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Univariate and Multivariate Ostrowski-Type Inequalities Using Atangana–Baleanu Caputo Fractional Derivative

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Abstract: In this paper, we obtain some univariate and multivariate Ostrowski-type inequalities using the Atangana–Baleanu fractional derivative in the sense of Liouville–Caputo (ABC). The results obtained for both left and right ABC fractional derivatives can be applied to study further fractional inequalities and estimate various non-local function problems since the operator consists of a non-singular kernel. The obtained results are more generalized in nature.

Keywords: Ostrowski inequalities; ABC fractional derivative; AB mean value theorem

MSC: 26A33; 26D07; 26D10; 26D15

Citation: Desta, H.D.; Pachpatte, D.B.; Mijena, J.B.; Abdi, T. Univariate and Multivariate Ostrowski-Type Inequalities Using Atangana–Baleanu Caputo Fractional Derivative. *Axioms* **2022**, *11*, 482. https://doi.org/10.3390/axioms11090482

Academic Editor: Christophe Chesneau

Received: 22 August 2022
Accepted: 14 September 2022
Published: 19 September 2022

1. Introduction

Ostrowski, in the year 1938, presented the following inequality [1]:

Let $\mathcal{Y} : [\mu, \lambda] \to \mathbb{R}$ be continuous on $[\mu, \lambda]$ and differentiable on (μ, λ) with derivative $\mathcal{Y}' : (\mu, \lambda) \to \mathbb{R}$ being bounded on (μ, λ), i.e., $\|\mathcal{Y}'\|_\infty = \sup_{x \in (\mu,\lambda)} |\mathcal{Y}'(x)| < \infty$

$$\left| \mathcal{Y}(x) - \frac{1}{\lambda - \mu} \int_\mu^\lambda \mathcal{Y}(\xi) d\xi \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{\mu+\lambda}{2}\right)^2}{(\lambda - \mu)^2} \right] (\lambda - \mu) \|\mathcal{Y}'\|_\infty,$$

for all $x \in [\mu, \lambda]$. The constant $\frac{1}{4}$ is the best possible. Recently, fractional calculus was found to be the most rapidly growing area in the field of mathematics. It is the study of non-integer-order differentiation and integration, which has attracted a lot of attention from many scholars due to its widespread applications in different fields. Fractional calculus has a great deal of applications in different fields of science and engineering and control theory [2–7]; see also the recent survey-cum-expository review article [8,9].

Mathematical inequality plays a crucial part in the investigation of ordinary and partial fractional differential equations. They are useful in studying properties such as the uniqueness and stability of the solutions. For instance, in [10], the stability, existence and uniqueness of the solution of the fractional Langevin equation are studied using the generalized proportional Hadamard–Caputo fractional derivative. Certain inequalities are found to be useful in providing bounds in solving the problem. Lately, many interesting fractional differential and integral inequalities have been obtained by many researchers—for instance, the Minkowski inequality, Hermite–Hadamard inequality, Opial integral inequalities [11–14], and others. In recent years, results on inequalities involving the univariate and multivariate fractional Ostrowski inequalities using the Caputo, Canavati, and ψ-definitions have been studied (see [15–18] and references cited therein).

In [19], Atangana and Baleanu introduced a new definition of fractional-order derivative called Atangana–Baleanu (AB) using the Mittag–Leffler function. AB derivatives are useful in the study of fractional dynamics because the fractional derivative of a function is given by a definite integral. The AB fractional derivative operator consists of a non-singular kernel, which is efficient in solving non-local problems. Since the kernel is non-local and non-singular, this operator has an additional benefit as compared to the others. In [20], the authors have given some generalizations of the Ostrowski inequality using Hölder's inequality and used the AB fractional integral operator.

Motivated by the above results and the scope of such inequalities in their application in numerical analysis and probability theory, we have established the Ostrowski-type univariate and multivariate inequalities using the right and left ABC fractional derivative operator and generalized the classical inequalities.

