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Partial Residuated Implications Induced by Partial Triangular Norms and Partial Residuated Lattices

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Abstract: This paper reveals some relations between fuzzy logic and quantum logic on partial residuated implications (PRIs) induced by partial t-norms as well as proposes partial residuated monoids (PRMs) and partial residuated lattices (PRLs) by defining partial adjoint pairs. First of all, we introduce the connection between lattice effect algebra and partial t-norms according to the concept of partial t-norms given by Borzooei, together with the proof that partial operation in any commutative quasiresiduated lattice is partial t-norm. Then, we offer the general form of PRI and the definition of partial fuzzy implication (PFI), give the condition that partial residuated implication is a fuzzy implication, and prove that each PRI is a PFI. Next, we propose PRLs, study their basic characteristics, discuss the correspondence between PRLs and lattice effect algebras (LEAs), and point out the relationship between LEAs and residuated partial algebras. In addition, like the definition of partial t-norms, we provide the notions of partial triangular conorms (partial t-conorms) and corresponding partial co-residuated lattices (PcRLs). Lastly, based on partial residuated lattices, we define well partial residuated lattices (wPRLs), study the filter of well partial residuated lattices, and then construct quotient structure of PRMs.

Keywords: fuzzy logic; lattice effect algebra; partial residuated implication; partial fuzzy implication; partial residuated lattice; filter

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1. Introduction

In 1965, Zadeh first proposed fuzzy sets in [1], and then gradually established fuzzy logic. In fuzzy logic, the research on t-norm and t-conorm emerges in an endless stream. T-norm and t-conorm, as traditional binary operations, are introduced in the study of probabilistic metric space (see [2,3]). However, there may be some "undefined" cases in the practical application of fuzzy logic. In this regard, some scholars explored in [4–7] from the perspective of partial membership functions and fuzzy partial logic (logical connectors are partial operations). For example, Běhounek et al. considered the fuzzy partial logic in [4] and defined binary primitive conjunctions by Tables 1 and 2, and using the special value "*" to explain semantics such as "undefined", "meaningless", "non-applicable", etc.:

Table 1. Definability of connectives (1).

c	β	*
α	$\alpha c \beta$	0
*	*	*

Table 2. Definability of connectives (2).

$\wedge$	$\mathbf{0}$	δ	*
0	0	0	0
γ	0	$\gamma \wedge \delta$	*
*	0	*	*

In [8,9], Burmeister et al. dealt with the "undefined" situation from the viewpoint of the aggregate function and partial algebra, respectively. In 1994, Foulis et al. put forward the effect algebra that can describe imprecise quantum phenomena in [10]. Therefore, partial operations offer an important research direction in fuzzy logic and quantum logic. Borzooei et al. proposed partial t-norms in [11]. Wei considered partial t-implications in [12]. In [13], Chajda et al. investigated the natural implication in lattice effect algebra and explored the effect implication algebra. Furthermore, some scholars also studied the fuzzy implication and residuated implication in effect algebras (see [12,14–16]). Baets provided the concept of residuated implication in [17]. Regarding the algebraic structure of partial t-norms, Sheng and Zhang considered the partial algebraic structure: regular partial residuated lattice in [18]. In addition, some scholars studied the relationship between fuzzy logic and quantum logic (see [19–21]), including the filter and congruence relationship between residuated lattices and effect algebras (see [22–24]).

In light of the above inspiration, this paper focuses on the following contents. First, we present the general form of PRIs induced by partial t-norms as well as reasonably define PFIs. Second, the concept of partial adjoint pairs (PAPs) is properly defined. On this basis, partial residuated monoids and partial residuated lattices are defined. We also develop the induction relationship between lattice effect algebra and PRL. Finally, the filter and quotient structure of PRMs are established.

2. Lattice Effect Algebras and Partial t-Norms

We briefly review the concepts of lattice effect algebras, quasiresiduated lattices and partial t-norms, construct partial t-norms in lattice effect algebras, and prove that the operation $\odot$ in commutative quasiresiduated lattices is a partial t-norm.

Definition 1 ([10,12,14]). *A partial algebra $(E, +, ', 0, 1)$ is called an effect algebra, where + is a partial operation and $'$ is a unary operation such that for any $x, y, z \in E$:*

(E1) *$x + y$ is defined iff $y + x$ is defined, and then $x + y = y + x$;*
(E2) *$x + y$ and $(x + y) + z$ are defined iff $y + z$ and $x + (y + z)$ are defined, and then $(x + y) + z = x + (y + z)$;*
(E3) *For every $x \in E$, there exists a unique $x' \in E$ such that $x + x' = 1$;*
(E4) *If $x + 1$ is defined, then $x = 0$.*

$(E; \leq)$ is a partial ordered set, where $\leq$ is a partial ordered relation on E through $x \leq y$ iff there exists $z \in E$ and $x + z = y$. If $(E; \leq)$ is a lattice, we call it is a lattice effect algebra (LEA).

Theorem 1 ([14]). *Let $(E, \leq, +, ', 0, 1)$ be an LEA. Then, for any $x, y, z \in E$:*

(1) *$x + y$ is defined iff $x \leq y'$;*
(2) *If $x \leq y$ and $y + z$ is defined, then $x + z$ is defined and $x + z \leq y + z$;*
(3) *If $x \leq y$, then $x + (x + y')' = y$.*

Definition 2 ([14]). *A partial algebra $(C, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a commutative quasiresiduated lattice (cQL), where $(C, \vee, \wedge, 0, 1)$ is a bounded lattice, $\odot$ is a partial operation, and $\rightarrow$ is a full operation such that, for any $x, y, z \in C$:*

(i) *$(C, \odot, 1)$ is a commutative partial monoid and $x \odot y$ is defined iff $x' \leq y$;*
(ii) *$x'' = x$, if $x \leq y$ then $y' \leq x'$;*
(iii) *$(x \vee y') \odot y \leq y \wedge z$ iff $x \vee y' \leq y \rightarrow z$.*

Here, x' is an abbreviation for $x \to 0$.

Theorem 2 ([16])**.** *Let $(C, \vee, \wedge, \odot, \to, 0, 1)$ be a cQL. Then, for any $x, y, z \in C$:*

(1) If $x' \leq y$, then $x \odot y \leq y$;
(2) If $x' \leq y$, then $x \leq y \to (x \odot y)$;
(3) If $x' \leq y$ and $z \leq y$, then $x \odot y \leq z$ iff $x \leq y \to z$.

Definition 3 ([11])**.** *Let L be a bounded lattice. A partial binary operation $\odot$ on L is called a partial t-norm (pt-norm), if for any $x, y, z, h, k \in L$:*

(p1) $x \odot 1 = 1$;
(p2) If $x \odot y$ is defined, then $y \odot x$ is defined and $x \odot y = y \odot x$;
(p3) If $y \odot z$ and $x \odot (y \odot z)$ are defined, then $x \odot y$ and $(x \odot y) \odot z$ are defined and $x \odot (y \odot z) = (x \odot y) \odot z$;
(p4) If $x \leq y$, $h \leq k$ and $x \odot h$, $y \odot k$ are defined, then $x \odot h \leq y \odot k$.

Example 1. *Define the operation $\odot$ as follows:*

$$a \odot b := \begin{cases} undefined & if\ a, b \in [0, 0.5] \\ min\{a, b\} & others \end{cases} \tag{1}$$

Then, the operation $\odot$ is a pt-norm $(a, b \in [0, 1])$.

Example 2. *Define the operation $\odot$ as follows:*

$$a \odot b := \begin{cases} min\{a, b\} & if\ a, b \in [0.5, 1] \\ undefined & others \end{cases} \tag{2}$$

Then, the operation $\odot$ is a pt-norm $(a, b \in [0, 1])$.

Example 3. *Assume that $L = \{0, l, m, n, 1\}$. The Hasse diagram of $(L; \leq)$ is shown in Figure 1, and the operation $\odot$ is defined by Table 3. Then, $\odot$ is a pt-norm.*

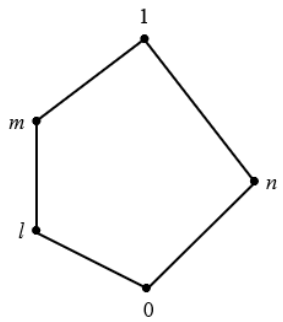

Figure 1. Lattice order relation on L.

Table 3. The partial operation $\odot$.

$\odot$	**0**	***l***	***m***	***n***	**1**
0					0
l			0		l
m		0	l		m
n				0	n
1	0	l	m	n	1

Example 4. *Assume that $L = \{0, l, m, n, 1\}$. The Hasse diagram of $(L; \leq)$ is shown in Figure 2, and the operation $\odot$ is defined by Table 4. Then, $\odot$ is a pt-norm.*

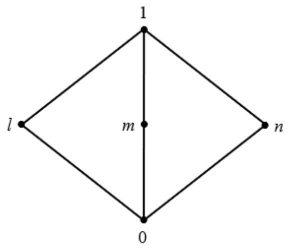

Figure 2. Lattice order relation on L.

