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Solving Fractional Volterra–Fredholm Integro-Differential Equations via A^{**} Iteration Method

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Abstract: In this article, we develop a faster iteration method, called the A^{**} iteration method, for approximating the fixed points of almost contraction mappings and generalized α-nonexpansive mappings. We establish some weak and strong convergence results of the A^{**} iteration method for fixed points of generalized α-nonexpansive mappings in uniformly convex Banach spaces. We provide a numerical example to illustrate the efficiency of our new iteration method. The weak w^2-stability result of the new iteration method is also studied. As an application of our main results, we approximate the solution of a fractional Volterra–Fredholm integro-differential equation. Our results improve and generalize several well-known results in the current literature.

Keywords: almost contraction mapping; weak and strong convergence; weak w^2-stability; Fractional Volterra–Fredholm Integro-Differential Equations

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1. Introduction

The fixed-point theory is important to many applied and theoretical fields, such as linear and variational inequality, nonlinear analysis, approximation theory, dynamic system theory, mathematical modelling, mathematics of fractals, mathematical economics (equilibrium problems, game theory and optimization problems), differential and integral equations. Let $\mathbb{J}$ be a nonempty subset of a Banach space $\mathbb{M}$, $\mathbb{N}$ the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The fixed point of a self-mapping $\mathcal{H}$ defined on $\mathbb{J}$ is a point $g \in \mathbb{J}$ satisfying $g = \mathcal{H}g$. The set of all fixed point of $\mathcal{H}$ is denoted by $F_{\mathcal{H}} = \{g \in \mathbb{J} : g = \mathcal{H}g\}$. The mapping $\mathcal{H}$ is said to be nonexpansive if $\|\mathcal{H}g - \mathcal{H}t\| \leq \|g - t\|$, for all $g, t \in \mathbb{J}$ and it is said to be quasi-nonexpansive if $\|\mathcal{H}g - m^*\| \leq \|g - m^*\|$, for all $g \in \mathbb{J}$ and $m^* \in F_{\mathcal{H}}$.

In the past few years, the fixed-point theory for nonexpansive mappings has attracted several authors as a results of their vast applications in integral equations, differential equations, convex optimization, control theory, signal processing, game theory, and many more. The first result concerning the existence of fixed points of nonexpansive mappings was given in Hilbert spaces by Browder [1]. In [2,3], Browder and Göhde independently extended the result of Browder [1] to uniformly convex Banach spaces. In [4,5], Goebel and Kirk further extended the result of Browder [1] to reflexive Banach spaces. Several extensions and generalizations of the class of nonexpansive mappings have been studied in the past two decades. One of these important generalizations of the class of nonexpansive mappings, known as Suzuki generalized nonexpansive mappings, was provided by Suzuki [6]. This class of mappings is also known as mappings satisfying the condition (C). In [7], Aoyama and Kohsaka introduced the class nonexpansive-type mappings known as α-nonexpansive mappings. In [8], Pant and Shukla considered another generalized nonexpansive-type mapping called generalized α-nonexpansive mappings. The authors showed that this class of mappings is more general than the class of mappings satisfying the condition (C). In [9], Pant and Pandey developed the class of Reich–Suzuki nonexpansive mappings and proved that this class of mappings is more general than the class of mappings satisfying the condition (C). Furthermore, they proved some existence and fixed-point results for such mappings.

Recently, Pandy et al. [10] combined the classes of generalized α-nonexpansive mappings and Reich–Suzuki nonexpansive mappings and defined a new class of mappings as follows:

Definition 1 ([10])**.** *Let $\mathcal{H}$ be a self-mapping defined on a nonempty subset $\mathbb{J}$ of a Banach space $\mathcal{M}$. Then, $\mathcal{H}$ is said to be a generalized α-Reich–Suzuki nonexpansive mapping if for all $g, t \in \mathbb{J}$, there exists $\alpha \in [0, 1)$ such that*

$$\frac{1}{2}\|g - \mathcal{H}g\| \leq \|g - t\| \text{ implies } \|\mathcal{H}g - \mathcal{H}t\| \leq \max\{\mathbb{W}(g,t), \mathbb{T}(g,t)\},$$

where

$$\mathbb{W}(g,t) = \alpha\|g - \mathcal{H}g\| + \alpha\|t - \mathcal{H}t\| + (1 - 2\alpha)\|g - t\|,$$

and

$$\mathbb{T}(g,h) = \alpha\|g - \mathcal{H}t\| + \alpha\|t - \mathcal{H}g\| + (1 - 2\alpha)\|g - t\|.$$

The Banach contraction theorem, also known contraction principle, is one of the fundamental results in metric spaces. This theorem was established in 1922 by Banach [11] and it works with the Picard iteration method for contraction mappings in a complete metric space. The Banach contraction principle guarantees the existence and uniqueness of fixed point of a given contraction mapping. It has been shown that the Picard iteration method always converges to the fixed points of some higher classes of mappings than contraction mappings even when there exist fixed points of such mappings. Thus, the Banach contraction principle has some drawbacks. Based on simplicity and better rate of convergence, many iteration methods have recently been developed by many authors to overcome the drawback in the Banach contraction principle (see [12–20] and the references therein).

