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The Existence, Uniqueness, and Multiplicity of Solutions for Two Fractional Nonlocal Equations

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Abstract: This paper establishes the existence of unique and multiple solutions to two nonlocal equations with fractional operators. The main results are obtained using the variational method and algebraic analysis. The conclusion is that there exists a constant $\lambda^* > 0$ such that the equations have only three, two, and one solution, respectively, for $\lambda \in (0, \lambda^*)$, $\lambda = \lambda^*$, and $\lambda > \lambda^*$. The main conclusions fill the gap in the knowledge of this kind of fractional-order problem.

Keywords: fractional nonlocal equation; variational method; algebraic analysis

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1. Work Spaces and Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with a smooth boundary. In this paper, we will prove the existence of solutions for two nonlocal equations with the fractional Laplacian $(-\Delta)^s$ in the fractional Hilbert space $H_0^s(\Omega)$, where $s \in (0,1)$ is the fractional order. From the introduction in [1], we know that there are many studies that have made useful efforts to define the fractional operator with a fractional order $s \in (0,1)$. Here, the fractional Sobolev-type spaces are also called Aronszajn spaces [2], Gagliardo spaces [3], or Slobodeckij spaces [4].

1.1. The First Definition of the Fractional Hilbert Space and Fractional Laplacian Operator

Now, let us introduce the first definition of the fractional Laplacian operator $(-\Delta)^s$ with the fixed scalar $s \in (0,1)$, which relates to the eigenvalue problem of $-\Delta$ in the general Hilbert space $H_0^1(\Omega)$. As is well known, for the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

there exists a sequence $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and a function sequence $\{\varphi_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$, such that

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k, & \text{in} \quad \Omega, \\ \varphi_k = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, it holds that $\varphi_1 > 0$ in Ω and

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots, \ \lim_{k \to \infty} \lambda_k = +\infty.$$

By the general normalization, the L^2-norm can be normalized as $\|\varphi_k\|_{L^2(\Omega)} = 1$ for $k \geq 1$. Let $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ be the sequences in $\mathbb{R}$ defined by [5] (Section 1),

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$$H_0^s(\Omega) := \left\{ u := \sum_{k=1}^{\infty} \alpha_k \varphi_k \in L^2(\Omega)\ ;\ \sum_{k=1}^{\infty} \lambda_k^s \alpha_k^2 < +\infty,\ 0 < s < 1 \right\}.$$

If $u := \sum_{k=1}^{\infty} \alpha_k \varphi_k$ and $v := \sum_{k=1}^{\infty} \beta_k \varphi_k$ belong to $H_0^s(\Omega)$, then $H_0^s(\Omega)$ is a Hilbert space under a abstract inner product

$$\left\langle u, v \right\rangle_{H_0^s(\Omega)} = \left\langle \sum_{k=1}^{\infty} \alpha_k \varphi_k, \sum_{k=1}^{\infty} \beta_k \varphi_k \right\rangle_{H_0^s(\Omega)} := \sum_{k=1}^{\infty} \lambda_k^s \alpha_k \beta_k. \tag{1}$$

Based on the fractional Hilbert space $H_0^s(\Omega)$, define $(-\Delta)^s\ :\ H_0^s(\Omega) \to H_0^{-s}(\Omega)$ as

$$\sum_{k=1}^{\infty} \alpha_k \varphi_k = u \mapsto (-\Delta)^s u := \sum_{k=1}^{\infty} \lambda_k^s \alpha_k \varphi_k, \tag{2}$$

which is called the fractional Laplacian operator. Therefore, in view of the fractional Hilbert space $H_0^s(\Omega)$, the inner product can be stated as (1) and

$$\begin{aligned} \left\langle u, v \right\rangle_{H_0^s(\Omega)} &= \left\langle (-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v \right\rangle_{L^2(\Omega)} = \int_\Omega (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v \mathrm{d}x \\ &= \left\langle (-\Delta)^s u, v \right\rangle_{L^2(\Omega)} = \left\langle (-\Delta)^s v, u \right\rangle_{L^2(\Omega)} \end{aligned} \tag{3}$$

for any $u, v \in H_0^s(\Omega)$. Correspondingly, the norm is

$$\|u\|_{H_0^s(\Omega)} := \left(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^{\frac{1}{2}},\ u \in H_0^s(\Omega). \tag{4}$$

The space $H_0^s(\Omega)$ is the so-called fractional Hilbert space and is a reflective Banach space. Obviously, the fractional Hilbert space $H_0^s(\Omega)$ and the fractional Laplacian operator are defined by spectral theory here, which are related to a mathematical eigenvalue problem. For more details, see, for instance, [6,7].

1.2. The Second Definition of the Fractional Hilbert Space and Fractional Laplacian Operator

Fix the fractional exponent $s \in (0, 1)$. For any $p \in [1, +\infty)$, the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega)\ ;\ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y < +\infty \right\}$$

is defined as an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}} \tag{5}$$

is the natural norm of $u \in W^{s,p}(\Omega)$, where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}$$

is the so-called Gagliardo semi-norm of u. For $u \in W^{s,p}(\Omega)$, it holds that $|u|^{p-2}u \in [W^{s,p}(\Omega)]'$. The dual action between $v \in W^{s,p}(\Omega)$ and $|u|^{p-2}u \in [W^{s,p}(\Omega)]'$ is

$$\left\langle |u|^{p-2}u, v \right\rangle_{[W^{s,p}(\Omega)]'\times W^{s,p}(\Omega)} = \left\langle |u|^{p-2}u, v \right\rangle_{L^{\frac{p}{p-1}}(\Omega\times\Omega)\times L^p(\Omega\times\Omega)} + \\ + \left\langle \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{(\frac{N}{p}+s)(p-1)}}, \frac{v(x)-v(y)}{|x-y|^{\frac{N}{p}+s}} \right\rangle_{L^{\frac{p}{p-1}}(\Omega\times\Omega)\times L^p(\Omega\times\Omega)}.$$

