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Extremal Graphs to Vertex Degree Function Index for Convex Functions

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Abstract: The *vertex-degree function index* $H_f(\Gamma)$ is defined as $H_f(\Gamma) = \sum_{v \in V(\Gamma)} f(d(v))$ for a function $f(x)$ defined on non-negative real numbers. In this paper, we determine the extremal graphs with the maximum (minimum) vertex degree function index in the set of all n-vertex chemical trees, trees, and connected graphs. We also present the Nordhaus–Gaddum-type results for $H_f(\Gamma) + H_f(\overline{\Gamma})$ and $H_f(\Gamma) \cdot H_f(\overline{\Gamma})$.

Keywords: vertex degree function index; tree; cemical tree; connected graph; Nordhaus–Gaddum-type result

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1. Introduction

In this paper, the graphs we discuss are simple graphs without multiple edges and loops. The vertex and edge set of Γ will be denoted by $V(\Gamma)$ and $E(\Gamma)$, and the order and size of Γ will usually be denoted by n and m, respectively. Let a vertex $v \in V(\Gamma)$; we denote the *degree* of v by $d_\Gamma(v)$ in Γ. The *neighbors* of v in $V(\Gamma)$ are denoted by $N_\Gamma(v)$. For a graph Γ, we denote the maximum and minimum degree of Γ by $\Delta(\Gamma)$ and $\delta(\Gamma)$, respectively. A *leaf* $v \in V(\Gamma)$ is a vertex v satisfied $d_\Gamma(v) = 1$. We call a connected graph without a cycle a *tree*, denoted by T. A tree whose maximum degree is no more than 4 is called a *chemical tree*. The *star graph* with order n, denoted by S_n, is a tree with one center vertex and $n-1$ leaves. The *disjoint union* of two vertex-disjoint graphs Γ_1 and Γ_2 will be denoted by $\Gamma_1 \cup \Gamma_2$, whose vertex and edge sets are $V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1) \cup E(\Gamma_2)$, respectively. We denote the *union* of k copies of a graph Γ by $k\Gamma$. The *join* of Γ_1 and Γ_2 is obtained by joining edges between each vertex of Γ_1 and all vertices of Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$. For a graph Γ, the edge $uv \in E(\Gamma)$ and the vertex $w \in V(\Gamma)$, $\Gamma - uv$ mean removing uv from Γ and $\Gamma - w$, which means removing w from Γ .

A *universal vertex* of Γ with order n is a vertex v that have $d(v) = n-1$. An (n,m)-*graph* is the graph with n vertices and m edges. We denote by $\Gamma(n,m)$ the set of (n,m)-graphs. The *cyclomatic number* of a graph Γ is the minimum number of edges whose deletion transforms Γ into an acyclic graph, denoted by $\gamma(\Gamma)$. The set of graphs with order n and cyclomatic number γ is denoted by $\Gamma_{n,\gamma}$. We have $\gamma(\Gamma) = m - n + 1$ for a connected graph $\Gamma \in \Gamma_{n,\gamma}$.

The *vertex-degree function index* $H_f(\Gamma)$ is denoted by

$$H_f(\Gamma) = \sum_{v \in V(\Gamma)} f(d(v))$$

for a function $f(x)$ defined on non-negative real numbers in [1]. For example, the first Zagreb index [2] is defined as $M_1(\Gamma) = \sum_{v \in V(\Gamma)} d(v)^2$ when $f(x) = x^2$, and the forgotten topological index [3] is defined as $F(\Gamma) = \sum_{v \in V(\Gamma)} d(v)^3$ when $f(x) = x^3$. The general first

Zagreb index, denoted by ${}^0R_\alpha(\Gamma)$, was defined in [4,5] as ${}^0R_\alpha(\Gamma) = \sum_{v \in V(\Gamma)} d(v)^\alpha$, where α is a real number, $\alpha \notin \{0, 1\}$. For the mathematical properties of the above topological indices, see [6–11] and the references therein. Let v be a leaf of S_n, where $n \geq 3$. For $0 \leq \gamma \leq n-2$, the graph obtained from S_n by joining edges v with γ other pendant vertices is denoted by $H_{n,\gamma}$ in [12]. Deng [10] obtained the bounds of the Zagreb indices for trees, unicyclic graphs, and bicyclic graphs. Hu and Li determined the connected (n, m)-graphs with the minimum and maximum zeroth-order general Randić index in [13]. Li and Zheng [5] obtained a unified approach to the extremal trees for different indices. Some extremal results concerning the general zeroth-order Randić index were deduced in [14–16]; also see the survey [12].

In [17], Tomescu obtained that the function $f(x)$ has property $(P_\nearrow; P_\searrow)$ if $\varphi(i+1) > \varphi(i); \varphi(i+1) < \varphi(i)$, respectively, for every integer $i \geq 0$, where $\varphi(x) = f(x+2) + f(x) - 2f(x+1)$, and he obtained the maximum (minimum) vertex degree function index $H_f(\Gamma)$ in the set of all n-vertex connected graphs that have the cyclomatic number γ when $0 \leq \gamma \leq n-2$ if $f(x)$ is strictly convex (concave) and satisfies the property $P_\nearrow; P_\searrow$. Tomescu [18] obtained some structural properties of connected (n, m)-graphs which are maximum (minimum) with respect to vertex-degree function index $H_f(\Gamma)$, when $f(x)$ is a strictly convex (concave) function. In the same paper, it is also shown that the unique graph obtained from the star S_n by adding γ edges between a fixed pendant vertex v and γ other pendant vertices has the maximum general zeroth-order Randić index ${}^0R_\alpha$ in the set of all n-vertex connected graphs that have the cyclomatic number γ when $1 \leq \gamma \leq n-2$ and $\alpha \geq 2$.

Tomescu obtained the following results.

Theorem 1 ([18])**.** *In the set of connected (n, m)-graphs Γ that have $m \geq n$, the graph that maximizes (minimizes) $H_f(\Gamma)$ where $f(x)$ is strictly convex (concave) possesses the following properties:*

(1) Γ has a universal vertex v;

(2) The subgraph $\Gamma - v$ consists of some isolated vertices and a nontrivial connected component C, which is maximum (minimum) relatively to H_g, where $g(x) = f(x+1)$. C also contains a universal vertex and no induced subgraph isomorphic to P_4 or C_p, where $p \geq 4$.

