4\frac{\nu^{5\alpha}}{\Gamma(5\alpha+1)},\\ \Psi_5(\zeta,\eta,z,\nu) &= \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\frac{1}{36}\left[\zeta^2\frac{\partial^2\Psi_4}{\partial\zeta^2}+\eta^2\frac{\partial^2\Psi_4}{\partial\eta^2}+z^2\frac{\partial^2\Psi_4}{\partial z^2}\right]\right]\\ &= \zeta^4\eta^4z^4\frac{\nu^{6\alpha}}{\Gamma(6\alpha+1)},\\ &\;\;\vdots\end{aligned} \tag{42}$$

The approximate solution of Equation (42) is denoted by

$$
\begin{aligned}
\Psi(\zeta,\eta,z,\nu) &= \Psi_0+\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5+\Psi_6+\cdots \\
\Psi(\zeta,\eta,z,\nu) &= \zeta^4\eta^4z^4\left[\frac{\nu^{\alpha}}{\Gamma(\alpha+1)}+\frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)}\right] \\
&\quad +\zeta^4\eta^4z^4\left[\frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{\nu^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots\right]
\end{aligned} \tag{43}
$$

By letting $\alpha=1$ in Equation (43), we have

$$
\Psi(\zeta,\eta,z,\nu)=\zeta^4\eta^4z^4\left[\nu+\frac{\nu^2}{2}+\frac{\nu^3}{6}+\frac{\nu^4}{24}+\frac{\nu^5}{120}+\cdots\right] \tag{44}
$$

Therefore, the solution of Equation (36) is provided by

$$
\Psi(\zeta,\eta,z,\nu)=\zeta^4\eta^4z^4[e^{\nu}-1]
$$

Figure 5: The exact and approximate solutions $\Psi(\zeta,\eta,z,\nu)$ for Example 3 when $\alpha=2$.

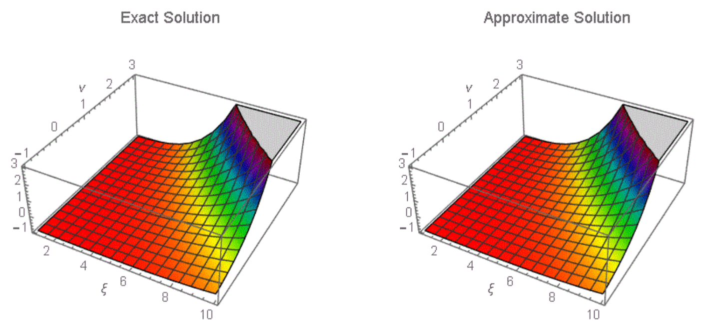

Figure 5. $\Psi(\zeta,\eta,z,\nu)=\zeta^4\eta^4z^4[e^{\nu}-1]$.

Figure 6: The exact and approximate solutions $\Psi(\zeta,\eta,z,\nu)$ for Example 3 when $\alpha=1, 0.80, 0.50$.

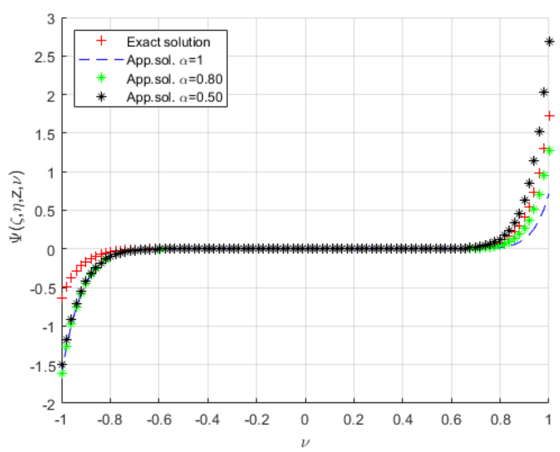

Figure 6. $\Psi(\zeta,\eta,z,\nu)=\zeta^4\eta^4z^4[e^{\nu}-1]$.

In the next example, we apply the natural transform decomposition method to solve the homogeneous time-fractional gas dynamics equation.

