Article

# Convergence Analysis of the Strang Splitting Method for the Degasperis-Procesi Equation 

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#### Abstract

This article is concerned with the convergence properties of the Strang splitting method for the Degasperis-Procesi equation, which models shallow water dynamics. The challenges of analyzing splitting methods for this equation lie in the fact that the involved suboperators are both nonlinear. In this paper, instead of building the second order convergence in $L^{2}$ for the proposed method directly, we first show that the Strang splitting method has first order convergence in $H^{2}$. In the analysis, the Lie derivative bounds for the local errors are crucial. The obtained first order convergence result provides the $H^{2}$ boundedness of the approximate solutions, thereby enabling us to subsequently establish the second order convergence in $L^{2}$ for the Strang splitting method.


Keywords: Degasperis-Procesi equation; strang splitting method; convergence analysis; lie derivative

MSC: 65M12; 65M15; 65J08

## 1. Introduction

Operator splitting methods are widely used for the numerical solution of both ordinary differential equations (ODEs) and partial differential equations (PDEs) by decomposing the complicated problems into simpler subequations. These subequations can be solved individually using algorithms that are more efficient. Amounts of research has been conducted on this topic. A comprehensive investigation of operator splitting methods is presented in [1,2], covering their construction, implementation, and theoretical analysis. Notably, these studies primarily focus on ODEs.

Additionally, the work by [3] is dedicated to the application of the operator splitting method for solving PDEs, specifically those that are convection dominated. However, this theory is limited to scalar and weakly coupled systems of equations. Previous studies have investigated the use of operator splitting for various equations, including the Korteweg-de Vries equation [4], the Schrödinger equation [5], partial differential equations with Burgers nonlinearity [6], the Burgers-Huxley equation [7], the Vlasov-Poisson equations [8,9], Fisher's equation and Benjamin-Bono-Mahony equations [10], the Allen-Cahn equation $[11,12$ ] and the Cahn-Hilliard equation [13-15].

The effectiveness of the operator splitting method relies on the interconnection between different subequations and the dynamics of the evolution problem. Specifically, a particular type of partial differential equations involving the Burgers term tends to introduce singularities, even when the initial data is smooth. When applying operator splitting methods to these equations, determining the appropriate time step becomes a delicate task. By introducing a new auxiliary time variable, the convergence of operator splitting method for KdV equation is analyzed in [4]. Further, based on the Lie-commutator bounds for the local error and conditional stability of error propagation, authors in [16] establish the second order convergence of the Strang splitting for Schrödinger-Poisson and cubic nonlinear Schrödinger equation. They identify the principal error terms of the local error as quadrature errors. This result is then extended to a type of partial differential equations
with Burgers nonlinearity in [6]. In these equations, one subequation is Burgers equation, while the other subequation is linear. However, there exist very few results available when both suboperators of the equations are nonlinear.

Inspired by the growing interest in operator splitting techniques applied to PDEs, Ref. [17] has proposed the implementation of the Strang splitting method specifically for the Degasperis-Procesi (DP) equations. This equation, serving as a model for capturing the behavior of shallow water dynamics, can be expressed in the following form [18]

$$
\begin{equation*}
u_{t}-u_{x x t}+3 k^{3} u_{x}+4 u u_{x}=3 u_{x} u_{x x}+u u_{x x x},\left.u(t, x)\right|_{t=0}=u_{0}(x), \tag{1}
\end{equation*}
$$

here $k$ is a real constant.
To design the temporal discretization of this equation based on the splitting strategy, we rewrite it as

$$
\left(1-\partial_{x x}\right)\left(u_{t}+u u_{x}\right)+3 k^{3} u_{x}+3 u u_{x}=0,\left.u(t, x)\right|_{t=0}=u_{0}(x) .
$$

Note that the inverse Helmholtz operator $\left(1-\partial_{x x}\right)^{-1}$ can be expressed as a convolution

$$
\begin{equation*}
\left(1-\partial_{x x}\right)^{-1} f=P * f, \text { for all } f \in L^{2}(\mathbb{R}) \tag{2}
\end{equation*}
$$

with $P(x):=\frac{1}{2} \mathrm{e}^{-|x|}$. Here the symbol $f * g$ denotes convolution of $f$ and $g$,i.e.,

$$
(f * g)(t)=\int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d} \tau, \quad f, g \in L^{2}(\mathbb{R})
$$

Thus the DP Equation (1) can be transformed into

$$
\begin{equation*}
u_{t}=C(u),\left.\quad u(t, x)\right|_{t=0}=u_{0}(x) \tag{3}
\end{equation*}
$$

with

$$
C(u)=-u u_{x}-3 P *\left(k^{3} u_{x}+u u_{x}\right) .
$$

Let $u(t)=\Phi_{C}^{t}\left(u_{0}\right)$ denote the exact solution of the initial value problem (3). Evidently, operator $C$ can be split into two suboperators $C=A+B$. Thus, we consider the following subsystems

$$
\begin{align*}
v_{t} & =A(v) & =-v v_{x}, & \left.v\right|_{t=0} \tag{4}
\end{align*}=v_{0}, ~=~\left(k^{3} \omega_{x}+\omega \omega_{x}\right),\left.\omega\right|_{t=0}=\omega_{0} .
$$

The first equation is known as the Burgers equation and the latter one is referred to as the Benjamin-Bona-Mahony (BBM) equation. Let us denote the exact solution operators of (4) and (5) by $\Phi_{A}^{t}$ and $\Phi_{B}^{t}$ respectively. Then, the operator splitting method, in its most basic form reads as follows:

$$
u_{n+1}=\Phi_{A}^{\tau} \circ \Phi_{B}^{\tau}\left(u_{n}\right), \quad n=0,1,2 \ldots,
$$

where $u_{n}$ is the approximation of $u(t)$ at $t=t_{n}=n \tau, \tau$ is the time step size. This method is called Lie splitting method. In this paper, we focus on another more refined operator splitting, known as Strang splitting, which is read as

$$
\begin{equation*}
u_{n+1}=\Psi^{\tau}\left(u_{n}\right)=\Phi_{A}^{\frac{\tau}{2}} \circ \Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{n}\right), \quad n=0,1,2 \ldots \tag{6}
\end{equation*}
$$

In [17] the efficiency of the Strang splitting method for the DP equation is demonstrated numerically. However, to our knowledge, there is as yet no rigorous convergence result in the literature for the splitting method for the DP equation. In the present study, we intend to analyze the convergence properties of the Strang splitting method for the DP equation.

The major difficulty in the numerical analysis of the splitting scheme above lies in the fact that both suboperators are nonlinear. The classical techniques suitable for only
one nonlinear operator are not directly applicable here. In this paper, instead of building second order convergence in the $L^{2}$-norm for the proposed method directly, we first show that the Strang splitting method has first order convergence in $H^{2}$. While this result may not seem attractive, it serves as the cornerstone for the $H^{2}$ boundedness of the approximate solutions. In the analysis, the Lie derivative bounds for the local errors are crucial. Finally, by applying the Lady Windermere's fan to estimate the global error, we prove second order convergence in $L^{2}$. A similar approach has been used by [16] when considering only a nonlinear suboperator of the Burgers type. We strive further and extend the analysis to the DP equation, where both suboperators are nonlinear.

The rest of the paper is organized as follows: Section 2 is devoted to the preliminaries, where the assumptions are made and the regularity properties of the DP equation and the two subequations are derived. The first order convergence is analyzed in Section 3. Furthermore, the approximate solutions are proven to be bounded, which plays an important role in the second order analysis. Finally, the second order convergence of the Strang splitting is presented in Section 4.

## 2. Preliminaries

In this section, we collect and prove the results which are crucial in the proof of first- and second order convergence analysis. For $1 \leq p \leq \infty$, the norm in the Lebesgue space $L^{p}=L^{p}(\mathbb{R})$ is denoted by $\|\cdot\|_{L^{p}}$, while for $s>0$, the norm in the Sobolev space $H^{s}=H^{s}(\mathbb{R})$ is denoted by $\|\cdot\|_{H^{s}}$.