Theorem 2 ([17])**.** *If $n \geq 3, 1 \leq \gamma \leq n-2$, $f(x)$ is strictly convex and has property $P_\nearrow$, and Γ is a connected n-vertex graph with cyclomatic number γ, then*

$$H_f(\Gamma) \leq f(n-1) + f(\gamma+1) + \gamma f(2) + (n-\gamma-2)f(1),$$

with equality if and only if $\Gamma \cong H_{n,\gamma} \cong K_1 \vee (K_{1,\gamma} \cup (n-\gamma-2)K_1)$.

In Section 2, we give upper and lower bounds for the vertex degree function index of connected graphs if $f(x)$ is a convex and increasing function that has property $P_\nearrow$. We obtain sharp upper and lower bounds for the vertex degree function index of trees and chemical trees if $f(x)$ is a convex and increasing function.

Let $f(\Gamma)$ be a graph invariant and n be a positive integer. The *Nordhaus–Gaddum Problem* is to determine sharp bounds for $f(\Gamma) + f(\overline{\Gamma})$ and $f(\Gamma) \cdot f(\overline{\Gamma})$ as Γ ranges over the class of all graphs of order n, and to characterize the extremal graphs, i.e., graphs that achieve the bounds. Nordhaus–Gaddum-type relations have received wide attention; see the recent survey [19] by Aouchiche and Hansen and the book chapter by Mao [20].

Denote by $\mathcal{G}(n)$ the class of connected graphs of order n whose complements are also connected. In Section 3, the upper and lower bounds for $H_f(\Gamma) + H_f(\overline{\Gamma})$ and $H_f(\Gamma) \cdot H_f(\overline{\Gamma})$ are given for $\Gamma \in \mathcal{G}(n)$.

2. Bounds on $H_f(\Gamma)$

At first, we give the following upper bound for $H_f(\Gamma)$.

Theorem 3. *Let Γ be an n-vertex $(n \geq 5)$, m-edge graph with a cyclomatic number γ such that $\gamma \in [2n-t-3, 3n-2t-7]$, where t $(1 \leq t \leq n-4)$ is the number of pendant vertices in Γ. If $f(x)$ is a strictly convex function that has property $P_{\nearrow}$, then*

$$\begin{aligned} H_f(\Gamma) \leq & f(n-1) + tf(1) + f(n-t) + f(\gamma-n+t+5) + (\gamma-n+t+4)f(3) \\ & + (2n-2t-\gamma-5)f(2) \end{aligned}$$

with equality if and only if $\Gamma \cong K_1 \vee \Big(\Big(K_1 \vee \Big(K_{1,\gamma_2} \cup (n_2-\gamma_2-2)K_1\Big)\Big) \cup tK_1\Big)$, where $n_2 = n-t-1, \gamma_2 = \gamma-n+t+2$.

Proof. Let $\Gamma \in \Gamma_{n,\gamma}$ such that $H_f(\Gamma)$ is maximum. By (1) of Theorem 1, a universal vertex $v_1 \in V(\Gamma)$ exists, and hence

$$H_f(\Gamma) = f(n-1) + H_{g_1}(\Gamma - v_1),$$

where $g_1(x) = f(x+1)$. By (2) of Theorem 1, $\Gamma - v_1$ consists of some isolated vertices and a nontrivial connected component C. Let $\Gamma_1 = \Gamma - v_1$. Note that t is the number of isolated vertices of $\Gamma - v_1$; we have

$$H_{g_1}(\Gamma - v_1) = tg_1(0) + H_{g_1}(C).$$

Suppose that m_1, n_1, γ_1 and m_2, n_2, γ_2 are the number edges, vertices, and cyclomatic number of Γ_1, Γ_2, respectively, where $\Gamma_1 = \Gamma - v_1$ and $\Gamma_2 = C$. Since $\gamma_2 = m_2 - n_2 + 1, m_2 = m_1 - n_1 + 1, n_2 = n_1 - t, m_1 = m - n + 1, n_1 = n - 1$, we have $m_2 = m - 2n + 3$ and $n_2 = n - t - 1$, it follows that $\gamma_2 = \gamma - 2n + t + 4$; note that $\gamma \in [2n-t-3, 3n-2t-7]$, so $1 \leq \gamma_2 \leq n_2 - 2$ and $n_2 \geq 3$, which implies $m_2 \geq n_2$. Then, we know that Γ_2 is a connected n_2-vertex graph with cyclomatic number γ_2 and $1 \leq \gamma_2 \leq n_2 - 2$, $n_2 \geq 3$. So, we can apply Theorem 2 for Γ_2 and we have

$$H_{g_1}(\Gamma_2) \leq g_1(n_2-1) + g_1(\gamma_2+1) + \gamma_2 g_1(2) + (n_2-\gamma_2-2)g_1(1)$$

with equality if only if $\Gamma_2 \cong K_1 \vee \Big(K_{1,\gamma_2} \cup (n_2-\gamma_2-2)K_1\Big)$.

Hence, we have

$$\begin{aligned} H_f(\Gamma) =& f(n-1) + tg_1(0) + H_{g_1}(\Gamma_2) \\ \leq & f(n-1) + tg_1(0) + g_1(n_2-1) + g_1(\gamma_2+1) + \gamma_2 g_1(2) + (n_2-\gamma_2-2)g_1(1) \\ =& f(n-1) + tf(1) + f(n-t) + f(\gamma-n+t+5) + (\gamma-n+t+4)f(3) \\ & + (2n-2t-\gamma-5)f(2) \end{aligned}$$

with equality if only if $\Gamma \cong K_1 \vee \Big(\Big(K_1 \vee \Big(K_{1,\gamma_2} \cup (n_2-\gamma_2-2)K_1\Big)\Big) \cup tK_1\Big)$, where $n_2 = n-t-1, \gamma_2 = \gamma-n+t+2$. □

A similar result holds for strictly concave functions $f(x)$, which have property $P_{\searrow}$: the minimum of $H_f(\Gamma)$ is reached in $\Gamma_{n,\gamma}$ if and only if $\Gamma \cong K_1 \vee \Big(\Big(K_1 \vee \Big(K_{1,\gamma_2} \cup (n_2-\gamma_2-2)K_1\Big)\Big) \cup tK_1\Big)$, where $n_2 = n-t-1, \gamma_2 = \gamma-n+t+2$.