The organization of the paper is as follows. In Section 2, we present the preliminary definition and results from the literature that will be used in our main results. In Sections 3–5, we obtain univariate and multivariate Ostrowski-type fractional integral inequalities using the ABC fractional derivative. Finally, Section 6 is devoted to the concluding remarks of our work.

2. Preliminaries

First, we discuss some key definitions of fractional derivatives and integrals that we will be using throughout the paper.

Definition 1 ([19,21])**.** *Let $\mathcal{Y} \in H^1(\mu, \lambda)$, $\mu < \lambda$ and $\delta \in (0,1)$. The left Atangana–Baleanu fractional derivative of $\mathcal{Y}$ in the Liouville–Caputo (ABC) sense with the Mittag–Leffler non-singular kernel of order δ is defined at $\xi \in (\mu, \lambda)$ by*

$$\left({}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right)(\xi) = \frac{\mathfrak{B}(\delta)}{1-\delta}\int_{\mu}^{\xi} \mathcal{Y}'(s)E_{\delta}\left[-\delta\frac{(\xi-s)^{\delta}}{1-\delta}\right]ds,$$

where E_δ is the Mittag–Leffler function defined by $E_\delta(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(n\delta+1)}$ and $\mathfrak{B}(\delta)$ is a normalizing positive function satisfying $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

The left Atangana–Baleanu fractional derivative of $\mathcal{Y}$ of order δ in the Riemann–Liouville sense is defined by

$$\left({}^{ABRL}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right)(\xi) = \frac{\mathfrak{B}(\delta)}{1-\delta}\frac{d}{d\xi}\int_{\mu}^{\xi} \mathcal{Y}(s)E_{\delta}\left[-\delta\frac{(\xi-s)^{\delta}}{1-\delta}\right]ds.$$

The associated fractional integral is

$$\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\mathcal{Y}\right)(\xi) = \frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi) + \frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}\mathcal{Y}\right)(\xi),$$

where

$$I^{\delta}_{\mu+}\mathcal{Y}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi} \mathcal{Y}(s)(\xi-s)^{\delta-1}ds,$$

is the left Riemann–Liouville integral.

Similarly, the right fractional derivative and integral are defined as follows:

Definition 2 ([22]). *Let $\mathcal{Y} \in H^1(\mu,\lambda)$, $\mu < \lambda$ and $\delta \in (0,1)$. The right Atangana–Baleanu fractional derivative of $\mathcal{Y}$ in the Liouville–Caputo sense (ABC) with the Mittag–Leffler non-singular kernel of order δ is defined at $\xi \in (\mu,\lambda)$ by*

$$\left({}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}\right)(\xi) = \frac{\mathfrak{B}(\delta)}{1-\delta}\int_{\xi}^{\lambda} \mathcal{Y}'(s)E_{\delta}\left[-\delta\frac{(s-\xi)^{\delta}}{1-\delta}\right]ds.$$

The right Atangana–Baleanu fractional derivative of $\mathcal{Y}$ of order α in the Riemann–Liouville sense is defined by

$$\left({}^{ABRL}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}\right)(\xi) = -\frac{\mathfrak{B}(\delta)}{1-\delta}\frac{d}{d\xi}\int_{\xi}^{\lambda} \mathcal{Y}(s)E_{\delta}\left[-\delta\frac{(\xi-s)^{\delta}}{1-\delta}\right]ds.$$

The associated fractional integral is

$$\left({}^{AB}\mathfrak{I}^{\delta}_{\lambda-}\mathcal{Y}\right)(\xi) = \frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi) + \frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\lambda-}\mathcal{Y}\right)(\xi),$$

where

$$I^{\delta}_{\lambda-}\mathcal{Y}(\xi) = \frac{1}{\Gamma(\delta)}\int_{\xi}^{\lambda}\mathcal{Y}(s)(s-\xi)^{\delta-1}ds,$$

is the right Riemann–Liouville integral. The properties of the fractional derivatives with the Mittag–Leffler function can be found in [23].