### 2.1. Setting

For the well-posedness of the DP equation, we recall the results in [19]: if $u_{0} \in H^{s}, s>\frac{3}{2}$, then there exists a maximal time $T$, such that the DP equation has a unique strong solution $u=u\left(\cdot, u_{0}\right) \in C\left([0, T) ; H^{s}\right) \cap C^{1}\left([0, T) ; H^{s-1}\right)$, and the solution depends continuously on the initial data. Moreover, we have

$$
\begin{equation*}
\|u(t)\|_{L^{2}} \leq 2\left\|u_{0}\right\|_{L^{2}}, \quad\|u(t)\|_{L^{\infty}} \leq c\left\|u_{0}\right\|_{H^{1}}, \tag{7}
\end{equation*}
$$

where the constant $c$ only depends on $T$. For convenience, we use $c$ to stand for a generic constant. It may have different values even in the same line.

In order to carry out the error analysis we make the following further assumption on the DP equation. We assume that on $t \in[0, T]$, the solution $u(t)$ is in $H^{3}$ and there exists a constant $\rho>0$ such that $u(t)$ is uniformly bounded as

$$
\begin{equation*}
\|u(t)\|_{H^{3}} \leq \rho \tag{8}
\end{equation*}
$$

for $0 \leq t \leq T$.
Since the analysis in this paper heavily depends on the nonlinear variation-of-constants formula, which needs to calculate Lie derivatives, we first list some results about the Lie derivative. Denote $\varphi_{F}^{t}(v)$ as the solution at time $t$ of the differential equation $\dot{\varphi}=F(\varphi)$ with initial data $\varphi(0)=v$, then, for any unbounded vector field $G$ on $H^{1}$ and $v \in H^{1}$, the Lie derivative $D_{F}$ associated with $F$ is defined by

$$
\left(D_{F} G\right)(v)=\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} G\left(\varphi_{F}^{t}(v)\right)=\mathrm{d} G(v)[F(v)]
$$

where $\mathrm{d} G$ is the Fréchet-derivative. Specially, for the identity operator $G=\mathrm{Id}$, it follows that $\left(D_{F} \mathrm{Id}\right)(v)=F(v)$.

The exponential operator $\mathrm{e}^{t D_{F}}$ on $G$ is defined as

$$
\left(\mathrm{e}^{t D_{F}} G\right)(v)=G\left(\varphi_{F}^{t}(v)\right)
$$

Obviously, for the identity operator $G=I d,\left(\mathrm{e}^{t D_{F}} \mathrm{Id}\right)(v)=\varphi_{F}^{t}(v)$. For derivatives, we have the rule

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\mathrm{e}^{t D_{F}} G\right)(v)=\left(D_{F} \mathrm{e}^{t D_{F}} G\right)(v)=\left(\mathrm{e}^{t D_{F}} D_{F} G\right)(v)
$$

For composition, we have

$$
\left(\varphi_{t}^{[2]} \circ \varphi_{s}^{[1]}\right)\left(y_{0}\right)=\left(\mathrm{e}^{s D_{1}} \mathrm{e}^{t D_{2}} \mathrm{Id}\right)\left(y_{0}\right)
$$

where $\varphi_{s}^{[1]}$ and $\varphi_{t}^{[2]}$ are the flows of the differential equations $\dot{y}=f^{[1]}(y)$ and $\dot{y}=f^{[2]}(y)$, respectively, $D_{1}, D_{2}$ are Lie derivatives associated to $f^{[1]}, f^{[2]}$ respectively.

Therefore, the Strang splitting method (6) can be written as

$$
\begin{equation*}
u_{n+1}=\Psi^{\tau}\left(u_{n}\right)=\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{e}^{\tau D_{B}} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{n}\right), \quad n=0,1,2 \ldots \tag{9}
\end{equation*}
$$

Define Lie commutator of two nonlinear operators $G$ and $H$ as

$$
[G, H](v)=\mathrm{d} G(v)[H(v)]-\mathrm{d} H(v)[G(v)]
$$

then we have the following property

$$
\left\{D_{G}, D_{H}\right\}(v):-D_{G} D_{H}(v)-D_{H} D_{G}(v)=-[G, H](v) .
$$

With the help of Lie derivative, we can express the exact solution of the nonlinear equation into the similar form of the variation-of-constants formula, which is called nonlinear variation-of-constants formula. The convergence analysis heavily depends on this formula.

Lemma 1 (Nonlinear variation-of-constants formula). The exact solution of the following initial value problem

$$
\left\{\begin{array}{l}
u^{\prime}(t)=C(u(t))=A(u(t))+B(u(t)), \quad 0 \leq t \leq T \\
u(0)=u_{0}
\end{array}\right.
$$

has the form

$$
u(t)=\left(\mathrm{e}^{t D_{C}} \mathrm{Id}\right)\left(u_{0}\right)=\left(\mathrm{e}^{t D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\int_{0}^{t}\left(\mathrm{e}^{(t-s) D_{C}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s
$$

Proof. Define function $\varphi(s)=\left(\mathrm{e}^{(t-s) D_{C}} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)$, according to the formula $\varphi(t)-\varphi(0)=\int_{0}^{t} \dot{\varphi}(s) \mathrm{d} s$ and $D_{C}=D_{A}+D_{B}$, we have

$$
\begin{aligned}
& \left(\mathrm{e}^{t D_{A}} \mathrm{Id}\right)\left(u_{0}\right)-\left(\mathrm{e}^{t D_{C}} \mathrm{Id}\right)\left(u_{0}\right) \\
= & -\int_{0}^{t}\left(\mathrm{e}^{(t-s) D_{C}} D_{C} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s+\int_{0}^{t}\left(\mathrm{e}^{(t-s) D_{C}} D_{A} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s \\
= & -\int_{0}^{t}\left(\mathrm{e}^{(t-s) D_{C}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s .
\end{aligned}
$$

This completes the proof.
Moreover, we note that the special convolution (2) is involved in the DP equation, in order to simplify the analysis, we list the properties of convolution which are used extensively in the convergence analysis.

Lemma 2. The convolution (2) has the following properties:
(1) If $f, g \in H^{m}$ for integer $m \geq 0$, then $\|P * g\|_{H^{m}} \leq\|g\|_{H^{m}}$ and $\left\|P *(f g)_{x}\right\|_{H^{m}} \leq$ $c\|f\|_{H^{m}}\|g\|_{H^{m}}$.
(2) $\partial_{x}^{2}(P * g)=P * g-g$, for $g \in H^{2}$.
(3) $\left|P *\left(f g_{x}\right)\right| \leq\left(\|f\|_{L^{\infty}}+\left\|f_{x}\right\|_{L^{\infty}}\right) P *|g|$, for $f, g \in H^{2}$.

Proof. (1) See Lemmas 2.3 and 2.4 in [10].
(2) Integrating by parts yields

$$
\partial_{x}^{2}(P * g)=\frac{1}{2} \int_{-\infty}^{x} \mathrm{e}^{-x+\eta} \mathrm{d} g_{\eta}(\eta)+\frac{1}{2} \int_{x}^{+\infty} \mathrm{e}^{x-\eta} \mathrm{d} g_{\eta}(\eta)=P * g-g .
$$

(3) Integrating by parts gives

$$
\begin{aligned}
\left|P *\left(f g_{x}\right)\right| & =\left|\frac{1}{2} \int_{-\infty}^{x} \mathrm{e}^{-x+\eta} f \mathrm{~d} g(\eta)+\frac{1}{2} \int_{x}^{+\infty} \mathrm{e}^{x-\eta} f \mathrm{~d} g(\eta)\right| \\
& =\left|-\frac{1}{2} \int_{-\infty}^{x} \mathrm{e}^{-x+\eta}\left(f+f_{\eta}\right) g \mathrm{~d} \eta-\frac{1}{2} \int_{x}^{+\infty} \mathrm{e}^{x-\eta}\left(-f+f_{\eta}\right) g \mathrm{~d} \eta\right| \\
& \leq\left(\|f\|_{L^{\infty}}+\left\|f_{x}\right\|_{L^{\infty}}\right) P *|g| .
\end{aligned}
$$

This completes the proof.