Lemma 1. *If $f(x)$ is a convex function, then $f(x) - f(x-a) \geq f(x-b) - f(x-b-a)$ with equality if and only if $b = 0$, where $a, b \geq 0$.*

Proof. Let $h(x) = f(x) - f(x-a)$. Since $f(x)$ is a convex function, it follows that $f'(x)$ is an increasing function and $h'(x) = f'(x) - f'(x-a) \geq 0$. So, $h(x)$ is an increasing function and $h(x) \geq h(x-b)$ with equality if and only if $b=0$, and therefore $f(x)-f(x-a) \geq f(x-b)-f(x-b-a)$. $\square$

We now give a lower bound for $H_f(T)$.

Theorem 4. *Let T be a tree of order n $(n \geq 4)$. If $f(x)$ is a convex function, then $H_f(T) \geq (n-2)f(2)+2f(1)$ with equality if and only if $T \cong P_n$.*

Proof. If $n=4$, then $T \cong S_4$ or $T \cong P_4$. One can easily check that

$$H_f(S_4) = 3f(1)+f(3) > 2\,f(2)+2\,f(1) = H_f(P_4)$$

as $f(3)-f(2) > f(2)-f(1)$, by Lemma 1. The result holds for $n=4$.

We now suppose that $n \geq 5$. We prove this result by the induction on n. Assume that the result holds for $n-1$ and prove it for n. Let T' be a tree of order $n-1$ such that $T-v_j = T'$, where $d_T(v_j)=1$, $v_i = N_T(v_j)$ and $d_{T'}(v_i) = d_T(v_i)-1 = p-1$. Thus, we have $H_f(T') \geq (n-3)f(2)+2\,f(1)$ with equality if and only if $T' \cong P_{n-1}$. One can easily see that

$$H_f(T) = H_f(T') + f(p) - f(p-1) + f(1).$$

Since $f(x)$ is a convex function, it follows from Lemma 1 that $f(p)-f(p-1) \geq f(2)-f(1)$ with equality if and only if $p=2$. Therefore, by the induction hypothesis with the above results, we obtain

$$\begin{aligned}
H_f(T) &= H_f(T') + f(p) - f(p-1) + f(1) \\
&\geq (n-3)f(2) + 3\,f(1) + f(p) - f(p-1) \\
&\geq (n-2)f(1) + 2\,f(1)
\end{aligned}$$

and the result holds by induction. Moreover, the equality holds if and only if $T' \cong P_{n-1}$ and $d_T(v_i) = p = 2$, that is, if and only if $T \cong P_n$. $\square$

Corollary 1. *Let T be a chemical tree of order n $(n \geq 4)$. If $f(x)$ is a convex function, then $H_f(T) \geq (n-2)\,f(2)+2\,f(1)$ with equality if and only if $T \cong P_n$.*

Using Theorem 4, we obtain a lower bound for $H_f(\Gamma)$.

Theorem 5. *Let Γ be a connected graph of order n $(n \geq 4)$. If $f(x)$ is a convex and increasing function, then $H_f(\Gamma) \geq (n-2)f(2)+2f(1)$ with equality if and only if $\Gamma \cong P_n$.*

Proof. Since $f(x)$ is an increasing function, it follows that $f(x+1)+f(y+1) \geq f(x)+f(y)$, and hence $H_f(\Gamma+e) \geq H_f(\Gamma)$, where e is an edge joining between two non-adjacent vertices in Γ. Clearly, for the graph Γ of order n, we have $H_f(\Gamma) \geq H_f(T)$, where T is a tree of order n. This result with Theorem 4, we obtain $H_f(\Gamma) \geq H_f(T) \geq (n-2)\,f(2)+2\,f(1)$. Moreover, the equality holds if and only if $T \cong P_n$. $\square$

A complete split graph $CS(n,\alpha)$ is defined as the graph join $\overline{K}_\alpha \vee K_{n-\alpha}$, where α is the independence number of graph $CS(n,\alpha)$.

Theorem 6. *Let Γ be a connected graph of order n $(n \geq 4)$ with independence number α. If $f(x)$ is a strictly increasing function, then $H_f(\Gamma) \leq (n-\alpha)\,f(n-1)+\alpha\,f(n-\alpha)$ with equality if and only if $\Gamma \cong CS(n,\alpha)$.*

Proof. Since $f(x)$ is a strictly increasing function, it follows that $f(x+1)+f(y+1) > f(x)+f(y)$, and hence $H_f(\Gamma+e) > H_f(\Gamma)$, where e is an edge joining between two non-adjacent vertices in Γ. Since Γ is a graph of order n with independence number α, we must have that Γ is a subgraph of $CS(n,\alpha)$. If $\Gamma \cong CS(n,\alpha)$, then $H_f(\Gamma) = (n-\alpha)f(n-1) + \alpha f(n-\alpha)$; hence, the equality holds. Otherwise, $\Gamma \ncong CS(n,\alpha)$. Since Γ is a subgraph of $CS(n,\alpha)$ and $H_f(\Gamma+e) > H_f(\Gamma)$, we obtain $H_f(\Gamma) < H_f(\Gamma+e) < \cdots < H_f(CS(n,\alpha)-e_1) < H_f(CS(n,\alpha)) = (n-\alpha)f(n-1)+\alpha f(n-\alpha)$, where e_1 is an edge in $CS(n,\alpha)$. This completes the proof of the theorem. □

Let C be the set of pendant vertices, and let A be the set of non-leaf vertices that have at least 2 neighbor vertices, each of which are not leaves. Let B be the set of non-leaf vertices that have only one neighbor vertex, which is not a leaf. Note that $V(\Gamma) = A \cup B \cup C$.

Lemma 2. *Let Γ be a graph of order n, and $f(x)$ be a convex function.*

(1) *If $u \in A$, $w \in B$, and $xw \in E(\Gamma)$ such that $d_\Gamma(x) = 1$, $d_\Gamma(u) = 2$ or 3, $d_\Gamma(w) = 2$ or 3, then $H_f(\Gamma_1) \geq H_f(\Gamma)$, where $\Gamma_1 = \Gamma - wx + ux$.*

(2) *If $u \in A$, $w \in B$, and $xw, yw \in E(\Gamma)$ such that $d_\Gamma(x) = d_\Gamma(y) = 1$, $d_\Gamma(u) = 2$, $d_\Gamma(w) = 4$, then $H_f(\Gamma_2) = H_f(\Gamma)$, where $\Gamma_2 = \Gamma - wx - wy + ux + uy$.*

(3) *If $u \in A$, $w \in B$, and $xw \in E(\Gamma)$ such that $d_\Gamma(x) = 1$, $d_\Gamma(u) = 3$, $d_\Gamma(w) = 4$, then $H_f(\Gamma_3) = H_f(\Gamma)$, where $\Gamma_3 = \Gamma - wx + ux$.*