Table 4. The partial operation $\odot$.

$\odot$	**0**	***l***	***m***	***n***	**1**
0					0
l				0	l
m			0		m
n		0			n
1	0	l	m	n	1

Proposition 1. *Let $(E, \leq, +, ', 0, 1)$ be an LEA. Define the binary operation $\odot$ on E as follows (for any $x, y \in E$):*

$$x \odot y := (x' + y') \text{ iff } x' \leq y.$$

Then, $\odot$ is a partial t-norm.

Proof. (1) Since $x' \leq 1$, $x \odot 1$ is defined, then $x \odot 1 = (x' + 1')' = (x' + 0)' = x$.

(2) If $x \odot y$ is defined, then $x' \leq y$, so $y' \leq x$, and $(x' + y')' = (y' + x')'$, i.e., $y \odot x$ is defined. Thus, the exchange law is established.

(3) Suppose $y \odot z$, $x \odot (y \odot z)$ are defined, we have $y' \leq z$ and $x' \leq (y' + z')'$. Applying Theorem 1 (3), $y' + (y' + z')' = z$. By Theorem 1 (1), $y' \leq (y' + z')''$. On the other hand, $(y' + z')'' \leq x''$. Thus, $y' \leq (y' + z')'' \leq x''$, that is, $x' \leq y$. Moreover, above, we have $(x' + y')'' = y' + x' \leq y' + (y' + z')' = z$. Hence, we have $(x \odot y) \odot z = (x' + y')'' + z' = x' + (y' + z')'' = x \odot (y \odot z)$. Thus, the associative law is established.

(4) For any $x, y, h, k \in E$, if $x \leq y$, $h \leq k$, and $x \odot h$, $y \odot k$ are defined, then $y' \leq x'$, $k' \leq h'$, $x' \leq h$, $y' \leq k$. Applying Theorem 1 (2), $y' + k' \leq x' + k' \leq x' + h'$, $(x' + h') \leq (y' + k')'$, $x \odot h \leq y \odot k$.

Therefore, $\odot$ is a partial t-norm. $\square$

Proposition 2. *Let $(C, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a cQL. Then, the partial operation $\odot$ is a partial t-norm on C.*

Proof. If $x \leq y$, $x \odot z$ and $y \odot z$ are defined, then applying Theorem 2 (2), $y \leq z \rightarrow (y \odot z)$. Thus, $x \leq z \rightarrow (y \odot z)$. Moreover, we have $y \odot z \leq z$, by Theorem 2 (1) and (3), $x \odot z \leq y \odot z$. Then, when $x \leq y$, $h \leq k$, and $x \odot h$, $y \odot k$ are defined, this implies $x \odot h \leq y \odot h \leq y \odot k$. Therefore, $\odot$ is a partial t-norm. $\square$

3. Partial Residuated Implications (PRIs) Derived from Partial t-Norms

Many scholars have studied the residuated implication induced by t-norm. In [9], Borzooei gave the concept of partial t-norm but did not make further research. In this section, we will study the residuated implication derived by partial t-norms and call it partial residuated implications.

Definition 4. *Let L be a bounded lattice and $\odot$ be a pt-norm on L. A partial operation $\rightarrow_\odot$ induced by $\odot$ is called a partial residuated implication (PRI) such that for any $a, b \in L$:*

$$a \rightarrow_\odot b := \begin{cases} \sup\{x \in L \mid a \odot x \text{ is defined and } a \odot x \leq b\} & \text{if } S \neq \emptyset \text{ and } \sup S \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases} \quad (3)$$

where $S = \{x \in L \mid a \odot x \, is \, defined \, and \, a \odot x \leq b\}$.

Example 5. *Assume that $(E, \leq, +, ', 0, 1)$ is an LEA. Define Sasaki arrow $\rightarrow_S$ on E as follows:*

$$x \rightarrow_S y = x' + (x \wedge y) \tag{4}$$

Then, Sasaki arrow $\rightarrow_S$ is a PRI on E.

Example 6. *Assume that $(E, \leq, +, ', 0, 1)$ is an LEA. Define the function I_S on E as follows:*

$$I_S(x, y) := \begin{cases} 1 & if x \leq y \\ a' & if \text{ the interval } E[0, x] \text{ is totally ordered, has an atom } a \text{ and } x - (x \wedge y) = a \\ 0 & otherwise \end{cases} \tag{5}$$

Then, $I_S(x, y)$ is a PRI on E.

Example 7. *Assume that $L = \{0, l, m, n, 1\}$. The Hasse diagram of $(L; \leq)$ is shown in Figure 3 and the operations $\odot$ and $\rightarrow_\odot$ are defined by Tables 5 and 6. Then, $\odot$ is a pt-norm and $\rightarrow_\odot$ is a PRI induced by $\odot$.*

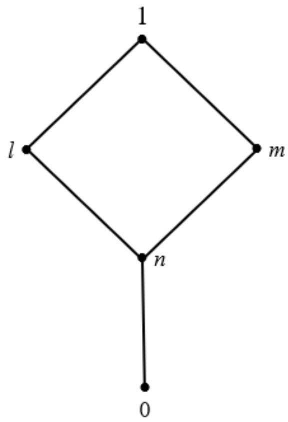

Figure 3. Lattice order relation on L.

Table 5. The partial operation $\odot$.

$\odot$	**0**	***l***	***m***	***n***	**1**
0					0
l		n	n	m	l
m		n	n	m	m
n		m	m	n	n
1	0	l	m	n	1

Table 6. The partial operation $\rightarrow_\odot$.

$\rightarrow_\odot$	**0**	***l***	***m***	***n***	**1**
0	1	1	1	1	1
l		1	1	1	1
m		1	1	1	1
n		1	1	1	1
1	0	l	m	n	1

Theorem 3. *Let L be a bounded lattice, $\odot$ be a pt-norm on L and $\rightarrow_\odot$ be a PRI derived from $\odot$. The following statements are equivalent:*

(i) $\odot$ *is infinitely $\vee$-distributive, i.e., if* $\vee_{i\in I}x_i$ *and* $\vee_{i\in I}(x\odot x_i)$ *are existing, then* $x\odot(\vee_{i\in I}x_i) = \vee_{i\in I}(x\odot x_i)$;

(ii) $x\odot z$ *is defined and* $x\odot z\leq y$ *iff* $x\rightarrow_\odot y$ *is defined and* $z\leq x\rightarrow_\odot y$;

(iii) *If* $x\rightarrow_\odot y$ *and* $x\odot(x\rightarrow_\odot y)$ *are defined, then* $x\odot(x\rightarrow_\odot y)\leq y$;

(iv) *If* $\{a\in L\mid x\odot a$ *is defined and* $x\odot a\leq y\}$ *is not empty, then the set has the maximum element.*

Proof. $(i)\Rightarrow(ii)$: If $x\odot z$ is defined and $x\odot z\leq y$, then $z\in\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$, hence $x\rightarrow_\odot y=\vee\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$, so $z\leq x\rightarrow_\odot y$. Conversely, if $z\leq x\rightarrow_\odot y$, from Definition 3 (4), we obtain $x\odot z\leq x\odot(x\rightarrow_\odot y)=x\odot(\vee\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\})=\vee\{x\odot v\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}=y$.

$(ii)\Rightarrow(iii)$: We know $x\rightarrow_\odot y\odot x\rightarrow_\odot y$, then $x\odot(x\rightarrow_\odot y)\leq y$.

$(iii)\Rightarrow(iv)$: If $x\odot(x\rightarrow_\odot y)\leq y$ and $\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$ is a nonempty set, then $x\rightarrow_\odot y\in\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$ $(iv)\Rightarrow(i)$: If $\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$ is a nonempty set, let $x_i\in\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq y\}$, from Definition 3 (4), we know $\vee_{i\in I}(x\odot x_i)\leq x\odot(\vee_{i\in I}x_i)$. Next, we only need to prove $x\odot(\vee_{i\in I}x_i)\leq\vee_{i\in I}(x\odot x_i)$. Let $u=\vee_{i\in I}(x\odot x_i)$, then $x\odot x_i\leq u$, we have $x_i\in\{v\in L\mid x\odot v\ is\ defined\ and\ x\odot v\leq u\}$, for every $x_i\in L$, hence $x_i\leq x\rightarrow_\odot u$ and $\vee_{i\in I}x_i\leq x\rightarrow_\odot u$, hence, $x\odot(\vee_{i\in I}x_i)\leq\vee_{i\in I}(x\odot x_i)$. In conclusion, $x\odot(\vee_{i\in I}x_i)=\vee_{i\in I}(x\odot x_i)$. □

Corollary 1. *Let L be a bounded lattice and $\odot$ be a pt-norm on L. If $\odot$ is infinitely $\vee$-distributive and* $\{a\in L\mid x\odot a$ *is defined and* $x\odot a\leq y\}$ *is a nonempty set, then PRI* $\rightarrow_\odot$ *is a fuzzy implication.*

Proof. It follows from Theorem 3. □

4. Partial Fuzzy Implications (PFIs) and Partial Residuated Lattices (PRLs)

We propose the definition of partial fuzzy implication, and define partial residuated monoid and partial residuated lattice by defining partial adjoint pairs. We also prove that partial residuated lattices are partial algebraic structures corresponding to pt-norms and PRIs. Finally, the related properties of partial residuated lattices are studied.