Let $W_0^{s,p}(\Omega)$ be the completion of the space $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$ under the norm defined by (5), then $\|u\|_{W^{s,p}(\Omega)}$ and $[u]_{W^{s,p}(\Omega)}$ are the equivalent norms in $W_0^{s,p}(\Omega)$, which has an equivalent dual

$$\left\langle |u|^{p-2}u, v \right\rangle_{[W^{s,p}(\Omega)]'\times W^{s,p}(\Omega)} \\ = \left\langle \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{(\frac{N}{p}+s)(p-1)}}, \frac{v(x)-v(y)}{|x-y|^{\frac{N}{p}+s}} \right\rangle_{L^{\frac{p}{p-1}}(\Omega\times\Omega)\times L^p(\Omega\times\Omega)}$$

between $W_0^{s,p}(\Omega)$ and $[W_0^{s,p}(\Omega)]' = W^{-s,\frac{p}{p-1}}(\Omega)$. The space $W_0^{s,2}(\Omega)$ is a reflective Banach space, denoted here by $W_0^{s,2}(\Omega) := \mathcal{H}_0^s(\Omega)$, which is called the fractional Hilbert space. Therefore, in an equivalent form, the norm is

$$\|u\|_{\mathcal{H}_0^s(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

and the inner product is

$$\left\langle u, v \right\rangle_{\mathcal{H}_0^s(\Omega)} = \left\langle \frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}}, \frac{v(x)-v(y)}{|x-y|^{\frac{N}{2}+s}} \right\rangle_{L^2(\Omega\times\Omega)}, \ \forall\, u, v \in \mathcal{H}_0^s(\Omega).$$

A fractional Laplacian with an exterior zero condition can be defined as

$$\begin{aligned} (-\Delta)^s u(x) &= C(N,s)\ P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\ &= C(N,s) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \\ &= -\frac{1}{2} C(N,s) \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathrm{d}y,\ \ y \in \mathbb{R}^N,\ \ \forall x \in \mathbb{R}^N, \end{aligned} \tag{6}$$

where $P.V.$ is the Cauchy's principal value and the constant

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} \mathrm{d}\zeta \right)^{-1} > 0,\ \zeta = (\zeta_1, \cdots, \zeta_N) \in \mathbb{R}^N.$$

Let $\complement\Omega = \mathbb{R}^N \setminus \Omega$, $\Pi = \mathbb{R}^{2N} \setminus (\complement\Omega \times \complement\Omega)$, and consider $u, v \in \mathcal{H}_0^s(\Omega)$. Then,

$$\begin{aligned} \left\langle (-\Delta)^s u, v \right\rangle_{L^2(\Omega)} &= C(N,s)\ P.V. \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\ &= C(N,s)\ P.V. \iint\limits_{\Pi} \frac{[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\ &= \left\langle (-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v \right\rangle_{L^2(\Omega)}. \end{aligned}$$

Now, define the equivalent norm as

$$\|u\|_{\mathcal{H}_0^s(\Omega)} = \left(C(N,s)\ P.V. \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} \tag{7}$$

and the equivalent inner product as

$$\begin{aligned}\left\langle u,v\right\rangle_{\mathcal{H}_0^s(\Omega)} &= C(N,s)\ P.V.\int_\Omega\int_\Omega \frac{[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x \\ &= \left\langle (-\Delta)^{\frac{s}{2}}u,(-\Delta)^{\frac{s}{2}}v\right\rangle_{L^2(\Omega\times\Omega)} = \int_\Omega (-\Delta)^{\frac{s}{2}}u\cdot(-\Delta)^{\frac{s}{2}}v\mathrm{d}x \\ &= \left\langle (-\Delta)^s u,v\right\rangle_{L^2(\Omega)} = \left\langle (-\Delta)^s v,u\right\rangle_{L^2(\Omega)}.\end{aligned}$$

Then, it is easy to see that the norms (4) and (7) are equivalent, and the inner products have the same form as

$$\left\langle u,v\right\rangle_H = \int_\Omega (-\Delta)^{\frac{s}{2}}u\cdot(-\Delta)^{\frac{s}{2}}v\mathrm{d}x,$$

for $H = H_0^s(\Omega)$ or $H = \mathcal{H}_0^s(\Omega)$.

The fractional Laplacian in (6) is defined by a Fourier transformation and inverse Fourier transform, the details of which can be found, for instance, in [1] and ([8], Section 4).

Many researchers have explained the differences between the two definitions of the fractional Hilbert space and the fractional Laplacian operator, for example, [5,6]. Indeed, from ([5] Section 1) we know that

$$H_0^s(\Omega) := \left\{u = \sum_{k=1}^{\infty}\alpha_k\varphi_k \in L^2(\Omega)\ ;\ \sum_{k=1}^{\infty}\lambda_k^s\alpha_k^2 < +\infty,\ 0<s<1\right\}.$$

Moreover, from ([9] Section 2) we know that

$$\mathcal{H}_0^s(\Omega) = \left\{u\in H^s(\mathbb{R}^N);\ u(x)=0 \text{ a.e. } x\in\mathbb{R}^N\backslash\Omega\right\}.$$

A clear result is that $u(x) = 0$ a.e. in $\mathbb{R}^N\backslash\Omega$ is not necessary for $H_0^s(\Omega)$, but $u(x) = 0$ a.e. in $\mathbb{R}^N\backslash\Omega$ is necessary due to the non-local character of the operator for $\mathcal{H}_0^s(\Omega)$.