(4) *If $u, v \in B$, and $xu \in E(\Gamma)$ such that $d_\Gamma(x) = 1$, $d_\Gamma(u) = 2$ or 3, $d_\Gamma(v) = 3$, then $H_f(\Gamma_4) \geq H_f(\Gamma)$, where $\Gamma_4 = \Gamma - ux + vx$.*

(5) *If $u, v, w \in B$, and $xu, yv \in E(\Gamma)$ such that $d_\Gamma(x) = d_\Gamma(y) = 1$, $d_\Gamma(u) = d_\Gamma(v) = d_\Gamma(w) = 2$, then $H_f(\Gamma_5) \geq H_f(\Gamma)$, where $\Gamma_5 = \Gamma - xu - yv + wx + wy$.*

(6) *If $u, v \in B$, and $xu \in E(\Gamma)$ such that $d_\Gamma(x) = 1$, $d_\Gamma(u) = 2$ and $d_\Gamma(v) = 2$, then $H_f(\Gamma_6) \geq H_f(\Gamma)$, where $\Gamma_6 = \Gamma - ux + vx$.*

Proof. Suppose that Γ is the graph of order n and $f(x)$ is convex.
For (1), from Lemma 1, $f(d_\Gamma(u)+1) - f(d_\Gamma(u)) + f(d_\Gamma(w)-1) - f(d_\Gamma(w)) \geq 0$ holds for $d_\Gamma(u) = 2,3$ and $d_\Gamma(w) = 2,3$, and hence

$$H_f(\Gamma_1) = H_f(\Gamma) + f(d_\Gamma(u)+1) - f(d_\Gamma(u)) + f(d_\Gamma(w)-1) - f(d_\Gamma(w)) \geq H_f(\Gamma).$$

For (2), we can easily obtain

$$\begin{aligned} H_f(\Gamma_2) &= H_f(\Gamma) + f(d_\Gamma(u)+2) - f(d_\Gamma(u)) + f(d_\Gamma(w)-2) - f(d_\Gamma(w)) \\ &= H_f(\Gamma) + f(4) - f(2) + f(2) - f(4) = H_f(\Gamma). \end{aligned}$$

For (3), we have

$$\begin{aligned} H_f(\Gamma_3) &= H_f(\Gamma) + f(d_\Gamma(u)+1) - f(d_\Gamma(u)) + f(d_\Gamma(w)-1) - f(d_\Gamma(w)) \\ &= H_f(\Gamma) + f(4) - f(3) + f(3) - f(4) = H_f(\Gamma). \end{aligned}$$

For (4), from Lemma 1, we know that $f(d_\Gamma(v)+1) - f(d_\Gamma(v)) + f(d_\Gamma(u)-1) - f(d_\Gamma(u)) \geq 0$ holds for $d_\Gamma(u) = 2,3$ and $d_\Gamma(v) = 3$, and hence

$$H_f(\Gamma_4) = H_f(\Gamma) + f(d_\Gamma(v)+1) - f(d_\Gamma(v)) + f(d_\Gamma(u)-1) - f(d_\Gamma(u)) \geq H_f(\Gamma).$$

For (5), since $f(x)$ is a convex function, it follows that $f(1) + f(3) \geq 2f(2)$. From Lemma 1, we have $f(1) + f(4) \geq f(2) + f(3)$, and hence $2f(1) + f(4) \geq 3f(2)$. Then,

$$\begin{aligned} H_f(\Gamma_5) =& H_f(\Gamma) + f(d_\Gamma(v)-1) - f(d_\Gamma(v)) + f(d_\Gamma(u)-1) - f(d_\Gamma(u)) \\ & + f(d_\Gamma(w)+2) - f(d_\Gamma(w)) \\ =& H_f(\Gamma) + 2f(1) - 2f(2) + f(4) - f(2) \\ =& H_f(\Gamma) + 2f(1) + f(4) - 3f(2) \geq H_f(\Gamma). \end{aligned}$$

For (6), from Lemma 1, we know that $f(d_\Gamma(v)+1) - f(d_\Gamma(v)) + f(d_\Gamma(u)-1) - f(d_\Gamma(u)) \geq 0$ holds for $d_\Gamma(u) = 2$ and $d_\Gamma(v) = 2$, and hence

$$H_f(\Gamma_6) = H_f(\Gamma) + f(d_\Gamma(v)+1) - f(d_\Gamma(v)) + f(d_\Gamma(u)-1) - f(d_\Gamma(u)) \geq H_f(\Gamma).$$

□

For chemical trees, we have the following upper bound.

Theorem 7. *Let T be a chemical tree of order n ($n \geq 5$). If $f(x)$ is a convex function, then three integers (m_1, m_2, m_3) exist such that*

$$H_f(T) \leq \begin{cases} m_1 f(4) + (n - m_1 - 1) f(1) + f(2) & \text{if } i = 0, \\ m_2 f(4) + (n - m_2 - 1) f(1) + f(3) & \text{if } i = 1, \\ m_3 f(4) + (n - m_3) f(1) & \text{if } i = 2 \end{cases}$$

with equality if and only if T contains only one 2-degree vertex but contains no 3-degree vertices for $i = 0$; T contains only one 3-degree vertex but contains no 2-degree vertices for $i = 1$; and T only contains 1-degree vertices and 4-degree vertices for $i = 2$, where $n \equiv i \pmod 3$.

Proof. Suppose that T is a chemical tree of order n and $f(x)$ is a convex function. By operations (1), (2), and (3) of Lemma 2, we can obtain a new tree T' with $V(T') = V(T)$ containing no 2-degree vertices or 3-degree vertices in A. That is to say, all of the 2-degree vertices and 3-degree vertices are in B. Suppose that $n \equiv i \pmod 3$ and n_1, n_2, n_3, n_4 are the number of vertices with degree $1, 2, 3, 4$, respectively, in T'.

Note that $H_f(T') \geq H_f(T)$. We distinguish the following cases to show this theorem.

Case 1. $i = 0$.

We claim that $n_2 \neq 0$ or $n_3 \neq 0$; otherwise, T' contains only 1-degree and 4-degree vertices. Since $n_1 + n_4 = n$ and $n_1 + 4n_4 = 2(n-1)$, we have $n = 3n_4 + 2$, contradicting the fact that $n \equiv 0 \pmod 3$.