Definition 5 ([12]). *Let L be a bounded lattice. The function* $I: L\times L\rightarrow L$ *is called a fuzzy implication, if, for any* $x,y,x_1,x_2,y_1,y_2\in L$, *the following conditions are satisfied:*

(i) *If* $x_1\leq x_2$, *then* $I(x_2,y)\leq I(x_1,y)$;

(ii) *If* $y_1\leq y_2$, *then* $I(x,y_1)\leq I(x,y_2)$;

(iii) $I(0,0)=I(1,1)=1$, $I(1,0)=0$.

Definition 6 ([25]). *Let L be a bounded lattice. The function* $N: L\rightarrow L$ *is called a negation, if for any* $x,y\in L$, *the following conditions are satisfied:*

(i) $N(0)=1$ *and* $N(1)=0$;

(ii) *If* $x\leq y$, *then* $N(y)\leq N(x)$.

Definition 7. *Let L be a bounded lattice. The function* $PI: L\times L\rightarrow L$ *is called a partial fuzzy implication (PFI), if for any* $x,y,x_1,x_2,y_1,y_2\in L$, *the following conditions are satisfied:*

(PI1) *If* $x_1\leq x_2$, $PI(x_1,y)$ *and* $PI(x_2,y)$ *are defined, then* $PI(x_2,y)\leq PI(x_1,y)$;

(PI2) *If* $y_1\leq y_2$, $PI(x,y_1)$ *and* $PI(x,y_2)$ *are defined, then* $PI(x,y_1)\leq PI(x,y_2)$;

(PI3) $PI(0,0)=PI(1,1)=1$, $PI(1,0)=0$.

Example 8. *Let L be a bounded lattice, PI is a PFI on L. Define the operation PI_N as follows (for any $x, y \in L$):*

$$PI_N(x,y) := \begin{cases} PI(N(y), N(x)) & if PI(N(y), N(x)) \text{ } is \text{ } defined \\ undefined & otherwise \end{cases} \tag{6}$$

Then, PI_N is a PFI on L, where N is a negation.

Example 9. *Let L be a bounded lattice, PI is a PFI on L. Define the operation PI_N^m as follows (for any $x, y \in L$):*

$$PI_N^m(x,y) := \begin{cases} min\{PI(x,y) \vee N(x), PI_N(x,y) \vee y\} & if \text{ } PI(x,y) \text{ } and \text{ } PI_N(x,y) \text{ } are \text{ } defined \\ undefined & otherwise \end{cases} \tag{7}$$

Then, PI_N^m is a PFI on L, where N is a negation.

Example 10. *Let $L = [0,1]$, PI_1 and PI_2 are two PFIs on L. Define the operation $PI_{PI_1-PI_2}$ as follows (for any $x, y, a \in L$):*

$$PI_{PI_1-PI_2}(x,y) := \begin{cases} 1 & if \text{ } x = 0 \\ a \cdot PI_1(x, \frac{y}{a}) & if \text{ } PI_1 \text{ } is \text{ } defined \text{ } and \text{ } x > 0 \text{ }, \text{ } y \leq a \\ a + (1-a) \cdot PI_2(x, \frac{y-a}{1-a}) & if \text{ } PI_2 \text{ } is \text{ } defined \text{ } and \text{ } x > 0 \text{ }, \text{ } x > a \\ undefined & otherwise \end{cases} \tag{8}$$

Then, $PI_{PI_1-PI_2}$ is a PFI on L.

Theorem 4. *Let L be a bounded lattice, $\odot$ be a pt-norm on L and $\rightarrow_\odot$ be a PRI induced by $\odot$. Then, $\rightarrow_\odot$ is the PFI.*

Proof. (PI1) If $a \rightarrow_\odot c$ and $b \rightarrow_\odot c$ are defined, then $a \rightarrow_\odot c = sup\{x_1 \in L \mid a \odot x_1 \text{ } is \text{ } defined \text{ } and \text{ } a \odot x_1 \leq c\}$, $b \rightarrow_\odot c = sup\{x_2 \in L \mid b \odot x_2 \text{ } is \text{ } defined \text{ } and \text{ } b \odot x_2 \leq c\}$, i.e., $\exists x_2$, s.t., $b \odot x_2$ is defined and $b \odot x_2 \leq c$, hence $b \leq x_2 \rightarrow_\odot c$. In addition, when $a \leq b$, we have $a \leq x_2 \rightarrow_\odot c$, so $a \odot x_2$ is defined and $a \odot x_2 \leq c$, then $x_2 \in \{x_1 \in L \mid a \odot x_1 \text{ } is \text{ } defined \text{ } and \text{ } a \odot x_1 \leq c\}$, and $\{x_2 \in L \mid b \odot x_2 \text{ } is \text{ } defined \text{ } and \text{ } b \odot x_2 \leq c\} \subseteq \{x_1 \in L \mid a \odot x_1 \text{ } is \text{ } defined \text{ } and \text{ } a \odot x_1 \leq c\}$, hence $sup\{x_2 \in L \mid b \odot x_2 \text{ } is \text{ } defined \text{ } and \text{ } b \odot x_2 \leq c\} \subseteq sup\{x_1 \in L \mid a \odot x_1 \text{ } is \text{ } defined \text{ } and \text{ } a \odot x_1 \leq c\}$, i.e., $b \rightarrow_\odot c \leq a \rightarrow_\odot c$.

(PI2) Similar to (PI1), we can obtain $a \rightarrow_\odot b \leq a \rightarrow_\odot c$.

(PI3) $0 \rightarrow_\odot 0 = sup\{a \in L \mid 0 \odot a \text{ } is \text{ } defined \text{ } and \text{ } 0 \odot a \leq 0\} = sup\{a \in L \mid 0 \odot a \text{ } is \text{ } defined \text{ } and \text{ } 0 \odot a = 0\} = 1$, i.e., $PI(0,0) = 1$;

$1 \rightarrow_\odot 1 = sup\{a \in L \mid 1 \odot a \text{ } is \text{ } defined \text{ } and \text{ } 1 \odot a \leq 1\} = 1$, i.e., $PI(1,1) = 1$;

$1 \rightarrow_\odot 0 = sup\{a \in L \mid 1 \odot a \text{ } is \text{ } defined \text{ } and \text{ } 1 \odot a \leq 0\} = sup\{a \in L \mid 1 \odot a \text{ } is \text{ } defined \text{ } and \text{ } 1 \odot a = 0\} = 0$, i.e., $PI(1,0) = 0$. □

Definition 8. *A pair $(\otimes, \rightarrow)$ on a poset $(P, \leq)$ is called a partial adjoint pair (PAP) where $\otimes$ and $\rightarrow$ are two partial operations, if for any $x, y, z \in P$, the following conditions are satisfied:*

(PA1) The operation $\otimes$ is isotone, i.e., if $x \leq y$, $x \otimes z$ and $y \otimes z$ are defined, then $x \otimes z \leq y \otimes z$; if $x \leq y$, $z \otimes x$ and $z \otimes y$ are defined, then $z \otimes x \leq z \otimes y$.

(PA2) The operation $\rightarrow$ is antitone in the first variable, i.e., if $x \leq y$, $x \rightarrow z$ and $y \rightarrow z$ are defined, then $y \rightarrow z \leq x \rightarrow z$; $\rightarrow$ is isotone in the second variable, i.e., if $x \leq y$, $z \rightarrow x$ and $z \rightarrow y$ are defined, then $z \rightarrow x \leq z \rightarrow y$.

(PA3) If $x \otimes y$ and $x \rightarrow z$ are defined, then $x \otimes y \leq z$ iff $y \leq x \rightarrow z$.

Definition 9. *A partial algebra* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is called a partial residuated monoid (PRM) where* $(L; \leq, 0, 1)$ *is a bounded partial ordered set,* $\otimes$ *and* $\rightarrow$ *are two partial operations, if for any* $x, y, z \in L$*, the following conditions are satisfied:*

(M1) If $x \otimes y$ *is defined, then* $y \otimes x$ *is defined and* $x \otimes y = y \otimes x$*;*
(M2) If $y \otimes z$*,* $x \otimes (y \otimes z)$ *are defined, then* $x \otimes y$*,* $(x \otimes y) \otimes z$ *are defined and* $x \otimes (y \otimes z) = (x \otimes y) \otimes z$*;*
(M3) $x \otimes 1$ *is defined and* $x \otimes 1 = x$*;*
(M4) $(\otimes, \rightarrow)$ *is a PAP on L.*

If $(L; \leq, 0, 1)$ *is a bounded lattice, then* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is called a partial residuated lattice (PRL).*

Example 11. *Assume that* $L = \{0, l, m, 1\}$*. The Hasse diagram of* $(L; \leq)$ *is shown in Figure 4, and the operations* $\otimes$ *and* $\rightarrow$ *are defined by Tables 7 and 8. Then, L is a PRL.*

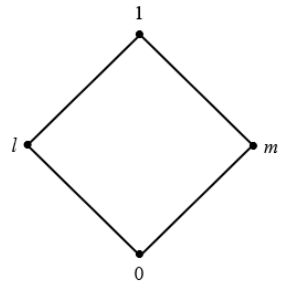

Figure 4. Lattice order relation on L.

