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# Irreducible Characters with Cyclic Anchor Group

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**Abstract:** We consider $G$ to be a finite group and $p$ as a prime number. We fix $\psi$ to be an irreducible character of $G$ with its restriction to all $p$-regular elements of $G$ and $\psi^0$ to be an irreducible Brauer character. The main aim of this paper is to describe and investigate the relationship between cyclic anchor group of $\psi$ and the defect group of a $p$-block which contains $\psi$. Our methods are to study and generalize some facts for the cyclic defect groups of a $p$-block $B$ to the case of a cyclic anchor group of irreducible characters which belong to $B$. We establish and prove a criteria for an irreducible character to have a cyclic anchor group.

**Keywords:** finite group; group algebra; character; block; defect group; vertex theory

**MSC:** 20C20; 20C15; 20C05

## 1. Introduction

We let $p$ be a prime number. We let $G$ be a finite group. We let $B$ be a $p$-block of group $G$ with defect group $D$. We let $(k, \mathcal{R}, F)$ be a $p$-modular system which consists of a complete discrete valuation ring $\mathcal{R}$ with field of fractions $k$ of characteristic 0. We let $v_p$ be a valuation on field $k$ such that $v_p(p) = 1$; then, we have residue field $F = \mathcal{R}/J(\mathcal{R})$ which is of characteristic $p$, where $J(\mathcal{R})$ is the Jacobson radical of ring $\mathcal{R}$. We let $Irr(G)$ be the set of all ordinary irreducible characters of $G$ which corresponds to the set of simple $kG$-modules. We let $IBr(G)$ be the set of all irreducible Brauer characters of $G$. We write $\psi^0$ to mean the restriction of the ordinary irreducible character $\psi$ to the set of all $p$-regular elements ($p$ does not divide the order of the elements) of $G$; see [1], Chapter 3, Section 6 and [2], Chapter 2.

In 1941, R. Brauer studied the ordinary irreducible character theory in the $p$-block $B$ of defect one as in [3]. Then, in 1946, he offered a definition of the defect group of a $p$-block in [4]. In 1959, J. A. Green [5] proved that if $N$ is an indecomposable $FG$-module and $V$ is a vertex of $N$ which is a subset of the Sylow $p$-subgroup $P$ of $G$, then, the index $[P : V]$ divides $dim_F(N)$, where $dim_F(N)$ is the dimension of $N$ as an $F$-space (see Section 2 below for the definitions of the defect group and the vertex group). In 1966, E. C. Dade [6] circulated all results that appeared in [3] to the $p$-block with a cyclic defect group by using Thompson's method as in [7]. Already in 1971, Brauer [8] offered a definition of the inertial index $\mathcal{T}$ of a $p$-block $B$, if $B$ is the $p$-block of $G$ with defect group $D$. Let us write $C_G(D)$ for the centralizer of the defect group $D$ in the group $G$. If $\mathcal{B}$ is the $p$-block of $DC_G(D)$ with inertial group $I(\mathcal{B}) := \{x \in G | \mathcal{B}^x = \mathcal{B}\}$, then the inertial index is the natural number $\mathcal{T} = [I(\mathcal{B}) : DC_G(D)]$.

In 1976, G. O. Michler [9] introduced a generalization of Brauer's concept of the inertial index. In the case that the defect group of $B$ is cyclic and the assumption that $F$ (of characteristic $p$) is a splitting field for the group $G$ and its subgroups, then Michler's concept of $\mathcal{T}$ and Brauer's concept of $\mathcal{T}$ are equal.

We let $\psi \in Irr(G)$. Then, $\psi$ can be uniquely extended to an algebra map $\psi : kG \to k$ by the rule $\psi(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g \psi(g)$. We consider the element

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$$e_\psi = \frac{\psi(1)}{|G|} \sum_{g \in G} \psi(g^{-1})g,$$

which is the unique central primitive idempotent in $kG$ such that $\psi(e_\psi) \neq 0$; see [10], Theorem 3.3.1. The algebra $(\mathcal{R}G) \cdot e_\psi$ is a primitive $G$-interior $\mathcal{R}$-algebra ([11], p. 76) because the center $Z((\mathcal{R}G) \cdot e_\psi)$ is a subring of the center $Z((kG) \cdot e_\psi)$. The anchor group of an irreducible character $\psi$ of $G$ is defined as the defect group of the primitive $G$-interior $\mathcal{R}$-algebra $(\mathcal{R}G) \cdot e_\psi$. This notion appears in [12]. The main motivation of this paper is to describe and investigate the relationship between a cyclic anchor group of $\psi \in Irr(G)$ with $\psi^0 \in IBr(G)$ and a defect group of a $p$-block containing $\psi$. Our results are to study and generalize some facts for the cyclic defect groups of a $p$-block $B$ to the case of cyclic anchor group of irreducible characters which belong to $B$. The most important references that are concerned with the block theory, defect theory and vertex theory [4,10,11,13–18].

The organization of this paper is as follows. Section 2 contains the preliminaries of the anchor group of irreducible character of $G$ as in [12]. We study D. G. Higman's Theorem [19] which classifies the modular group algebra $FG$ to be finite representation type.

In Section 3, we focus our attention on the basic facts and introduce the main theorems of the cyclic anchor group of an irreducible character $\psi$ of $G$ in the case that $\psi^0 \in IBr(G)$. Our first main goal is to show that the anchor group, say, $A_\psi$, of $\psi$ is cyclic if and only if the defect group of the $p$-block, which contains $\psi$, is cyclic.

Our second main goal is to reformulate G. O. Michler's Theorem which generalizes the results in this approach for Dade [6], Rothschild [20], Kupisch [21] and Janusz [22].