Since $n_1 + n_2 + n_3 + n_4 = n$ and $n_1 + 2n_2 + 3n_3 + 4n_4 = 2(n-1)$, we have $n_2 + 2n_3 \equiv 1 \pmod 3$, and so $n_2 - n_3 \equiv 1 \pmod 3$ and $n_3 - n_2 \equiv 2 \pmod 3$.

If $n_2 \geq n_3$, then it follows from (4) of Lemma 2 that

$$\begin{aligned} H_f(T') &= n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \\ &\leq (n_1 + n_3) f(1) + (n_2 - n_3) f(2) + (n_4 + n_3) f(4). \end{aligned}$$

Suppose that $n_2 - n_3 = 3k_1 + 1$. From (5) of Lemma 2, we have

$$H_f(T') \leq (n_1 + n_3 + 2k_1) f(1) + f(2) + (n_4 + n_3 + k_1) f(4).$$

Let $m_1 = n_4 + n_3 + k_1$, and thus we are done.

If $n_2 < n_3$, then it follows from (4) of Lemma 2 that $H_f(T') = n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \leq (n_1 + n_2) f(1) + (n_3 - n_2) f(3) + (n_4 + n_2) f(4)$. Suppose that $n_3 - n_2 = 3\ell_1 + 2$. From (4) of Lemma 2, we have

$$\begin{aligned} H_f(T') &\leq (n_1 + n_2) f(1) + (\ell_1 + 1) f(2) + \ell_1 f(3) + (n_4 + n_2 + \ell_1 + 1) f(4) \\ &\leq (n_1 + n_2 + \ell_1) f(1) + f(2) + (n_4 + n_2 + 2\ell_1 + 1) f(4). \end{aligned}$$

Let $m_1 = n_4 + n_2 + 2\ell_1 + 1$, and thus we are done.

Case 2. $i = 1$.

We claim that $n_2 \neq 0$ or $n_3 \neq 0$; otherwise, T' contains only 1-degree and 4-degree vertices. Since $n_1 + n_4 = n$ and $n_1 + 4n_4 = 2(n-1)$, we have $n = 3n_4 + 2$, contradicting the fact that $n \equiv 1 \pmod 3$.

Since $n_1 + n_2 + n_3 + n_4 = n$ and $n_1 + 2n_2 + 3n_3 + 4n_4 = 2(n-1)$, we have $n_2 + 2n_3 \equiv 2 \pmod 3$, and so $n_2 - n_3 \equiv 2 \pmod 3$, $n_3 - n_2 \equiv 1 \pmod 3$.

If $n_2 \geq n_3$, then it follows from (4) of Lemma 2 that

$$\begin{aligned} H_f(T') &= n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \\ &\leq (n_1 + n_3) f(1) + (n_2 - n_3) f(2) + (n_4 + n_3) f(4). \end{aligned}$$

If $n_2 - n_3 = 3k_2 + 2$, it follows from (5) and (6) of Lemma 2 that

$$\begin{aligned} H_f(T') &\leq (n_1 + n_3 + 2k_2) f(1) + 2f(2) + (n_4 + n_3 + k_2) f(4) \\ &\leq (n_1 + n_3 + 2k_2 + 1) f(1) + f(3) + (n_4 + n_3 + k_2) f(4). \end{aligned}$$

Let $m_2 = n_4 + n_3 + k_2$, and thus we are done.

If $n_2 < n_3$, then it follows from (4) of Lemma 2 that

$$\begin{aligned} H_f(T') &= n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \\ &\leq (n_1 + n_2) f(1) + (n_3 - n_2) f(3) + (n_4 + n_2) f(4). \end{aligned}$$

If $n_3 - n_2 = 3\ell_2 + 1$, then it follows from (4) of Lemma 2 that

$$\begin{aligned} H_f(T') &\leq (n_1 + n_2) f(1) + (3\ell_2 + 1) f(3) + (n_4 + n_2) f(4) \\ &\leq (n_1 + n_2) f(1) + \ell_2 f(2) + (\ell_2 + 1) f(3) + (n_4 + n_2 + \ell_2) f(4) \\ &\leq (n_1 + n_2 + \ell_2) f(1) + f(3) + (n_4 + n_2 + 2\ell_2) f(4). \end{aligned}$$

Let $m_2 = n_4 + n_2 + 2\ell_2$, and thus we are done.

Case 3. $i = 2$.

Since $n_1 + n_2 + n_3 + n_4 = n$ and $n_1 + 2n_2 + 3n_3 + 4n_4 = 2(n-1)$, we have $n_2 + 2n_3 \equiv 0 \pmod 3$, and so $n_2 - n_3 \equiv 0 \pmod 3$, $n_3 - n_2 \equiv 0 \pmod 3$.

If $n_2 \geq n_3$, then it follows from (4) of Lemma 2 that $H_f(T') = n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \leq (n_1 + n_3) f(1) + (n_2 - n_3) f(2) + (n_4 + n_3) f(4)$. Suppose that $n_2 - n_3 = 3k_3$. By (5) of Lemma 2, we have

$$H_f(T') \leq (n_1 + n_3 + 2k_3) f(1) + (n_4 + n_3 + k_3) f(4).$$

Let $m_3 = n_4 + n_3 + k_3$, and thus we are done.

If $n_2 < n_3$, then it follows from (4) of Lemma 2 that

$$\begin{aligned} H_f(T') =& n_1 f(1) + n_2 f(2) + n_3 f(3) + n_4 f(4) \\ \leq& (n_1 + n_2) f(1) + (n_3 - n_2) f(3) + (n_4 + n_2) f(4). \end{aligned}$$

Suppose that $n_3 - n_2 = 3\ell_3$. By (4) of Lemma 2, we have

$$\begin{aligned} H_f(T') &\leq (n_1 + n_2) f(1) + 3\ell_3 f(3) + (n_4 + n_2) f(4) \\ &\leq (n_1 + n_2) f(1) + \ell_3 f(2) + \ell_3 f(3) + (n_4 + n_2 + \ell_3) f(4) \\ &\leq (n_1 + n_2 + \ell_3) f(1) + (n_4 + n_2 + 2\ell_3) f(4). \end{aligned}$$

Let $m_3 = (n_4 + n_2 + 2\ell_3)$, and thus we are done. $\square$

For trees, we have the following upper bound.

Theorem 8. *Let T be a tree of order n $(n \geq 4)$. If $f(x)$ is a convex function, then $H_f(T) \leq (n-1)f(1) + f(n-1)$ with equality if and only if $T \cong S_n$.*

Proof. If $n = 4$, then by the proof of Theorem 4, we obtain $H_f(S_4) > H_f(P_4)$. The result holds for $n = 4$.

