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Fuzzy Differential Subordination for Meromorphic Function Associated with the Hadamard Product

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Abstract: This paper is related to fuzzy differential subordinations for meromorphic functions. Fuzzy differential subordination results are obtained using a new operator which is the combination Hadamard product and integral operator for meromorphic function.

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1. Introduction, Preliminaries and Definitions

Assume that $\mathbb{H}(\Lambda)$ is the class of functions analytic in $\Lambda = \{\chi : \chi \in \mathbb{C} \text{ and } |\chi| < 1\}$ and $\mathbb{H}[c, j]$ is the subclass of $\mathbb{H}(\Lambda)$ consisting of functions of the form

$$\mathbb{H}[c, j] = \left\{\mathcal{F} : \mathcal{F} \in \mathbb{H}(\Lambda) \text{ and } \mathcal{F}(\chi) = c + a_j\chi^j + a_{j+1}\chi^{j+1} + \cdots, \quad (\chi \in \Lambda\,).\right\} \tag{1}$$

and $\mathbb{H}[1, 1]$=$\mathbb{H}$. For $\Omega \subset \mathbb{C}$, we denote by $\mathbb{H}(\Omega)$ the class of meromorphic function in Ω. For $t \in \mathbb{N}$, denoted by Σ_t the class of meromorphic function given by

$$\Sigma_t := \left\{\mathcal{F} : \mathcal{F} \in \mathbb{H}(\Lambda) \text{ and } \mathcal{F}(\chi) = \frac{1}{\chi} + \sum_{j=t+1}^{\infty} a_j\chi^j \quad (\chi \in \Lambda^* = \Lambda \setminus \{0\};\ t \in \mathbb{N}).\right\} \tag{2}$$

where Λ^* is the punctured unit disc defined by

$$\Lambda^* := \{\chi : \chi \in \mathbb{C} \text{ and } \ 0 < |\chi| < 1\}.$$

In particular, we write $\Sigma := \Sigma_1$ has the following form

$$\mathcal{F}(\chi) = \frac{1}{\chi} + \sum_{j=2}^{\infty} a_j\chi^j, \tag{3}$$

which are analytic and univalent in the punctured unit disc Λ^*

The functions $\mathcal{F} \in \Sigma$ given by (3) and $\mathcal{G} \in \Sigma$ given by

$$\mathcal{G}(\chi) = \frac{1}{\chi} + \sum_{j=2}^{\infty} b_j\chi^j, \tag{4}$$

The Hadamard or convolution product of $\mathcal{F}$ and $\mathcal{G}$ is defined as follows

$$(\mathcal{F} * \mathcal{G})(\chi) := \frac{1}{\chi} + \sum_{j=2}^{\infty} a_j b_j \chi^j. \tag{5}$$

Let $\mathcal{S}^*_\Sigma$ and $\mathcal{C}_\Sigma$ be the subclasses of Σ, which are meromorphic starlike and meromorphic convex in Λ^*, respectively, and defined by

$$\mathcal{S}^*_\Sigma := \left\{\mathcal{F} : \mathcal{F} \in \Sigma \text{ and } -\mathfrak{Re}\left(\frac{\chi\mathcal{F}'(\chi)}{\mathcal{F}(\chi)}\right) > 0 \quad (\chi \in \Lambda^*)\right\},$$

and

$$\mathcal{C}_\Sigma := \left\{\mathcal{F} : \mathcal{F} \in \pm \text{ and } -\mathfrak{Re}\left(1 + \frac{\chi\mathcal{F}''(\chi)}{\mathcal{F}'(\chi)}\right) > 0 \quad (\chi \in \Lambda^*)\right\}.$$