**Theorem 1.** *Let $\chi \in Irr(G)$ with $\chi^0 \in IBr(G)$ and a cyclic anchor group $A_\chi$. Assume that $\chi$ belongs to the $p$-block $B$ with defect group $D$. Write $|A_\chi|$ for the order of $A_\chi$ and $\mathcal{T}$ for the inertial index of the $p$-block $B$. Then, the following hold:*

1. *The defect group of $B$ is $A_\chi$.*
2. *The inertial index $\mathcal{T}$ divides $p-1$.*
3. *The $p$-block $B$ contains $\mathcal{T}$ non-isomorphic simple $FG$-modules.*
4. *The $p$-block $B$ contains $\mathcal{T}|A_\chi|$ non-isomorphic indecomposable $FG$-modules.*
5. *The $p$-group $A_\chi$ is a vertex of every simple $FG$-module that belongs the $p$-block $B$.*

Our third main goal is to prove that if $A_\psi$ is a cyclic anchor group of $\psi$, which is contained in a Sylow $p$-subgroup, say, $P$, then index $[P : A_\psi]$ divides the degree $\psi(1)$. Which is considered the relative version of Theorem 9 in [5]. Our Theorems 7 and 8 describe the anchor group of any irreducible character $\theta$ of a normal subgroup $Q$ of $G$ lying under $\psi$ (respectively, under $\xi \in Irr(I(C))$ with $\xi^0 \in IBr(I(C))$) with a cyclic anchor group, where $C$ is a $p$-block of $Q$. In Theorem 9, we investigate the anchor group of any irreducible character $\widetilde{\psi}$ of the quotient group $G/Q$ such that $\psi$ is the lift of $\widetilde{\psi}$ to $G$ with a cyclic anchor group. We let $P$ be a $p$ subgroup of $G$ such that $N_G(P) \leq Q \leq G$. We deduce that the cyclic anchor group is invariant under the process of induction of irreducible character of $N_G(P)$ (respectively, of $Q$) which belongs to the $p$-block of $N_G(P)$ (respectively, of $Q$) of defect group $P$ to the irreducible character of $G$ in Theorem 10 and Theorem 11, respectively.

The last main goal of this paper is to establish and prove criteria for an irreducible character to have a cyclic anchor group as in the following results:

**Proposition 1.** *Let $G$ be a finite group of order $|G| = p^a m$ such that $gcd(p, m) = 1$, $a, m \in \mathbb{Z}^+$ where $p$ is a fixed prime number. Let $\psi \in Irr(G)$ such that $\psi_p(1) = p^{a-1}$. Then, the anchor group of $\psi$ is cyclic group.*

**Theorem 2.** *Consider a finite group $G$ to be a semidirect product of a normal cyclic group $P$ of order $p^a$ and a $p$-complement $Q$ of order prime to $p$. If $\chi \in Irr(G)$, then the anchor group of $\chi$ is a cyclic group. In particular, the anchor group of any $\chi \in Irr(G)$ is a Sylow $p$ subgroup.*

Our main methods to accomplish the results are block theory and character theory, which includes induction, restrictions, inner product of characters and orthogonality relations; see [1]. In fact, character theory is a vast area of research in group theory and generalizations; see [23–26]. Also, the Clifford theory is the main theorem to study character theory and $p$-blocks of finite groups, in fact, given a normal subgroup $Q$ of a finite group $G$ and $\theta \in Irr(Q)$. We suppose that $I_G(\theta) = \{g \in G | \theta^g = \theta\} < G$ is the inertia group of $\theta$ in $G$; then, the Clifford theory can be used to create a bijection between any $\chi \in Irr(G|\theta)$ and $\eta \in Irr(I_G(\theta)|\theta)$ such that $\chi = Ind_{I_G(\theta)}^G(\eta)$ and $Res_{I_G(\theta)}^G(\eta) = e\theta$, where $e$ is a nonnegative integer. The integer $e$ is said to be the ramification index of $\chi$ relative to $Q$ which satisfies $Res_Q^G(\chi) = e(\sum_{g \in [G/I_G(\theta)]} \theta^g)$. Here, $\theta^g$ is the conjugate character of $\theta$ such that $\theta^g(s) = \theta(gsg^{-1})$ for all $s \in Q$. We refer to [1,27,28] for further information on Clifford theory. For the finite groups that appear in this paper, we assume, in general, that $k$ and $F$ are splitting fields.

## 2. Definitions and Preliminaries

We consider $G$ to be a finite group of order $|G| = p^a m$ such that $gcd(p, m) = 1$, $a, m \in \mathbb{Z}^+$ for a fixed prime number $p$. We let $\psi \in Irr(G)$. If $p^n$ is the highest power of $p$ which divides the natural number $\frac{|G|}{\psi(1)}$, then $p^n = \frac{|G|_p}{\psi(1)_p}$, where $r_p$ denotes the $p$-part of an integer $r$. Then, the natural number $n$ is called the $p$ defect of the irreducible character $\psi$ and we denote it by $def(\psi)$. If $def(\psi) = a$, we say that $\psi$ is of full defect. The maximal $p$ defect of irreducible character which belongs to the $p$-block $B$ is called the defect number of $B$ and is denoted by $def(B)$. The height of $\psi$ is obtained by subtracting the defect number of $B$ and the defect of $\psi$. We write $h(\psi) = def(B) - def(\psi)$ to mean the height of $\psi$. We write that $D^\Delta := \{(x, x) : x \in Q\}$ for $D$ is a subgroup of $G$. We let $e_B \in Z(\mathcal{R}G)$ be a$p$-block of $G$; then, $(\mathcal{R}G) \cdot e_B$ is an indecomposable $\mathcal{R}[G \times G]$-module via the following natural action: $(g_1, g_2)a = g_1 a g_2^{-1}$, where $a \in \mathcal{R}G$, $g_1$, $g_2 \in G$. We say that $D$ is a defect group of $p$-block $e_B$ of $\mathcal{R}G$ if $D^\Delta$ is a vertex of $(\mathcal{R}G) \cdot e_B$ as an indecomposable $\mathcal{R}[G \times G]$-module which is uniquely defined up to $G$-conjugacy. We remind the reader that the vertex of an indecomposable $\mathcal{R}G$-module $M$ is a unique (up to $G$-conjugacy) minimal $p$-subgroup $V$ of $G$ such that $M$ is $V$-projective of $G$. This is equivalent to that $M$ is a direct summand of the induced $Ind_V^G(N)$ for some $\mathcal{R}V$-module $N$. We refer to [1,10,11,14–16] for further theory on the defect group.