### 2.2. Properties of the Exact Solutions

Now, we are in the position to estimate the properties of the DP equation and the two subequations. We first show that the solution dependence on the initial data is Lipschitz continuous in a weaker topology.

Lemma 3. Let $u(t), v(t)$ be the exact solutions of the DP Equation (1) with initial data $u_{0}$, $v_{0}$ respectively. If $\|u(t)\|_{H^{3}} \leq r,\|v(t)\|_{H^{3}} \leq r$ for $0 \leq t \leq T$, then there exists a constant $K=K(r, k, T)>1$ such that

$$
\begin{equation*}
\|u(t)-v(t)\|_{H^{m}} \leq K\left\|u_{0}-v_{0}\right\|_{H^{m}}, m=0,1,2 \tag{10}
\end{equation*}
$$

Proof. Set $\delta=u-v$ and $\delta_{0}=u_{0}-v_{0}$, then it follows that

$$
\begin{equation*}
\delta_{t}=-u \delta_{x}-v_{x} \delta-3 P *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right) \tag{11}
\end{equation*}
$$

Taking the first and second order derivatives with respect to $x$ of this equation yields

$$
\begin{equation*}
\delta_{x t}=-\left(u_{x}+v_{x}\right) \delta_{x}-u \delta_{x x}-v_{x x} \delta-3 P_{x} *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right) \tag{12}
\end{equation*}
$$

and

$$
\begin{align*}
\delta_{x x t}= & -\left(u_{x x}+2 v_{x x}\right) \delta_{x}-\left(2 u_{x}+v_{x}\right) \delta_{x x}-u \delta_{x x x}-v_{x x x} \delta \\
& -3 P *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right)-\left(k^{3}+u\right) \delta_{x}-v_{x} \delta . \tag{13}
\end{align*}
$$

To estimate $\|\delta\|_{H^{m}}$ ( $m=0,1,2$ ), multiplying Equations (11)-(13) by $\delta, \delta_{x}$ and $\delta_{x x}$ respectively and integrating gives

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{-\infty}^{\infty} \delta^{2} \mathrm{~d} x & =-\int_{-\infty}^{\infty} u \delta \delta_{x} \mathrm{~d} x-\int_{-\infty}^{\infty} v_{x} \delta^{2} \mathrm{~d} x-3 \int_{-\infty}^{\infty} \delta P *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right) \mathrm{d} x \\
& =\int_{-\infty}^{\infty} u_{x} \delta^{2} \mathrm{~d} x-\int_{-\infty}^{\infty} v_{x} \delta^{2} \mathrm{~d} x-3 \int_{-\infty}^{\infty} \delta P *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right) \mathrm{d} x \\
& \leq c\left(\left\|u_{x}\right\|_{L^{\infty}}+\|u\|_{L^{\infty}}+\left\|v_{x}\right\|_{L^{\infty}}+k^{3}\right)\|\delta\|_{L^{2}}^{2} \leq c\left(r+k^{3}\right)\|\delta\|_{L^{2}}^{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{-\infty}^{\infty} \delta_{x}^{2} \mathrm{~d} x= & -\int_{-\infty}^{\infty}\left(\frac{1}{2} u_{x}+v_{x}\right) \delta_{x}^{2} \mathrm{~d} x-\int_{-\infty}^{\infty} v_{x x} \delta \delta_{x} \mathrm{~d} x \\
& -3 \int_{-\infty}^{\infty} \delta_{x} P *\left(\left(k^{3}+u\right) \delta_{x x}+u_{x} \delta_{x}\right) \mathrm{d} x-3 \int_{-\infty}^{\infty} \delta_{x} P_{x} *\left(v_{x} \delta\right) \mathrm{d} x \\
\leq & c\left(\left\|u_{x}\right\|_{L^{\infty}}+\|u\|_{L^{\infty}}+\left\|v_{x}\right\|_{L^{\infty}}+k^{3}\right)\left\|\delta_{x}\right\|_{L^{2}}^{2} \\
& +\|\delta\|_{L^{\infty}}\left\|v_{x x}\right\|_{L^{2}}\left\|\delta_{x}\right\|_{L^{2}}+c\left\|v_{x}\right\|_{L^{2}}\|\delta\|_{L^{2}}\left\|\delta_{x}\right\|_{L^{2}} \\
\leq & c\left(r+k^{3}\right)\|\delta\|_{H^{1}}^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{-\infty}^{\infty} \delta_{x x}^{2} \mathrm{~d} x= & -\int_{-\infty}^{\infty}\left(u_{x x}+2 v_{x x}\right) \delta_{x} \delta_{x x} \mathrm{~d} x-\int_{-\infty}^{\infty}\left(\frac{3}{2} u_{x}+v_{x}\right) \delta_{x x}^{2} \mathrm{~d} x-\int_{-\infty}^{\infty} v_{x x x} \delta \delta_{x x} \\
& -3 \int_{-\infty}^{\infty} \delta_{x x} P *\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta_{x}\right) \mathrm{d} x-3 \int_{-\infty}^{\infty} \delta_{x x}\left(\left(k^{3}+u\right) \delta_{x}+v_{x} \delta\right) \mathrm{d} x \\
\leq & c\left(\left\|u_{x x}\right\|_{L^{\infty}}+\left\|v_{x x}\right\|_{L^{\infty}}+\|u\|_{L^{\infty}}+\left\|u_{x}\right\|_{L^{\infty}}+\left\|v_{x}\right\|_{L^{\infty}}+k^{3}\right)\left\|\delta_{x}\right\|_{L^{2}}\left\|\delta_{x x}\right\|_{L^{2}} \\
& +c\left(\left\|u_{x}\right\|_{L^{\infty}}+\left\|v_{x}\right\|_{L^{\infty}}\right)\left\|\delta_{x x}\right\|_{L^{2}}^{2}+c\|\delta\|_{L^{\infty}}\left\|v_{x x x}\right\|_{L^{2}}\left\|\delta_{x x}\right\|_{L^{2}} \\
+ & c\left(\|u\|_{L^{\infty}}+\left\|u_{x}\right\|_{L^{\infty}}+\left\|v_{x}\right\|_{L^{\infty}}+k^{3}\right)\|\delta\|_{L^{2}}\left\|\delta_{x x}\right\|_{L^{2}} \\
\leq & c\left(r+k^{3}\right)\|\delta\|_{H^{2}}^{2}
\end{aligned}
$$

where Lemma 2 is used in the estimations. Combining them together, we obtain

$$
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|\delta\|_{H^{m}}^{2} \leq c\left(r+k^{3}\right)\|\delta\|_{H^{m}}^{2}
$$

which leads to

$$
\|\delta\|_{H^{m}} \leq \mathrm{e}^{c\left(r+k^{3}\right) t}\left\|\delta_{0}\right\|_{H^{m}, m}=0,1,2
$$

Then the proof is complete.
Lemma 4. If the initial data of the DP Equation (1) satisfies $\left\|u_{0}\right\|_{H^{m}} \leq M$ for $m \geq 2$, then there exists $\bar{t}(M)>0$ such that the solution $u(t)$ of the DP equation has $\|u(t)\|_{H^{m}} \leq 2 M$, for $0 \leq t \leq \bar{t}(M)$.