A general statement found in [1] shows that the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is pre-compact for $q\in[1,p_s^*)$ and continuous for $q\in[1,p_s^*]$, where $p_s^* = \frac{Np}{N-sp}$ if $N > sp$ and $p_s^* = \infty$ if $N \le sp$. That is to say, $\mathcal{H}_0^s(\Omega) \hookrightarrow L^q(\Omega)$ is pre-compact for $q\in[1,2_s^*)$ and continuous for $q\in[1,2_s^*]$. In the above-mentioned work, the authors still claim that this property is fit for $H_0^s(\Omega)$. We refer readers to the statements in [5–13] for the fractional Hilbert space and fractional Laplacian operator, and some useful applications using Caputo's fractional-order problems can be found in [14–19], which extend the possibilities of these problems. Indeed, there exists much information on the properties of the Sobolev and Hilbert Spaces with different fractional orders in the works by [8,11–13]. Moreover, as mentioned above, some conclusions for the space $H_0^s(\Omega)$ can be found in [5–7]. Yang and Yu [5] considered a nonlinear elliptic equation in $H_0^s(\Omega)$ with a singular domain using the variational method and determined the existence of infinitely many solutions with some necessary conditions. Servadei and Valdinoci [6] gave the difference between two spectra of two fractional operators. Wang, Yang, and Zhou [7] studied a fractional-order problem with Hardy's term and Sobolev's critical exponent and obtained infinitely many solutions in $\{u_n\}\subset H_0^s(\Omega)$ by applying the perturbation method. Similarly, Fiscella and Mishra [9] obtained two solutions in $\mathcal{H}_0^s(\Omega)$ by a Nehari manifold for a fractional Kirchhoff Hardy problem with singular and critical Sobolev nonlinearities, where $a,b>0$. Wang, Wang, and Fečkan [14] obtained the economic growth modeling of the Group of Seven using a BP neural network and fractional-order gradient. In [15], Wang, Fečkan, and Wang applied the fractional-order gradient approach to forecast the economic growth of the Group of Seven. Sathiyaraj, Wang, and Balasubramaniam [16] researched a class of time-delayed fractional stochastic integro-differential systems, and controllability and optimal control were obtained by applying the fixed point theorem. A forecasting model was obtained in [17] by Liao, Wang, and Wang, where the main tool is the fractional-order gray gradient. With the impulses, Yang, Fečkan, and Wang [18] concluded the consensus of linear conformable fractional-order multi-agent systems, and Wang, Liu, and Fečkan [19] made clear applications for the fractional-order control of equations by iterative learning.

2. Main Results and Background

Let the domain Ω be bounded in $\mathbb{R}^N(N \geq 1)$ with a smooth boundary, the constants a and b belong to $\mathbb{R}$, and the parameters $\lambda > 0$ and $f(x) \in L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} > 0$. In this paper, we consider the existence of a solution in $H_0^s(\Omega)$ for the following problem:

$$\begin{cases} \left(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)(-\Delta)^s u = \lambda f(x), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

where $(-\Delta)^s$ is defined by (2). Moreover, we still consider the problem

$$\begin{cases} \left(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)(-\Delta)^s u = \lambda f(x), & \text{in} \quad \Omega, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{9}$$

where $(-\Delta)^s$ is defined by (6). Here, we aim to study the existence of solutions for Equations (8) and (9). The solution belongs to $H_0^s(\Omega)$ in Equation (8) and $\mathcal{H}_0^s(\Omega)$ in Equation (9). So, we denote the work space by $H = H_0^s(\Omega)$ in the considered equation, Equation (8), and $H = \mathcal{H}_0^s(\Omega)$ in the considered equation, Equation (9), and denote the norm in H with $\|\cdot\|$ and the dual action between H and H' with $\langle\cdot,\cdot\rangle$.

Definition 1. *A function $u \in H$ is called a solution of Equation (8) or Equation (9) if*

$$\left(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\right)\int_\Omega (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}\varphi \mathrm{d}x = \lambda \int_\Omega f(x)\varphi \mathrm{d}x, \ \forall \varphi \in H.$$

The main results can be stated as follows:

Theorem 1. *Assume that $|a| + |b| \neq 0$ and $\|f\|_{L^2(\Omega)} \geq 0$, then, for any $\lambda \geq 0$, Problems (8) and (9) have at least one solution.*

Theorem 2. *Assume that $ab \geq 0$, $a + b \neq 0$ and $\|f\|_{L^2(\Omega)} \geq 0$, then, for any $\lambda \geq 0$, Problems (8) and (9) only have one solution.*

Theorem 3. *Assume that $ab < 0$ and $\lambda\|f\|_{L^2(\Omega)} = 0$, then, Problems (8) and (9) have infinitely many solutions.*

Theorem 4. *Assume that $ab < 0$ and $\|f\|_{L^2(\Omega)} > 0$, then, there exists $\lambda^* > 0$ such that Problems (8) and (9) have only one, two, or three solutions, respectively, when $\lambda > \lambda^*$, $\lambda = \lambda^*$, or $\lambda \in (0, \lambda^*)$.*

Problem (8) is related to the following Kirchhoff model [20]:

$$\varrho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = f(t,x,u), \tag{10}$$

where every symbol has its physical significance and $f(t,x,u)$ is an abstract function. In the form of expression, Equation (10) contains a global integration $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x$ so it is also called a nonlocal problem. In general, a steady state of Equation (10) can be rewritten as

$$-\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u), \tag{11}$$

where a, b are constants and $\Omega \subset \mathbb{R}^N$ and $f(x,u)$ are abstract functions, which, in this case, is the so-called Kirchhoff-type equation. If $ab > 0$, it is a generalization of the D'Alembert equation. If $ab < 0$, it can be described as the interaction of forces between two atoms and in this case, the minus Young's modulus is clearly shown. For more details, see the introduction in [21–23] for this problem and another negative modulus in [24,25]. Indeed, there are many results that have been obtained by researchers in the past few years and we refer readers to the summary in [23], where the context and research status of this problem can be seen. Considering the Kirchhoff-type equation, there are many results for Problem (11) (see, for instance, [26–37] for $ab < 0$ and [38–48] for $a, b > 0$). In addition, other cases can be rewritten in the form mentioned above.