Lemma 1 ((AB Mean Value Theorem) [24]). *Let $0 < \delta < 1$, $\mu < \lambda$ in $\mathbb{R}$ and $\mathcal{Y} : [\mu,\lambda] \to \mathbb{R}$ differentiable such that $\mathcal{Y}' \in L^1[\mu,\lambda]$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \in C[\mu,\lambda]$. Then, for any $\xi \in [\mu,\lambda]$, there exists $\omega \in [\mu,\xi]$ such that*

$$\mathcal{Y}(\xi) = \mathcal{Y}(\mu) + \frac{1-\delta}{\mathfrak{B}(\delta)}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi) + \frac{(\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\omega).$$

Similarly, the AB mean value theorem can be stated for the right Atangana–Baleanu fractional derivative as follows:
Let $0 < \delta < 1$, $\mu < \lambda$ in $\mathbb{R}$ and $\mathcal{Y} : [\mu,\lambda] \to \mathbb{R}$ differentiable such that $\mathcal{Y}' \in L^1[\mu,\lambda]$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y} \in C[\mu,\lambda]$. Then, for any $\xi \in [\mu,\lambda]$, there exists $\omega \in [\xi,\lambda]$ such that

$$\mathcal{Y}(\xi) = \mathcal{Y}(\lambda) + \frac{1-\delta}{\mathfrak{B}(\delta)}{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}(\xi) + \frac{(\lambda-\xi)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)}{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}(\omega).$$

Lemma 2 ((AB Newton–Leibniz Theorem) [23]). *The AB integral and derivatives of Liouville–Caputo type satisfy the following inversion relation*

$${}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi) = \mathcal{Y}(\xi) - \mathcal{Y}(\mu),$$

and

$${}^{AB}\mathfrak{I}^{\delta}_{\lambda-}\,{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}(\xi) = \mathcal{Y}(\xi) - \mathcal{Y}(\lambda),$$

for $0 < \delta < 1$, $\mu < \xi < \lambda$ in $\mathbb{R}$ and $\mathcal{Y} : [\mu,\lambda] \to \mathbb{R}$ is differentiable such that $\mathcal{Y}'$, ${}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}$ are in $L^1[\mu,\lambda]$.

Consider the norm $\|.\|_{\infty} : C([\mu,\lambda]) \to \mathbb{R}$ and

$$\left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty} = \sup_{\xi\in(\mu,\lambda)} \left| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi) \right| < +\infty,$$

and

$$\left\| {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y} \right\|_{\infty} = \sup_{\xi\in(\mu,\lambda)} \left| {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}(\xi) \right| < +\infty.$$

3. Main Results

Ostrowski-type inequalities with left and right ABC-fractional derivatives are given next:

Theorem 1. *Let $\mathcal{Y} : [\mu,\lambda] \to \mathbb{R}$ be differentiable, $[\mu,\lambda] \subset \mathbb{R}$ with $\mathcal{Y}' \in L^1(\mu,\lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \in C[\mu,\lambda]$ and $0 < \delta < 1$; then, for any $\xi \in [\mu,\lambda]$, we have*

$$\left| \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} \mathcal{Y}(\xi)d\xi - \mathcal{Y}(\mu) \right| \le \left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{(\lambda-\mu)^{\delta}}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\delta)} \right] \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty}. \tag{1}$$

Proof. We have from Lemma 1

$$\begin{aligned}
&|\mathcal{Y}(\xi) - \mathcal{Y}(\mu)| \\
&= \left| \frac{1-\delta}{\mathfrak{B}(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi) + \frac{(\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\omega) \right| \\
&\le \frac{1-\delta}{\mathfrak{B}(\delta)} \left| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi) \right| + \frac{(\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \left| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\omega) \right| \\
&\le \left(\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{(\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \right) \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty}.
\end{aligned} \tag{2}$$