We describe the anchor group of an irreducible character of $G$, which was first introduced by R. Kessar, B. Külshammer and M. Linckelmann in [12].

**Definition 1.** *Consider $G$ to be a finite group and $\psi \in Irr(G)$. The defect group of the primitive $G$-interior $\mathcal{R}$-algebra $(\mathcal{R}G) \cdot e_\psi$ is called the anchor group of $\psi$.*

The following remark appears in [29] as the anchor group is a defect group.

**Remark 1.** *If $A_\chi$ is an anchor group of an irreducible character $\chi$ of $G$, then $A_\chi$ is a $p$ subgroup of $G$.*

The following proposition is crucial in this work and we extract it from paper [12].

**Proposition 2.** *Let $G$ be a finite group and $\psi \in Irr(G)$ which belongs to the $p$-block $B$ of $\mathcal{R}G$. Suppose that $A_\psi$ is an anchor group of $\psi$. Let $L$ be an indecomposable $\mathcal{R}G$-lattice affording $\psi$. The following hold:*

*(i) a defect group of $B$ contains $A_\psi$.*
*(ii) a vertex of $L$ is contained in $A_\psi$.*
*(iii) if $B$ has an abelian defect group $D$, then $D$ is an anchor group of $\psi$.*

An algebra is of finite representation type if there are only a finite number of isomorphism classes of finite dimensional indecomposable modules; see [30], Chapter 4,

Section 4 and [31], Chapter 11, Section 4. Considering the assumption that $G$ is a finite group and $(k, \mathcal{R}, F)$ is a $p$-modular system, we use the following result which appears in [19] as a tool to investigate the required issue.

**Theorem 3.** *(D.G. Higman) The modular group algebra $FG$ possesses finite representation type if and only if any Sylow $p$-subgroup of $G$ is cyclic.*

## 3. Main Results

We focus our attention on studying and generalizing some basic facts of the cyclic anchor group of an irreducible character $\psi$ of $G$ in the case $\psi^0 \in IBr(G)$. We describe and investigate the relationship between a cyclic anchor group of $\psi$ and a defect group of a $p$-block containing $\psi$. We reformulate G. O. Michler's Theorem for the $p$-blocks with a cyclic defect group in [9] to the $p$-blocks that contain an irreducible character with a cyclic anchor group. We prove that if $\psi$ is with a cyclic anchor group, then the $\mathcal{R}$-algebra $(\mathcal{R}G) \cdot e_\psi$ is of finite representation type. We show that the index of the cyclic anchor group of $\psi$ in a Sylow $p$ subgroup of $G$ that divides the degree of $\psi$. We describe the anchor group of any irreducible character $\theta$ of a normal subgroup $Q$ of $G$ lying under an irreducible character $\psi$ of $G$ with a cyclic anchor group. We establish and prove criteria for an irreducible character to have a cyclic anchor group.

**Theorem 4.** *Consider $G$ to be a finite group. Let $\psi \in Irr(G)$ which belongs to $p$-block $B$ of $\mathcal{R}G$ with a defect group $D$. Suppose that $\psi^0 \in IBr(G)$. Then, the anchor group, say, $A_\psi$, of $\psi$ is cyclic if and only if the defect group $D$ is cyclic. In particular, if $A_\psi$ is cyclic, then it is the defect group of $B$.*

**Proof.** Suppose that $D$ is a cyclic defect group. By Proposition 2 (i), $A_\psi$ is contained in the defect group $D$. Then, $A_\psi$ is cyclic, since every subgroup of a cyclic group is cyclic. Conversely, assume that $A_\psi$ is a cyclic group; then, from Proposition 2 (ii), $A_\psi$ contains a vertex of an indecomposable $\mathcal{R}G$-lattice $L$ affording $\psi$. Thus, a vertex of $L$ is cyclic. Since $\psi^0 \in IBr(G)$, there is a unique $\mathcal{R}G$-lattice $L$ affording $\psi$, up to isomorphism by [12], Proposition 3.6. However, the vertex of a simple $FG$-module with Brauer character $\psi^0$ is contained in every vertex of the $\mathcal{R}G$-lattice $L$ affording $\psi$. But there is a vertex of $L$, which is cyclic. Thus, a vertex of a simple $FG$-module is cyclic. Therefore, by Erdmann's Theorem in [32], the defect group of the $p$-block of $FG$ is cyclic. Since the $p$-blocks of $\mathcal{R}G$ and $FG$ are in one-to-one correspondence, under reduction modulo, the maximal ideal $J(\mathcal{R})$ of $\mathcal{R}$ by [31], Proposition 12.2.1. Therefore, it is the same thing to study the $p$-block of $\mathcal{R}G$ and of $FG$. We let $\bar{e}_B \in Z(FG)$. Likewise, it is possible to define a defect group of $\bar{e}_B$ of modular group algebra $FG$ as it is a vertex of the indecomposable $F[G \times G]$-module $(FG) \cdot \bar{e}_B$. From [31], Exercise 11.10.21, the reduction modulo $J(\mathcal{R})$ preserves the vertices of indecomposable modules. Thus, the vertex of $(\mathcal{R}G) \cdot e_B$ and the vertex $(FG) \cdot \bar{e}_B$ are the same. It follows that the reduction modulo $J(\mathcal{R})$ preserves the defect groups of the $p$-blocks of $\mathcal{R}G$ and $FG$. Also, see [18], Theorem 6.1.6. Hence, the defect group of the $p$-block $B$ of $\mathcal{R}G$ is cyclic. Furthermore, if $A_\psi$ is cyclic, then the defect group of $B$ is abelian. From Proposition 2 (iii), $A_\psi$ is the defect group of $B$. □