As far as we know, there is no article discussing all $a, b \in \mathbb{R}$ for Equations (8) and (9), although there are many results that consider the fractional Kirchhoff-type equation. Here, we refer readers to some recent papers, for instance, [49–55]. Motivated by the above-mentioned papers, we consider the existence of a unique solution to Problems (8) and (9). Theorems 1 to 4 are based on the variational method and algebraic analysis and those results can be extended to the case of $f \in H^{-s}(\Omega)$.

3. Proof of Main Results

3.1. Proof of Theorem 1

Proof. Consider the functional $I : H \mapsto \mathbb{R} \cup \{\infty\}$ defined by

$$I(u) := \frac{a}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\Big)^2 - \lambda \int_\Omega f(x)u\mathrm{d}x.$$

Then, we can easily verify that for any $\varphi \in H$, it holds that

$$\begin{aligned}\frac{d}{dt}I(u+t\varphi)\Big|_{t=0} &= \Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}\varphi \mathrm{d}x - \lambda\int_\Omega f(x)\varphi\mathrm{d}x \\ &:= \Big\langle I'(u), \varphi\Big\rangle.\end{aligned}$$

Which shows that $I(u)$ is a Gâteaux's differentiable functional and its critical points are the solutions to Problem (8) if $H = H_0^s(\Omega)$ and the solutions to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$.

If $\lambda = 0$, we can see that $u = 0$ is a solution to Problem (8) if $H = H_0^s(\Omega)$ and a solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$. So, next, we let $\lambda > 0$.

If $b = 0 \neq a$, the functional $I(u)$ is equivalent to the functional

$$\begin{aligned}J(u) &= \int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \frac{\lambda}{a}\int_\Omega f(x)u\mathrm{d}x = \|u\|^2 - \frac{\lambda}{a}\int_\Omega f(x)u\mathrm{d}x \\ &\geq \|u\|^2 - \Big|\frac{\lambda}{a}\int_\Omega f(x)u\mathrm{d}x\Big| \geq \|u\|^2 - \Big|\frac{\lambda}{a}\Big|\|f\|_{L^2(\Omega)}\|u\| \\ &\geq -\frac{\lambda^2}{4a^2}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

So, $J(u)$ is a coercive lower-bound functional. Denote by c the infimum of $J(u)$, then, there is a minimizing sequence $\{u_n\}_{n=1}^\infty \subset H$ such that

$$-\frac{\lambda^2}{4a^2}\|f\|^2_{L^2(\Omega)} \leq \lim_{n\to\infty} J(u_n) = \inf_{u\in H} J(u) = c < +\infty.$$

Moreover, $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence and H is a reflexive Banach space [1]. Hence, there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and a function $u_* \in H$ such that

$$\begin{cases} \lim\limits_{k\to\infty} \int_\Omega (-\Delta)^{\frac{s}{2}}(u_{n_k} - u_*) \cdot (-\Delta)^{\frac{s}{2}}\varphi \mathrm{d}x = 0, \ \forall\, \varphi \in H, \\ \lim\limits_{k\to\infty} \int_\Omega |u_{n_k} - u_*|^p \mathrm{d}x = 0, \ \forall\, p \in [1, 2_s^*), \\ \lim\limits_{k\to\infty} u_{n_k}(x) = u_*(x), \text{ a.e. } x \in \Omega, \end{cases} \tag{12}$$

where the sign of $u_{n_k}(x)$ is determined by the sign of $\frac{\lambda}{a} f(x)$. Indeed, the sign of $u_{n_k}(x)$ is same as the sign of $\frac{\lambda}{a} f(x)$. Therefore, one has

$$\begin{aligned} \lim_{k\to\infty} \Big| \int_\Omega f(x)(u_k - u_*)\mathrm{d}x \Big| &\le \lim_{k\to\infty} \Big(\int_\Omega |f|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_\Omega |u_k - u_*|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \\ &= 0 \end{aligned} \tag{13}$$

by the fact that $2_s^* = \frac{2N}{N-2s} > 2$ if $N > 2s$ and $2_s^* = \infty > 2$ if $N \le 2s$. By using the Ekeland's variational principle in ([56], Lemma 1.1), there exists a minimizing sequence $\{u_{k_j}\}_{n=1}^{\infty} \subset \{u_{n_k}\}_{n=1}^{\infty}$, such that

$$J(u_{k_j}) \le \inf_{u\in H} J(u) + \varepsilon \quad \text{and} \quad J(u_{k_j}) \le J(v) + \varepsilon d(v, u_{k_j}), \forall\, v \in H \tag{14}$$

for any $\varepsilon > 0$ and positive integral k_j, where $d(v, u_{k_j}) = \|v - u_n\|$ is an equivalent metric in H. For each k_j and any $\varphi \in H$ with $\|\varphi\| = 1$, we let $v = u_{k_j} + t(\pm\varphi)$ and find

$$\begin{aligned} \Big\langle J'(u_{k_j}), \varphi \Big\rangle &= \lim_{t\to 0^+} \frac{J(u_{k_j} + t\varphi) - J(u_{k_j})}{t} \ge -\varepsilon\|t\varphi\| \ge -\varepsilon, \\ \Big\langle J'(u_{k_j}), -\varphi \Big\rangle &= \lim_{t\to 0^+} \frac{J(u_{k_j} + t(-\varphi)) - J(u_{k_j})}{t} \ge -\varepsilon\|t(-\varphi)\| \ge -\varepsilon, \end{aligned}$$