Example 4. *Consider the following homogeneous time-fractional gas dynamics equation [27]:*

$$\frac{\partial^{\alpha}\Psi(\zeta,\nu)}{\partial\nu^{\alpha}} + \Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta} - \Psi(\zeta,\nu)[1-\Psi(\zeta,\nu)] = 0,\ \nu > 0, 0 < \alpha \leq 1, \tag{45}$$

with the initial condition

$$\Psi(\zeta,\nu) = e^{-\zeta}, \quad \zeta \in \mathbb{R}. \tag{46}$$

Taking the NTM of Equation (45) and the initial condition in Equation (46), we have

$$\begin{aligned}
\mathbb{N}^{+}\left[\frac{\partial^{\alpha}\Psi(\zeta,\nu)}{\partial\nu^{\alpha}}\right] &= \mathbb{N}^{+}\left[-\Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta} + \Psi(\zeta,\nu) - \Psi^{2}(\zeta,\nu)\right] \\
\mathbb{N}^{+}[\Psi(\zeta,\nu)] &= \frac{1}{s}e^{-\zeta} + \frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\right] \\
&+ \frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\Psi(\zeta,\nu) - \Psi^{2}(\zeta,\nu)\right]
\end{aligned} \tag{47}$$

Taking the inverse NTM of Equation (47), we obtain

$$\begin{aligned}
\Psi(\zeta,\nu) &= e^{-\zeta} - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\right]\right] \\
&+ \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[\Psi(\zeta,\nu)]\right] - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\Psi^{2}(\zeta,\nu)\right]\right]
\end{aligned} \tag{48}$$

Now, we assume an infinite series solution for the Equation (48) given by the form

$$\begin{aligned}
\Psi(\zeta,\nu) &= \sum_{n=0}^{\infty}\Psi_{n}(\zeta,\nu) \\
\Psi(\zeta,\nu) &= e^{-\zeta} - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}\right]\right] \\
&+ \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[\Psi(\zeta,\nu)]\right] - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\Psi^{2}(\zeta,\nu)\right]\right]
\end{aligned} \tag{49}$$

The nonlinear terms $\Psi(\zeta,\nu)\frac{\partial\Psi(\zeta,\nu)}{\partial\zeta}$, and $\Psi^{2}(\zeta,\nu)$ are denoted by

$$\Psi\Psi_{\zeta} = \sum_{n=0}^{\infty}K_{n}, \qquad \Psi^{2} = \sum_{n=0}^{\infty}M_{n} \tag{50}$$

where K_n and M_n are Adomian polynomials. Then, Equation (49) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Psi_{n}(\zeta,\nu) = e^{-\zeta} - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}K_{n}\right]\right] \\
&+ \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[\Psi_{n}]\right] - \mathbb{N}^{-1}\left[\frac{\mu^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}M_{n}\right]\right],
\end{aligned} \tag{51}$$

The nonlinear term K_n and M_n are expressed as

$$\begin{aligned}
K_{0} &= \Psi_{0}\Psi_{0\zeta}, \\
K_{1} &= \Psi_{0}\Psi_{1\zeta} + \Psi_{1}\Psi_{0\zeta}, \\
K_{2} &= \Psi_{0}\Psi_{2\zeta} + \Psi_{1}\Psi_{1\zeta} + \Psi_{2}\Psi_{0\zeta}, \\
K_{3} &= \Psi_{0}\Psi_{3\zeta} + \Psi_{1}\Psi_{2\zeta} + \Psi_{2}\Psi_{1\zeta} + \Psi_{3}\Psi_{0\zeta}, \\
&\vdots
\end{aligned} \tag{52}$$

and

$$\begin{aligned} M_0 &= \Psi_0^2, \\ M_1 &= 2\Psi_0\Psi_1, \\ M_2 &= 2\Psi_0\Psi_2 + \Psi_1^2, \\ &\vdots \end{aligned} \tag{53}$$

Making both sides of Equation (51) equivalent, we can obtain

$$\begin{aligned} \Psi_0(\zeta,\nu) &= e^{-\zeta}, \\ \Psi_1(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\Psi_0\Psi_{0\zeta}\right]\right] \\ &\quad +\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_0]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\Psi_0^2\right]\right] \\ &= e^{-\zeta}\frac{\nu^\alpha}{\Gamma(\alpha+1)}, \\ \Psi_2(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\Psi_0\Psi_{1\zeta} + \Psi_1\Psi_{0\zeta}\right]\right] \\ &\quad +\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_1]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[2\Psi_0\Psi_1]\right] \\ &= e^{-\zeta}\frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)}, \\ \Psi_3(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\Psi_0\Psi_{2\zeta} + \Psi_1\Psi_{1\zeta} + \Psi_2\Psi_{0\zeta}\right]\right] \\ &\quad +\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_3]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_0\Psi_3 + \Psi_1\Psi_2]\right] \\ &= e^{-\zeta}\frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)}, \\ \Psi_4(\zeta,\nu) &= -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[\Psi_0\Psi_{3\zeta} + \Psi_1\Psi_{2\zeta} + \Psi_2\Psi_{1\zeta} + \Psi_3\Psi_{0\zeta}\right]\right] \\ &\quad +\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+[\Psi_3]\right] - \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{s^\alpha}\mathbb{N}^+\left[2\Psi_0\Psi_2 + \Psi_1^2\right]\right] \\ &= e^{-\zeta}\frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)}, \\ &\vdots \end{aligned} \tag{54}$$