Thus, we have

$$|\mathcal{Y}(\xi) - \mathcal{Y}(\mu)| \le \left(\frac{(1-\delta)\Gamma(\delta) + (\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \right) \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty}, \tag{3}$$

for $\xi \in [\mu,\lambda]$. We have

$$\begin{aligned}
\left| \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} \mathcal{Y}(\xi)d\xi - \mathcal{Y}(\mu) \right| &= \left| \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} \Big(\mathcal{Y}(\xi) - \mathcal{Y}(\mu) \Big) d\xi \right| \\
&\le \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} |\mathcal{Y}(\xi) - \mathcal{Y}(\mu)| d\xi \\
&\le \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} \frac{(1-\delta)\Gamma(\delta) + (\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty} d\xi \\
&= \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty} \frac{1}{\lambda-\mu} \int_{\mu}^{\lambda} \frac{(1-\delta)\Gamma(\delta) + (\xi-\mu)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} d\xi \\
&= \left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{(\lambda-\mu)^{\delta}}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\delta)} \right] \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y} \right\|_{\infty}.
\end{aligned} \tag{4}$$

□

Similarly, for the right fractional derivative, we have

Theorem 2. *Let $\mathcal{Y} : [\mu, \lambda] \to \mathbb{R}$ be differentiable $[\mu, \lambda] \subset \mathbb{R}$ with $\mathcal{Y}' \in L^1(\mu, \lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y} \in C[\mu, \lambda]$ for $0 < \delta < 1$; then, for any $\xi \in [\mu, \lambda]$, we have*

$$\left| \frac{1}{\lambda - \mu} \int_{\mu}^{\lambda} \mathcal{Y}(\xi) d\xi - \mathcal{Y}(\lambda) \right| \leq \left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{(\lambda - \mu)^{\delta}}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\delta)} \right] \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y} \right\|_{\infty}. \quad (5)$$

Proof. This can be proven by following similar steps as in Theorem 1. □

Now, in our next theorem, we prove the result on the ABC fractional Ostrowski inequality, in which we have considered both the left and right ABC fractional derivatives of any point between μ and λ.

Theorem 3. *Let $\mathcal{Y} : [\mu, \lambda] \to \mathbb{R}$ be differentiable $[\mu, \lambda] \subset \mathbb{R}$ with $\mathcal{Y}' \in L^1(\mu, \lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y}$, ${}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y} \in C[\mu, \lambda]$ for $0 < \delta < 1$. Then, for any $\xi, \xi_0 \in [\mu, \lambda]$,*

$$\begin{aligned} \left| \frac{1}{\lambda - \mu} \int_{\mu}^{\lambda} \mathcal{Y}(\xi) d\xi - \mathcal{Y}(\xi_0) \right| \leq \frac{1}{\lambda - \mu} \Bigg\{ (\mu - \xi_0) \left(\frac{\delta - 1}{\mathfrak{B}(\delta)} - \frac{(\mu - \xi_0)^{\delta}}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\delta)} \right) \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y} \right\|_{\infty} \\ +(\xi_0 - \lambda) \left(\frac{\delta - 1}{\mathfrak{B}(\delta)} - \frac{(\xi_0 - \lambda)^{\delta}}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\alpha)} \right) \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y} \right\|_{\infty} \Bigg\}. \end{aligned} \quad (6)$$

Proof. From Lemma 1, we have, for the left ABC fractional derivative,

$$\mathcal{Y}(\xi) - \mathcal{Y}(\xi_0) = \frac{\delta - 1}{\mathfrak{B}(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y}(\xi) + \frac{(\xi - \xi_0)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y}(\omega), \quad (7)$$

for $\xi \in [\xi_0, \lambda]$, and for the right ABC fractional derivative,

$$\mathcal{Y}(\xi) - \mathcal{Y}(\xi_0) = \frac{\delta - 1}{\mathfrak{B}(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y}(\xi) + \frac{(\xi_0 - \xi)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y}(\omega), \quad (8)$$

for $\xi \in [\mu, \xi_0]$.