Definition 1 ([1,2])**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be in $\mathbb{H}(\Lambda)$. The function $\mathcal{F}_1$ is subordinate to $\mathcal{F}_2$, or $\mathcal{F}_2$ is superordinate to $\mathcal{F}_1$, if there exists a function ϖ analytic in Λ with $\varpi(0) = 0$ and $|\varpi(\chi)| < 1$ $(\chi \in \Lambda)$, such that $\mathcal{F}_1(\chi) = \mathcal{F}_2(\varpi(\chi))$. Also, we write $\mathcal{F}_1(\chi) \prec \mathcal{F}_2(\chi)$. If $\mathcal{F}_2$ is univalent, then $\mathcal{F}_1 \prec \mathcal{F}_2$, if and only if $\mathcal{F}_1(0) = \mathcal{F}_2(0)$ and $\mathcal{F}_1(\Lambda) \subset \mathcal{F}_2(\Lambda)$.*

We introduce definitions and propositions for fuzzy differential subordination:

Definition 2 ([3])**.** *Let $\mathcal{Z}$ be a nonempty set, then $\Upsilon : \mathcal{Z} \to [0,1]$ is fuzzy subset and a pair $(\mathcal{Y}, \Upsilon_{\mathcal{Y}})$, such that $\Upsilon_{\mathcal{Y}} : \mathcal{Z} \to [0,1]$ and*

$$\mathcal{Y} = \left\{x \in \mathcal{Z} : 0 < \Upsilon_{\mathcal{K}(x)} \leq 1\right\} = \sup(\mathcal{K}, \Upsilon_{\mathcal{K}}) \tag{6}$$

is fuzzy set. A function $\Upsilon_{\mathcal{Y}}$ is a function of the fuzzy set $(\mathcal{Y}, \Upsilon_{\mathcal{Y}})$.

Definition 3 ([4])**.** *Assuming that $\Upsilon : \mathbb{C} \to \mathbb{R}_+$ is a function such that*

$$\Upsilon_{\mathbb{C}}(\chi) = |\Upsilon(\chi)| \qquad (\chi \in \mathbb{C}).$$

Denote by

$$\Upsilon_{\mathbb{C}}(\mathbb{C}) = \{\chi : \chi \in \mathbb{C} \textit{ and } 0 < |\Upsilon(\chi)| \leq 1\} := Supp\ (\mathbb{C}, \Upsilon_{\mathbb{C}}(\chi)).$$

Also, we call the following subset:

$$\Upsilon_{\mathbb{C}}(\mathbb{C}) = \{\chi : \chi \in \mathbb{C} \textit{ and } 0 < |\Upsilon(\chi)| \leq 1\} := \Lambda_{\Upsilon}(0,1),$$

the fuzzy unit disk.

Proposition 1 ([5])**.** *(i) If $(\mathcal{Y}, \Upsilon_{\mathcal{Y}}) = (\Lambda, \Upsilon_\Lambda)$, then we have $\mathcal{Y} = \Lambda$, where $\mathcal{Y} = \sup(\mathcal{Y}, \Upsilon_{\mathcal{Y}})$ and $\Lambda = \sup(\Lambda, \Upsilon_\Lambda)$.*

(ii) If $(\mathcal{Y}, \Upsilon_{\mathcal{Y}}) \subseteq (\Lambda, \Upsilon_\Lambda)$, then we have $\mathcal{Y} \subseteq \Lambda$, where $\mathcal{Y} = \sup(\mathcal{Y}, \Upsilon_{\mathcal{Y}})$ and $\Lambda = \sup(\Lambda, \Upsilon_\Lambda)$.