which shows that for every k_j, $-\varepsilon < \langle J'(u_{k_j}), \varphi \rangle < \varepsilon$ for any $\varepsilon > 0$ and any $\varphi \in H$ with $\|\varphi\| = 1$. This implies that $\|J'(u_{k_j})\|_{H'} < \varepsilon$. Hence, we have $J'(u_{k_j}) \to 0$ as $j \to \infty$ by the arbitrariness of $\varepsilon > 0$. Therefore, we have

$$\begin{aligned} \lim_{j\to\infty} \Big\langle J'(u_{k_j}), u_{k_j} - u_* \Big\rangle &= \lim_{j\to\infty} \Big[\int_\Omega (-\Delta)^{\frac{s}{2}} u_{k_j} \cdot (-\Delta)^{\frac{s}{2}} (u_{k_j} - u_*)\mathrm{d}x - \frac{\lambda}{a} \int_\Omega f(x)(u_{k_j} - u_*)\mathrm{d}x \Big] \\ &= \lim_{k\to\infty} \Big[\int_\Omega (-\Delta)^{\frac{s}{2}} u_k \cdot (-\Delta)^{\frac{s}{2}} (u_k - u_*)\mathrm{d}x - \frac{\lambda}{a} \int_\Omega f(x)(u_k - u_*)\mathrm{d}x \Big] \\ &= \lim_{k\to\infty} \int_\Omega (-\Delta)^{\frac{s}{2}} u_k \cdot (-\Delta)^{\frac{s}{2}} (u_k - u_*)\mathrm{d}x \\ &= 0 \end{aligned}$$

from the results of (12) and (13). Taking $\varphi = u_*$ we find that

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega |(-\Delta)^{\frac{s}{2}}(u_k - u_*)|^2 \mathrm{d}x = &\lim_{k\to\infty} \int_\Omega (-\Delta)^{\frac{s}{2}} u_k \cdot (-\Delta)^{\frac{s}{2}} (u_k - u_*)\mathrm{d}x \\ &+ \lim_{k\to\infty} \int_\Omega (-\Delta)^{\frac{s}{2}} u_* \cdot (-\Delta)^{\frac{s}{2}} (u_* - u_k)\mathrm{d}x \\ =&0. \end{aligned}$$

Therefore, we have $J'(u_*) = 0$ and

$$\Big\langle I'(u_*), \varphi \Big\rangle = a\Big\langle J'(u_*), \varphi \Big\rangle = a \int_\Omega (-\Delta)^{\frac{s}{2}} u_* \cdot (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x - \lambda \int_\Omega f(x)\varphi \mathrm{d}x = 0, \forall \varphi \in H,$$

which shows that u_* is a solution to Problem (8) if $H = H_0^s(\Omega)$ and u_* is a solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$.

If $b \neq 0$, without loss of generality, we can assume $b > 0$ in $I(u)$. Otherwise, we can consider the functional $J(u) = -I(u)$. Let $b > 0$ in $I(u)$, then,

$$\begin{aligned} I(u) &= \frac{a}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x\Big)^2 - \lambda\int_\Omega f(x)u\mathrm{d}x \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\|f\|_{L^2(\Omega)}\|u\|. \end{aligned}$$

It is easy to see that $I(u)$ is a coercive lower-bound functional. Denote by c the infimum of $I(u)$, then, there is a minimizing sequence $\{u_n\}_{n=1}^\infty \subset H$ such that

$$-\infty < \lim_{n\to\infty} J(u_n) = \inf_{u\in H} J(u) = c < +\infty.$$

Similarly, we find that $\{u_n\}_{n=1}^\infty$ is a bounded sequence in H and there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and a function $u_* \in H$ such that (12) and (13) hold. The Ekeland's variational principle implies that there exists a minimizing sequence $\{u_{k_j}\}_{n=1}^\infty \subset \{u_{n_k}\}_{n=1}^\infty$, such that

$$I(u_{k_j}) \leq \inf_{u\in H} I(u) + \varepsilon \quad \text{and} \quad I(u_{k_j}) \leq I(v) + \varepsilon d(v, u_{k_j}), \forall\, v \in H$$

As the proof as $J'(u_{k_j}) \to 0$, we have $I'(u_{k_j}) \to 0$ as $j \to \infty$ by the arbitrariness of $\varepsilon > 0$. So,

$$\begin{aligned} &\lim_{j\to\infty}\Big\langle I'(u_{k_j}), u_{k_j} - u_*\Big\rangle \\ &= \lim_{j\to\infty}\Big[\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u_{k_j}|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u_{k_j}\cdot(-\Delta)^{\frac{s}{2}}(u_{k_j} - u_*)\mathrm{d}x \\ &\qquad\qquad - \lambda\int_\Omega f(x)(u_{k_j} - u_*)\mathrm{d}x\Big] \\ &= \lim_{k\to\infty}\Big[\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u_k\cdot(-\Delta)^{\frac{s}{2}}(u_k - u_*)\mathrm{d}x \\ &= 0 \end{aligned}$$

from the results of (12) and (13). We have

$$\lim_{k\to\infty}\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\Big) = 0 \tag{15}$$

or

$$\lim_{k\to\infty}\int_\Omega (-\Delta)^{\frac{s}{2}}u_k\cdot(-\Delta)^{\frac{s}{2}}(u_k - u_*)\mathrm{d}x = 0. \tag{16}$$

We claim that the case in (15) is an impossible event. For the case in (15), it can be deduced that $\|u_k\|^2 \to -\frac{a}{b}$. This shows that $\int_\Omega f(x)u_k\mathrm{d}x \to 0$ as $k \to \infty$ by $\langle I'(u_k), u_k\rangle \to 0$. That is to say,

$$c = \lim_{k\to\infty}\Big[\frac{a}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\Big)^2 - \lambda\int_\Omega f(x)u_k\mathrm{d}x\Big] = -\frac{a^2}{4b}.$$