We now suppose that $n \geq 5$. We prove this result by induction on n. Assume that the result holds for $n-1$ and prove it for n. Let T' be a tree of order $n-1$ such that $T - v_j = T'$, where $d_T(v_j) = 1$, $v_i = N_T(v_j)$ and $d_{T'}(v_i) = d_T(v_i) - 1 = p - 1$, (say). Thus, we have $H_f(T') \leq (n-2)f(1) + f(n-2)$ with equality if and only if $T' \cong S_{n-1}$. One can easily see that

$$H_f(T) = H_f(T') + f(p) - f(p-1) + f(1).$$

Since $f(x)$ is a convex function, it follows from Lemma 1 that $f(n-1) - f(n-2) \geq f(p) - f(p-1)$ with equality if and only if $p = n-1$. Therefore, by the induction hypothesis with the above results, we obtain

$$\begin{aligned} H_f(T) &= H_f(T') + f(p) - f(p-1) + f(1) \\ &\leq (n-1)f(1) + f(n-2) + f(p) - f(p-1) \\ &\leq (n-1)f(1) + f(n-1) \end{aligned}$$

and the result holds by induction. Moreover, the equality holds if and only if $T' \cong S_{n-1}$ and $d_T(v_i) = p = n-1$, that is, if and only if $T \cong S_n$. □

Remark 1. *If $f(x)$ is a convex function, then by Theorems 4 and 8, we conclude that the path P_n gives the minimum $H_f(T)$ and the star gives the maximum $H_f(T)$ among all trees of order n.*

3. Nordhaus–Gaddum-Type Results

In this section, we give the Nordhaus–Gaddum-type results for the vertex degree function index.

Theorem 9. *Let Γ be a graph of order n. If $f(x)$ is a convex function, then*

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \geq \begin{cases} 2n\, f(\frac{n-1}{2}) & \text{if } n \text{ is odd,} \\ n\left[f(\frac{n}{2}) + f(\frac{n}{2} - 1)\right] & \text{if } n \text{ is even.} \end{cases}$$

Moreover, the equality holds if and only if Γ is a $\lfloor \frac{n}{2} \rfloor$-regular graph.

Proof. We have

$$\begin{aligned} H_f(\Gamma) + H_f(\overline{\Gamma}) &= \sum_{i=1}^{n} f(d_\Gamma(v_i)) + \sum_{i=1}^{n} f(n-1-d_\Gamma(v_i)) \\ &= \sum_{i=1}^{n} \left[f(d_\Gamma(v_i)) + f(n-1-d_\Gamma(v_i))\right]. \end{aligned} \tag{1}$$

We consider two cases.

Case 1. n is odd.

First, we assume that $\frac{n-1}{2} \leq d_\Gamma(v_i) \leq n-1$. Setting $x = d_\Gamma(v_i)$, $a = b = d_\Gamma(v_i) - \frac{n-1}{2}$ in Lemma 1, we obtain

$$f(d_\Gamma(v_i)) - f\left(\frac{n-1}{2}\right) \geq f\left(\frac{n-1}{2}\right) - f(n - d_\Gamma(v_i) - 1),$$

that is,

$$f(d_\Gamma(v_i)) + f(n - d_\Gamma(v_i) - 1) \geq 2\,f\left(\frac{n-1}{2}\right)$$

with equality if and only if $d_\Gamma(v_i) = \frac{n-1}{2}$. From (1), we obtain

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \geq 2\sum_{i=1}^{n} f\left(\frac{n-1}{2}\right) = 2nf\left(\frac{n-1}{2}\right)$$

with equality if and only if Γ is an $\frac{(n-1)}{2}$-regular graph, that is, if and only if Γ is a $\lfloor \frac{n}{2} \rfloor$-regular graph.

Next, we assume that $0 \leq d_\Gamma(v_i) \leq \frac{n-1}{2} - 1$, that is, $\frac{n-1}{2} < d_{\overline{\Gamma}}(v_i) \leq n-1$. Setting $x = d_{\overline{\Gamma}}(v_i)$, $a = b = d_{\overline{\Gamma}}(v_i) - \frac{n-1}{2}$ in Lemma 1, we obtain

$$f\Big(d_{\overline{\Gamma}}(v_i)\Big) - f\left(\frac{n-1}{2}\right) \geq f\left(\frac{n-1}{2}\right) - f\Big(n - d_{\overline{\Gamma}}(v_i) - 1\Big),$$

that is,

$$f\Big(d_{\overline{\Gamma}}(v_i)\Big) + f\Big(n - d_{\overline{\Gamma}}(v_i) - 1\Big) \geq 2\,f\left(\frac{n-1}{2}\right)$$

with equality if and only if $d_{\overline{\Gamma}}(v_i) = \frac{n-1}{2}$. Hence, $H_f(\Gamma) + H_f(\overline{\Gamma}) = \sum_{i=1}^{n} \Big[f\Big(d_\Gamma(v_i)\Big) + f\Big(d_{\overline{\Gamma}}(v_i)\Big)\Big] = \sum_{i=1}^{n} \Big[f\Big(n-1-d_{\overline{\Gamma}}(v_i)\Big) + f\Big(d_{\overline{\Gamma}}(v_i)\Big)\Big] \geq 2n\,f\left(\frac{n-1}{2}\right)$ with equality if and only if Γ is an $\frac{(n-1)}{2}$-regular graph, that is, if and only if Γ is a $\lfloor \frac{n}{2} \rfloor$-regular graph.

Case 2. n is even.

In this case, first we assume that $\frac{n}{2} \leq d_\Gamma(v_i) \leq n-1$. Setting $x = d_\Gamma(v_i)$, $a = d_\Gamma(v_i) - \frac{n}{2} + 1$, $b = d_\Gamma(v_i) - \frac{n}{2}$ in Lemma 1, we obtain

$$f(d_\Gamma(v_i)) - f\left(\frac{n}{2} - 1\right) \geq f\left(\frac{n}{2}\right) - f(n - d_\Gamma(v_i) - 1)$$

with equality if and only if $d_\Gamma(v_i) = \frac{n}{2}$, and hence

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \geq \sum_{i=1}^{n}\left(f\left(\frac{n}{2}\right) + f\left(\frac{n}{2} - 1\right)\right) \geq n\left[f\left(\frac{n}{2}\right) + f\left(\frac{n}{2} - 1\right)\right]$$

with equality if and only if Γ is an $\frac{n}{2}$-regular graph, that is, if and only if Γ is a $\lfloor \frac{n}{2} \rfloor$-regular graph.