Table 7. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	**1**
0			0	0
l		l		l
m	0		m	m
1	0	l	m	1

Table 8. The partial operation $\rightarrow$.

$\rightarrow$	**0**	***l***	***m***	**1**
0	1			
l	m	1		
m	0			
1	0	l		

Example 12. *Assume that* $L = \{0, l, m, n, 1\}$*. The Hasse diagram of* $(L; \leq)$ *is shown in Figure 1, and the operations* $\otimes$ *and* $\rightarrow$ *are defined by Tables 9 and 10. Then, L is a PRL.*

Table 9. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	***n***	**1**
0					0
l				0	l
m			m		m
n		0		n	n
1	0	l	m	n	1

Table 10. The partial operation $\rightarrow$.

$\rightarrow$	$\mathbf{0}$	$\boldsymbol{l}$	$\boldsymbol{m}$	$\boldsymbol{n}$	$\mathbf{1}$
0	1	1	1	1	1
l	n	1	1		1
m	n		1		1
n	m	m		1	1
1	0	l	m	n	1

Theorem 5. *Let $(L;\leq,\otimes,\rightarrow,0,1)$ be a PRL. Then, for any $x,y\in L$:*

(1) $x\rightarrow x=1$;
(2) $x\rightarrow 1=1$;
(3) $1\rightarrow x=x$;
(4) *If $x\rightarrow y$ is defined, then $x\rightarrow y=1$ iff $x\leq y$.*

Proof. (1) We know $x\otimes 1\leq x$, by (PA3), and we obtain $1\leq x\rightarrow x$, then $x\rightarrow x=1$.

(2) We know $x\otimes 1\leq 1$, then $1\leq x\rightarrow 1$, further $x\rightarrow 1=1$.

(3) Since $1\otimes x\leq x$, $x\leq 1\rightarrow x$. In addition, $1\rightarrow x\leq 1\rightarrow x$, then $(1\rightarrow x)\otimes 1\leq x$, so $1\rightarrow x=x$.

(4) $(\Rightarrow)$ For all $x,y\in L$, $1\leq x\rightarrow y$, so $x\otimes 1\leq y$, hence, $x\leq y$.

$(\Leftarrow)$ For all $x,y\in L$, $x\otimes 1\leq y$, so $1\leq x\rightarrow y$, hence $x\rightarrow y=1$. □

Theorem 6. *Let L be a bounded lattice, $\odot$ be a partial t-norm on L and $\rightarrow_{\odot}$ be a PRI derived from $\odot$. Then, $(L;\leq,\odot,\rightarrow_{\odot},0,1)$ is a PRL.*

Proof. By Definitions 3 and 4, we can clearly know that (PA1), (PA3), (M1), (M2) and (M3) are true; next, we prove (P2).

For any $x,y,z\in L$, suppose $x\leq y$, if $a\in L$, $z\odot a\leq x$, then $z\odot a\leq x\leq y$. That is, $\{a\in L\mid z\odot a\ is\ defined\ and\ z\odot a\leq x\}\subseteq\{b\in L\mid z\odot b\ is\ defined\ and\ z\odot b\leq y\}$, hence $sup\{a\in L\mid z\odot a\ is\ defined\ and\ z\odot a\leq x\}\leq sup\{b\in L\mid z\odot b\ is\ defined\ and\ z\odot b\leq y\}$. Thus, $z\rightarrow_{\odot} x\leq z\rightarrow_{\odot} y$. Seemingly, we can obtain $y\rightarrow_{\odot} z\leq x\rightarrow_{\odot} z$. □

Theorem 7. *Let $(E;\leq,+,',0,1)$ be an LEA. Define two binary operations $\odot$ and $\rightarrow$ as follows (for any $x,y\in E$):*

$$x\odot y:=(x'+y')'\ iff\ x'\leq y$$

$$x\rightarrow y:=x'+y\ iff\ y\leq x$$

Then, $(E;\leq,\odot,\rightarrow,0,1)$ is a PRL.

Proof. It follows from Proposition 1 that $\odot$ is a partial t-norm, then (M1), (M2) and (M3) hold, we only need to prove (M4). It is obvious that (PA1) holds, next, we will prove (PA2) and (PA3).

(PA2) On the one hand, if $x\leq y$, then $y'\leq x'$. In addition, $x\rightarrow z=x'+z$, $y\rightarrow z=y'+z$. Hence, $y'+z\leq x'+z$, $y\rightarrow z\leq x\rightarrow z$. On the other hand, we can obtain similar results: $y\rightarrow z\leq x\rightarrow z$.

(PA3) First of all, we know that, if $x\odot y\leq z$, then $(x'+y')'\leq z$; hence, $z'\leq x'+y'$. In other words, there exists $u\in E$, $u+z'=x'+y'$, so $(u+z')'\leq z$. From the properties of lattice effect algebra, $y'=(x'+(u+z')')'\Leftrightarrow y=x'+(u+z')'$, so $x'+(u+z')'\leq x'+z$. Thus, $y\leq x'+z\Leftrightarrow y\leq x\rightarrow z$. In addition, then, if $y\leq x\rightarrow z$, then $y\leq x'+z$. In other words, there exists $v\in E$, $y+v=x'+z$, so $(y+v)'\leq y'$. From the properties of lattice effect algebra, $z=(x'+(y+v)')'\Leftrightarrow z'=x'+(y+v)'$, so $x'+(y+v)'\leq x'+y'$. Thus, $z'\leq x'+y'\Leftrightarrow(x'+y')'\leq z\Leftrightarrow x\odot y\leq z$.

Hence, $(E;\leq,+,',0,1)$ is a PRL. □

Definition 10. *A pair* $(\otimes, \rightarrow)$ *on a poset* $(P, \leq)$ *is called a special partial adjoint pair (sPAP) where* $\otimes$ *and* $\rightarrow$ *are two partial operations, if, for any* $x, y, z \in P$*, the following conditions are satisfied:*

(sA1) The operation $\otimes$ *is isotone, i.e., if* $x \leq y$*,* $x \otimes z$ *and* $y \otimes z$ *are defined, then* $x \otimes z \leq y \otimes z$*; if* $x \leq y$*,* $z \otimes x$ *and* $z \otimes y$ *are defined, then* $z \otimes x \leq z \otimes y$*.*
(sA2) The operation $\rightarrow$ *is antitone in the first variable, i.e., if* $x \leq y$ *and* $x \rightarrow z$ *is defined, then* $y \rightarrow z$ *is defined and* $y \rightarrow z \leq x \rightarrow z$*;* $\rightarrow$ *is isotone in the second variable, i.e., if* $x \leq y$ *and* $z \rightarrow y$ *is defined, then* $z \rightarrow x$ *is defined and* $z \rightarrow x \leq z \rightarrow y$*.*
(sA3) $x \otimes y$ *is defined and* $x \otimes y \leq z$ *iff* $x \rightarrow z$ *is defined and* $y \leq x \rightarrow z$*.*

Definition 11. *A partial algebra* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is called a special partial residuated lattice (sPRL) where* $(L; \leq, 0, 1)$ *is a bounded lattice,* $\otimes$ *and* $\rightarrow$ *are two partial operations, if, for any* $x, y, z \in L$*, the following conditions are satisfied:*

(sP1) If $x \otimes y$ *is defined, then* $y \otimes x$ *is defined and* $x \otimes y = y \otimes x$*;*
(sP2) If $y \otimes z$*,* $x \otimes (y \otimes z)$ *are defined, then* $x \otimes y$*,* $(x \otimes y) \otimes z$ *are defined and* $x \otimes (y \otimes z) = (x \otimes y) \otimes z$*;*
(sP3) $x \otimes 1$ *is defined and* $x \otimes 1 = x$*;*
(sP4) $(\otimes, \rightarrow)$ *is an sPAP on L.*

Theorem 8. *Let* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *be an sPRL. Then,* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is a residuated lattice.*

Proof. (1) For all $x \in L$, $x \otimes 1 \leq x$, so $1 \leq x \rightarrow x$; furthermore, $x \rightarrow x = 1$.

(2) For all $x \in L$, we have $x \leq 1 = 0 \rightarrow 0$, so $x \otimes 0$ is defined and $x \otimes 0 \leq 0$, so $x \otimes 0 = 0$.