This contradicts the results that

$$\begin{aligned}
c = \inf_{u\in H} I(u) &= \inf_{u\in H}\Big[\frac{a}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)^2 - \lambda\int_\Omega f(x)u\mathrm{d}x\Big] \\
&< \inf_{u\in H}\Big[\frac{a}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{b}{4}\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)^2\Big] \\
&< \begin{cases} 0, & \text{if} \quad a \geq 0, b > 0; \\ -\frac{a^2}{4b}, & \text{if} \quad a < 0 < b. \end{cases}
\end{aligned}$$

For the case in (16), it is easy to verify $u_k \to u_*$ as $k \to \infty$. Therefore, we have $I'(u_*) = 0$ and

$$\begin{aligned}
\Big\langle I'(u_*), \varphi\Big\rangle &= \Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u_*|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u_* \cdot (-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x - \lambda\int_\Omega f(x)\varphi\mathrm{d}x \\
&= 0, \forall\varphi \in H.
\end{aligned}$$

This implies that u_* is a solution to Problem (8) if $H = H_0^s(\Omega)$ and u_* is a solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$.

By using the same method, it is clear that Problems (8) and (9) have at least one solution when $|a| + |b| \neq 0$, $\|f\|_{L^2(\Omega)} \geq 0$ and $\lambda \geq 0$. The proof is completed. □

3.2. Proof of Theorem 2

Proof. From the results in Theorem 1, Problems (8) and (9) have at least one solution under the assumptions in Theorem 2. Now, we just need the uniqueness of the solution. Let u, v be two solutions to Problem (8) or Problem (9), with $ab > 0$ and $\|f\|_{L^2(\Omega)} \geq 0$, then, for any $\lambda \geq 0$, we have

$$\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x = \lambda\int_\Omega f(x)\varphi\mathrm{d}x, \ \ \forall\ \varphi \in H$$

and

$$\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}v|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}v \cdot (-\Delta)^{\frac{s}{2}}\varphi\mathrm{d}x = \lambda\int_\Omega f(x)\varphi\mathrm{d}x, \ \ \forall\ \varphi \in H.$$

Let $\varphi = u - v$ to find

$$\begin{aligned}
&\Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}(u-v)\mathrm{d}x \\
&= \Big(a + b\int_\Omega |(-\Delta)^{\frac{s}{2}}v|^2\mathrm{d}x\Big)\int_\Omega (-\Delta)^{\frac{s}{2}}v \cdot (-\Delta)^{\frac{s}{2}}(u-v)\mathrm{d}x.
\end{aligned}$$

That is to say,

$$\begin{aligned}
a\int_\Omega |(-\Delta)^{\frac{s}{2}}(u-v)|^2\mathrm{d}x + b\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\int_\Omega (-\Delta)^{\frac{s}{2}}u \cdot (-\Delta)^{\frac{s}{2}}(u-v)\mathrm{d}x& \\
- b\int_\Omega |(-\Delta)^{\frac{s}{2}}v|^2\mathrm{d}x\int_\Omega (-\Delta)^{\frac{s}{2}}v \cdot (-\Delta)^{\frac{s}{2}}(u-v)\mathrm{d}x& \\
=0.&
\end{aligned}$$

Therefore, it is easy to verify that $u(x) = v(x)$ for $a \neq 0 = b$ in the sense of H. If $ab > 0$ or $a = 0 \neq b$, we have

$$
\begin{aligned}
0 \leq & \frac{a}{b} \int_\Omega |(-\Delta)^{\frac{s}{2}}(u-v)|^2 \mathrm{d}x \\
= & -\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_\Omega (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}}(u-v) \mathrm{d}x \\
& + \int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \int_\Omega (-\Delta)^{\frac{s}{2}} v \cdot (-\Delta)^{\frac{s}{2}}(u-v) \mathrm{d}x \\
= & -\Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \Big)^2 + \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_\Omega (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v \mathrm{d}x \\
& - \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big)^2 + \int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \int_\Omega (-\Delta)^{\frac{s}{2}} v \cdot (-\Delta)^{\frac{s}{2}} u \mathrm{d}x \\
= & \int_\Omega (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v \mathrm{d}x \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big) \\
& \qquad - \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \Big)^2 - \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big)^2 \\
\leq & \frac{1}{2} \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x + \int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big)^2 \\
& - \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \Big)^2 - \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big)^2 \\
= & -\frac{1}{2} \Big(\int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x - \int_\Omega |(-\Delta)^{\frac{s}{2}} v|^2 \mathrm{d}x \Big)^2 \\
\leq & 0.
\end{aligned}
$$

This means that $u(x) = v(x)$ in the sense of H. The proof is completed. □

3.3. Proof of Theorem 3

Proof. Assume that $ab < 0$ and $\lambda \|f\|_{L^2(\Omega)} = 0$, then, for any $v \in H$ with $\|v\|^2 = 1$, it is easy to verify that

$$u(x) = \sqrt{-\frac{a}{b}} \cdot v(x)$$

is the solution to Problem (8) if $H = H_0^s(\Omega)$ and $u(x)$ is the solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$. Hence, Problems (8) and (9) have infinitely many solutions by the arbitrariness of v. □

3.4. Proof of Theorem 4

Proof. Let $\|f\|_{L^2(\Omega)} > 0$. For the following problem

$$\begin{cases} (-\Delta)^s u = f(x), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{17}$$

with $(-\Delta)^s$ in (2) and the problem

$$\begin{cases} (-\Delta)^s u = f(x), & \text{in} \quad \Omega, \\ u = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega \end{cases} \tag{18}$$

with $(-\Delta)^s$ in (6), we know that Problems (17) and (18) have only one solution by the results in Theorem 1. We denote by U the unique solution to Problem (17) or Problem (18). Now, we consider the following algebraic problem

$$\Big(a + b \int_\Omega |\nabla(tU)|^2 \mathrm{d}x \Big) (-\Delta)^s (tU) = \lambda f(x).$$

Then, tU is a solution to Problem (8) if $H = H_0^s(\Omega)$ and tU is a solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$.

