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Convergence of Inexact Iterates of Monotone Nonexpansive Mappings with Summable Errors

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Abstract: In our 2006 paper with D. Butnariu, it was shown that the convergence of iterates of a nonexpansive self-mapping of a complete metric space is stable in the presence of summable computational errors. In the present paper, we establish such results for monotone nonexpansive mappings.

Keywords: complete metric space; fixed point; inexact iterate; monotone nonexpansive mapping

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1. Introduction

For more than 60 years now, there has been considerable research activity regarding the fixed point theory of various classes of nonexpansive operators [1–15]. The starting point of these efforts is Banach's seminal result [16] on the existence of a unique fixed point for a strict contraction. It also concerns the convergence of iterates of a nonexpansive operator to one of its fixed points. Since that classical theorem, many developments have taken place in this field. See, for example, [15,17–20].

In our 2006 paper with D. Butnariu [3], it was shown that the convergence of iterates of a nonexpansive self-mapping of a complete metric space is stable in the presence of summable computational errors. In the present paper, our goal is to establish such results for *monotone* nonexpansive mappings. Note that the study of monotone nonexpansive mapping is a well-established area of research. For pertinent examples and applications of solving matrix and ordinary differential equations, see [21,22]. The results of [3] and the present paper show that the convergence of iterates remains in force even when small computational errors are taken into account. Needless to say, such errors always occur in calculations.

Assume that (X, ρ) is a complete metric space. For every point $\xi \in X$ and each non-empty set $D \subset X$, put

$$\rho(\xi, D) := \inf\{\rho(\xi, \eta) : \eta \in D\}.$$

For every point $\xi \in X$ and every positive number Δ, put

$$B(\xi, \Delta) := \{\eta \in X : \rho(\xi, \eta) \leq \Delta\}.$$

Finally, for every operator mapping $T : X \to X$, let $T^0\xi = \xi$ for every point $\xi \in X$.

In [3] the authors analyzed the convergence of orbits of nonexpansive operators in complete metric spaces in the presence of computational errors and obtained the following result (see also Theorem 2.72 on page 97 of [13]).

Theorem 1. *Assume that a mapping $T : X \to X$ satisfies*

$$\rho(T(x), T(y)) \leq \rho(x, y) \textit{ for all } x, y \in X$$

and assume that for every point $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges in (X, ρ).

Assume further that the sequences $\{x_n\}_{n=0}^{\infty} \subset X$ and $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfy the conditions

$$\sum_{n=0}^{\infty} r_n < \infty$$

and

$$\rho(x_{n+1}, T(x_n)) \le r_n, \ n = 0, 1, \ldots.$$

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T in (X, ρ).

Theorem 1 has found interesting applications and is an important ingredient in the study of superiorization and perturbation resilience of algorithms. See, for example, [23–27] and references mentioned therein.

2. The Main Results

Assume that (X, ρ) is a complete metric space equipped with an order $\le$, such that $x \le x$ for each $x \in X$, if $x, y \in X$ satisfy $x \le y$ and $y \le x$, then $x = y$, and if $x, y, z \in X$ satisfy $x \le y$ and $y \le z$, then $x \le z$.

Assume that a mapping $T : X \to X$ satisfies

$$T(x) \le T(y) \text{ for each } x, y \in X \text{ such that } x \le y \tag{1}$$

and

$$\rho(T(x), T(y)) \le \rho(x, y) \text{ for each } x, y \in X \text{ such that } x \le y. \tag{2}$$

In this paper, we establish the following results which are proved in Section 4.

Theorem 2. *Assume that for every point $x \in X$, the sequence $\{T^i(x)\}_{i=1}^{\infty}$ converges. Let a sequence $\{x_i\}_{i=0}^{\infty}$ satisfy the conditions*

$$\sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)) < \infty \tag{3}$$

and

$$x_{i+1} \ge T(x_i) \text{ for each integer } i \ge 0. \tag{4}$$

Then the sequence $\{x_i\}_{i=0}^{\infty}$ converges. If, in addition, T is continuous, then its limit is a fixed point of T.

Theorem 3. *Assume that F is a non-empty subset of X, for every point $x \in X$,*

$$\lim_{i \to \infty} \rho(T^i(x), F) = 0$$

and that a sequence $\{x_i\}_{i=0}^{\infty}$ satisfies (3) and (4). Then $\lim_{i \to \infty} \rho(x_i, F) = 0$.

Theorem 4. *Assume that for every point $x \in X$, there exists a compact set $E(x) \subset X$, such that*

$$\lim_{i \to \infty} \rho(T^i(x), E(x)) = 0$$

and that a sequence $\{x_i\}_{i=0}^{\infty}$ satisfies (3) and (4). Then, there exists a compact set $E \subset X$, such that $\lim_{i \to \infty} \rho(x_i, E) = 0$.