Let $\mathcal{F}, \mathcal{G} \in \mathbb{H}(\Lambda)$. We denote

$$\mathcal{F}(\Lambda) = \left\{\mathcal{F}(\chi) : 0 < \left|\Upsilon_{\mathcal{F}(\Lambda)}\mathcal{F}(\chi)\right| \leq 1,\ \chi \in \Lambda\right\} = \sup\left(\mathcal{F}(\Lambda), \Upsilon_{\mathcal{F}(\Lambda)}\right) \tag{7}$$

and

$$\mathcal{G}(\Lambda) = \left\{\mathcal{G}(\chi) : 0 < \left|\Upsilon_{\mathcal{G}(\Lambda)}\mathcal{G}(\chi)\right| \leq 1,\ \chi \in \Lambda\right\} = \sup\left(\mathcal{G}(\Lambda), \Upsilon_{\mathcal{G}(\Lambda)}\right). \tag{8}$$

Definition 4 ([5]). *Let $\chi_0 \in \Lambda$ and $\mathcal{F}, \mathcal{G} \in \mathbb{H}(\Lambda)$. The function $\mathcal{F}$ is fuzzy subordinate to $\mathcal{G}$, written as $\mathcal{F} \prec_{\Upsilon} \mathcal{G}$ or $\mathcal{F}(\chi) \prec_{\Upsilon} \mathcal{G}(\chi)$, when the followig conditions are satisfied:*

$$\begin{aligned} (i) \quad \mathcal{F}(\chi_0) &= \mathcal{G}(\chi_0) \\ (ii) \quad \left|\Upsilon_{\mathcal{F}(\Lambda)}\mathcal{F}(\chi)\right| &\leq \left|\Upsilon_{\mathcal{G}(\Lambda)}\mathcal{G}(\chi)\right|, \ \chi \in \Lambda. \end{aligned}$$

Proposition 2 ([5]). *Assuming that $\chi_0 \in \Lambda$ and $\mathcal{F}, \mathcal{G} \in \mathbb{H}(\Lambda)$. If $\mathcal{F}(\chi) \prec_{\Upsilon} \mathcal{G}(\chi)$, $\chi \in \Lambda$, then*

$$\begin{aligned} (i) \quad \mathcal{F}(\chi_0) &= \mathcal{G}(\chi_0) \\ (ii) \quad \mathcal{F}(\Lambda) &\subseteq \mathcal{G}(\Lambda) \text{ and } \left|\Upsilon_{\mathcal{F}(\Lambda)}\mathcal{F}(\chi)\right| \leq \left|\Upsilon_{\mathcal{G}(\Lambda)}\mathcal{G}(\chi)\right|, \ \chi \in \Lambda, \end{aligned}$$

where $\mathcal{F}(\Lambda)$ and $\mathcal{G}(\Lambda)$ are defined by (7) and (8), respectively.

Definition 5 ([6]). *For $\Phi : \mathbb{C}^3 \times \Lambda \to \mathbb{C}$ and $\mathbb{H}$ is an analytic function such that $\Phi(c, 0, 0, 0) = \mathbb{H}(0) = c$. Assuming that p is analytic in $*$ with $p(0) = c$ and satisfies the second order fuzzy differential subordination*

$$\left|\Upsilon_{\Psi(\mathbb{C}^3\times\Lambda)}\Psi\Big(\big(\varphi(\chi), \chi\varphi'(\chi), \chi^2\varphi''(\chi); \chi\big)\Big)\right| \leq \left|\Upsilon_{\mathbb{H}(\Lambda)}\mathbb{H}(\chi)\right|, \tag{9}$$

i.e.,

$$\Psi\Big(\big(\varphi(\chi), \chi\varphi'(\chi), \chi^2\varphi''(\chi); \chi\big)\Big) \prec_{\Upsilon} \mathbb{H}(\chi).$$

then φ is a fuzzy solution of the fuzzy differential subordination.
A function $\boldsymbol{q}$ is a fuzzy dominant for the fuzzy differential subordination if

$$\left|\Upsilon_{\varphi(\Lambda)}\varphi(\chi)\right| \leq \left|\Upsilon_{\boldsymbol{q}(\Lambda)}\boldsymbol{q}(\chi)\right|, \text{ i.e., } \varphi(\chi) \prec_{\Upsilon} \boldsymbol{q}(\chi), \ \zeta \in \Lambda.$$

for all φ satisfying (9). A fuzzy dominant $\widetilde{\boldsymbol{q}}$ satisfies that