Theorem 4 can be strengthened to $A_\psi$ being abelian.

**Proof of Theorem 1.** 1. From Theorem 4, if $A_\chi$ is a cyclic group, then a defect group of $B$ is cyclic. From Proposition 2 (iii), $A_\chi$ is the defect group of the $p$-block $B$, and the result holds. The remaining results from (1); the defect group of the $p$-block $B$ is $A_\chi$. Then, when [9], Theorem 10.1 if applied, the results are proven. □

**Corollary 1.** *Suppose that $B$ is a $p$-block of $FG$ such that $B := (FG) \cdot e^*$. Here, $e^*$ is a central primitive idempotent in $FG$, while $e$ is a central primitive idempotent in $\mathcal{R}G$. Let $\psi \in Irr(G)$ with $\psi^0 \in IBr(G)$ and a cyclic anchor group such that $ee_\psi \neq 0$. Then, $B$ is of finite representation type. In particular, $(\mathcal{R}G) \cdot e_\psi$ is of finite representation type.*

**Proof.** The condition that $ee_\psi \neq 0$ is equivalent to $\psi$ belongs to the $p$-block $B$. Then the defect group $\mathcal{D}$ of the $p$-block $B$ is cyclic by Theorem 4. The modular group algebra $F\mathcal{D}$ has only $|\mathcal{D}|$ isomorphism classes of indecomposable modules from ([33], p. 24, 25). Since every indecomposable $FG$-module which is contained in the $p$-block $B$ possesses vertex contained in the defect group $\mathcal{D}$ of $B$. So, there are only finite number of sources in the $p$-block $B$. Thus, there are only finite number of isomorphism classes of indecomposable modules in $B$. Hence, the $p$-block $B := (FG) \cdot e^*$ is of finite representation type. In particular, as $A_\psi$ is the cyclic defect group of the primitive $G$-interior $\mathcal{R}$-algebra $(\mathcal{R}G) \cdot e_\psi$, then $(\mathcal{R}G) \cdot e_\psi$ is of finite representation type. □

**Theorem 5.** *Suppose that $\psi \in Irr(G)$ with cyclic anchor group $A_\psi$ which is contained in $P$. Let $\psi^0 \in IBr(G)$. Then, index $[P : A_\psi]$ divides $\psi(1)$, where $P$ is a Sylow $p$-subgroup of $G$.*

**Proof.** Suppose that $\psi$ belongs to the $p$-block $B$ with defect group $D$. Suppose that $W$ is an indecomposable $\mathcal{R}G$-lattice which affords $\psi$. From Proposition 2 (ii), $A_\psi$ contains a vertex of $W$. Therefore, a vertex of $W$ is a cyclic group. Since a vertex of the unique simple $FG$-module $\overline{W}$ affording $\psi^0$ is contained in every vertex of indecomposable $\mathcal{R}G$-lattice affording $\psi$. Thus, a vertex of the simple $FG$-module is cyclic. Hence, by [9], Theorem 10.1 and Erdmann's Theorem in [32], $D$ is a vertex of $\overline{W}$. As in proof of Theorem 4, the reduction modulo $J(\mathcal{R})$ preserves the defect groups of the $p$-blocks of $\mathcal{R}G$ and $FG$. It can be concluded, in this case, that the vertex of the simple $FG$-module $\overline{W}$ affording $\psi^0$ is equal to the anchor group of an irreducible character $\psi$ which is equal to the defect group of the $p$-block $B$. Then, the result is obtained from [5], Theorem 9. □

**Theorem 6.** *Suppose that $\psi \in Irr(G)$ with cyclic anchor group $A_\psi$ such that $N_G(A_\psi)$ is a $p$-solvable group. Let $\psi^0 \in IBr(G)$. Then, $\psi(1)_p = [G : A_\psi]_p$.*

**Proof.** Let $L$ be the indecomposable $\mathcal{R}G$-lattice which affords $\psi$. Then, by Proposition 2 (ii), a vertex of $L$ is contained in $A_\psi$. Hence, a vertex of $\mathcal{R}G$-lattice $L$ is cyclic. It follows that the unique simple $FG$-module which affords $\psi^0$ has a cyclic vertex. In this case, as in the proof of Theorem 5, the vertex of the simple $FG$-module which affords $\psi^0$ is equal to the anchor group of irreducible character $\psi$. Then, the result is obtained from [34]. □