(3) By (2), we know $x \otimes 0 = 0$, so $x \otimes 0 \leq y$, then $x \rightarrow y$ is defined and $0 \leq x \rightarrow y$.

(4) By (1), we know $x \leq 1 = y \rightarrow y$, so $x \otimes y$ is defined and $x \otimes y \leq y$.

To sum up, $\otimes$ and $\rightarrow$ are full operations, then $(L; \leq, \otimes, \rightarrow, 0, 1)$ is a residuated lattice. □

Definition 12. *A PRL* $(L; \leq, \otimes, \rightarrow, 0, 1)$ *is called a well partial residuated lattice (wPRL), if for any* $x, y \in L$*:*

(W) If $x \rightarrow y$ *is defined, then* $x \otimes (x \rightarrow y)$ *is defined.*

Example 13. *Assume that* $L = \{0, l, m, 1\}$*. The Hasse diagram of* $(L; \leq)$ *is shown in Figure 4, and the operations* $\otimes$ *and* $\rightarrow$ *are defined by Tables 11 and 12. Then, L is a wPRL.*

Table 11. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	**1**
0			0	0
l			0	l
m		0		m
1	0	l	m	1

Table 12. The partial operation $\rightarrow$.

$\rightarrow$	**0**	***l***	***m***	**1**
0	1			1
l		1		1
m		l	1	1
1	0	l	m	1

Example 14. *Assume that* $L = \{0, l, m, n, 1\}$*. The Hasse diagram of* $(L; \leq)$ *is shown in Figure 1, and the operations* $\otimes$ *and* $\rightarrow$ *are defined by Tables 13 and 14. Then, L is a wPRL.*

Table 13. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	***n***	**1**
0					0
l				0	l
m			m	0	m
n		0	0	n	n
1	0	l	m	n	1

Table 14. The partial operation $\to$.

$\to$	**0**	***l***	***m***	***n***	**1**
0	1	1	1	1	1
l	n	1	1		1
m	n		1		1
n				1	1
1	0	l	m	n	1

Example 15. *Assume that* $L = \{0, l, m, n, p, 1\}$*. The Hasse diagram of* $(L; \leq)$ *is shown in Figure 5, and the operations* $\otimes$ *and* $\to$ *are defined by Tables 15 and 16. Then, L is a wPRL.*

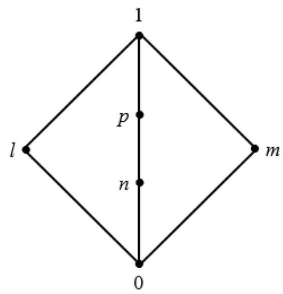

Figure 5. Lattice order relation on L.

Table 15. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	***n***	***p***	**1**
0	0			0	0	0
l						l
m						m
n	0			0	0	n
p	0			0	p	p
1	0	l	m	n	p	1

Table 16. The partial operation $\to$.

$\to$	**0**	***l***	***m***	***n***	***p***	**1**
0	1	1	1	1	1	1
l		1				1
m			1			1
n	p			1	1	1
p	n			n	1	1
1	0	l	m	n	p	1

Theorem 9. *Let* $(L; \leq, \otimes, \to, 0, 1)$ *be a wPRL. Then, for any* $x, y \in L$*:*

(1) *If* $x \otimes y$ *is defined, then* $x \otimes y \leq x \wedge y$*;*

(2) *If* $x \to y$ *is defined, then* $x \otimes (x \to y) \leq y$ *and* $x \leq (x \to y) \to y$*.*

Proof. (1) Since $x \leq 1 = y \to y$, then $x \otimes y \leq y$; obviously, $x \otimes y \leq x$. Hence, $x \otimes y \leq x \wedge y$.

(2) Based on assumptions and by Definition 12 (W), $x \otimes (x \to y)$ is defined, since $x \to y \leq x \to y$, by Definition 8 (PA3), $x \otimes (x \to y) \leq y$. In addition, applying Definition 9 (M1) and Definition 12 (W), $(x \to y) \to y$ is defined, applying Definition 8 (PA3), we obtain $x \leq (x \to y) \to y$. □

5. Partial t-Conorms and Partial Co-Residuated Lattices

In [19], Zhou and Li further investigate the relationship between residuated structures and some quantum structures from the perspective of partial algebra, and introduce the concept of partial residuated lattice. In order to avoid ambiguity, we call it a $\mathcal{ZL}$-partial residuated lattice ($\mathcal{ZL}$-PRL). In this section, we introduce partial co-residuated lattices and reveal the relationship between them and $\mathcal{ZL}$-PRL.

Definition 13 ([26]). *Let S be a bounded lattice. A binary operation $\oplus$ on S is called a t-conorm, if for any $x, y, z, h, k \in S$:*

(i) $0 \oplus x = x$;

(ii) $x \oplus y = y \oplus x$;

(iii) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(iv) *If $x \leq y$ and $h \leq k$, then $x \oplus h \leq y \oplus k$.*

Definition 14 ([27]). *A pair $(\oplus, \ominus)$ on a poset $(P, \leq)$ is called a co-adjoint pair where $\oplus$ and $\ominus$ are two binary operations, if, for any $x, y, z \in P$:*

(cA1) The operation $\oplus$ is isotone, i.e., if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $z \oplus x \leq z \oplus y$.

(cA2) The operation $\ominus$ is isotone in the first argument, i.e., if $x \leq y$, then $x \ominus z \leq y \ominus z$; $\ominus$ is antitone in the second argument, i.e., if $x \leq y$, then $z \ominus y \leq z \ominus x$.

(cA3) $z \leq x \oplus y$ iff $z \ominus y \leq x$.

Definition 15 ([27]). *A structure $(S; \leq, \oplus, \ominus, 0, 1)$ is called a co-residuated lattice where $\oplus$ and $\ominus$ are two binary operations, if, for any $x, y, z \in S$:*

(cR1) $(S; \oplus, 0)$ is a commutative semigroup;

(cR2) For all $x \in S$, $x \oplus 0 = x$;

(cR3) $(\oplus, \ominus)$ is a co-adjoint pair on S.

Definition 16. *Let S be a bounded lattice. A partial operation $\circledast$ on S is called a partial t-conorm, if, for any $x, y, h, k \in S$:*

(i) $0 \circledast x = x$;

(ii) *If $x \circledast y$ is defined, then $y \circledast x$ is defined and $x \circledast y = y \circledast x$;*

(iii) *If $y \circledast z$ and $x \circledast (y \circledast z)$ are defined, then $x \circledast y$ and $(x \circledast y) \circledast z$ are defined and $x \circledast (y \circledast z) = (x \circledast y) \circledast z$;*

(iv) *If $x \leq y$, $h \leq k$, $x \circledast h$ and $y \circledast k$ are defined, then $x \circledast h \leq y \circledast k$.*

Example 16. *Define the operation $\circledast$ as follows:*

$$a \circledast b := \begin{cases} undefined & if\ a, b \in [0.5, 1] \\ max\{a, b\} & others \end{cases} \tag{9}$$

Then, the operation $\circledast$ is a partial t-conorm $(a, b \in [0, 1])$.

Example 17. *Define the operation $\circledast$ as follows:*

$$a \circledast b := \begin{cases} max\{a, b\} & if\ a, b \in [0, 0.5] \\ undefined & others \end{cases} \tag{10}$$

Then, the operation $\circledast$ is a partial t-conorm $(a, b \in [0, 1])$.

Example 18. *Define the operation* $\circledast$ *as follows:*

$$a \circledast b := \begin{cases} a \vee b & if\ a + b \leq \alpha\ or\ a = 0\ or\ b = 0 \\ undefined & others \end{cases} \tag{11}$$

Then, the operation $\circledast$ *is a partial t-conorm* $(a, b, \alpha \in [0,1])$.

Example 19. *Assume that* $S = \{0, l, m, n, 1\}$. *The Hasse diagram of* $(S; \leq)$ *is shown in Figure 1, and the operation* $\circledast$ *is defined by Table 17. Then,* $\circledast$ *is a partial t-conorm.*

Table 17. The partial operation $\circledast$.

$\circledast$	**0**	***l***	***m***	***n***	**1**
0	0	l	m	n	1
l	l	m	n		
m	m	m	m		
n	n			n	
1	1				

Definition 17. *Let S be a bounded lattice and* $\circledast$ *be a partial t-conorm on S. A partial operation* $\rightsquigarrow_{\circledast}$ *induced by* $\circledast$ *is called a partial residuated co-implication such that, for any* $a, b \in S$:

$$a \rightsquigarrow_{\circledast} b := \begin{cases} \inf\{x \in S \mid a \circledast x\ \ is\ \ defined\ \ and\ \ a \circledast x \geq b\} & if\ I \neq \emptyset\ \ and\ \ \inf I\ \ exists \\ undefined & otherwise \end{cases} \tag{12}$$

where $I = \{x \in S \mid a \circledast x$ *is defined and* $a \circledast x \geq b\}$.