$$\left|\Upsilon_{\widetilde{\boldsymbol{q}}(\Lambda)}\widetilde{\boldsymbol{q}}(\chi)\right| \leq \left|\Upsilon_{\boldsymbol{q}(\Lambda)}\boldsymbol{q}(\chi)\right|, \text{ i.e., } \widetilde{\boldsymbol{q}}(\chi) \prec_{\Upsilon} \boldsymbol{q}(\chi), \ \chi \in \Lambda.$$

for all fuzzy dominant $\boldsymbol{q}$ of (9) is the fuzzy best dominant of (9).

If $\mathcal{F}, \mathcal{G} \in \Sigma$ has the form (3) and (4), we define the integral operator $\mathcal{N}_m^{\alpha} : \Sigma \to \Sigma$, with $m > 0$, $\alpha \geq 0$, by

$$\mathcal{N}_m^0(\mathcal{F} * \mathcal{G})(\chi) := (\mathcal{F} * \mathcal{G})(\chi),$$

and

$$\mathcal{N}_m^{\alpha}(\mathcal{F} * \mathcal{G})(\chi) := \frac{m^{\alpha}}{\Gamma(\alpha)\chi^{m+1}} \int_0^{\chi} t^m \left(\log \frac{\chi}{t}\right)^{\alpha-1} (\mathcal{F} * \mathcal{G})(t)dt, \tag{10}$$

where all the powers are at the principal value.

It can be easily verified that

$$\mathcal{N}_m^{\alpha}(\mathcal{F} * \mathcal{G})(\chi) = \frac{1}{\chi} + \sum_{j=2}^{\infty} \left(\frac{m}{m+j+1}\right)^{\alpha} a_j b_j \chi^j, \ \chi \in \Lambda^*. \tag{11}$$

The extended operator $\mathcal{J}_m^{\alpha,\mu} : \Sigma \to \Sigma$ is defined by the following convolution formula

$$\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) * \mathcal{N}_m^{\alpha}(\mathcal{F} * \mathcal{G})(\chi) = \frac{1}{\chi(1-\chi)^{\mu+1}}, \ \chi \in \Lambda^*,$$

where the power is at the principal value, and we have

$$\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) = \frac{1}{\chi} + \sum_{j=2}^{\infty}\binom{\mu+j}{j}\left(\frac{m+j+1}{m}\right)^{\alpha} a_j b_j \chi^j,\ \chi \in \Lambda^*, \tag{12}$$

for $m > 0$, $\alpha \geq 0$, and $\mu \geq 0$. From (12), it is easy to verify that

$$\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)' = m\mathcal{J}_m^{\alpha+1,\mu}\mathcal{F}(\chi) - (m+1)\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi),\ \chi \in \Lambda^*. \tag{13}$$

To investigate main results, we need the following Lemmas:

Lemma 1 ([1])**.** *Assume that $\mathcal{E} \in \Sigma$ and*

$$\mathcal{K}(\chi) = \frac{1}{\chi}\int_0^{\chi}\mathcal{E}(t)dt \quad (\chi \in \Lambda^*).$$

If

$$-\mathfrak{Re}\left(1 + \frac{\chi\mathcal{E}'(\chi)}{\mathcal{E}'(\chi)}\right) > -\frac{1}{2}\ (\chi \in \Lambda^*),$$

then $\mathcal{K} \in \mathbb{C}$.