Equivalently, we consider the existence of the zero point t for the function $g : \mathbb{R} \to \mathbb{R}$,

$$t \mapsto g(t) = \Big(a + bt^2 \int_\Omega |\nabla U|^2 \mathrm{d}x\Big)t - \lambda,$$

where $\lambda > 0$ is a parameter. Obviously, $g \in C^\infty(\mathbb{R})$, considering that the zero point t of g, tU is the solution to Problem (8) or Problem (9).

If $ab \geq 0$ and $a + b \neq 0$, we can see that $g(t)$ is strictly monotonically increasing and

$$g(-\infty) = -\infty < g(+\infty) = +\infty, \text{ if } a, b \geq 0$$

or strictly monotonically decreasing on $\mathbb{R}$ and

$$g(-\infty) = +\infty > g(+\infty) = -\infty, \text{ if } a, b \leq 0.$$

So, the function $g(t)$ has only one zero point t_1. That is to say, $t_1 U$ is the unique solution to Problem (8) if $H = H_0^s(\Omega)$ and $t_1 U$ is the unique solution to Problem (9) if $H = \mathcal{H}_0^s(\Omega)$.

Next, we set the proof as two steps.

Step 1.We prove that for the conditions $ab < 0$ and $\|f\|_{L^2(\Omega)} > 0$, there exists $\lambda_* > 0$, such that Problems (8) and (9) have at least one, two, or three solutions, respectively, when $\lambda > \lambda^*$, $\lambda = \lambda^*$, or $\lambda \in (0, \lambda^*)$.

If $ab < 0$, it holds that

$$g'(t) = a + 3bt^2 \int_\Omega |\nabla U|^2 \mathrm{d}x = a + 3bt^2 \|U\|^2.$$

For the case $a > 0 > b$, we have $g(-\infty) = +\infty > g(+\infty) = -\infty$,

$$g'(t) \begin{cases} < 0, & \text{if} \quad t \in \big(-\infty, -\sqrt{\frac{-a}{3b\|U\|^2}}\big), \\ = 0, & \text{if} \quad t = -\sqrt{\frac{-a}{3b\|U\|^2}}, \\ > 0, & \text{if} \quad t \in \big(-\sqrt{\frac{-a}{3b\|U\|^2}}, \sqrt{\frac{-a}{3b\|U\|^2}}\big), \\ = 0, & \text{if} \quad t = \sqrt{\frac{-a}{3b\|U\|^2}}, \\ < 0, & \text{if} \quad t \in \big(\sqrt{\frac{-a}{3b\|U\|^2}}, +\infty\big). \end{cases}$$

This implies that $g(t)$ achieves its local minimum at $t_L = -\sqrt{\frac{-a}{3b\|U\|^2}}$ and achieves its local maximum at $t_R = \sqrt{\frac{-a}{3b\|U\|^2}}$. Let t_1, t_2, t_3 be the zero points of $g(t)$, then, a general algebraic analysis shows that

$$\begin{cases} t_1 < t_L; t_2, t_3 \text{ non-existent} & \text{if} \quad g(t_L) < g(t_R) < 0, \\ t_1 < t_L < t_2 < t_R < t_3 & \text{if} \quad g(t_L) < 0 < g(t_R), \\ t_1 = t_L = t_2 < t_R < t_3 & \text{if} \quad g(t_L) = 0 < g(t_R), \\ t_1 < t_L < t_2 = t_R = t_3 & \text{if} \quad g(t_L) < 0 = g(t_R), \\ t_3 > t_R; t_1, t_2 \text{ non-existent} & \text{if} \quad 0 < g(t_L) < g(t_R). \end{cases}$$

Since

$$g(t_L) = -\Big(a + b\|U\|^2 \cdot \frac{-a}{3b\|U\|^2}\Big)\sqrt{\frac{-a}{3b\|U\|^2}} - \lambda = \frac{-2a}{3}\sqrt{\frac{-a}{3b\|U\|^2}} - \lambda < 0$$

and

$$g(t_R) = \left(a + b\|U\|^2 \cdot \frac{-a}{3b\|U\|^2}\right)\sqrt{\frac{-a}{3b\|U\|^2}} - \lambda = \frac{2a}{3}\sqrt{\frac{-a}{3b\|U\|^2}} - \lambda,$$

there exists $\lambda^* = \frac{2a}{3}\sqrt{\frac{-a}{3b\|U\|^2}}$ such that the zero points t_1, t_2, t_3 of $g(t)$ satisfy

$$\begin{cases} t_1 < -\sqrt{\frac{-a}{3b\|U\|^2}}; t_2, t_3 \text{ non-existent} & \text{if} \quad \lambda \in (\lambda^*, +\infty), \\ t_1 < -\sqrt{\frac{-a}{3b\|U\|^2}} < t_2 = \sqrt{\frac{-a}{3b\|U\|^2}} = t_3 & \text{if} \quad \lambda = \lambda^*, \\ t_1 < -\sqrt{\frac{-a}{3b\|U\|^2}} < t_2 < \sqrt{\frac{-a}{3b\|U\|^2}} < t_3 & \text{if} \quad \lambda \in (0, \lambda^*). \end{cases}$$

For the case $a < 0 < b$, we have $g(-\infty) = +\infty > g(+\infty) = -\infty$,

$$g'(t) \begin{cases} > 0, & \text{if} \quad t \in \big(-\infty, -\sqrt{\frac{-a}{3b\|U\|^2}}\big), \\ = 0, & \text{if} \quad t = -\sqrt{\frac{-a}{3b\|U\|^2}}, \\ < 0, & \text{if} \quad t \in \big(-\sqrt{\frac{-a}{3b\|U\|^2}}, \sqrt{\frac{-a}{3b\|U\|^2}}\big), \\ = 0, & \text{if} \quad t = \sqrt{\frac{-a}{3b\|U\|^2}}, \\ > 0, & \text{if} \quad t \in \big(\sqrt{\frac{-a}{3b\|U\|^2}}, +\infty\big). \end{cases}$$