**Remark 2.** *Theorem 6 need not hold if the anchor group $A_\chi$ of irreducible character $\chi$ is not cyclic. Let $G$ be the general linear group $GL(2,3)$, $p = 2$. Let $\chi \in Irr(G)$ with degree two which appears in [12], Example 7.2. The anchor group of $\chi$ is the Sylow 2 subgroup of $G$ which is not cyclic. Hence, the $p$-part of the index $[G : A_\chi]_p$ is one while $\psi(1)_p = 2$.*

We let $Q$ be a normal subgroup of $G$. If $C$ is a $p$-block of $Q$, then $C^g = g^{-1}Cg$ is a $p$-block of $Q$ for every $g \in G$. We remind the reader that the inertial group $I(C)$ of the $p$-block $C$ is a subgroup of $G$. It is

$$I(C) := \{g \in G : C^g = C\}.$$

It is clear that $Q \leq I(C) \leq G$. If $\{C_1, \ldots, C_s\}$ is a $G$-conjugacy class of $C$, then we have $\sum_{i=1}^{s} e_{C_i}$, which is a central idempotent in $\mathcal{R}G$, since the set $\{C_1, \ldots, C_s\}$ is invariant under $G$ and every $e_{C_i}$ is an idempotent of $\mathcal{R}Q$. We have

$$\{e_\psi : \psi \in Irr(G)\},$$

which is the set of all central primitive idempotents of $kG$ from [1], Theorem 6.22. Now, we apply [35], Corollary 1.17 (c) to $kG$; we infer that $\sum_{i=1}^{s} e_{C_i}$ is a central idempotent of $kG$

consisting of a sum of $e_\psi$ for some $\psi \in Irr(G)$. Thus, by [2], Theorem 3.9, there is set $\{B_1, \ldots, B_t\}$ of blocks of $G$ such that

$$\sum_{i=1}^{s} e_{C_i} = \sum_{i=1}^{t} e_{B_i}.$$

In this case, we say that the $p$-block $B_i$ covers $C$, and the set $\{B_1, \ldots, B_t\}$ of blocks of $G$, which covers $C$, is denoted by $BL(G|C)$. If $\theta \in Irr(Q)$ such that $\theta \in Irr(C)$, the inertial group $I(\theta)$ of $\theta$ is defined by

$$I(\theta) := \{g \in G : \theta^g = \theta\}.$$

It is clear by the definition of the inertial group that we have $Q \leq I(\theta) \leq I(C)$. The set of all irreducible characters of $G$ which lie over $\theta$ is the set $\{\psi \in Irr(G) | \langle Res^G_Q(\psi), \theta\rangle \neq 0\}$ which is denoted by $Irr(G|\theta)$. We use $=_G$ to denote equality up to $G$-conjugacy.

**Theorem 7.** *Assume that $Q$ is a normal subgroup of $G$. Let $\psi \in Irr(G)$ with $\psi^0 \in IBr(G)$ and a cyclic anchor group $A_\psi$ such that $\langle Res^G_Q(\psi), \theta\rangle \neq 0$, for some irreducible character $\theta$ of $Q$. Then, an anchor group of $\theta$ is cyclic.*

**Proof.** Suppose that $\psi$ belongs to the $p$-block $B$ of $G$ with defect group $D_B$ and $\theta$ belongs to the $p$-block $C$ of $Q$ with defect group $D_C$. Since $\langle Res^G_Q(\psi), \theta\rangle \neq 0$, then, by [2], Theorem 9.2, this is equivalent to $B$ covering $C$. Since $A_\psi$ is a cyclic anchor group, $A_\psi$ is a defect group of $B$. Therefore, from [1], Chapter 5, Theorem 5.10 (v), Theorem 5.16 (ii), $A_\psi \leq I(C)$. It follows that $Q \cap A_\psi =_Q D_C$. Thus, the defect group $D_C$ of the $p$-block $C$ of $Q$ is cyclic. It follows that the anchor group of $\theta$, say, $A_\theta$, is the defect group of $C$ by Theorem 4. □

In the following result, we study a situation of an anchor group of irreducible character of $Q$ lying under an irreducible character of $I(C)$ with a cyclic anchor group.

**Theorem 8.** *As the notion above, suppose that $\xi \in Irr(I(C))$ with cyclic anchor group $A_\xi$ such that $\langle Res^{I(C)}_Q(\xi), \theta\rangle \neq 0$, for some irreducible character $\theta$ of $Q$. Let $\xi^0 \in IBr(I(C))$. Then, the anchor group of $\xi$ equals to the anchor group of $Ind^G_{I(C)}(\xi)$ up to $G$-conjugacy.*

**Proof.** Suppose that $\xi$ belongs to the $p$-block $B$ of $I(C)$ with defect group $D_B$, and $\theta$ belongs to the $p$-block $C$ of $Q$. There is $\langle Res^{I(C)}_Q(\xi), \theta\rangle \neq 0$. Then, by [2], Theorem 9.2, this is equivalent to $B$ covering $C$. Since $A_\xi$ is a cyclic anchor group, $A_\xi$ is a defect group of $B$. Since $\theta \in Irr(C)$, $I(\theta) \leq I(C)$. From Clifford's Theorem ([1], Chapter 3, Theorem 3.8 (iii)), there is a one-to-one correspondence $Irr(I(C)|\theta) \to Irr(G|\theta)$ sending $\xi$ to $Ind^G_{I(C)}(\xi)$. Therefore, there is a one-to-one correspondence between $Irr(BL(I(C)|C))$ and $Irr(BL(G|C))$. Therefore, since $\xi \in Irr(BL(I(C)|C))$, then $Ind^G_{I(C)}(\xi) \in Irr(B^G|C)$. By [1], Chapter 5, Theorem 5.10 (iv), $D_B =_G D_{B^G}$. It follows that $A_\xi =_G A_{Ind^G_{I(C)}(\xi)}$. □