Proof. Set $u=\Phi_{C}^{t}\left(u_{0}\right)$, we have

$$
\|u\|_{H^{m}} \frac{\mathrm{~d}}{\mathrm{~d} t}\|u\|_{H^{m}}=\left(u, u_{t}\right)_{H^{m}}=-\sum_{k=0}^{m} \int_{-\infty}^{\infty} \partial_{x}^{k} u \partial_{x}^{k}\left(u u_{x}+3 P_{x} *\left(k^{3} u+\frac{1}{2} u^{2}\right)\right) \mathrm{d} x .
$$

We estimate the Burgers term and the convolution term separately. First for the Burgers term we have

$$
\partial_{x}^{k} u \partial_{x}^{k}\left(u u_{x}\right)=\sum_{j=0}^{k} C_{k}^{j} \partial_{x}^{k} u \partial_{x}^{j} u \partial_{x}^{k+1-j} u .
$$

Note that when $k<m$, we have $j \leq k<m$ and $k+1-j \leq m$, thus

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{k} u \partial_{x}^{j} u \partial_{x}^{k+1-j} u \mathrm{~d} x\right| \leq\left\|\partial_{x}^{k} u\right\|_{L^{\infty}}\left\|\partial_{x}^{j} u\right\|_{L^{2}}\left\|\partial_{x}^{k+1-j} u\right\|_{L^{2}} \leq c\|u\|_{H^{m}}^{3} .
$$

When $k=m$ and $j>0$, we have $\xi=\min \{j, k+1-j\}<m$ and $\eta=\max \{j, k+1-$ $j\} \leq m$, this gives

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{k} u \partial_{x}^{j} u \partial_{x}^{k+1-j} u \mathrm{~d} x\right| \leq\left\|\partial_{x}^{\xi} u\right\|_{L^{\infty}}\left\|\partial_{x}^{k} u\right\|_{L^{2}}\left\|\partial_{x}^{\eta} u\right\|_{L^{2}} \leq c\|u\|_{H^{m}}^{3} .
$$

While in the remaining case $k=m$ and $j=0$, we have

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{m} u u \partial_{x}^{m+1} u \mathrm{~d} x\right|=\left|\frac{1}{2} \int_{-\infty}^{\infty} u \mathrm{~d}\left(\partial_{x}^{m} u\right)^{2}\right|=\left|\frac{1}{2} \int_{-\infty}^{\infty} \partial_{x} u\left(\partial_{x}^{m} u\right)^{2} \mathrm{~d} x\right| \leq c\|u\|_{H^{m}}^{3} .
$$

For the convolution term, it follows that

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{k} u \partial_{x}^{k+1}\left(P *\left(k^{3} u+\frac{1}{2} u^{2}\right)\right) \mathrm{d} x\right| \leq\left\|\partial_{x}^{k} u\right\|_{L^{2}}\left\|\partial_{x}^{k+1}\left(P *\left(k^{3} u+\frac{1}{2} u^{2}\right)\right)\right\|_{L^{2}}
$$

According to Lemma 2, it is easy to obtain that

$$
\left\|\partial_{x}^{k+1}\left(P *\left(k^{3} u+\frac{1}{2} u^{2}\right)\right)\right\|_{L^{2}} \leq c\left(\|u\|_{H^{m}}+\|u\|_{H^{m}}^{2}\right)
$$

Then, we have

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{k} u \partial_{x}^{k+1}\left(P *\left(k^{3} u+\frac{1}{2} u^{2}\right)\right) \mathrm{d} x\right| \leq c\left(\|u\|_{H^{m}}^{2}+\|u\|_{H^{m}}^{3}\right) .
$$

It follows that

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\|u\|_{H^{m}} \leq c\left(\|u\|_{H^{m}}+\|u\|_{H^{m}}^{2}\right)
$$

Noting that the solution of differential equation $y^{\prime}(t)=a\left(y+y^{2}\right), a>0$ with initial data $y(0)=y_{0}$ is given by

$$
y(t)=\frac{y_{0} \mathrm{e}^{a t}}{1+\left(1-\mathrm{e}^{a t}\right) y_{0}} .
$$

Hence, there exists $\bar{t}(M)>0$, if $0<t<\bar{t}(M)$, then $y(t) \leq 2 y_{0}$. Therefore the result is obtained.

We can get similar results of Lemma 4 for Equations (4) and (5), which is useful in the convergence analysis and we list them below.

Lemma 5. For some $m \geq 2$,
(1) if $\left\|v_{0}\right\|_{H^{m}} \leq M$, then there exists $\bar{t}_{1}(M)>0$ such that $\left\|\Phi_{A}^{t}\left(v_{0}\right)\right\|_{H^{m}} \leq 2 M$, for $0 \leq t \leq$ $\bar{t}_{1}(M)$;
(2) if $\left\|\omega_{0}\right\|_{H^{m}} \leq M$, then there exists $\bar{t}_{2}(M)>0$ such that $\left\|\Phi_{B}^{t}\left(\omega_{0}\right)\right\|_{H^{m}} \leq 2 M$, for $0 \leq t \leq$ $\bar{t}_{2}(M)$.

In the analysis of convergence, it is necessary to establish the boundedness of the approximate solution. The following Lemma significantly contributes to the derivation of such boundedness.

Lemma 6. For Equations (4) and (5),
(1) if $\left\|\Phi_{A}^{t}\left(v_{0}\right)\right\|_{H^{2}} \leq \alpha$ holds for $0 \leq t \leq \tau_{1}$, then $\left\|\Phi_{A}^{t}\left(v_{0}\right)\right\|_{H^{3}} \leq \mathrm{e}^{c_{1} \alpha t}\left\|v_{0}\right\|_{H^{3}}$ for $0 \leq t \leq \tau_{1}$, where $c_{1}$ is independent of $v_{0}$ and $t$;
(2) if $\left\|\Phi_{B}^{t}\left(\omega_{0}\right)\right\|_{H^{2}} \leq \beta$ holds for $0 \leq t \leq \tau_{2}$, then $\left\|\Phi_{B}^{t}\left(\omega_{0}\right)\right\|_{H^{3}} \leq \mathrm{e}^{c_{2}(1+\beta) t}\left\|\omega_{0}\right\|_{H^{3}}$ for $0 \leq t \leq \tau_{2}$, where $c_{2}$ is independent of $\omega_{0}$ and $t$.

Proof. (1) Denote $v=\Phi_{A}^{t}\left(v_{0}\right)$, we have

$$
\begin{aligned}
\|v\|_{H^{3}} \frac{\mathrm{~d}}{\mathrm{~d} t}\|v\|_{H^{3}} & =-\sum_{k=0}^{3} \sum_{j=0}^{k} C_{k}^{j} \int_{-\infty}^{\infty} \partial_{x}^{k} v \partial_{x}^{j+1} v \partial_{x}^{k-j} v \mathrm{~d} x \\
& \leq \sum_{k=0}^{3} \sum_{j=0}^{k} C_{k}^{j}\left|\int_{-\infty}^{\infty} \partial_{x}^{k} v \partial_{x}^{j+1} v \partial_{x}^{k-j} v \mathrm{~d} x\right|
\end{aligned}
$$

For $k \leq 2, j \leq k$, it follows that

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{k} v \partial_{x}^{j+1} v \partial_{x}^{k-j} v \mathrm{~d} x\right| \leq\left\|\partial_{x}^{k} v\right\|_{L^{\infty}}\left\|\partial_{x}^{j+1} v\right\|_{L^{2}}\left\|\partial_{x}^{k-j} v\right\|_{L^{2}} \leq c\|v\|_{H^{2}}\|v\|_{H^{3}}^{2} .
$$

For $k=3$ and $j<3$, we have $\xi=\min \{j+1,3-j\}<3$ and $\eta=\max \{j+1,3-j\} \leq 3$, then it follows that

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{3} v \partial_{x}^{j+1} v \partial_{x}^{3-j} v \mathrm{~d} x\right| \leq\left\|\partial_{x}^{\xi} v\right\|_{L^{\infty}}\left\|\partial_{x}^{3} v\right\|_{L^{2}}\left\|\partial_{x}^{\eta} v\right\|_{L^{2}} \leq c\|v\|_{H^{2}}\|v\|_{H^{3}}^{2} .
$$

For the last case, $k=3$ and $j=3$, integrating by parts yields

$$
\left|\int_{-\infty}^{\infty} \partial_{x}^{3} v \partial_{x}^{4} v v \mathrm{~d} x\right| \leq \frac{1}{2}\left|\int_{-\infty}^{\infty}\left(\partial_{x}^{3} v\right)^{2} v_{x} \mathrm{~d} x\right| \leq \frac{1}{2}\left\|v_{x}\right\|_{L^{\infty}}\left\|\partial_{x}^{3} v\right\|_{L^{2}}^{2} \leq c\|v\|_{H^{2}}\|v\|_{H^{3}}^{2} .
$$

Therefor, we have

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\|v\|_{H^{3}} \leq c_{1} \alpha\|v\|_{H^{3}}
$$

This ends the proof.
(2) Similarly, the statement of (2) can be verified by using the convolution property given in Lemma 2.