Next, we assume that $0 \leq d_\Gamma(v_i) \leq \frac{n}{2} - 1$, that is, $\frac{n}{2} \leq d_{\overline{\Gamma}}(v_j) \leq n-1$. Setting $x = d_{\overline{\Gamma}}(v_j)$, $a = d_{\overline{\Gamma}}(v_j) - \frac{n}{2} + 1$, $b = d_{\overline{\Gamma}}(v_j) - \frac{n}{2}$ in Lemma 1, we obtain

$$f\Big(d_{\overline{\Gamma}}(v_j)\Big) - f\left(\frac{n}{2} - 1\right) \geq f\left(\frac{n}{2}\right) - f\Big(n - d_{\overline{\Gamma}}(v_j) - 1\Big),$$

that is,

$$f\Big(d_{\overline{\Gamma}}(v_j)\Big) + f\Big(n - d_{\overline{\Gamma}}(v_j) - 1\Big) \geq f\left(\frac{n}{2}\right) + f\left(\frac{n}{2} - 1\right)$$

with equality if and only if $d_{\overline{\Gamma}}(v_j) = \frac{n}{2}$. Hence,

$$\begin{aligned} H_f(\Gamma) + H_f(\overline{\Gamma}) = \sum_{i=1}^{n} \Big[f\Big(d_\Gamma(v_i)\Big) + f\Big(d_{\overline{\Gamma}}(v_i)\Big)\Big] &= \sum_{i=1}^{n} \Big[f\Big(n-1-d_{\overline{\Gamma}}(v_i)\Big) + f\Big(d_{\overline{\Gamma}}(v_i)\Big)\Big] \\ &\geq n\left[f\left(\frac{n}{2}\right) + f\left(\frac{n}{2} - 1\right)\right]. \end{aligned}$$

with equality if and only if Γ is an $\frac{n}{2}$-regular graph, that is, if and only if Γ is a $\lfloor \frac{n}{2} \rfloor$-regular graph. □

Theorem 10. *Let Γ be a graph of order n with maximum degree Δ. If $f(x)$ is a convex function, then*

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \leq n\left[f(\Delta) + f(n-1-\Delta)\right]$$

with equality if and only if Γ is a regular graph or graph Γ has only two type of degrees Δ and $n-1-\Delta$ $(\Delta > (n-1)/2)$.

Proof. Setting $x = \Delta$, $a = \Delta - d_\Gamma(v_i)$ and $b = \Delta + d_\Gamma(v_i) - (n-1)$, by Lemma 1 we obtain

$$f(d_\Gamma(v_i)) + f(n-1-d_\Gamma(v_i)) \leq f(\Delta) + f(n-1-\Delta)$$

with equality if and only if $d_\Gamma(v_i) = n-1-\Delta$ or $i = 1$. Hence,

$$\begin{aligned} H_f(\Gamma) + H_f(\overline{\Gamma}) &= \sum_{i=1}^{n} f(d_\Gamma(v_i)) + \sum_{i=1}^{n} f(n-1-d_\Gamma(v_i)) \\ &\leq \sum_{i=1}^{n} \left[f(\Delta) + f(n-1-\Delta)\right] = n\left[f(\Delta) + f(n-1-\Delta)\right]. \end{aligned}$$

Moreover, the equality holds if and only if $d_\Gamma(v_i) = \Delta$ or $d_\Gamma(v_i) = n-1-\Delta$ for any vertex $v_i \in V(\Gamma)$, that is, if and only if Γ is a regular graph or graph Γ has only two type of degrees Δ and $n-1-\Delta$ $(\Delta > (n-1)/2)$. □

Corollary 2. *Let Γ be a graph of order n. If $f(x)$ is a convex function, then*

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \leq n\left[f(n-1) + f(0)\right]$$

with equality if and only if Γ is a complete graph or Γ is an empty graph.

Proof. Setting $x = n-1$, $a = n-1-\Delta$ and $b = \Delta$, by Lemma 1 we obtain

$$f(\Delta) + f(n-1-\Delta) \leq f(n-1) + f(0)$$

with equality if and only if $\Delta = 0$ or $\Delta = n-1$. Using this result with Theorem 10, we obtain the result. Moreover, the equality holds if and only if Γ is a complete graph or Γ is an empty graph. □

The following is the well-known Jensen inequality.

Lemma 3 (Jensen Inequality [21])**.** *If $f(x)$ is convex function, for all $x_1, x_2, \ldots, x_n \in [a, b]$, then*

$$\sum_{i=0}^{n} f(x_i) \geq n f\left(\frac{\sum_{i=0}^{n} x_i}{n}\right),$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 11. *Let Γ be a graph of order n and size m. If $f(x)$ is a convex function, then*

$$H_f(\Gamma) + H_f(\overline{\Gamma}) \geq n\, f\left(\frac{2m}{n}\right) + n\, f\left(\frac{n(n-1) - 2m}{n}\right)$$

and

$$H_f(\Gamma) \cdot H_f(\overline{\Gamma}) \geq n^2 f\left(\frac{2m}{n}\right) \cdot f\left(\frac{n(n-1)-2m}{n}\right),$$

with equality if and only if Γ *is a regular graph.*

Proof. Since $f(x)$ is a convex function and $\sum\limits_{i=1}^{n} d_\Gamma(v_i) = 2m$, it follows from Lemma 3 that

$$\sum_{i=1}^{n} f(d_\Gamma(v_i)) \geq n\, f\left(\frac{\sum\limits_{i=1}^{n} d_\Gamma(v_i)}{n}\right) = n\, f\left(\frac{2m}{n}\right)$$

and

$$\sum_{i=1}^{n} f(d_{\overline{\Gamma}}(v_i)) \geq n\, f\left(\frac{\sum\limits_{i=1}^{n} d_{\overline{\Gamma}}(v_i)}{n}\right) = n\, f\left(\frac{n(n-1)-2m}{n}\right).$$

Hence,

$$H_f(\Gamma) + H_f(\overline{\Gamma}) = \sum_{i=1}^{n} f(d_\Gamma(v_i)) + \sum_{i=1}^{n} f(d_{\overline{\Gamma}}(v_i)) \geq n\, f\left(\frac{2m}{n}\right) + n\, f\left(\frac{n(n-1)-2m}{n}\right)$$

and

$$H_f(\Gamma) \cdot H_f(\overline{\Gamma}) = \sum_{i=1}^{n} f(d_\Gamma(v_i)) \cdot \sum_{i=1}^{n} f(d_{\overline{\Gamma}}(v_i)) \geq n^2 f\left(\frac{2m}{n}\right) \cdot f\left(\frac{n(n-1)-2m}{n}\right).$$