Lemma 2 ([7])**.** *Consider that the convex function $\mathcal{E}$ satisfies $\mathcal{E}(0) = c$, let $\lambda \in \mathbb{C}^*$ such that $\mathfrak{Re}(\lambda) \geq 0$. If $\mathbb{P} \in \mathbb{H}[c,j]$ with $\mathbb{P}(0) = c$ and $\Omega : (\mathbb{C}^2 \times \Lambda) \to \mathbb{C}$, $\Omega(\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi)) = \mathbb{P}(\chi) + \frac{1}{\lambda}\chi\mathbb{P}'(\chi)$ is holomorphic in Λ; then,*

$$\left|\Upsilon_{\Omega(\mathbb{C}^2\times\Lambda)}\left(\mathbb{P}(\chi) + \frac{1}{\lambda}\chi\mathbb{P}'(\chi)\right)\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda)}\mathcal{E}(\chi)\right| \Longrightarrow \mathbb{P}(\chi) + \frac{1}{\lambda}\chi\mathbb{P}'(\chi) \prec_{\Upsilon} \mathcal{E}(\chi) \quad (\chi \in \Lambda),$$

implies

$$\Upsilon_{\mathbb{P}(\Lambda)}\mathbb{P}(\chi) \leq \Upsilon_{\boldsymbol{q}(\Lambda)}\boldsymbol{q}(\chi) \leq \Upsilon_{\mathcal{E}(\Lambda)}\mathcal{E}(\chi).$$

i.e.,

$$\mathbb{P}(\chi) \prec_{\Upsilon} \boldsymbol{q}(\chi),$$

where

$$\boldsymbol{q}(\chi) = \frac{\lambda}{n\chi^{\lambda/n}}\int_0^{\chi} t^{\frac{\lambda}{n}-1}\mathcal{E}(t)dt,$$

is convex and best dominant.

Lemma 3 ([7])**.** *Consider $\boldsymbol{q}$ is convex function in Λ, let $\mathcal{E}(\chi) = \boldsymbol{q}(\chi) + n\vartheta\chi\boldsymbol{q}'(\chi)$, $\vartheta > 0$ and $j \in \mathbb{N}$. If $\mathbb{P} \in \mathbb{H}[\boldsymbol{q}(0), j]$ and $\Omega : (\mathbb{C}^2 \times \Lambda) \to \mathbb{C}$, $\Omega(\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi)) = \mathbb{P}(\chi) + \vartheta\chi\mathbb{P}'(\chi)$ in Λ, then*

$$\left|\Upsilon_{\mathbb{P}(\Lambda)}((\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi)))\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda)}\mathcal{E}(\chi)\right| \Longrightarrow \mathbb{P}(\chi) + \vartheta\chi\mathbb{P}'(\chi) \prec_{\Upsilon} \boldsymbol{q}(\chi),$$

then

$$\left|\Upsilon_{\mathbb{P}(\Lambda)}\mathbb{P}(\chi)\right| \leq \left|\Upsilon_{\boldsymbol{q}(\Lambda)}\boldsymbol{q}(\chi)\right|, \quad \chi \in \Lambda$$

implies that

$$\mathbb{P}(\chi) \prec_{\Upsilon} \boldsymbol{q}(\chi).$$

and $\boldsymbol{q}$ is the best fuzzy dominant.

Recently, El-Deeb et al. [8], Srivastava and El-Deeb [9], El-Deeb and Oros [10], Lupaş [4,7,11–13], Oros [5,6,14–17], El-Deeb and Lupas [18] and Wanas [19–21] obtained fuzzy differential subordination results.

In Section 2 below, we obtain several fuzzy differential subordinations for meromorphic functions that are associated with the operator $\mathcal{J}_m^{\alpha,\mu}$ by using the method of fuzzy differential subordination.

2. Main Results

Theorem 1. *Let the convex function ϕ in Λ^*, such that $\phi(0) = 1$.*

$$\mathcal{E}{=}\phi(\chi) + \chi\phi'(\chi) \qquad \chi \in \Lambda^*$$

For $\mathcal{F} \in \sum$ and satisfies the following fuzzy differetial subordination:

$$\left|\Upsilon_{\Omega(\mathbb{C}^2\times\Lambda^*)}\left[m\mathcal{J}_m^{\alpha+1,\mu}\mathcal{F}(\chi) - (m+1)\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) + \chi\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)'\right]\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right|, \tag{14}$$

implies

$$\left[m\mathcal{J}_m^{\alpha+1,\mu}\mathcal{F}(\chi) - (m+1)\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) + \chi\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)'\right] \prec_\Upsilon \mathcal{E}(\chi)$$

then

$$\left|\Upsilon_{\chi(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right|$$

equivalently with

$$\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)' \prec_\Upsilon \phi(\chi).$$