## 3. First Order Convergence

It is known that Strang splitting method has the convergence order of two. Nevertheless, in order to demonstrate this convergence, the utilization of Lady Windermere's fan is necessary, which relies on the boundedness of the approximate solutions. Consequently, this section will focus on proving the first order convergence to ensure the boundedness of the approximate solutions. Our investigation shall commence by examining the error estimates for the local error.

Lemma 7. The local error of the Strang splitting method is bounded in $H^{2}$ by

$$
\left\|\Psi^{\tau}\left(u_{0}\right)-\Phi_{C}^{\tau}\left(u_{0}\right)\right\|_{H^{2}} \leq c_{3} \tau^{2}
$$

where $c_{3}$ only depends on $\left\|u_{0}\right\|_{H^{3}}$.
Proof. First we represent the exact solution $u(t)=\Phi_{C}^{\tau}\left(u_{0}\right)$ by the nonlinear variation-ofconstants formula (see Lemma 1)

$$
\begin{equation*}
u(\tau)=\left(\mathrm{e}^{\tau D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\int_{0}^{\tau}\left(\mathrm{e}^{(\tau-s) D_{C}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s \tag{14}
\end{equation*}
$$

Using this formula again for the integrand, we obtain

$$
u(\tau)=\left(\mathrm{e}^{\tau D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\int_{0}^{\tau}\left(\mathrm{e}^{(\tau-s) D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s+\varepsilon_{1}
$$

with

$$
\begin{equation*}
\varepsilon_{1}=\int_{0}^{\tau} \int_{0}^{\tau-s}\left(\mathrm{e}^{(\tau-s-\sigma) D_{C}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} \sigma \mathrm{~d} s \tag{15}
\end{equation*}
$$

For sufficiently small $\tau$, according to Lemmas 2, 4 and 5, we have

$$
\begin{aligned}
\left\|\varepsilon_{1}\right\|_{H^{2}} \leq & c \int_{0}^{\tau} \int_{0}^{\tau-s}\left\|\left(D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}} \mathrm{~d} \sigma \mathrm{~d} s \\
\leq & c \int_{0}^{\tau} \int_{0}^{\tau-s}\left\|\left(\mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}+\left\|\left(\mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}^{2} \mathrm{~d} \sigma \mathrm{~d} s \\
\leq & c \int_{0}^{\tau} \int_{0}^{\tau-s}\left\|\left(D_{B} \mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}+\left\|\left(D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}^{2} \mathrm{~d} \sigma \mathrm{~d} s \\
\leq & c \int_{0}^{\tau} \int_{0}^{\tau-s}\left\|\left(\mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}+\left\|\left(\mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}^{2} \\
& +\left(\left\|\left(\mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}+\left\|\left(\mathrm{e}^{s D_{A}} \operatorname{Id}\right)\left(u_{0}\right)\right\|_{H^{2}}^{2}\right)^{2} \mathrm{~d} \sigma \mathrm{~d} s \\
\leq & c\left(\left\|u_{0}\right\|_{H^{2}}\right) \tau^{2} .
\end{aligned}
$$

On the other hand, using the first order Taylor expansion with the remainder in the integral form to the exact solution of BBM equation yields

$$
\begin{equation*}
\mathrm{e}^{\tau D_{B}}=\operatorname{Id}+\tau D_{B}+\tau^{2} \int_{0}^{1}(1-\theta) \mathrm{e}^{\theta \tau D_{B}} D_{B}^{2} \mathrm{~d} \theta \tag{16}
\end{equation*}
$$

Inserting it into the numerical scheme (9), we obtain

$$
u_{1}=\left(\mathrm{e}^{\tau D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\tau\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} D_{B} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\varepsilon_{2}
$$

with

$$
\begin{equation*}
\varepsilon_{2}=\tau^{2} \int_{0}^{1}(1-\theta)\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{e}^{\theta \tau D_{B}} D_{B}^{2} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} \theta \tag{17}
\end{equation*}
$$

Following the same procedure as for $\varepsilon_{1}$ and using Lemmas 2 and 5 , we can also show that

$$
\left\|\varepsilon_{2}\right\|_{H^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right) \tau^{2}
$$

holds for sufficiently small $\tau$. Thus, we get

$$
\begin{aligned}
u_{1}-u(\tau) & =\tau\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} D_{B} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{0}\right)-\int_{0}^{\tau}\left(\mathrm{e}^{(\tau-s) D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) d s+\varepsilon_{2}-\varepsilon_{1} \\
& =\varepsilon_{0}+\varepsilon_{2}-\varepsilon_{1}
\end{aligned}
$$

We find that $\varepsilon_{0}$ is just the quadrature error of the midpoint rule applied to the integral over $[0, \tau]$ of the function $f(s)=\left(\mathrm{e}^{(\tau-s) D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)$. We express this quadrature error in the first order Peano form,

$$
\varepsilon_{0}=\tau f\left(\frac{1}{2} \tau\right)-\int_{0}^{\tau} f(s) d s=\tau^{2} \int_{0}^{1} \kappa_{1}(\theta) f^{\prime}(\theta \tau) \mathrm{d} \theta
$$

where $\kappa_{1}$ is the real-valued, bounded Peano kernel of the midpoint rule. We find

$$
f^{\prime}(s)=-\left(\mathrm{e}^{(\tau-s) D_{A}}\left\{D_{A}, D_{B}\right\} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)=\left(\mathrm{e}^{(\tau-s) D_{A}}[A, B] \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)
$$

Calculating the Fréchet-derivatives of operators $A$ and $B$ gives

$$
\begin{aligned}
\mathrm{d} A(\omega)[v] & =-(\omega v)_{x}, & \mathrm{~d} B(\omega)[v] & =-3 P *\left(k^{3} v+\omega v\right)_{x} \\
\mathrm{~d}^{2} A(\omega)[\mu, v] & =-(\mu v)_{x}, & \mathrm{~d}^{2} B(\omega)[\mu, v] & =-3 P *(v \mu)_{x}
\end{aligned}
$$

then

$$
\begin{aligned}
& {[A, B](v)=\mathrm{d} A(v)[B(v)]-\mathrm{d} B(v)[A(v)] } \\
= & 3 v_{x} P_{x} *\left(k^{3} v+\frac{1}{2} v^{2}\right)+3 v P *\left(k^{3} v+\frac{1}{2} v^{2}\right)-\frac{3}{2} k^{3} v^{2}-\frac{1}{2} v^{3}-3 P *\left(\frac{1}{2} k^{3} v^{2}+\frac{1}{3} v^{3}\right) .
\end{aligned}
$$

where Lemma 2 is used in the last equality. According to Lemmas 2 and 5, it follows that

$$
\left\|f^{\prime}(s)\right\|_{H^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{3}}\right) .
$$

Hence, the quadrature error $\varepsilon_{0}$ is $\mathcal{O}\left(\tau^{2}\right)$ in the $H^{2}$ norm for $u_{0} \in H^{3}$. This completes the proof.