The above equation becomes

$$\begin{aligned} \Psi(\zeta,\nu) &= \sum_{n=0}^{\infty}\Psi_n(\zeta,\nu) \\ &= \Psi_0(\zeta,\nu) + \Psi_1(\zeta,\nu) + \Psi_2(\zeta,\nu) + \Psi_3(\zeta,\nu) + \Psi_4(\zeta,\nu) + \cdots \\ &= e^{-\zeta}\left[1 + \frac{\nu^\alpha}{\Gamma(\alpha+1)} + \frac{\nu^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\nu^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\nu^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right] \\ &= e^{-\zeta}\sum_{n=0}^{\infty}\frac{\nu^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{55}$$

take $\alpha = 1$, the approximate solution of Equation (55) given by

$$\Psi(\zeta,\nu) = e^{-\zeta}\left[1 + \frac{\nu}{1!} + \frac{\nu^2}{2!} + \frac{\nu^3}{3!} + \frac{\nu^4}{4!} + \cdots\right]$$

the solution of Equation (45) is denoted by

$$\Psi(\zeta,\nu) = e^{-\zeta+\nu}$$

Figure 7: The exact and approximate solutions $\Psi(\zeta,\nu)$ for Example 4 when $\alpha = 2$.

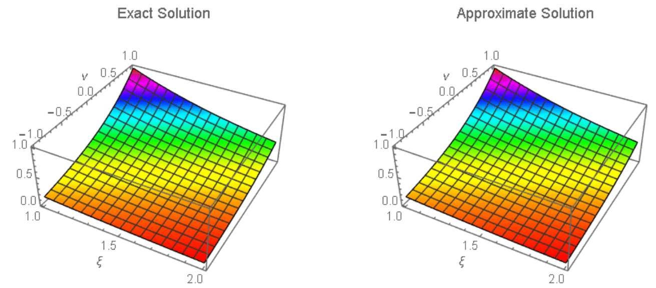

Figure 7. $\Psi(\zeta,\nu) = e^{-\zeta+\nu}$.

Figure 8: The exact and approximate solutions $\Psi(\zeta,\nu)$ for Example 4 when $\alpha = 1, 0.80, 0.50$.

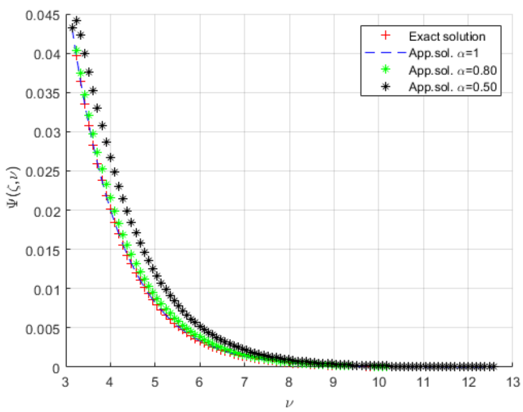

Figure 8. $\Psi(\zeta,\nu) = e^{-\zeta+\nu}$.

The introduced paper's purpose is to obtain analytical and numerical solutions for (NTDM) more precisely. In Tables 1–4, we show a comparison between the absolute errors for the obtained numerical results and the exact solution.

Table 1. Comparison of the absolute errors for the obtained numerical results and the exact solution for Example 1, for $\alpha = 2, 1.80$, and 1.5.

ζ	ν	$\alpha = 2$	$\alpha = 1.8$	$\alpha = 1.5$	Exact	Absolute Error
	0.5	0.0299640962	0.0293796164	0.0280835396	0.0299640962	0
0.25	0.75	0.0513947958	0.0452272241	0.0384946707	0.0426024225	0.0087923732
	1	0.0734500746	0.0501614436	0.0460926237	0.0010907754	0.0723592991
	0.5	0.1302738263	0.1175184656	0.1123341583	01198563846	0.0104174417
0.50	0.75	0.2055791830	0.1809088961	0.153976827	0.1704096900	0.035169493
	1	0.2938002984	0.2006457422	0.184370449	0.0043631016	0.2894371968
	0.5	0.5210953055	0.4700738629	0.4493366329	0.4794255386	0.0416697669
1	0.75	0.8223167320	0.7236355851	0.6159147310	0.6816387600	0.140677972
	1	1.175201193	0.8025829685	0.7374819794	0.0174524064	1.157748787

Table 2. Comparison of the absolute errors for the obtained numerical results and the exact solution for Example 2, when $\zeta = \eta = 0.25, 0.50$, and 1.