Proof. Let

$$\mathbb{P}(\chi) = \chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'. \tag{15}$$

From (12) and (15), we have

$$\begin{aligned}
\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi) &= m\left(\frac{1}{\chi} + \sum_{j=2}^{\infty}\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha+1}a_jb_j\chi^j\right) \\
&\quad -(m+1)\left(\frac{1}{\chi} + \sum_{j=2}^{\infty}\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha}a_jb_j\chi^j\right) \\
&\quad +\left(\frac{1}{\chi} + \sum_{j=2}^{\infty}j^2\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha}a_jb_j\chi^j\right) \\
&= m\mathcal{J}_m^{\alpha+1,\mu}\mathcal{F}(\chi) - (m+1)\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) + \chi\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)'.
\end{aligned} \tag{16}$$

From (14) and (16), we obtain

$$\left|\Upsilon_{\chi(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right|,$$

which implies that

$$\left|\Upsilon_{\Omega(\mathbb{C}^2\times\Lambda^*)}\left(\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi)\right)\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\left(\phi(\chi) + \chi\phi'(\chi)\right)\right|.$$

Thus, by applying Lemma 2 with $\lambda = 1$, we obtain

$$\left|\Upsilon_{\mathbb{P}(\Lambda^*)}\mathbb{P}(\chi)\right| \leq \left|\Upsilon_{\phi(*)}\phi(\chi)\right| \Longrightarrow \left|\Upsilon_{\chi(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)\right| \leq \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right|$$

i.e.,

$$\chi\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)' \prec_{\Upsilon} \phi(\chi).$$

The proof of the theorem is completed. □

Theorem 2. *Let ϕ be the convex function in Λ^*, such that $\phi(0) = 1$, and*

$$\mathcal{E}{=}\phi(\chi) + \chi\phi'(\chi) \qquad \chi \in \Lambda^*$$

Let $\mathcal{F} \in \sum$ and satisfies the following fuzzy differetial subordination:

$$\left|\Upsilon_{(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right| \le \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right| \Longrightarrow \left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)' \prec_{\Upsilon} \mathcal{E}(\chi) \tag{17}$$

then

$$\left|\Upsilon_{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)\right| \le \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right| \Longrightarrow \mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) \prec_{\Upsilon} \phi(\chi).$$

Proof. Assume that

$$\mathbb{P}(\chi) = \mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) \tag{18}$$

From (12) and (18), we have

$$\begin{aligned}\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi) &= \left(\frac{1}{\chi} + \sum_{j=2}^{\infty}\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha}a_jb_j\chi^j\right) + \left(-\frac{1}{\chi} + \sum_{j=2}^{\infty}j\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha}a_jb_j\chi^j\right)\\ &= \sum_{j=2}^{\infty}(j+1)\binom{\mu+j}{k}\left(\frac{m+j+1}{m}\right)^{\alpha}a_jb_j\chi^j.\end{aligned} \tag{19}$$

We obtain

$$\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi) = \left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'.$$

We obtain

$$\left|\Upsilon_{(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right)\right| \le \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right|,$$

which implies that

$$\left|\Upsilon_{\Psi(\mathbb{C}^2\times\Lambda^*)}\left(\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi)\right)\right| \le \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right| \le \left|\Upsilon_{\phi(\Lambda^*)}\left(\phi(\chi) + \chi\phi'(\chi)\right)\right|.$$