We are now in a position to state our first main result regarding the boundedness of the approximate solutions, which is built along with the first order convergence result. This boundedness is necessary for the refined second order error estimate.

Theorem 1 (First-order convergence in $H^{2}$ ). Let Assumption (8) be fulfilled. Further, let $u(t)$ be the solution of the DP Equation (1), $u_{n}$ be the numerical solution given by the Strang splitting method (6). Then, there exists $\bar{\tau}>0$ such that for $\tau \leq \bar{\tau}$ and $t_{n}=n \tau \leq T$,

$$
\left\|u_{n}-u\left(t_{n}\right)\right\|_{H^{2}} \leq c_{5} \tau,\left\|u_{n}\right\|_{H^{2}} \leq R:-2 \rho,\left\|u_{n}\right\|_{H^{3}} \leq \mathrm{e}^{c_{6} n \tau}\left\|u_{0}\right\|_{H^{3}} \leq \Lambda:-\mathrm{e}^{c_{6} T}\left\|u_{0}\right\|_{H^{3}} .
$$

Here, $\bar{\tau}, c_{5}$ only depend on $\left\|u_{0}\right\|_{H^{3}}, R$ and $T$, while $c_{6}=5 c_{1} R+c_{2}+4 c_{2} R$ is independent of $u_{0}$ and $\tau$.

Proof. The induction method is employed during the process of establishing the proof.
For $n=1$, Lemma 7 indicates that $\left\|u_{1}-u\left(t_{1}\right)\right\| \leq c_{5} \tau$ holds true. For the boundedness of $u_{1}$, we have

$$
\left\|u_{1}\right\|_{H^{2}}=\left\|u_{1}-u\left(t_{1}\right)\right\|_{H^{2}}+\left\|u\left(t_{1}\right)\right\|_{H^{2}} \leq c_{5} \tau+\rho \leq R
$$

with $\tau$ sufficient small enough such that $c_{5} \tau \leq \rho$. For $\left\|u_{1}\right\|_{H^{3}}$, note that

$$
\begin{aligned}
\left\|u_{1}\right\|_{H^{3}} & =\left\|\Phi_{A}^{\frac{\tau}{2}} \circ \Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}} \\
& \leq \mathrm{e}^{c_{1} \frac{\tau}{2}\left\|\Phi_{A}^{\frac{\tau}{2}} \circ \Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}}\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}} .} .
\end{aligned}
$$

In the last inequality, Lemma 6 is used. From Lemma 5, we have

$$
\begin{aligned}
\left\|\Phi_{A}^{\frac{\tau}{2}} \circ \Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}} & \leq 2\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}} \\
& \leq 4\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}} \leq 8\left\|u_{0}\right\|_{H^{2}} \leq 8 \rho .
\end{aligned}
$$

The same logic is adopted to bound $\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}}$ and $\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}}$.

$$
\begin{aligned}
\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}} & \leq \mathrm{e}^{c_{2} \tau\left(1+\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}}\right)}\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}} \\
\left\|\Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}} & \leq 4\left\|u_{0}\right\|_{H^{2}} \leq 4 \rho, \\
\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{3}} & \leq \mathrm{e}^{c_{1} \frac{1}{2} \tau\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}}\left\|u_{0}\right\|_{H^{3}},} \\
\left\|\Phi_{A}^{\frac{\tau}{2}}\left(u_{0}\right)\right\|_{H^{2}} & \leq 2\left\|u_{0}\right\|_{H^{2}} \leq 2 \rho .
\end{aligned}
$$

Combing above estimations together, we have

$$
\left\|u_{1}\right\|_{H^{3}} \leq \mathrm{e}^{\left(5 c_{1} \rho+c_{2}+4 c_{2} \rho\right) \tau}\left\|u_{0}\right\|_{H^{3}} \leq \mathrm{e}^{c_{6} \tau}\left\|u_{0}\right\|_{H^{3}} \leq \Lambda
$$

Assume that the results are true for $k \leq n-1$, i.e.,

$$
\left\|u_{k}-u\left(t_{k}\right)\right\|_{H^{2}} \leq c_{0} \tau, \quad\left\|u_{k}\right\|_{H^{2}} \leq R, \quad\left\|u_{k}\right\|_{H^{3}} \leq \mathrm{e}^{c_{6} k \tau}\left\|u_{0}\right\|_{H^{3}} .
$$

We intend to show that the above results are also true for $k=n$. Using Lady Windermere's fan argument, the global error can be expressed as

$$
\begin{aligned}
& \left\|u_{n}-u\left(t_{n}\right)\right\|_{H^{2}}=\left\|\Phi_{C}^{0}\left(u_{n}\right)-\Phi_{C}^{n \tau}\left(u_{0}\right)\right\|_{H^{2}} \\
\leq & \sum_{k=0}^{n-1}\left\|\Phi_{C}^{k \tau}\left(u_{n-k}\right)-\Phi_{C}^{(k+1) \tau}\left(u_{n-k-1}\right)\right\|_{H^{2}} \\
= & \sum_{k=0}^{n-2}\left\|\Phi_{C}^{(n-1-k) \tau}\left(u_{k+1}\right)-\Phi_{C}^{(n-1-k) \tau}\left(\Phi_{C}^{\tau}\left(u_{k}\right)\right)\right\|_{H^{2}}+\left\|u_{n}-\Phi_{C}^{\tau}\left(u_{n-1}\right)\right\|_{H^{2}} .
\end{aligned}
$$

The Lipschitz condition (10) is subsequently utilized to estimate the error. However, such an approach necessitates that we have to prove for some $r>0$,

$$
\left\|\Phi_{C}^{(n-1-k) \tau}\left(u_{k+1}\right)\right\|_{H^{3}} \leq r,\left\|\Phi_{C}^{(n-k) \tau}\left(u_{k}\right)\right\|_{H^{3}} \leq r \text { for } k \leq n-2 .
$$

From the recursive assumption, it is easy to get $\left\|u_{k}\right\|_{H^{3}} \leq \Lambda$ for $k \leq n-1$. According to Lemma 4 with $m=3$, Lipschitz condition in Lemma 10 with $r=2 \Lambda$ is available. Then the global error is

$$
\begin{aligned}
& \left\|u_{n}-u\left(t_{n}\right)\right\|_{H^{2}} \\
\leq & \sum_{k=0}^{n-2}\left\|\Phi_{C}^{(n-1-k) \tau}\left(u_{k+1}\right)-\Phi_{C}^{(n-1-k) \tau}\left(\Phi_{C}^{\tau}\left(u_{k}\right)\right)\right\|_{H^{2}}+\left\|u_{n}-\Phi_{C}^{\tau}\left(u_{n-1}\right)\right\|_{H^{2}} \\
\leq & \sum_{k=0}^{n-2} K(r, k, T)\left\|u_{k+1}-\Phi_{C}^{\tau}\left(u_{k}\right)\right\|_{H^{2}}+c_{3} \tau^{2} \\
\leq & c\left(c_{3}\right) n K(r, k, T) \tau^{2} \leq c_{5} \tau .
\end{aligned}
$$

Similarly, we have

$$
\begin{gathered}
\left\|u_{n}\right\|_{H^{2}} \leq\left\|u_{n}-u\left(t_{n}\right)\right\|_{H^{2}}+\left\|u\left(t_{n}\right)\right\|_{H^{2}} \leq c_{5} \tau+\rho \leq R \\
\left\|u_{n}\right\|_{H^{3}} \leq\left\|\Phi_{A}^{\frac{\tau}{2}} \circ \Phi_{B}^{\tau} \circ \Phi_{A}^{\frac{\tau}{2}}\left(u_{n-1}\right)\right\|_{H^{3}} \leq \mathrm{e}^{c_{6} \tau}\left\|u_{n-1}\right\|_{H^{3}} \leq \mathrm{e}^{c_{6} n \tau}\left\|u_{0}\right\|_{H^{3}} \leq \Lambda .
\end{gathered}
$$

This completes the proof.