ζ	ν	$\alpha = 2$	$\alpha = 1.8$	$\alpha = 1.5$	Exact	Absolute Error
	0.5	0.0064403174	0.0066750912	0.0072013111	0.0064403174	1×10^{-12}
0.25	0.75	0.0082695312	0.0084681072	00096675856	0.0082695313	6.7×10^{-11}
	1	0.0106182872	0.0074431273	0.0128345819	0.0106182883	1.19×10^{-9}
	0.5	0.1030451762	0.1070299290	0.1152209783	0.0130450794	9.68×10^{-8}
0.50	0.75	0.1323130518	0.0885159904	0.1546813706	0.1323125010	5.508×10^{-7}
	1	0.1698925954	0.1190900375	0.2053533103	0.1698926143	1.89×10^{-8}
	0.5	1.648721270	1.708823359	1.843435654	1.648721270	0
1	0.75	2.117000000	2.232441653	2.474901931	2.117000017	1.7×10^{-8}
	1	2.718281526	1.905440600	3.285652963	2.718281828	3.02×10^{-7}

Table 3. Comparison of the absolute errors for the obtained numerical results and the exact solution for Example 3, when $\zeta = \nu = z = 0.25, 0.50, 1$.

ζ	ν	$\alpha = 1$	$\alpha = 0.80$	$\alpha = 0.50$	Exact	Absolute Error
	0.5	0.386654×10^{-7}	0.481388×10^{-7}	0.103833×10^{-6}	0.386668×10^{-7}	1.39203×10^{-12}
0.25	0.75	0.665619×10^{-7}	0.841673×10^{-7}	0.157573×10^{-6}	0.665783×10^{-7}	1.647378×10^{-11}
	1	0.102321×10^{-6}	0.129654×10^{-6}	0.219436×10^{-6}	1.024175×10^{-7}	9.62711×10^{-11}
	0.5	0.0001583735	0.0001971769	0.0004253023	1.583792×10^{-7}	5.7017×10^{-9}
0.50	0.75	0.0002726376	0.0003447496	0.0006454195	2.727050×10^{-7}	6.74765×10^{-8}
	1	0.0004191080	0.0005869575	0.0008988117	4.195023×10^{-7}	3.9432705×10^{-7}
	0.5	0.6486979167	0.8076367013	1.742038385	0.6487212707	2.3354×10^{-5}
1	0.75	1.116723633	1.412094400	2.643638290	1.117000017	2.76384×10^{-4}
	1	1.716666667	2.175242941	3.681533056	1.718281828	1.615161×10^{-3}

Table 4. Comparison of the absolute errors for the obtained numerical results and the exact solution for Example 4, with $\alpha = 1, 0.80$, and 0.5.

ζ	ν	$\alpha = 1$	$\alpha = 0.80$	$\alpha = 0.50$	Exact	Absolute Error
	0.5	1.284007229	1.501210297	2.135501642	1.284025417	1.8188×10^{-5}
0.25	0.75	1.648506023	1.968100996	2.837668354	1.648721271	2.15248×10^{-4}
	1	2.115742127	2.553826345	3.645981610	2.117000017	1.25789×10^{-3}
	0.5	0.999985352	1.169143755	1.663130351	1	1.415448×10^{-5}
0.50	0.75	1.283857782	1.532758597	2.209978336	1.284025417	1.67635×10^{-4}
	1	1.647741626	1.988921957	2.839493333	1.648721271	9.79645×10^{-4}
	0.5	0.6065220684	0.7091215329	1.008739549	0.6065306597	8.5913×10^{-6}
1	0.75	0.7786991072	0.9296650831	1.340419618	0.7788007831	1.01659×10^{-4}
	1	0.9994058152	1.206342147	1.722239764	1	5.941848×10^{-4}

5. Conclusions

The authors in this work successfully executed the natural transform decomposition method (NTDM) to acquire the approximate solutions of (1+3)-dimensional fractional nonlinear partial differential equations. We have also offered three test problems. The simplicity and high precision of the method show that this technique can be involved in many nonlinear partial differential equations. The NTDM presents a significant improvement in the field over the existing methods such as the optimal homotopy asymptotic method (OHAM) and fractional homotopy analysis transform method (FHATM). In addition to the currently presented methods, it is noteworthy to highlight the discontinuous Galerkin method [28] as a novel and efficient alternative for solving fractional-order linear and nonlinear partial differential equations. Its application to these equations holds substantial potential and can produce promising outcomes. Mathematica software package was applied to obtain the numerical results and graphs.

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