Applying Lemma 3, we obtain

$$\left|\Upsilon_{\mathbb{P}(\Lambda^*)}\mathbb{P}(\chi)\right| \le \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right| \Longrightarrow \left|\Upsilon_{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)\right| \le \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right|,$$

which implies that

$$\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) \prec_{\Upsilon} \phi(\chi).$$

The proof of Theorem 2 is completed. □

Theorem 3. *For $\mathcal{E} \in \mathcal{H}(\Lambda^*)$ with $\mathcal{E}(0) = 1$, where*

$$-\Re\left(1 + \frac{\chi\mathcal{E}'(\chi)}{\mathcal{E}(\chi)}\right) > -\frac{1}{2} \ (\chi \in \Lambda^*).$$

If $\mathcal{F} \in \sum$ and the following fuzzy differential subordination holds true:

$$\left|\Upsilon_{(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi))'}\left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'\right| \le \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right| \Longrightarrow \left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)' \prec_{\mathcal{F}} \mathcal{E}(\chi) \tag{20}$$

then

$$\left|\Upsilon_{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\left(\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)\right| \leq \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right| \Longrightarrow \mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi) \prec_{\mathcal{F}} \phi(\chi).$$

where the function $\phi(\chi)$ defined as follows

$$\phi(\chi) = \frac{1}{\chi}\int_0^{\chi}\mathcal{E}(t)dt$$

is convex and is the best fuzzy dominant.

Proof. Let

$$\mathbb{P}(\chi) = \mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi). \tag{21}$$

It is clear that $\mathbb{P}(\chi) \in \mathbb{H}[1,1]$. Suppose that $\mathcal{E} \in \mathcal{H}(\Lambda^*)$ with $\mathcal{E}(0) = 1$, such that

$$-\mathfrak{Re}\left(1 + \frac{\chi\mathcal{E}'(\chi)}{\mathcal{E}(\chi)}\right) > -\frac{1}{2} \ \ (\chi \in \Lambda^*).$$

From Lemma 1, we have

$$\phi(\chi) = \frac{1}{\chi}\int_0^{\chi}\mathcal{E}(t)dt,$$

is convex and satisfies the fuzzy differetial subordination (20). Since

$$\mathcal{E}(\chi) = \phi(\chi) + \chi\phi'(\chi) \quad (\chi \in \Lambda^*).$$

We have

$$\begin{aligned} \mathbb{P}(\chi) + \chi\mathbb{P}'(\chi) &= \sum_{j=2}^{\infty}(j+1)\binom{\mu+j}{j}\left(\frac{m+j+1}{m}\right)^{\alpha}a_j b_j \chi^j \\ &= \left(\chi\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)\right)'. \end{aligned} \tag{22}$$

From (22), the fuzzy differential subordination (20) is

$$\left|\Upsilon_{\mathbb{P}(\Lambda)}(\mathbb{P}(\chi) + \chi\mathbb{P}'(\chi))\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda)}\mathcal{E}(\chi)\right|.$$

By applying Lemma 3 with $v = 1$, we obtain

$$\left|\Upsilon_{\mathbb{P}(\Lambda^*)}\mathbb{P}(\chi)\right| \leq \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right|.$$

Which complete the proof. □

Setting

$$\mathcal{E}(\chi) = \frac{1 + (2\rho - 1)\chi}{1 + \chi} \quad (\chi \in \Lambda^*)$$

in Theorem 3, we obtain the following corollary.

Corollary 1. *Assume that*

$$\mathcal{E}(\chi) = \frac{1 + (2\rho - 1)\chi}{1 + \chi} \quad (\chi \in \Lambda^*)$$

is convex function in Λ^* such that $\mathcal{E}(0) = 1$ and $0 \leq \rho < 1$. The function $\mathcal{F} \in \sum$ satisfies the following fuzzy differential subordination:

$$\begin{aligned} \left| \Upsilon_{\left(\chi \mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)\right)'} \left(\chi \mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \right)' \right| &\leq \left| \Upsilon_{\mathcal{E}(\Lambda^*)} \mathcal{E}(\chi) \right| \\ &\Longrightarrow \left(\chi \mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \right)' \prec_{\Upsilon} \mathcal{E}(\chi), \end{aligned} \tag{23}$$

then the function $\phi(\chi)$ is

$$\phi(\chi) = 2\rho - 1 + \frac{2(1-\rho)}{\chi} \log(1+\chi),$$

is convex and is the fuzzy best dominant.