Definition 18. *A pair* $(\circledast, \rightsquigarrow)$ *on a poset* $(P, \leq)$ *is called a partial co-adjoint pair (cPAP), where* $\circledast$ *and* $\rightsquigarrow$ *are two partial operations, if for any* $x, y, z \in P$, *the following conditions are satisfied:*

(cPA1) The operation $\circledast$ *is isotone, i.e., if* $x \leq y$, $x \circledast z$ *and* $y \circledast z$ *are defined, then* $x \circledast z \leq y \circledast z$; *if* $x \leq y$, $z \circledast x$ *and* $z \circledast y$ *are defined, then* $z \circledast x \leq z \circledast y$.
(cPA2) The operation $\rightsquigarrow$ *is isotone in the first argument, i.e., if* $x \leq y$, $x \rightsquigarrow z$ *and* $y \rightsquigarrow z$ *are defined, then* $x \rightsquigarrow z \leq y \rightsquigarrow z$; $\rightsquigarrow$ *is antitone in the second variable, i.e., if* $x \leq y$, $z \rightsquigarrow x$ *and* $z \rightsquigarrow y$ *are defined, then* $z \rightsquigarrow y \leq z \rightsquigarrow x$.
(cPA3) If $x \circledast y$ *and* $z \rightsquigarrow y$ *are defined, then* $z \leq x \circledast y$ *iff* $z \rightsquigarrow y \leq x$.

Definition 19. *A structure* $(S; \leq, \circledast, \rightsquigarrow, 0, 1)$ *is called a partial co-residuated lattice (PcRL) where* $(S; \leq, 0, 1)$ *is a bounded lattice,* $\circledast$ *and* $\rightsquigarrow$ *are two partial operations, if for any* $x, y, z \in S$, *the following conditions are satisfied:*

(cPR1) If $x \circledast y$ *is defined, then* $y \circledast x$ *is defined and* $x \circledast y = y \circledast x$.
(cPR2) If $y \circledast z$, $x \circledast (y \circledast z)$ *are defined, then* $x \circledast y$, $(x \circledast y) \circledast z$ *are defined and* $x \circledast (y \circledast z) = (x \circledast y) \circledast z$.
(cPR3) $x \circledast 0$ *is defined and* $x \circledast 0 = x$.
(cPR4) $(\circledast, \rightsquigarrow)$ *is a cPAP on S.*

We can know that the partial co-residuated lattice and the partial residuated lattice are dual.

Example 20. *Assume that* $S = \{0, l, m, n, 1\}$. *The Hasse diagram of* $(S; \leq)$ *is shown in Figure 1, and the operations* $\circledast$ *and* $\rightsquigarrow$ *are defined by Tables 18 and 19. Then, L is a PcRL.*

Table 18. The partial operation $\circledast$.

$\circledast$	**0**	***l***	***m***	***n***	**1**
0	0	l	m	n	1
l	l				
m	m				
n	n				
1	1				

Table 19. The partial operation $\rightsquigarrow$.

$\rightsquigarrow$	**0**	***l***	***m***	***n***	**1**
0	0	0	0	0	0
l	l				
m			m		m
n		m	m	n	0
1					1

Theorem 10. *Let $(S; \leq, \circledast, \rightsquigarrow, 0, 1)$ be a PcRL. Then, for any $x, y \in S$:*

(1) If $x \rightsquigarrow 0$ is defined, then $x \rightsquigarrow 0 = x$.
(2) If $x \rightsquigarrow y$ is defined, then $x \rightsquigarrow y = 0$ iff $x \leq y$.
(3) If $x \circledast y$ and $(x \circledast y) \rightsquigarrow y$ are defined, then $(x \circledast y) \rightsquigarrow y \leq x$.
(4) If $x \rightsquigarrow y$ and $(x \rightsquigarrow y) \circledast y$ are defined, then $x \leq (x \rightsquigarrow y) \circledast y$.

Proof. (1) If $x \rightsquigarrow 0$ is defined, and we have $x \leq x \circledast 0$, then $x \rightsquigarrow 0 \leq x$. In addition, for any $a \in S$, $x \rightsquigarrow 0 \leq a$, $x \leq a \circledast 0 = a$. Let $a = x \rightsquigarrow 0$. Thus, $x \rightsquigarrow 0 = x$.

(2) $(\Rightarrow)$ If $x \rightsquigarrow y$ is defined, and we have $x \rightsquigarrow y \leq x \rightsquigarrow y$, $0 \leq x \rightsquigarrow y$, so $x \circledast 0 \leq y$, hence $x \leq y$.

$(\Leftarrow)$ We have $x \leq y \circledast 0$, then $x \rightsquigarrow y \leq 0$, so, $x \rightsquigarrow y = 0$.

(3) We know $x \circledast y \leq x \circledast y$, applying (cPA3), $(x \circledast y) \rightsquigarrow y \leq x$.

(4) We know $x \rightsquigarrow y \leq x \rightsquigarrow y$, applying (cPA3), $x \leq (x \rightsquigarrow y) \circledast y$. □

Definition 20 ([19]). *A structure $(S; \leq, \oplus, \ominus, 0, 1)$ is called a partial residuated lattice where $\oplus$ and $\ominus$ are two partial operations, if the following conditions are satisfied:*

(i) $(S; \leq, 0, 1)$ is a bounded lattice.
(ii) $(S, \oplus, 0)$ is a partial commutative monoid, its unit element is 0.
(iii) $(\oplus, \ominus)$ is a partial adjoint pair on S.

In order to distinguish, we call the partial residuated lattice in Definition 20 is $\mathcal{ZL}$-PRL.

Theorem 11. *Let $(S; \leq, \oplus, \ominus, 0, 1)$ be a $\mathcal{ZL}$-PRL. If we define the order relation $\preceq$ and the constants i, θ as follows:*

$$a \preceq b \triangleq b \leq a \ (\forall a, b \in L),$$
$$i \triangleq 0,$$
$$\theta \triangleq 1.$$

Then, $(S; \preceq, \oplus, \ominus, \theta, i)$ is a partial co-residuated lattice.

Proof. Obviously, if we want to prove that it is a partial co-residuated lattice, we only need to prove that (cPA3) and (cPR3) are true. Thus, we have:

(1) For all $x, y, z \in S$, if $x \oplus y$ and $z \ominus y$ are defined, then $z \triangleq x \oplus y$ iff $z \ominus y \triangleq x$.

(2) For all $x \in S$, $x \oplus i$ is defined and $x \oplus i = i$.

It is easy to obtain that $(S; \preceq, \oplus, \ominus, \theta, i)$ is a PcRL. □

Corollary 2. *Let $(S; \preceq, \oplus, \ominus, \theta, i)$ be a PcRL. Then, it is a co-residuated lattice.*

Proof. It can be proved by Theorems 8 and 11. □

6. Filters in Well Partial Residuated Lattices (wPRLs)

We propose filters and strong filters of wPRLs, construct the quotient structure $(L/\sim_F ;\leq, \otimes, \rightarrow, [0]_F, [1]_F)$, and proved that it is a partial residuated monoid.

Definition 21. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a well partial residuated lattice (wPRL). $F \subseteq L$ and $F \neq \emptyset$, which is called a filter, if,*

(F1) $1 \in F$.
(F2) *If $x \in F$, $y \in L$ and $x \leq y$, then $y \in F$.*
(F3) *If $x, y \in F$ and $x \otimes y$ is defined, then $x \otimes y \in F$.*

If $F \neq L$, then F is called the proper filter.

Example 21. *Let $L = \{0, l, m, 1\}$ be a wPRL in Example 13. Then, the proper filters are: $\{1\}$, $\{l, 1\}$ and $\{m, 1\}$.*

Example 22. *Let $L = \{0, l, m, n, 1\}$ be a wPRL in Example 14. Then, the proper filters are: $\{1\}$, $\{m, 1\}$, $\{n, 1\}$ and $\{l, m, 1\}$.*

Example 23. *Let $L = \{0, l, m, n, p, 1\}$ be a wPRL in Example 15. Then, the proper filters are: $\{1\}$, $\{l, 1\}$, $\{m, 1\}$, $\{p, 1\}$, $\{l, m, 1\}$, $\{l, p, 1\}$ and $\{m, p, 1\}$.*

Example 24. *Define two partial operations $\otimes$ and $\rightarrow$ as follows:*

$$a \otimes b := \begin{cases} undefined & if\ a, b \in [0, 0.5] \\ min\{a, b\} & others \end{cases} \tag{13}$$

$$a \rightarrow b := \begin{cases} undefined & if\ a, b \in [0, 0.5]\ and\ a > b \\ 1 & if\ a \leq b \\ b & else \end{cases} \tag{14}$$

Then, $(L; \leq, \otimes, \rightarrow, 0, 1)$ is a PRL $(a, b \in [0, 1])$, the proper filters are: $F1 = \{1\}$, $F2 = [x, 1]$, where $x \in [0, 1]$.