## 4. Second Order Convergence

In Section 3, first order convergence is analyzed and the boundedness of the approximate solution is obtained in $H^{2}$-norm, which makes the proof of second order convergence available. Next, we prove the Strang splitting method is second order convergent in $L^{2}$-norm.

Theorem 2 (Second-order convergence in $L^{2}$ ). Let Assumption (8) be fulfilled. Further, let $u(t)$ be the solution of the DP Equation (1), $u_{n}$ be the approximate solution given by the Strang splitting method (6). Then, there exists $\bar{\tau}>0$ such that for $\tau \leq \bar{\tau}$ and $t_{n}=n \tau \leq T$,

$$
\left\|u_{n}-u\left(t_{n}\right)\right\|_{L^{2}} \leq c_{7} \tau^{2}
$$

with constants $\bar{\tau}, c_{7}$ only depend on $\left\|u_{0}\right\|_{H^{2}}, R$ and $T$.
Proof. The idea is similar to that for Lemma 7 and Theorem 1, but we expand the solution to one more higher order to achieve the second order convergence. This is possible due to the boundedness of the solution.

Using the same notations introduced in the proof of Lemma 7, for the local error we have

$$
\begin{equation*}
u_{1}-u(\tau)=\varepsilon_{0}+\varepsilon_{2}-\varepsilon_{1} . \tag{18}
\end{equation*}
$$

We now give refined estimates for three terms.
Recall the definition of $\varepsilon_{2}$ given by (17), we further have

$$
\varepsilon_{2}=\frac{\tau^{2}}{2}\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} D_{B}^{2} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\tilde{\varepsilon}_{2},
$$

where $\tilde{\varepsilon}_{2}$ takes the form

$$
\tilde{\varepsilon}_{2}=\tau^{3} \int_{0}^{1} \int_{0}^{1}(1-\theta) \theta(1-s)\left(\mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{e}^{s \theta \tau D_{B}} D_{B}^{3} \mathrm{e}^{\frac{\tau}{2} D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} s \mathrm{~d} \theta
$$

On the other side, for error $\varepsilon_{1}$ given by (15), using the nonlinear variation-of-constants formula again, we obtain

$$
\varepsilon_{1}=\int_{0}^{\tau} \int_{0}^{\tau-s}\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) d \sigma d s+\tilde{e}_{1},
$$

with

$$
\tilde{\varepsilon}_{1}=\int_{0}^{\tau} \int_{0}^{\tau-s} \int_{0}^{\tau-s-\sigma}\left(\mathrm{e}^{(\tau-s-\sigma-\delta) D_{C}} D_{B} \mathrm{e}^{\delta D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \mathrm{d} \delta \mathrm{~d} \sigma \mathrm{~d} s
$$

Define function $g(s, \sigma)=\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)$, then we have

$$
\varepsilon_{2}-\varepsilon_{1}=\frac{\tau^{2}}{2} g\left(\frac{\tau}{2}, 0\right)-\int_{0}^{\tau} \int_{0}^{\tau-s} g(s, \sigma) d \sigma d s+\tilde{\varepsilon}_{2}-\tilde{\varepsilon}_{1} .
$$

According to the quadrature error of a first order two-dimensional quadrature formula, we have

$$
\left\|\frac{\tau^{2}}{2} g\left(\frac{\tau}{2}, 0\right)-\int_{0}^{\tau} \int_{0}^{\tau-s} g(s, \sigma) \mathrm{d} \sigma \mathrm{~d} s\right\|_{L^{2}} \leq c \tau^{3}\left(\max \left\|\frac{\partial g}{\partial s}\right\|_{L^{2}}+\max \left\|\frac{\partial g}{\partial \sigma}\right\|_{L^{2}}\right)
$$

where the maxima are taken over the triangle $\{(s, \sigma): 0 \leq \sigma \leq s \leq \tau\}$. For the partial derivatives, we have

$$
\begin{aligned}
\frac{\partial g}{\partial s}= & -\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{A} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} D_{A} \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \\
& -\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{A} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}} D_{B} D_{A} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) \\
= & \left(\mathrm{e}^{(\tau-s-\sigma) D_{A}}[A, B] \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)+\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}} D_{B} \mathrm{e}^{\sigma D_{A}}[A, B] \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right) .
\end{aligned}
$$

With Lemmas 2 and 5, as the estimation of $\varepsilon_{1}$ in Lemma 7, we have

$$
\begin{gathered}
\left\|\frac{\partial g}{\partial s}\right\|_{L^{2}} \leq c\left\|\left([A, B] \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)\right\|_{L^{2}}+c\left\|\left(D_{B} \mathrm{e}^{\sigma D_{A}}[A, B] \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right) . \\
\text { For } \frac{\partial g}{\partial \sigma} \text {, we get }
\end{gathered}
$$

$$
\frac{\partial g}{\partial \sigma}=\left(\mathrm{e}^{(\tau-s-\sigma) D_{A}}[A, B] \mathrm{e}^{\sigma D_{A}} D_{B} \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right),
$$

then with Lemmas 2 and 5, it follows that

$$
\left\|\frac{\partial g}{\partial \sigma}\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right)
$$

Noting that, for sufficiently small $\tau$, according to the well-posedness result (7) and Lemma 5, we have

$$
\left\|\tilde{\varepsilon}_{1}\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{L^{2}}\right) \tau^{3} .
$$

Similarly, Lemma 5 yields

$$
\left\|\tilde{\varepsilon}_{2}\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{L^{2}}\right) \tau^{3} .
$$

Therefore, we have the bound

$$
\left\|\varepsilon_{2}-\varepsilon_{1}\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right) \tau^{3}
$$

Next, we show that similar third order bound also holds for $\varepsilon_{0}$. To that end, we write the error term $\tau f\left(\frac{1}{2} \tau\right)-\int_{0}^{\tau} f(s) d s$ in its second order Peano form

$$
\varepsilon_{0}=\tau f\left(\frac{1}{2} \tau\right)-\int_{0}^{\tau} f(s) d s=\tau^{3} \int_{0}^{1} \kappa_{2}(\theta) f^{\prime \prime}(\theta \tau) \mathrm{d} \theta
$$

with the second order Peano kernel $\kappa_{2}$ of the midpoint rule and $f$ is the same function in Lemma 7. We have

$$
f^{\prime \prime}(s)=\left(\mathrm{e}^{(\tau-s) D_{A}}[A,[A, B]] \mathrm{e}^{s D_{A}} \mathrm{Id}\right)\left(u_{0}\right)
$$

where

$$
\begin{aligned}
{[A,[A, B]](v)=} & (\mathrm{d} A(v))^{2}[B(v)]-2 \mathrm{~d} A(v) \mathrm{d} B(v)[A(v)]-\mathrm{d}^{2} A(v)[B(v), A(v)] \\
& +\mathrm{d}^{2} B(v)[A(v), A(v)]+\mathrm{d} B(v) \mathrm{d} A(v)[A(v)]
\end{aligned}
$$