We let $Q$ be a normal subgroup of $G$ and $\widetilde{G} = G/Q$. We suppose that $\widetilde{\psi} \in Irr(\widetilde{G})$; we say that the character $\psi$ is the lift of $\widetilde{\psi}$ to $G$, if it satisfies $\psi(x) = \widetilde{\psi}(xQ)$ for $x \in G$. From [36], Theorem 17.3, if $\widetilde{\psi} \in Irr(\widetilde{G})$, then $\psi \in Irr(G)$ if $Q \leq Ker\psi$. Therefore, we view $Irr(\widetilde{G})$ as a subset of $Irr(G)$. From [2, p.137], if $\widetilde{B}$ is a $p$-block of $G/Q$, then there is a unique $p$-block $B$ of $G$ such that $Irr(\widetilde{B}) \subseteq Irr(B)$.

**Theorem 9.** *Let $Q \lhd G$ and $\widetilde{G} = G/Q$. Suppose that $\widetilde{\psi} \in Irr(\widetilde{G})$ with anchor group $A_{\widetilde{\psi}}$ such that $\psi \in Irr(G)$ is the lift of $\widetilde{\psi}$ to $G$ with a cyclic anchor group $A_\psi$. Assume that $\psi^0 \in IBr(G)$. Then, $A_{\widetilde{\psi}} \leq A_\psi Q/Q$. If $Q$ is a normal $p$-subgroup of $G$, and then $A_{\widetilde{\psi}} \leq A_\psi/Q$.*

**Proof.** Suppose that $\widetilde{\psi} \in Irr(\widetilde{G})$ which belongs to the $p$-block $\widetilde{B}$ and $\psi \in Irr(G)$ which belongs to the $p$-block $B$. Therefore, $B$ contains a $p$-block $\widetilde{B}$. From [2], Theorem 9.9 (a), if $\widetilde{D}$ is a defect group of $\widetilde{B}$, then there exists a defect group $D$ of $B$ such that $\widetilde{D} \leq DQ/Q$. The anchor group $A_{\widetilde{\psi}}$ is contained in $\widetilde{D}$, and $A_\psi$ is a defect group of $B$ by Proposition 2 (i) and Theorem 4, respectively. Therefore,

$$A_{\widetilde{\psi}} \leq \widetilde{D} \leq A_\psi Q/Q.$$

In case $Q$ is a normal $p$-subgroup of $G$, from [2, Theorem 9.9 (b)], any $p$-block $B$ of $G$ contains a $p$-block $\widetilde{B}$ such that defect group $\widetilde{D}$ of $\widetilde{B}$ is of the form $D/Q$, where $D$ ia a defect group of $B$. Therefore,

$$A_{\widetilde{\psi}} \leq \widetilde{D} \leq A_\psi/Q.$$

□

Let $P$ be a $p$-subgroup of $G$ such that $N_G(P) \leq Q \leq G$. Deduce from the following two theorems that the cyclic anchor group is invariant under the process of induction of irreducible character of $N_G(P)$ (respectively, of $Q$) which belongs to the $p$-block of $N_G(P)$ (respectively, of $Q$) of defect group $P$ to the irreducible character of $G$.

**Theorem 10.** *Let $P$ be a $p$-subgroup of $G$. Suppose that $\theta \in Irr(N_G(P))$ which belongs to the $p$-block of $N_G(P)$ with defect group $P$. Let $\theta^0 \in IBr(N_G(P))$. If $\theta$ has a cyclic anchor group and $Ind_{N_G(P)}^G(\theta) \in Irr(G)$, then $P$ is the anchor group of $Ind_{N_G(P)}^G(\theta)$.*

**Proof.** Suppose that $\theta \in Irr(N_G(P))$, which belongs to the $p$-block $\mathcal{B}$ of $N_G(P)$ with defect group $P$. Since $\theta$ has a cyclic anchor group and $\theta^0 \in IBr(N_G(P))$, then by Proposition 2(iii) and Theorem 4, $P$ is the anchor group of $\theta$. By Brauer's First Main Theorem in [2] (Theorem 4.17), the map which sends the $p$-block $\mathcal{B}$ to $\mathcal{B}^G$ defines a bijection from $BL(N_G(P)|P)$ to $BL(G|P)$. Now, if $Ind_{N_G(P)}^G(\theta) \in Irr(G)$, then it belongs to the $p$-block $\mathcal{B}^G$ of $G$ by [2], Corollary 6.2. Thus, $\mathcal{B}^G$ has a cyclic defect group $P$. Therefore, $P$ is the anchor group of $Ind_{N_G(P)}^G(\theta)$. □

**Theorem 11.** *Let $P$ be a $p$-subgroup of $G$ such that $N_G(P) \leq Q \leq G$. Suppose that $\theta \in Irr(Q)$ which belongs to the $p$-block of $Q$ with defect group $P$. Assume that $\theta^0 \in IBr(Q)$. If $\theta$ has a cyclic anchor group and $Ind_Q^G(\theta) \in Irr(G)$; then, $P$ is the anchor group of $Ind_Q^G(\theta)$.*

**Proof.** Suppose that $\theta \in Irr(Q)$ which belongs to the $p$-block $\mathcal{B}$ of $Q$ with defect group $P$. Since $\theta$ has a cyclic anchor group and $\theta^0 \in IBr(Q)$, then, by Theorem 4, $P$ is the anchor group of $\theta$. From [1], Chapter 5, Theorem 3.8, the map which is sending $\mathcal{B}$ to $\mathcal{B}^G$ defines a bijection from $BL(Q|P)$ to $BL(G|P)$. □

The following example explains our results. We use the algebra package GAP [37] to find the degree of the irreducible characters, the structure of the defect group of a $p$-block of $G$, and its normalizer in the group $G$.