In the following contents, unless otherwise specified, it means that the contents are valid under the condition of definition.

Proposition 3. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL and F be a filter of L. Then,*

$$(x \in F, y \in L, x \rightarrow y \in F) \Rightarrow y \in F$$

Proof. By $x \in F, y \in L, x \rightarrow y \in F$, applying Definition 12 (W) and Definition 21 (F3), we obtain $x \otimes (x \rightarrow y) \in F$, and by Theorem 9 (2), we have $x \otimes (x \rightarrow y) \leq y$, so, applying Definition 21 (F2), $y \in F$. □

Definition 22. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL. A filter F of L is called a strong filter, if for any $x, y, z \in L$, the following conditions are satisfied:*

(s1) *If $z \rightarrow x$, $z \rightarrow y$ are defined and $x \rightarrow y \in F$, then $(z \rightarrow x) \rightarrow (z \rightarrow y) \in F$;*
(s2) *If $y \rightarrow z$, $x \rightarrow z$ are defined and $x \rightarrow y \in F$, then $(y \rightarrow z) \rightarrow (x \rightarrow z) \in F$;*
(s3) *If $(x \otimes y) \rightarrow z$ is defined and $x \rightarrow (y \rightarrow z) \in F$, then $(x \otimes y) \rightarrow z \in F$;*
(s4) *If $x \otimes z$, $y \otimes z$ are defined and $x \rightarrow y \in F$, then $(x \otimes z) \rightarrow (y \otimes z) \in F$.*

Example 25. *Assume that $L = \{0, l, m, 1\}$. The Hasse diagram of $(L; \leq)$ is shown in Figure 4, and the operations $\otimes$ and $\rightarrow$ are defined by Tables 20 and 21. Then, $(L; \leq, \otimes, \rightarrow, 0, 1)$ is a wPRL. The filters are: $\{1\}$, $\{l, 1\}$ and $\{m, 1\}$; they are not strong filters (Because if $F = \{1\}$, it does not meet (s2) and (s4). Thus, $\{l, 1\}$ and $\{m, 1\}$ are not strong filters either).*

Table 20. The partial operation $\otimes$.

$\otimes$	**0**	***l***	***m***	**1**
0			0	0
l		l	0	l
m		0		m
1	0	l	m	1

Table 21. The partial operation $\rightarrow$.

$\rightarrow$	**0**	***l***	***m***	**1**
0	1	1		1
l	m	1		1
m	l		1	1
1	0	l	m	1

Example 26. *Let $L = \{0, l, m, 1\}$ be a wPRL in Example 13. Then, the proper filters are: $\{1\}$, $\{l, 1\}$ and $\{m, 1\}$; they are not strong filters.*

Example 27. *Let $L = \{0, l, m, n, 1\}$ be a wPRL in Example 14. Then, the proper filters are: $\{1\}$, $\{m, 1\}$, $\{n, 1\}$ and $\{l, m, 1\}$, where $\{1\}$ and $\{n, 1\}$ are strong filters and $\{m, 1\}$ and $\{l, m, 1\}$ are not strong filters (they do not satisfy (s2) and (s4).)*

Example 28. *Let $L = \{0, l, m, n, p, 1\}$ be a wPRL in Example 15. Then, the proper filters are: $\{1\}$, $\{l, 1\}$, $\{m, 1\}$, $\{p, 1\}$, $\{l, m, 1\}$, $\{l, p, 1\}$ and $\{m, p, 1\}$; they are all strong filters.*

Proposition 4. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL and F be a strong filter of L. Then, for any $x, y, z \in L$:*

$$(x \otimes y) \rightarrow z \in F, \text{ implies } x \rightarrow (y \rightarrow z) \in F.$$

Proof. Applying Definition 12 (W), we obtain that $(x \otimes y) \otimes ((x \otimes y) \rightarrow z)$ is defined, so $((x \otimes y) \rightarrow z) \otimes (x \otimes y) \leq z$, we have $((x \otimes y) \rightarrow z) \otimes x \leq y \rightarrow z$, hence, $(x \otimes y) \rightarrow z \leq x \rightarrow (y \rightarrow z)$. Since $(x \otimes y) \rightarrow z \in F$, then $x \rightarrow (y \rightarrow z) \in F$. $\square$

Definition 23. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL, F be a filter of L. Define a binary relation $\sim_F$ (for any $x, y \in L$):*

$$x \sim_F y \text{ when and only when } x \rightarrow y \in F \text{ and } y \rightarrow x \in F.$$

Theorem 12. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL, F be a strong filter of L and $\sim_F$ be a binary relation. Then, $\sim_F$ is an equivalence relation on L.*

Proof. (1) For any $x \in L$, we know that $x \rightarrow x = 1 \in F$, so $x \sim_F x$.

(2) Applying Definition 23, $\sim_F$ is symmetric.

(3) Assume that $x \sim_F y$ and $y \sim_F z$. For one thing, $x \rightarrow y \in F$, when $x \rightarrow z$ is defined, by Definition 22 (s2), $(y \rightarrow z) \rightarrow (x \rightarrow z) \in F$, $y \rightarrow z \in F$, so $x \rightarrow z \in F$. For another, $z \rightarrow y$, $y \rightarrow x$ are defined, $z \rightarrow y \in F$, when $z \rightarrow x$ is defined, similarly, $(y \rightarrow x) \rightarrow (z \rightarrow x) \in F$, so, $z \rightarrow x \in F$. Hence, $x \sim_F z$. $\square$

Definition 24. *Let $(L; \leq, \otimes, \rightarrow, 0, 1)$ be a wPRL, $\sim$ be a binary relation of L, which is called a congruence relation, if, for any $x, y, x_1, y_1 \in L$,*

(C1) $\sim$ is an equivalence relation;
(C2) If $x \sim x_1$, $y \sim y_1$, $x \otimes y$ and $x_1 \otimes y_1$ are defined, then $(x \otimes y) \sim (x_1 \otimes y_1)$;
(C3) If $x \sim x_1$, $y \sim y_1$, $x \rightarrow y$ and $x_1 \rightarrow y_1$ are defined, then $(x \rightarrow y) \sim (x_1 \rightarrow y_1)$.

Theorem 13. *Let* $(L; \leq, \otimes, \to, 0, 1)$ *be a wPRL and* F *be a strong filter of* L*. Then,* $\sim_F$ *is the congruence relation.*

Proof. Applying Theorem 12, $\sim_F$ is an equivalence relation.

Suppose that $x \sim_F x_1, y \sim_F y_1$ and $x \to x_1 \in F$, then by Definition 22 (s4), $(x \otimes y) \to (x_1 \otimes y) \in F$. Similarly, $(x_1 \otimes y) \to (x \otimes y) \in F$ can be derived. Thus, $(x \otimes y) \sim_F (x_1 \otimes y)$. For the same reason, $(x_1 \otimes y) \sim_F (x_1 \otimes y_1)$. In conclusion, $(x \otimes y) \sim_F (x_1 \otimes y_1)$. This means that Definition 24 (C2) holds.

From $x \sim x, y \sim y$ and $y \to y_1 \in F$, applying Definition 22 (s1), we have $(x \to y) \to (x \to y_1) \in F$. Similarly, we can obtain $(x \to y_1) \to (x \to y) \in F$. Hence, $(x \to y) \sim_F (x \to y_1)$. Similraly, applying Definition 22 (s2), we can obtain $(x \to y_1) \sim_F (x_1 \to y_1)$. Thus, $(x \to y) \sim_F (x_1 \to y_1)$. This means Definition 24 (C3) holds. □

We noted that $[x]_F$ is the equivalent class of x, $L/\sim_F$ is the quotient set.

Theorem 14. *Let* $(L; \leq, \otimes, \to, 0, 1)$ *be a PRM,* F *be a strong filter and* $\sim_F$ *be a congruence relation. Define the following binary relation and binary operations on* $L/\sim_F$ *(for any* $x, y \in L$*):*

$$[x]_F \otimes [y]_F := \begin{cases} [x \otimes y]_F, & \forall h \in [x]_F, k \in [y]_F \text{ and } h \otimes k \text{ is defined} \\ [x]_F & \text{if } [y]_F = [1]_F \\ undefined, & \exists h \in [x]_F, k \in [y]_F \text{ and } h \otimes k \text{ is undefined} \end{cases} \tag{15}$$

$$[x]_F \to [y]_F := \begin{cases} [x \to y]_F, & \forall h \in [x]_F, k \in [y]_F \text{ and } h \to k \text{ is defined} \\ undefined, & \exists h \in [x]_F, k \in [y]_F \text{ and } h \to k \text{ is undefined} \end{cases} \tag{16}$$

$$If\, x \to y \text{ is defined, then } [x]_F \leq [y]_F \text{ when and only when } [x]_F \to [y]_F := [1]_F \tag{17}$$

Then, $(L/\sim_F; \leq, \otimes, \to, [0]_F, [1]_F)$ *is a PRM.*

Proof. By Definition 24, we know that the above definition of $\leq$ on $L/\sim_F$ is feasible.