After some tedious calculus and simplification, we get that

$$
\begin{aligned}
(\mathrm{d} A(v))^{2}[B(v)]= & -\frac{9}{2}\left(v^{2}\right)_{x}\left(P *\left(k^{3} v+\frac{1}{2} v^{2}\right)-k^{3} v-\frac{1}{2} v^{2}\right)-\frac{3}{2}\left(v^{2}\right)_{x x} P_{x} *\left(k^{3} v+\frac{1}{2} v^{2}\right) \\
& -3 v^{2}\left(P_{x} *\left(k^{3} v+\frac{1}{2} v^{2}\right)-\left(k^{3} v+\frac{1}{2} v^{2}\right)_{x}\right), \\
-2 \mathrm{~d} A(v) \mathrm{d} B(v)[A(v)]= & 6 v_{x}\left(P *\left(\frac{1}{2} k^{3} v^{2}+\frac{1}{3} v^{3}\right)-\frac{1}{2} k^{3} v^{2}-\frac{1}{3} v^{3}\right) \\
& +6 v\left(P_{x} *\left(\frac{1}{2} k^{3} v^{2}+\frac{1}{3} v^{3}\right)-k^{3} v v_{x}-v^{2} v_{x}\right), \\
-\mathrm{d}^{2} A(v)[B(v), A(v)]= & 3\left(v v_{x x}+v_{x}^{2}\right) P_{x} *\left(k^{3} v+\frac{1}{2} v^{2}\right)+3 v v_{x}\left(P *\left(k^{3} v+\frac{1}{2} v^{2}\right)-k^{3} v-\frac{1}{2} v^{2}\right), \\
\mathrm{d}^{2} B(v)[A(v), A(v)]= & -3 P_{x} *\left(v^{2} v_{x}^{2}\right), \\
\mathrm{d} B(v) \mathrm{d} A(v)[A(v)]= & -3 P_{x} *\left(k^{3}\left(2 v v_{x}^{2}+v^{2} v_{x x}\right)+2 v^{2} v_{x}^{2}+v^{3} v_{x x}\right) .
\end{aligned}
$$

Using Lemma 2, this yields

$$
\|[A,[A, B]](v)\|_{L^{2}} \leq c\left(\|v\|_{H^{2}}\right) .
$$

Similarly, we have

$$
\left\|f^{\prime \prime}(s)\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right)
$$

Hence, $\varepsilon_{0}$ is also bounded by $c \tau^{3}$. Combining the above estimations together, for the local error, we obtain

$$
\begin{equation*}
\left\|u_{1}-u(\tau)\right\|_{L^{2}} \leq c\left(\left\|u_{0}\right\|_{H^{2}}\right) \tau^{3} . \tag{19}
\end{equation*}
$$

Here, the local error depends on the $H^{2}$-norm of the approximate solution.
Finally, noting that the boundedness of the approximate solutions has been established in Theorem 1, the second order error estimate can be built by the Lady Windermere's fan argument.

$$
\begin{aligned}
& \left\|u_{n}-u\left(t_{n}\right)\right\|_{L^{2}}=\left\|\Phi_{C}^{0}\left(u_{n}\right)-\Phi_{C}^{n \tau}\left(u_{0}\right)\right\|_{L^{2}} \\
\leq & \sum_{k=0}^{n-1}\left\|\Phi_{C}^{k \tau}\left(u_{n-k}\right)-\Phi_{C}^{(k+1) \tau}\left(u_{n-k-1}\right)\right\|_{L^{2}} \\
= & \sum_{k=0}^{n-2}\left\|\Phi_{C}^{(n-1-k) \tau}\left(u_{k+1}\right)-\Phi_{C}^{(n-1-k) \tau}\left(\Phi_{C}^{\tau}\left(u_{k}\right)\right)\right\|_{L^{2}}+\left\|u_{n}-\Phi_{C}^{\tau}\left(u_{n-1}\right)\right\|_{L^{2}} \\
\leq & c K(r, k, T) n \tau^{3} \leq c K(r, k, T) T \tau^{2},
\end{aligned}
$$

where Lemma 3 and the third order estimate of the local error (19) are used. This completes the proof.

## 5. Conclusions

This paper presents the error estimates for the Strang splitting method applied to the Degasperis-Procesi equation. The nonlinearity of the two sub-operators after the operator splitting renders the classical error analysis method inapplicable. Our main technical contribution lies in initially establishing the first order convergence result in the Sobolev space $H^{2}$. This result is essential in proving the $H^{2}$ boundedness of the approximate solutions. Finally, by utilizing the Lie derivative bounds for the local error and the boundedness of the approximate solution, we are able to derive the desired second order convergence result in $L^{2}$.

The present convergence theory aligns with the numerical experimental findings documented in [17]. It is worth noting that the two subequations in this paper are solved accurately. Moving forward, the forthcoming study will focus on investigating the convergence analysis of the complete discretization splitting scheme when numerical approximations are employed for the two subequations.

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## References

1. Hairer, E.; Lubich, C.; Wanner, G. Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2006.
2. McLachlan, R.I.; Quispel, G.R.W. Splitting methods. Acta Numer. 2002, 11, 341-434. [CrossRef]
3. Holden, H.; Karlsen, K.H.; Lie, K.A.; Risebro, H. Spfferlitting Methods for Partial Diential Equations with Rough Solutions: Analysis and MATLAB Programs; European Mathematical Society Publishing: Helsinki, Finland, 2010.
4. Holden, H.; Karlsen, K.H.; Risebro, N.H.; Tao, T. Operator splitting for Korteweg-de Vries equation. Math. Comput. 2011, 80, 821-846. [CrossRef]
5. Koch, O.; Neuhauser, C.; Thalhammer, M. Error analysis of high-order splitting methods for nonlinear evolutionary Schrödinger equations and application to the MCTDHF equations in electron dynamics. ESAIM Math. Model. Num. 2013, 47, 1265-1286. [CrossRef]
6. Holden, H.; Lubich, C.; Risebro, N. Operator splitting for partial differential equations with Burgers nonlinearity. Math. Comput. 2013, 82, 173-185. [CrossRef]
7. Çiçek, Y.; Tanoğlu, G. Strang splitting method for Burgers-Huxley equation. Appl. Math. Comput. 2016, 276, 454-467. [CrossRef]
8. Einkemmer, L.; Ostermann, A. Convergence analysis of a discontinuous Galerkin/Strang splitting approximation for the Vlasov-Poisson equations. SIAM J. Numer. Anal. 2014, 52, 757-778. [CrossRef]
9. Einkemmer, L.; Ostermann, A. Convergence analysis of Strang splitting for Vlasov-type equations. SIAM J. Numer. Anal. 2014, 52, 140-155. [CrossRef]
10. Zürnacı, F. Convergence Analysis and Numerical Solutions of the Fisher's and Benjamin-Bono-Mahony Equations by Operator Splitting Method. Master's Thesis, İzmir Institute of Technology, İzmir, Turkey, 2014.
11. He, D.; Pan, K.; Hu, H. A spatial fourth-order maximum principle preserving operator splitting scheme for the multi-dimensional fractional Allen-Cahn equation. Appl. Numer. Math. 2020, 151, 44-63. [CrossRef]
12. Chen, H.; Sun, H. Second-order maximum principle preserving Strang's splitting schemes for anisotropic fractional Allen-Cahn equations. Numer. Algorithms 2022, 90, 749-771. [CrossRef]
13. Li, D.; Quan, C. The operator-splitting method for Cahn-Hilliard is stable. J. Sci. Comput. 2022, 90, 62. [CrossRef]
14. Zhai, S.; Wu, L.; Wang, J.; Weng, Z. Numerical approximation of the fractional Cahn-Hilliard equation by operator splitting method. Numer. Algorithms 2020, 84, 1155-1178. [CrossRef]
15. Zhang, K.; Alshehry, A.S.; Aljahdaly, N.H.; Shah, R.; Shah, N.A.; Ali, M.R. Efficient computational approaches for fractional-order Degasperis-Procesi and Camassa-Holm equations. Results Phys. 2023, 50, 106549. [CrossRef]
16. Lubich, C. On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations. Math. Comput. 2008, 77, 2141-2153. [CrossRef]
17. Feng, B.F.; Liu, Y. An operator splitting method for the Degasperis-Procesi equation. J. Comput. Phys. 2009, 228, 7805-7820. [CrossRef]
18. Degasperis, A.; Procesi, M. Asymptotic integrability. In Proceedings of the Symmetry and Perturbation Theory Rome, Italy, 16-22 December 1998; Degasperis, A., Gaeta, G., Eds.; World Scientific: Singapore, 1999; pp. 23-37.
19. Liu, Y.; Yin, Z. Global existence and blow-up phenomena for the Degasperis-Procesi equation. Commun. Math. Phys. 2006, 267, 801-820. [CrossRef]

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