**Example 1.** *In this example, $p = 3$ and $G = GL(2,3)$ are the general linear group of order $48 = 2^4.3$. $G$ is the group of invertible $2 \times 2$ matrices over the field of 3 elements. Then, considering the principal 3-block $B_0$, which has the Sylow 3-subgroup $P = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\}$ as the defect group by [18], Theorem 6.1.5. The group $G$ has four Sylow 3-subgroups, and $P$ is one of them. In particular, $P$ is the Heisenberg group; that is the unitary upper triangular matrices in $G$. Since all irreducible characters in the 3-block $B_1$ with defect one have the degree prime to three, the Sylow 3-subgroup $P$ is the their anchor group by [29], Corollary 10. The two irreducible characters of $GL(2,3)$ of degree three are with defect zero. Hence, from [18], p. 39, each one belongs to the singleton 3-block, and the defect group of the singleton 3-block is the trivial group $\{1_{GL(2,3)}\}$. From*

*Theorem 4, every irreducible character belonging to the principal 3-block $B_0$ or to the 3-block $B_1$ has a cyclic anchor group. The degrees of the irreducible characters of $GL(2,3)$ are*

| $\psi_i$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ | $\psi_7$ | $\psi_8$ |
|---|---|---|---|---|---|---|---|---|
| $\psi_i(1)$ | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |

*We note that for any $\psi_i \in Irr(GL(2,3))$ with $\psi_i^0 \in IBr(GL(2,3))$ and a cyclic anchor group $A_{\psi_i}$, the index $[P : A_{\psi_i}]$ divides $\psi_i(1)$. In particular, in this example, the Normalizer of $A_{\psi_i}$ in $G$; $N_G(A_{\psi_i})$ is a p-solvable group. Then, $\psi_i(1)_p = [P : A_{\psi_i}]_p$. Because if $\psi_i(1)_p = 1$, then the anchor group $A_{\psi_i}$ is a Sylow 3-subgroup. We have $N_G(A_{\psi_i}) = D_{12}$, the dihedral group of order 12. In the case $\psi_i(1)_p = 3$, $A_{\psi_i}$ is the trivial group $\{1_{GL(2,3)}\}$ and $N_G(A_{\psi_i}) = GL(2,3)$. We have $SL(2,3)$ a special linear group of order 24 which is a normal subgroup of $GL(2,3)$. We let $\psi$ be the two-dimensional irreducible character of $GL(2,3)$ with $\psi^0 \in IBr(GL(2,3))$ and a cyclic anchor. We let $\theta \in Irr(SL(2,3))$ such that $\langle Res^{GL(2,3)}_{SL(2,3)}(\psi), \theta\rangle \neq 0$. Then, $\theta$ is a non-principal irreducible character of degree one which belongs to the principal 3-block $b_0$ of $SL(2,3)$. Hence, the defect group of $b_0$ is a Sylow 3-subgroup which is cyclic. Thus, the anchor of $\theta$ is cyclic. We let $Q = C_2$ be a cyclic normal 3 subgroup of $GL(2,3)$. Then, $\widetilde{G} = G/C_2 \cong SL(2,3)$. We suppose that $\widetilde{\psi} \in Irr(\widetilde{G})$, where $\psi \in Irr(GL(2,3))$ is the lift of $\widetilde{\psi}$ to $GL(2,3)$. It is clear that anchor group $A_{\widetilde{\psi}} \leq A_\psi C_2 / C_2 \cong A_\psi$.*

We present proofs of a criteria for an irreducible character to have a cyclic anchor group.

**Proof of Proposition 1.** Suppose that $\psi$ belongs to the $p$-block $B$ with defect group $D$. There is $\psi \in Irr(G)$ such that $\psi_p(1) = p^{a-1}$. If $h(\psi) > 0$, then, by [1], Problems 3.24 (iv), p. 257, $h(\psi) \leq def(B) - 2$. This implies that $def(B) - def(\psi) \leq def(B) - 2$. Therefore, $v_p(\psi(1)) \leq a - 2$, which contradicts $v_p(\psi(1)) = a - 1$. Thus, $h(\psi) = 0$, and $def(B) = def(\psi)$ can be obtained. There is $p^{def(\psi)} = \frac{p^a}{p^{a-1}} = p^1$. Therefore, the defect group of the $p$-block $B$ is cyclic. Thus, from Proposition 2(i), the anchor group of $\psi$ is cyclic. □

**Proof of Theorem 2.** We have $G = P \rtimes Q$ where $|P| = p^a$ and $p \nmid |Q|$. The anchor group of any irreducible character of $G$ is a $p$-subgroup of $G$ by Remark 1 since $G$ has a cyclic Sylow $p$-subgroup which contains all $p$ subgroups of $G$. There is $G$, which has a normal cyclic Sylow $p$-subgroup; then, $G$ has a unique $p$-block $B$. By [27], Theorem 19.10, $G$ is a $p$-solvable group, and for all $\chi \in Irr(G)$, $p \nmid \chi(1)$. Thus, the defect group of $B$ is the Sylow $p$-subgroup which is cyclic. Therefore, by Proposition 2 (iii), a cyclic Sylow $p$-subgroup of $G$ is the anchor group of any $\chi \in Irr(G)$. □

Presence of a Nakayama algebra means that all indecomposable injective and projective modules over this algebra are uniserial. A uniserial module is a module which has a unique composition series. For more, see [31].