Moreover, the equality holds if and only if $d_\Gamma(v_i) = d_\Gamma(v_j)$ for $1 \leq i, j \leq n$, which means that Γ is a regular graph. □

Theorem 12. *Let* Γ *be a graph of order* n *with maximum degree* Δ *and minimum degree* δ*. If* $f(x)$ *is an increasing function, then*

$$\begin{aligned} H_f(\Gamma) \cdot H_f(\overline{\Gamma}) \leq & (n-1)^2 f(\Delta)\, f(n-1-\delta) + (n-1)\, f(\delta)\, f(n-1-\delta) \\ & + (n-1)\, f(\Delta)\, f(n-1-\Delta) + f(\delta)\, f(n-1-\Delta) \end{aligned}$$

with equality if and only if Γ *is a regular graph.*

Proof. Since Δ is the maximum degree and δ is the minimum degree, we can assume that $\Delta = d_\Gamma(v_1) \geq d_\Gamma(v_2) \geq \cdots \geq d_\Gamma(v_n) = \delta$. Moreover, we obtain

$$n-1-\delta \geq n-1-d_\Gamma(v_i) \geq n-1-\Delta \;\text{ for }\; v_i \in V(\Gamma).$$

Since $f(x)$ is an increasing function with the above results, we obtain

$$\begin{aligned} H_f(\Gamma) \cdot H_f(\overline{\Gamma}) = & \sum_{i=1}^{n} f(d_\Gamma(v_i)) \cdot \sum_{i=1}^{n} f(n-1-d_\Gamma(v_i)) = \left[f(\Delta) + \sum_{i=2}^{n-2} f(d_\Gamma(v_i)) + f(\delta)\right] \\ & \left[f(n-1-\delta) + \sum_{i=2}^{n-2} f(n-1-d_\Gamma(v_i)) + f(n-1-\Delta)\right] \\ \leq & \left[f(\delta) + (n-1)\, f(\Delta)\right] \left[f(n-1-\Delta) + (n-1)\, f(n-1-\delta)\right] \\ = & (n-1)^2 f(\Delta)\, f(n-1-\delta) + (n-1)\, f(\delta)\, f(n-1-\delta) \\ & + (n-1)\, f(\Delta)\, f(n-1-\Delta) + f(\delta)\, f(n-1-\Delta). \end{aligned}$$

Moreover, the above equality holds if and only if $d_\Gamma(v_i) = \Delta$ for all $v_i \in V(\Gamma)$ $(i = 1, 2, \ldots, n-1)$, and $d_\Gamma(v_i) = \delta$ for all $v_i \in V(\Gamma)$ $(i = 2, 3, \ldots, n)$, and hence $\Delta = \delta$, that is, if and only if Γ is a regular graph. □

Corollary 3. *Let Γ be a graph of order n with maximum degree Δ and minimum degree δ. If $f(x)$ is an increasing function, then*

$$H_f(\Gamma) \cdot H_f(\overline{\Gamma}) \leq n^2 f(\Delta) f(n-1-\delta)$$

with equality if and only if Γ is a regular graph.

Proof. Since $f(x)$ is an increasing function, we have $f(\delta) \leq f(\Delta)$ and $f(n-1-\Delta) \leq f(n-1-\delta)$. Using these results in Theorem 12, we obtain the required result. Moreover, the equality holds if and only if Γ is a regular graph. □

Theorem 13. *Let Γ be a graph of order n with maximum degree Δ and minimum degree δ. If $f(x)$ is an increasing function, then*

$$\begin{aligned} H_f(\Gamma) \cdot H_f(\overline{\Gamma}) \geq &(n-1)^2 f(\delta) f(n-1-\Delta) + (n-1) f(\delta) f(n-1-\delta) \\ &+ (n-1) f(\Delta) f(n-1-\Delta) + f(\Delta) f(n-1-\delta) \end{aligned}$$

with equality if and only if Γ is a regular graph.

Proof. The proof is similar to the proof of Theorem 12. Since $f(x)$ is an increasing function, we obtain

$$\begin{aligned} H_f(\Gamma) \cdot H_f(\overline{\Gamma}) = &\left[f(\Delta) + \sum_{i=2}^{n-2} f(d_\Gamma(v_i)) + f(\delta)\right] \left[f(n-1-\delta) + \sum_{i=2}^{n-2} f(n-1-d_\Gamma(v_i)) \right. \\ &\left. + f(n-1-\Delta)\right] \\ \geq &\left[f(\Delta) + (n-1) f(\delta)\right] \left[f(n-1-\delta) + (n-1) f(n-1-\Delta)\right] \\ = &(n-1)^2 f(\delta) f(n-1-\Delta) + (n-1) f(\delta) f(n-1-\delta) \\ &+ (n-1) f(\Delta) f(n-1-\Delta) + f(\Delta) f(n-1-\delta). \end{aligned}$$

Moreover, the equality holds if and only if Γ is a regular graph. □

Corollary 4. *Let Γ be a graph of order n with maximum degree Δ and minimum degree δ. If $f(x)$ is an increasing function, then*

$$H_f(\Gamma) \cdot H_f(\overline{\Gamma}) \geq n^2 f(\delta) f(n-1-\Delta)$$

with equality if and only if Γ is a regular graph.

4. Concluding Remarks

In this report, the *vertex-degree function index* $H_f(\Gamma)$ has been investigated for a different class of graphs. Tight bounds of the *vertex-degree function index* $H_f(\Gamma)$ have been set up for any n vertex-connected graphs, trees, and chemical trees. The extremal graphs where the bounds attain have also been identified. Moreover, we present the Nordhaus–Gaddum-type results for $H_f(\Gamma) + H_f(\overline{\Gamma})$ and $H_f(\Gamma) \cdot H_f(\overline{\Gamma})$, and the characterization of the extremal graphs. We now pose the following problem related to the work presented in this paper, as a potential topic for further research .

Problem 1. *To find the lower and upper bounds on the* vertex-degree function index $H_f(\Gamma)$ *and characterize corresponding extremal graphs for other significant classes of graphs such as bicyclic, tricyclic graphs, etc.*

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