Theorem 4. *Let ϕ be convex function in Λ^* and $\phi(0) = 1$,*

$$\mathcal{E}(\chi) = \phi(\chi) + \chi \phi'(\chi).$$

Let $\mathcal{F} \in \sum$, and $\left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)'$ be in Λ^*. If

$$\left| \Upsilon_{\left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)'} \left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)' \right| \leq \left| \Upsilon_{\mathcal{E}(\Lambda^*)} \mathcal{E}(\zeta) \right| \Longrightarrow \left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)' \prec_{\Upsilon} \mathcal{E}(\chi), \tag{24}$$

then

$$\left| \Upsilon_{\left(\frac{\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)} \left(\frac{\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right) \right| \leq \left| \Upsilon_{\phi(\Lambda^*)} \phi(\chi) \right|,$$

i.e.,

$$\frac{\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \prec_{\Upsilon} \phi(\chi).$$

Proof. Assuming that

$$\mathbb{P}(\chi) = \frac{\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \in \mathcal{H}[1,1]. \tag{25}$$

Differentiating both sides of (25) with respect to χ, we obtain

$$\mathbb{P}'(\chi) = \frac{\left(\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi) \right)'}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} - \mathbb{P}(\chi) \frac{\left(\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \right)'}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)}.$$

Then,

$$\begin{aligned} \mathbb{P}(\chi) + \chi \mathbb{P}'(\chi) &= \frac{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \left[\chi \left(\mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi) \right)' + \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi) \right] - \left(\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi) \right) \left(\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \right)'}{\left(\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi) \right)^2} \\ &= \left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu} \mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu} \mathcal{F}(\chi)} \right)'. \end{aligned} \tag{26}$$

Utilizing (26) in (24), we can obtain

$$\left|\Upsilon_{\left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu}\mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\right)'}\left(\frac{\chi \mathcal{J}_m^{\alpha-1,\mu}\mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\right)'\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right|,$$

which implies that

$$\left|\Upsilon_{\mathbb{P}(\Lambda)}(\mathbb{P}(\chi)+\chi\mathbb{P}'(\chi))\right| \leq \left|\Upsilon_{\mathcal{E}(\Lambda^*)}\mathcal{E}(\chi)\right| \leq \left|\Upsilon_{\phi(\Lambda^*)}(\phi(\chi)+\chi\phi'(\chi))\right|.$$

Thus, by applying Lemma 3 with $\vartheta = 1$, we obtain

$$\left|\Upsilon_{\left(\frac{\mathcal{J}_m^{\alpha-1,\mu}\mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\right)}\left(\frac{\mathcal{J}_m^{\alpha-1,\mu}\mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)}\right)\right| \leq \left|\Upsilon_{\phi(\Lambda^*)}\phi(\chi)\right|,$$

i.e.,

$$\frac{\mathcal{J}_m^{\alpha-1,\mu}\mathcal{F}(\chi)}{\mathcal{J}_m^{\alpha,\mu}\mathcal{F}(\chi)} \prec_{\Upsilon} \phi(\chi).$$

The proof of the theorem is completed. □

3. Conclusions

In our present investigation of the applications of fuzzy differential subordinations in the geometric function theory of complex analysis, we successfully made use of the integral operator $\mathcal{J}_m^{\alpha,\mu}$ for meromorphic function. Another avenue for further research on this subject is provided by the fact that, in the theory of differential subordinations and differential superordinations, there are differential subordinations and differential superordinations of the third and higher orders.

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