Hence, from (7), we have

$$|\mathcal{Y}(\xi) - \mathcal{Y}(\xi_0)| \leq \frac{(1-\delta)\Gamma(\delta) + (\xi - \xi_0)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y} \right\|_{\infty}, \quad (9)$$

for $\xi \in [\xi_0, \lambda]$.

Similarly, from (8), we have

$$|\mathcal{Y}(\xi) - \mathcal{Y}(\xi_0)| \leq \frac{(1-\delta)\Gamma(\delta) + (\xi_0 - \xi)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)} \left\| {}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y} \right\|_{\infty}, \quad (10)$$

for $\xi \in [\mu, \xi_0]$. From (9) and (10), we have

$$\left|\frac{1}{\lambda-\mu}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)d\xi-\mathcal{Y}(\xi_0)\right|=\left|\frac{1}{\lambda-\mu}\int_{\mu}^{\lambda}(\mathcal{Y}(\xi)-\mathcal{Y}(\xi_0))d\xi\right|$$
$$\leq\frac{1}{\lambda-\mu}\int_{\mu}^{\lambda}|\mathcal{Y}(\xi)-\mathcal{Y}(\xi_0)|d\xi$$
$$\leq\frac{1}{\lambda-\mu}\left\{\int_{\mu}^{\xi_0}|\mathcal{Y}(\xi)-\mathcal{Y}(\xi_0)|d\xi+\int_{\xi_0}^{\lambda}|\mathcal{Y}(\xi)-\mathcal{Y}(\xi_0)|d\xi\right\}$$
$$=\frac{1}{\lambda-\mu}\left\{\left(\int_{\mu}^{\tau_0}\frac{(1-\delta)\Gamma(\delta)+(\xi-\tau_0)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)}d\xi\right)\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y}\right\|_{\infty}\right.$$
$$\left.+\left(\int_{\xi_0}^{\lambda}\frac{(1-\delta)\Gamma(\delta)+(\xi_0-\xi)^{\delta}}{\mathfrak{B}(\delta)\Gamma(\delta)}d\xi\right)\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y}\right\|_{\infty}\right\}$$
$$=\frac{1}{\lambda-\mu}\left\{\left(\frac{\delta-1}{\mathfrak{B}(\delta)}(\mu-\xi_0)-\frac{1}{\mathfrak{B}(\delta)(\alpha+1)\Gamma(\delta)}(\mu-\xi_0)^{\delta+1}\right)\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\xi_0+}\mathcal{Y}\right\|_{\infty}\right.$$
$$\left.+\left(\frac{\delta-1}{\mathfrak{B}(\delta)}(\xi_0-\lambda)-\frac{1}{\mathfrak{B}(\delta)(\delta+1)\Gamma(\delta)}(\xi_0-\lambda)^{\delta+1}\right)\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\xi_0-}\mathcal{Y}\right\|_{\infty}\right\},$$

which proves (6). □

4. ABC Fractional Inequality of Two Variables

Now, we give the ABC fractional Ostrowski-type inequality in two variables.

Theorem 4. *Let $\mathcal{Y}, g : [\mu,\lambda] \to \mathbb{R}$ be differentiable $[\mu,\lambda] \subset \mathbb{R}$ with $\mathcal{Y}', g' \in L^1(\mu,\lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}, {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g \in C[\mu,\lambda]$. Then,*

$$2\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)d\xi-\int_{\mu}^{\lambda}\Big[\mathcal{Y}(\mu)g(\xi)+g(\mu)\mathcal{Y}(\xi)\Big]d\xi$$
$$\leq\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}g(\xi)\right]d\xi \tag{11}$$
$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}\mathcal{Y}(\xi)\right]d\xi,$$

for $\xi \in [\mu,\lambda]$.