So, $g(t)$ achieves its local maximum at $t_L = -\sqrt{\frac{-a}{3b\|U\|^2}}$ and achieves its local minimum at $t_R = \sqrt{\frac{-a}{3b\|U\|^2}}$. Let t_1, t_2, t_3 be the roots of $g(t)$, then, a general algebraic analysis shows that

$$\begin{cases} t_1 < t_L; t_2, t_3 \text{ non-existent} & \text{if} \quad g(t_L) > g(t_R) > 0, \\ t_1 < t_L < t_2 = t_R = t_3 & \text{if} \quad g(t_L) > 0 = g(t_R), \\ t_1 < t_L < t_2 < t_R < t_3 & \text{if} \quad g(t_L) > 0 > g(t_R), \\ t_1 = t_L = t_2 < t_R < t_3 & \text{if} \quad g(t_L) = 0 > g(t_R), \\ t_3 > t_R; t_1, t_2 \text{ non-existent} & \text{if} \quad 0 > g(t_L) > g(t_R). \end{cases}$$

Since

$$\begin{cases} g(t_L) = \dfrac{-2a}{3}\sqrt{\dfrac{-a}{3b\|U\|^2}} - \lambda, \\ g(t_R) = \dfrac{2a}{3}\sqrt{\dfrac{-a}{3b\|U\|^2}} - \lambda < 0, \end{cases}$$

there exists $\lambda^* = -\frac{2a}{3}\sqrt{\frac{-a}{3b\|U\|^2}}$ such that the zero points t_1, t_2, t_3 of $g(t)$ satisfy

$$\begin{cases} t_3 > \sqrt{\frac{-a}{3b\|U\|^2}};\ t_1, t_2 \text{ non-existent} & \text{if} \quad \lambda \in (\lambda^*, +\infty), \\ t_1 = -\sqrt{\frac{-a}{3b\|U\|^2}} = t_2 < \sqrt{\frac{-a}{3b\|U\|^2}} < t_3 & \text{if} \quad \lambda = \lambda^*, \\ t_1 < -\sqrt{\frac{-a}{3b\|U\|^2}} < t_2 < \sqrt{\frac{-a}{3b\|U\|^2}} < t_3 & \text{if} \quad \lambda \in (0, \lambda^*). \end{cases}$$

As a conclusion, we find that there exists $\lambda^* = \frac{2|a|}{3}\sqrt{\frac{-a}{3b\|U\|^2}} > 0$, such that Problems (8) and (9) have at least one, two, or three solutions, respectively, when $\lambda > \lambda^*$, $\lambda = \lambda^*$, or $\lambda \in (0, \lambda^*)$, where all solutions can be expressed as constant multiples of $U(x)$, U is the unique solution to Problem (17) if $H = H_0^s(\Omega)$, and U is the unique solution to Problem (18) if $H = \mathcal{H}_0^s(\Omega)$.

Step 2. The numbers of solutions in Step 1 are unique and linearly dependent.

Let u, v be two different solutions to Problem (8) or Problem (9), then, $(a + b\|u\|^2)$ and $(a + b\|v\|^2)$ are non-zero constants and

$$\left.\begin{aligned} &(a+b\|u\|^2)(-\Delta)^s u = (-\Delta)^s\left[(a+b\|u\|^2)u\right] \\ &(a+b\|v\|^2)(-\Delta)^s v = (-\Delta)^s\left[(a+b\|v\|^2)v\right] \end{aligned}\right\} = \lambda f(x), \not\equiv 0 \text{ in } L^2(\Omega).$$

Hence, one has

$$(a+b\|u\|^2)u = (a+b\|v\|^2)v,$$

This shows that all solutions to Problems (8) and (9) are linearly dependent.

From the process in Step 1, we know that the linearly dependent solutions to Problems (8) and (9) can be expressed as constant multiples of $U(x)$, where U is the unique solution to Problem (17) if $H = H_0^s(\Omega)$ and U is the unique solution to Problem (18) if $H = \mathcal{H}_0^s(\Omega)$. Therefore, all the solutions to Problem (8) or Problem (9) are t_1U, t_2U, t_3U, where t_1, t_2, t_3 are all the zero points of function $g(t)$. So, it is clear that Problems (8) and (9) have only one, two, or three solutions, respectively, when $\lambda > \lambda^*$, $\lambda = \lambda^*$, or $\lambda \in (0, \lambda^*)$ under the assumptions $ab < 0$ and $\|f\|_{L^2(\Omega)} > 0$. Furthermore, $\lambda^* = \frac{2|a|}{3}\sqrt{\frac{-a}{3b\|U\|^2}}$, where U is the unique solution to Problem (17) if $H = H_0^s(\Omega)$ and U is the unique solution to Problem (18) if $H = \mathcal{H}_0^s(\Omega)$. The proof is completed. □

4. Conclusions

In this paper, the Kirchhoff-type problem is considered with a negative modulus and different fractional Laplacian operators in an unified framework. First, we state the definitions of two fractional Laplacian operators and some properties of two fractional-order Hilbert spaces. Second, we list the main results for the existence, uniqueness, and multiplicity of the solutions for the considered problem. Moreover, we show the recent research on this kind of problem. Finally, we provide proof of the main results using the variational method and algebraic analysis. As a result, we obtain a constant λ^*, which makes the problems have only one, two, or three solutions, respectively, when $\lambda > \lambda^*$, $\lambda = \lambda^*$, or $0 < \lambda < \lambda^*$. In particular, we calculate the expression of the constant λ^*.

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