Firstly, we prove that $\leq$ is a partial ordered relation.

(1) Reflexivity is clearly established;

(2) For any $h \in [x]_F, k \in [y]_F, h \to k, k \to h$ are defined. If $[x]_F \leq [y]_F$ and $[y]_F \leq [x]_F$, then $[x]_F \to [y]_F = [x \to y]_F = [1]_F$, so, $1 \to (x \to y) = x \to y \in F$, and we know, $[y]_F \to [x]_F = [y \to x]_F = [1]_F$, so, $1 \to (y \to x) = y \to x \in F$. Hence, $[x]_F = [y]_F$. Antisymmetry is established.

(3) For any $h \in [x]_F, k \in [y]_F, l \in [z]_F, h \to k, h \to l$ and $k \to l$ are defined. If $[x]_F \leq [y]_F$ and $[y]_F \leq [z]_F$, then $[x]_F \to [y]_F = [x \to y]_F = [1]_F$, $[y]_F \to [z]_F = [y \to z]_F = [1]_F$, from this and applying (2), we have $x \to y \in F$ and $y \to z \in F$. Using Definition 22 (s1), $(x \to y) \to (x \to z) \in F$, so, $x \to z \in F$. Hence, $(x \to z) \to 1 = 1 \in F$, and $1 \to (x \to z) = x \to z \in F$ that is, $[x]_F \to [z]_F = [x \to z]_F = [1]_F$, for this reason, $[x]_F \leq [z]_F$. Transitive is established.

Secondly, we prove that $L/\sim_F$ is bounded:

We suppose that, for all $D, I, x \in L$, and $D \geq x$, $I \leq x$, then $x \to D$, $I \to x$ are defined and $x \to D = 1$, $I \to x = 1$. Thus, for any $[D]_F, [I]_F, [x]_F \in L/\sim_F$, if $d \in [D]_F$, $i \in [I]_F$, $h \in [x]_F$, $h \to d$ and $i \to h$ are defined, then $[x]_F \to [D]_F = [x \to D]_F = [1]_F$, $[I]_F \to [x]_F = [I \to x]_F = [1]_F$. Hence, $[x]_F \leq [D]_F, [I]_F \leq [x]_F$.

Finally, we prove the following:

(M1) (1) If for any $h \in [x]_F$, $k \in [y]_F$, $h \otimes k$ is defined, then $k \otimes h$ also is, hence, $[x]_F \otimes [y]_F = [x \otimes y]_F = [y \otimes x]_F = [y]_F \otimes [x]_F$.

(2) If $[y]_F = [1]_F$, then $[x]_F \otimes [y]_F = [x]_F = [1 \otimes x]_F = [1]_F \otimes [x]_F = [y]_F \otimes [x]_F$.

(M2) (1) If for any $h \in [x]_F$, $k \in [y]_F$, $l \in [z]_F$ and $k \otimes l$, $h \otimes (k \otimes l)$ are defined, then $h \otimes k$, $(h \otimes k) \otimes l$ are defined, hence $[x]_F \otimes ([y]_F \otimes [z]_F) = [x]_F \otimes ([y \otimes z]_F) = [x \otimes (y \otimes z)]_F = [(x \otimes y) \otimes z]_F = ([x \otimes y]_F) \otimes [z]_F = ([x]_F \otimes [y]_F) \otimes [z]_F$.

(2) ① If $[z]_F = [1]_F$, then $[x]_F \otimes ([y]_F \otimes [z]_F) = [x]_F \otimes [y]_F = [x \otimes y]_F = [(x \otimes y) \otimes z]_F = ([x]_F \otimes [y]_F) \otimes [z]_F$.

② If $[x]_F$ or $[y]_F = [1]_F$, similar proof can be obtained.

③ If $[x]_F$ and $[y]_F \neq [1]_F$, then $[y]_F \otimes [z]_F \neq [1]_F$, i.e., when $[x]_F \otimes ([y]_F \otimes [z]_F)$ is defined, $([x]_F \otimes [y]_F) \otimes [z]_F$ must defined.Hence, $[x]_F \otimes ([y]_F \otimes [z]_F) = ([x]_F \otimes [y]_F) \otimes [z]_F$.

(M3) For all $[x]_F \in L/\sim_F$, $[x]_F \otimes [1]_F = [x]_F$.

(M4) Now, we prove that $(\otimes, \to)$ is a PAP on $L/\sim_F$.

(PA1) If, for any $h \in [x]_F$, $k \in [y]_F$, $h \to k$ is defined and $[x]_F \leq [y]_F$, then $[x]_F \to [y]_F = [x \to y]_F = [1]_F$.

(1) If, for any $h \in [x]_F$, $k \in [y]_F$, $l \in [z]_F$ and $h \otimes l$, $k \otimes l$ are defined, then $[x]_F \otimes [z]_F = [x \otimes z]_F$, $[y]_F \otimes [z]_F = [y \otimes z]_F$. In addition, we know $x \to y \in F$, applying Definition 22 (s4), $(x \otimes z) \to (y \otimes z)$ is defined and $(x \otimes z) \to (y \otimes z) \in F = [1]_F$, so $[(x \otimes z) \to (y \otimes z)]_F = [1]_F$, i.e., $[x]_F \otimes [z]_F \leq [y]_F \otimes [z]_F$.

(2) If $[z]_F = [1]_F$, then $[x]_F \otimes [z]_F = [x]_F \leq [y]_F = [y]_F \otimes [z]_F$.

(PA2) For any $h \in [x]_F$, $k \in [y]_F$, $l \in [z]_F$, $h \to k$, $h \to l$ and $k \to l$ are defined. If $[x]_F \leq [y]_F$, then $[x]_F \to [y]_F = [x \to y]_F = [1]_F$. On the one hand, $[x]_F \to [z]_F = [x \to z]_F$, $[y]_F \to [z]_F = [y \to z]_F$, and we know $x \to y \in F$, applying Definition 22 (s2), $(y \to z) \to (x \to z) \in F = [1]_F$, so $[(y \to z) \to (x \to z)]_F = [1]_F$, i.e., $[y]_F \to [z]_F \leq [x]_F \to [z]_F$.

On the other hand, applying Definition 22 (s1), $[z]_F \to [x]_F \leq [z]_F \to [y]_F$.

(PA3) $(\Rightarrow)$ ① If, for any $h \in [x]_F$, $k \in [y]_F$, $l \in [z]_F$, $h \otimes k$, $h \to l$, $(h \otimes k) \to l$ are defined, then $[x]_F \otimes [y]_F = [x \otimes y]_F$, $[x]_F \to [z]_F = [x \to z]_F$. Thus, we can obtain that $[x]_F \otimes [y]_F \leq [z]_F \Leftrightarrow [x \otimes y]_F \leq [z]_F \Leftrightarrow [(x \otimes y) \to z]_F = [1]_F$, i.e., $(x \otimes y) \to z \in F$, applying Proposition 4, $y \to (x \to z) \in F = [1]_F$; hence, $[y \to (x \to z)]_F = [1]_F \Leftrightarrow [y]_F \to [x \to z]_F = [1]_F \Leftrightarrow [y]_F \leq [x]_F \to [z]_F$.

② If $[y]_F = [1]_F$, then $[x]_F \otimes [y]_F = [x]_F$. If for any $h \in [x]_F$, $l \in [z]_F$, $h \to l$ is defined, then $[x]_F \to [z]_F = [x \to z]_F$, and $[x]_F \otimes [y]_F \leq [z]_F \Leftrightarrow [x]_F \leq [z]_F \Leftrightarrow [x \to z]_F = [1]_F$, it follows that $[1]_F \leq [x \to z]_F \Leftrightarrow [y]_F \leq [x]_F \to [z]_F$.

$(\Leftarrow)$ By the same token, vice versa.

In conclusion, $L/\sim_F$ is a PRM. $\square$

7. Conclusions

Overall, in the first place, we analyze the partial residuated implication induced by partial t-norms, give the concepts of partial residuated monoids and partial residuated lattices by defining partial adjoint pairs, further construct partial residuated lattices from lattice effect algebras, as well as reveal the relationship between partial residuated lattices and lattice effect algebras. In the second place, notions of partial t-conorm and partial co-residuated lattice are given, and their properties are studied, which proves that $\mathcal{ZL}$-partial residuated lattice is a co-residuated lattice. Finally, we build the filter theory of well partial residuated lattice, which can be viewed as a natural extension of the filter theories for lattice effect algebras and many fuzzy logic algebras. It is noteworthy that we propose and discuss some new concepts of filters, strong filters, and congruence relations on wPRLs. In addition, we establish the quotient algebras structure $(L/\sim_F; \leq, \otimes, \to, [0]_F, [1]_F)$ of partial residuated monoid.

In future studies, the relationships among partial residuated lattices, general fuzzy set theories, and algebraic systems with applications can be considered (see [28–33]).

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