3. An Auxiliary Result

Lemma 1. *Assume that a mapping $T : X \to X$ satisfies (1) and (2) and that a sequence $\{x_i\}_{i=0}^{\infty}$ satisfies*

$$\sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)) < \infty \tag{5}$$

and

$$x_{i+1} \geq T(x_i) \text{ for each integer } i \geq 0, \tag{6}$$

that $n_0 \geq 0$ *is an integer, and that*

$$y_{n_0} = x_{n_0}$$

and

$$y_{i+1} = T(y_i) \text{for every integer} i > n_0. \tag{7}$$

Then, for each integer $n > n_0$*, we have*

$$x_n \geq y_n \tag{8}$$

and

$$\rho(x_n, y_n) \leq \sum_{i=n_0+1}^{n} \rho(x_i, T(x_{i-1})). \tag{9}$$

Proof. In view of (6) and (7),

$$x_{n_0+1} \geq T(x_{n_0}) = T(y_{n_0}) = y_{n_0+1},$$

$$\rho(y_{n_0+1}, x_{n_0+1}) = \rho(T(y_{n_0}), x_{n_0+1}) = \rho(x_{n_0+1}, T(x_{n_0}))$$

and relations (8) and (9) hold with $n = n_0 + 1$.

Assume that $n > n_0$ is a natural number and that Equations (8) and (9) hold. By (7), we have

$$\rho(x_{n+1}, y_{n+1}) \leq \rho(x_{n+1}, T(x_n)) + \rho(T(x_n), T(y_n)). \tag{10}$$

In view of (8),

$$\rho(T(x_n), T(y_n)) \leq \rho(x_n, y_n). \tag{11}$$

Relations (9) and (11) imply that

$$\rho(x_{n+1}, y_{n+1}) \leq \rho(x_{n+1}, T(x_n)) + \rho(x_n, y_n) \leq \sum_{i=n_0+1}^{n+1} \rho(x_i, T(x_{i-1})).$$

It follows from (2) and (6)–(8) that

$$x_{n+1} \geq T(x_n) \geq T(y_n) = y_{n+1}.$$

Thus, (8) and (9) hold for $n + 1$ too. Thus, the assumption made for n holds for $n + 1$ too. This completes the proof of Lemma 1. □

4. Proofs of Theorems 2–4

Proof of Theorem 2. Given $\epsilon > 0$, there exists a natural number n_0, such that

$$\sum_{i=n_0}^{\infty} \rho(x_i, T(x_{i-1})) < \epsilon/2. \tag{12}$$

Set

$$y_{n_0} = x_{n_0}$$

and

$$y_{i+1} = T(y_i) \text{for each integer} i \geq n_0. \tag{13}$$

Lemma 1 and relations (12) and (13) imply that for every natural number $n > n_0$, we have

$$\rho(x_n, y_n) \leq \epsilon/2. \tag{14}$$

In view of (13), there exists
$$y_* = \lim_{n\to\infty} y_n. \tag{15}$$
By (14) and (15), for all sufficiently large natural numbers n,
$$\rho(x_n, y_*) \le \rho(x_n, y_n) + \rho(y_n, y_*) \le \epsilon.$$
Thus $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists
$$x_* = \lim_{n\to\infty} x_n.$$
Clearly, if T is continuous, then x_* is a fixed point of T. Theorem 2 is proved. □

Proof of Theorem 3. Given a positive number ϵ, there exists a natural number n_0, such that Equation (12) holds. Define a sequence $\{y_i\}_{i=n_0}^{\infty}$ by (13). Lemma 1 and relations (12) and (13) imply that for every natural number $n > n_0$, we have
$$\rho(x_n, y_n) \le \epsilon/2.$$
In view of (13) and the above inequality, for every sufficiently large natural number n, we have
$$\rho(x_n, F) \le \rho(x_n, y_n) + \rho(y_n, F) < \epsilon.$$
This completes the proof of Theorem 3. □

Proof of Theorem 4. Given $\epsilon > 0$, there exists a natural number n_0 such that Equation (12) holds. Define a sequence $\{y_i\}_{i=n_0}^{\infty}$ by (13). Lemma 1 and relations (12) and (13) imply that for every natural number $n > n_0$, we have
$$\rho(x_n, y_n) \le \epsilon/2.$$
In view of (13), there exists a compact set $E_0 \subset X$, such that
$$\lim_{n\to\infty} \rho(y_n, E_0) = 0.$$
Clearly, for every sufficiently large natural number $n > n_0$, we have
$$\rho(x_n, E_0) \le \rho(x_n, y_n) + \rho(y_n, E_0) < \epsilon.$$
Thus, we have shown that there exists a compact set E, such that
$$\rho(x_n, E_0) < \epsilon$$
for each sufficiently large natural number $n > n_0$. We may assume that E_0 is finite. This implies that each subsequence of $\{x_i\}_{i=0}^{\infty}$ has a convergent subsequence. Denote by E the set of all limit points of the sequence $\{x_i\}_{i=0}^{\infty}$. It is not difficult to see that E is compact and that
$$\lim_{i\to\infty} \rho(x_i, E) = 0.$$
This completes the proof of Theorem 4. □

5. Conclusions

We have extended the convergence result of [3], which was established for inexact iterates of a nonexpansive self-mapping of a complete metric space, to monotone nonexpansive mappings. Such mappings have applications to solving matrix and ordinary differential equations (see [21,22]).

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