**Remark 3.** *The modular group algebra $FG$ for the group which is described above in Theorem 2 is a Nakayama algebra, since, from [31], Proposition 8.3.3, all indecomposable projective $FG$-modules are uniserial. From [31], Proposition 8.5.3, the projective $FG$-modules are the same as the injective $FG$-modules. Hence, all indecomposable injective and projective $FG$-modules are uniserial. Moreover, in this case, the modular group algebra $FG$ possesses a finite representation type by Theorem 3.*

## 4. Discussion

In [9], G. O. Michler proved that if $B$ is a $p$-block of $FG$ with a cyclic defect group $D$, then every simple module belonging to $B$ has $D$ as a vertex. The converse of the previous result, which states that "if $B$ is a $p$-block of $FG$ having a simple module with a cyclic vertex then the defect group of $B$ is cyclic", was shown by K. Erdmann in [32]. The $p$-block with a cyclic defect group is a renewed topic which appeared recently in many articles; some of them can be seen in [38,39]. In this work, our first main objective is to show that the

anchor group $A_\psi$ of irreducible character $\psi$ of $G$ is cyclic if and only if the defect group of the $p$-block which contains $\psi$ is cyclic. This relationship is achieved because one of the two directions is a direct result of [12], Proposition 3.1(i). The other direction is achieved by defining a defect group of the $p$-block $\bar{e}_B$ of the modular group algebra $FG$ as it is a vertex of the indecomposable $F[G \times G]$-module $(FG) \cdot \bar{e}_B$ and from [31], Exercise 11.10.21. We use the first result to reformulate G. O. Michler's Theorem in [9]. In [5], J. A. Green proved that if $N$ is an indecomposable $FG$-module and $V$ is a vertex of $N$, which is a subgroup of the Sylow $p$-subgroup $P$ of $G$, then the index $[P : V]$ divides $dim_F(N)$. In this work, we prove that if $A_\psi$ is a cyclic anchor group of irreducible character $\psi$ of $G$, which is contained in a Sylow $p$-subgroup $P$ of $G$, then the index $[P : A_\psi]$ divides the degree of $\psi$, $\psi(1)$. We also provide a description of the anchor group of any irreducible character $\theta$ of a normal subgroup $Q$ of $G$ lying under $\psi$ (respectively, under $\xi \in Irr(I(C))$ with $\xi^0 \in IBr(I(C))$) with a cyclic anchor group, where $C$ is a $p$-block of $Q$. We also study the anchor group of any irreducible character $\widetilde{\psi}$ of the quotient group $G/Q$ such that $\psi$ is the lift of $\widetilde{\psi}$ to $G$ with a cyclic anchor group. We let $P$ be a $p$ subgroup of $G$ such that $N_G(P) \leq Q \leq G$. We deduce that the cyclic anchor group is invariant under the process of induction of irreducible character of $N_G(P)$ (respectively, of $Q$) which belongs to the $p$-block of $N_G(P)$ (respectively, of $Q$) of defect group $P$ to irreducible character of $G$. To achieve the previous results, we use our first result in Theorem 4, the character theory, the Clifford theory and the block theory. Our final obectives in this paper are to develop criteria for an irreducible character to have a cyclic anchor group. The outcomes of this work are important because they contain a generalization of some properties of a defect group and a vertex group to the algebraic concept "anchor group of an irreducible character". Our work contains some results which can be extended and strengthened for more general cases. We plan to generalize more properties of the defect group of the $p$-block to the anchor group of an irreducible character.

## 5. Conclusions

We consider $G$ to be a finite group and $p$ to be a prime number. We fix $\psi \in Irr(G)$ with $\psi^0 \in IBr(G)$. In this work, we prove that the anchor group $A_\psi$ of $\psi$ is cyclic if and only if the defect group of the $p$-block which contains $\psi$ is cyclic. We reformulate Theorem 10.1 in [9]. We prove that the relative version of Theorem 9 in [5] is as follows: if $A_\psi$ is a cyclic anchor group of $\psi$ which is contained in a Sylow $p$-subgroup $P$ of $G$, then the index $[P : A_\psi]$ divides the degree of $\psi$. We let $Q$ be a normal subgroup of $G$, $P$ be a $p$-subgroup of $G$ such that $N_G(P) \leq Q \leq G$. We offer a description of the anchor group in the following cases:

- for any irreducible character $\theta$ of $Q$ lying under $\psi$ (respectively, under $\xi \in Irr(I(C))$ with $\xi^0 \in IBr(I(C))$) with a cyclic anchor group, where $C$ is a $p$-block of $Q$.
- for any irreducible character $\widetilde{\psi}$ of the quotient group $G/Q$ such that $\psi$ is the lift of $\widetilde{\psi}$ to $G$ with a cyclic anchor group.

We prove that the cyclic anchor group of $\theta \in Irr(N_G(P))$ with $\theta^0 \in IBr(N_G(P))$ (respectively, of $\theta \in Irr(Q)$ with $\theta^0 \in IBr(Q)$) is invariant under the process of induction of irreducible character of $N_G(P)$ (respectively, of $Q$ ) which belongs to the $p$-block of $N_G(P)$ (respectively, of $Q$) of defect group $P$ to the irreducible character of $G$. We establish and prove criteria for an irreducible character to have a cyclic anchor group. In future work, we will try to generalize more properties of the defect group of a $p$-block and a vertex group of an indecomposable $FG$-module to the anchor group of an irreducible character. We remark that there is a plan for future work to prove and obtain the conclusion of Theorem 4 in case that the anchor group is abelian. We believe that the future of character theory and its applications in group theory is bright and new methods, strategies and approaches appear day by day; see, for instance [40–42].

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