Proof. We have from Lemma 2

$$\mathcal{Y}(\xi)-\mathcal{Y}(\mu)={}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi), \tag{12}$$

and

$$g(\xi)-g(\mu)={}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi). \tag{13}$$

Multiplying (12) by $g(\xi)$ and (13) by $f(\xi)$, we have

$$\mathcal{Y}(\xi)g(\xi)-\mathcal{Y}(\mu)g(\xi)=g(\xi)\Big({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi)\Big), \tag{14}$$

$$\mathcal{Y}(\xi)g(\xi) - g(\mu)\mathcal{Y}(\xi) = \mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right). \tag{15}$$

Adding (14) and (15), we have

$$\begin{aligned} &2\mathcal{Y}(\xi)g(\xi) - \mathcal{Y}(\mu)g(\xi) - g(\mu)\mathcal{Y}(\xi) \\ &= g(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi)\right) + \mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right). \end{aligned} \tag{16}$$

Integrating the above Equation (16) from μ to λ with respect to ξ, we have

$$\begin{aligned} &2\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)d\xi - \int_{\mu}^{\lambda}\left[\mathcal{Y}(\mu)g(\xi) + g(\mu)\mathcal{Y}(\xi)\right]d\xi \\ &= \int_{\mu}^{\lambda} g(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi)\right)d\xi + \int_{\mu}^{\lambda}\mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right)d\xi \\ &\leq \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda} g(\xi)^{AB}\mathcal{J}^{\delta}_{\mu+}1d\xi + \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)^{AB}\mathcal{J}^{\delta}_{\mu+}1d\xi \\ &= \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda} g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}1\right)\right]d\xi \\ &+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}1\right)\right]d\xi \\ &= \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda} g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\alpha}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi}(\xi-s)^{\delta-1}ds\right]d\xi \\ &+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi}(\xi-s)^{\delta-1}ds\right]d\xi \\ &= \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda} g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\frac{(\xi-\mu)^{\delta}}{\delta}\right]d\xi \\ &+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)} + \frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\frac{(\xi-\mu)^{\delta}}{\delta}\right]d\xi \\ &= \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}g(\xi)\right]d\xi \\ &+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}\mathcal{Y}(\xi)\right]d\xi, \end{aligned}$$

which prove (11). □

Similarly, for the right ABC fractional Ostrowski inequality of two variables, the following holds.

Theorem 5. *Let $\mathcal{Y}, g : [\mu,\lambda] \to \mathbb{R}$ be differentiable $[\mu,\lambda] \subset \mathbb{R}$ with $\mathcal{Y}', g' \in L^1(\mu,\lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}, {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}g \in C[\mu,\lambda]$. Then,*

$$
\begin{aligned}
&2\int_{\mu}^{\lambda} \mathcal{Y}(\xi)g(\xi)d\xi - \int_{\mu}^{\lambda}\Big[\mathcal{Y}(\lambda)g(\xi) + g(\lambda)\mathcal{Y}(\xi)\Big]d\xi \\
&\leq \left\|{}^{ABC}\mathfrak{D}^{\alpha}_{\lambda-}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\lambda-\xi)^{\delta}g(\xi)\right]d\xi \\
&+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}g\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\lambda-\xi)^{\delta}\mathcal{Y}(\xi)\right]d\xi,
\end{aligned}
$$

for $\xi \in [\mu,\lambda]$.

Proof. The proof uses the same procedures as in Theorem 4. □

5. ABC Fractional Inequality of Three Variables

Now, we give the ABC fractional Ostrowski-type inequality in three variables as follows:

Theorem 6. *Let $\mathcal{Y}, g, h : [\mu,\lambda] \to \mathbb{R}$ be differentiable $[\mu,\lambda] \subset \mathbb{R}$ with $\mathcal{Y}', g', h' \in L^1(\mu,\lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}, {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g, {}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h \in C[\mu,\lambda]$. Then,*

$$
\begin{aligned}
&3\int_{\mu}^{\lambda} \mathcal{Y}(\xi)g(\xi)h(\xi)d\xi - \int_{\mu}^{\lambda}\Big[\mathcal{Y}(\mu)g(\xi)h(\xi) + g(\mu)h(\xi)\mathcal{Y}(\xi) + h(\mu)\mathcal{Y}(\xi)g(\xi)\Big]d\xi \\
&\leq \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\xi)h(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}g(\xi)h(\xi)\right]d\xi \\
&+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}h(\xi)\mathcal{Y}(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}h(\xi)\mathcal{Y}(\xi)\right]d\xi \\
&+ \left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi)g(\xi) + \frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}\mathcal{Y}(\xi)g(\xi)\right]d\xi.
\end{aligned}
\tag{17}
$$

Proof. We have from Lemma 2

$$\mathcal{Y}(\xi) - \mathcal{Y}(\mu) = {}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi), \tag{18}$$

$$g(\xi) - g(\mu) = {}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi), \tag{19}$$

$$h(\xi) - h(\mu) = {}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h(\xi). \tag{20}$$

Multiplying both sides of (18)–(20) by $g(\xi)h(\xi)$, $h(\xi)\mathcal{Y}(\xi)$ and $\mathcal{Y}(\xi)g(\xi)$, respectively, we obtain

$$\mathcal{Y}(\xi)g(\xi)h(\xi) - \mathcal{Y}(\mu)g(\xi)h(\xi) = g(\xi)h(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi)\right), \tag{21}$$

$$\mathcal{Y}(\xi)g(\xi)h(\xi) - g(\mu)h(\xi)f(\xi) = h(\xi)\mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right), \tag{22}$$

$$\mathcal{Y}(\xi)g(\xi)h(\xi) - h(\mu)\mathcal{Y}(\xi)g(\xi) = \mathcal{Y}(\xi)g(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h(\xi)\right). \tag{23}$$

Adding (21)–(23), we have

$$\begin{aligned}
&3\mathcal{Y}(\xi)g(\xi)h(\xi)-\mathcal{Y}(\mu)g(\xi)h(\xi)-g(\mu)h(\xi)\mathcal{Y}(\xi)-h(\mu)\mathcal{Y}(\xi)g(\xi)\\
&=h(\xi)\mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right)+h(\xi)\mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right)\\
&+\mathcal{Y}(\xi)g(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h(\xi)\right).
\end{aligned}\tag{24}$$

Integrating the above Equation (24) from μ to λ with respect to ξ, we have

$$\begin{aligned}
&3\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)h(\xi)d\xi-\int_{\mu}^{\lambda}\Big[\mathcal{Y}(\mu)g(\xi)h(\xi)+g(\mu)h(\xi)\mathcal{Y}(\xi)+h(\mu)\mathcal{Y}(\xi)g(\xi)\Big]d\xi\\
&=\int_{\mu}^{\lambda}g(\xi)h(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}(\xi)\right)d\xi+\int_{\mu}^{\lambda}h(\xi)\mathcal{Y}(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g(\xi)\right)d\xi\\
&+\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)\left({}^{AB}\mathfrak{I}^{\delta}_{\mu+}\,{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h(\xi)\right)d\xi\\
&\leq\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}g(\xi)h(\xi)\,{}^{AB}\mathfrak{I}^{\delta}_{\mu+}1d\xi+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}h(\xi)\mathcal{Y}(\xi)\,{}^{AB}\mathfrak{I}^{\delta}_{\mu+}1d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)\,{}^{AB}\mathfrak{I}^{\delta}_{\mu+}1d\xi\\
&=\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}g(\xi)h(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}1\right)\right]d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}h(\xi)\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}1\right)\right]d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\left(I^{\delta}_{\mu+}1\right)\right]d\xi\\
&=\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}g(\xi)h(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi}(\xi-s)^{\delta-1}ds\right]d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}h(\xi)\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi}(\xi-s)^{\delta-1}ds\right]d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\int_{\mu}^{\xi}(\xi-s)^{\delta-1}ds\right]d\xi\\
&=\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int_{\mu}^{\lambda}g(\xi)h(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\frac{(\xi-\mu)^{\delta}}{\delta}\right]d\xi\\
&+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int_{\mu}^{\lambda}h(\xi)\mathcal{Y}(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\frac{(\xi-\mu)^{\delta}}{\delta}\right]d\xi
\end{aligned}$$

$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int\limits_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)\left[\frac{1-\delta}{\mathfrak{B}(\delta)}+\frac{\delta}{\mathfrak{B}(\delta)}\frac{1}{\Gamma(\delta)}\frac{(\xi-\mu)^{\delta}}{\delta}\right]d\xi$$

$$=\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}\mathcal{Y}\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\xi)h(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}g(\xi)h(\xi)\right]d\xi$$

$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}g\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}h(\xi)\mathcal{Y}(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}h(\xi)\mathcal{Y}(\xi)\right]d\xi$$

$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\mu+}h\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi)g(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\xi-\mu)^{\delta}\mathcal{Y}(\xi)g(\xi)\right]d\xi,$$

which prove (17). □

Similarly, for the right ABC fractional Ostrowski inequality of three variables, the following holds:

Theorem 7. *Let $\mathcal{Y}, g, h : [\mu,\lambda] \to \mathbb{R}$ be differentiable $[\mu,\lambda] \subset \mathbb{R}$ with $\mathcal{Y}', g', h' \in L^1(\mu,\lambda)$ and ${}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}, {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}g, {}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}h \in C[\mu,\lambda]$. Then,*

$$3\int\limits_{\mu}^{\lambda}\mathcal{Y}(\xi)g(\xi)h(\xi)d\xi-\int\limits_{\mu}^{\lambda}\left[\mathcal{Y}(\lambda)g(\xi)h(\xi)+g(\lambda)h(\xi)\mathcal{Y}(\xi)+h(\lambda)\mathcal{Y}(\xi)g(\xi)\right]d\xi$$

$$\leq\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}\mathcal{Y}\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}g(\mathcal{Y})h(\mathcal{Y})+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\lambda-\xi)^{\delta}g(\xi)h(\xi)\right]d\xi$$

$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}g\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}h(\xi)\mathcal{Y}(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\lambda-\xi)^{\delta}h(\xi)\mathcal{Y}(\xi)\right]d\xi$$

$$+\left\|{}^{ABC}\mathfrak{D}^{\delta}_{\lambda-}h\right\|_{\infty}\int\limits_{\mu}^{\lambda}\left[\frac{1-\delta}{\mathfrak{B}(\delta)}\mathcal{Y}(\xi)g(\xi)+\frac{1}{\mathfrak{B}(\delta)\Gamma(\delta)}(\lambda-\xi)^{\delta}\mathcal{Y}(\xi)g(\xi)\right]d\xi.$$

Proof. The proof follows similar steps as in Theorem 6. □

6. Conclusions

In this paper, we have obtained the univariate and multivariate Ostrowski-type inequalities for the ABC fractional operator. These inequalities are obtained for one function and for products of two and three functions for both the left and right ABC fractional derivative operator. The results obtained are new and can be applied to study further fractional inequalities and estimate various non-local problems since the operator consists of a non-singular kernel. The obtained inequalities may be used in the future to study the estimate of the solution and other properties of fractional operators.

Author Contributions: Conceptualization, H.D.D. and D.B.P.; writing—original draft preparation, H.D.D. and D.B.P.; writing—review and editing, H.D.D., D.B.P., J.B.M. and T.A.; writing—response to reviewers' comments, H.D.D., D.B.P. and J.B.M.; supervision, T.A. and J.B.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank the anonymous referees and the academic editor whose comments and suggestions improved this final version of our paper. Moreover, the first author acknowledges Addis Ababa University, Department of Mathematics and International Science Program (ISP), Uppsala University for their support.

Conflicts of Interest: The authors declare no conflict of interest.

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