For control sequences $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ in $(0, 1)$, the following iteration methods are called Mann [21], Ishikawa [14], Noor [16], S [22], Abbas [23], Thakur [20], M [24] and F [25].

$$\begin{cases} g_1 \in \mathbb{J}, \\ g_{m+1} = (1 - a_m)g_m + a_m\mathcal{H}g_m, \end{cases} \qquad m \in \mathbb{N}. \tag{1}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ t_m = (1 - b_m)g_m + b_m\mathcal{H}g_m, \\ g_{m+1} = (1 - a_m)g_m + a_m\mathcal{H}t_m, \end{cases} \qquad m \in \mathbb{N}. \tag{2}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = (1 - c_m)g_m + c_m\mathcal{H}g_m \\ t_m = (1 - b_m)g_m + b_m\mathcal{H}p_m, \\ g_{m+1} = (1 - a_m)g_m + a_m\mathcal{H}t_m, \end{cases} \qquad m \in \mathbb{N}. \tag{3}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ t_m = (1-b_m)g_m + b_m\mathcal{H}g_m, \\ g_{m+1} = (1-a_m)\mathcal{H}g_m + a_m\mathcal{H}t_m, \end{cases} \quad m \in \mathbb{N}. \tag{4}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = (1-c_m)g_m + c_m\mathcal{H}g_m, \\ t_m = (1-b_m)\mathcal{H}g_m + b_m\mathcal{H}p_m, \\ g_{m+1} = (1-a_m)\mathcal{H}t_m + a_m\mathcal{H}p_m, \end{cases} \quad m \in \mathbb{N}. \tag{5}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = (1-b_m)g_m + c_m\mathcal{H}g_m, \\ t_m = \mathcal{H}((1-a_m)g_m + a_m p_m), \\ g_{m+1} = \mathcal{H}t_m, \end{cases} \quad m \in \mathbb{N}. \tag{6}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = (1-a_m)g_m + a_m\mathcal{H}g_m, \\ t_m = \mathcal{H}p_m, \\ g_{m+1} = \mathcal{H}t_m, \end{cases} \quad m \in \mathbb{N}. \tag{7}$$

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = \mathcal{H}((1-a_m)g_m + a_m\mathcal{H}g_m), \\ t_m = \mathcal{H}p_m, \\ g_{m+1} = \mathcal{H}t_m, \end{cases} \quad m \in \mathbb{N}. \tag{8}$$

In [25], Ali et al. showed that the F iterative method (8) converges faster than some of the above-mentioned iteration methods and several others in the literature.

Motivated by the ongoing research in this direction, we introduce a faster iteration method, called the A^{**} iteration method, as follows:

$$\begin{cases} g_1 \in \mathbb{J}, \\ p_m = \mathcal{H}((1-a_m)g_m + a_m\mathcal{H}g_m), \\ t_m = \mathcal{H}^2 p_m, \\ g_{m+1} = \mathcal{H}^2 t_m, \end{cases} \quad m \in \mathbb{N}, \tag{9}$$

where $\{a_m\}$ is a sequence in $(0,1)$. In this paper, we prove the weak and strong convergence theorems of the A^{**} iteration method for approximation of the fixed points of almost contraction mappings and generalized α-Reich–Suzuki nonexpansive mappings in Banach spaces. We provide a numerical example to show that our new iteration method (9) converges faster than the iteration methods (1)–(8). The stability results of (9) are also studied. As an application, we find the solution of a fractional Volterra–Fredholm integro-differential equation via A^{**} iteration method.

2. Preliminaries

We give some basic definitions and lemmas that will be useful in this article.

Definition 2 ([9])**.** *A Banach space $\mathbb{M}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $g, t \in \mathbb{M}$ with $\|g\| \leq 1$, $\|t\| \leq 1$ and $\|g - t\| > \epsilon$, implies $\left\|\frac{g+t}{2}\right\| < 1 - \delta$.*

Definition 3 ([24]). *We say Banach $\mathbb{M}$ enjoys Opial's condition if for any sequence $\{g_m\}$ in $\mathcal{M}$ which converges weakly to $g \in \mathbb{M}$ implies*

$$\limsup_{m\to\infty} \|g_m - g\| < \limsup_{m\to\infty} \|g_m - t\|, \ \forall\, t \in \mathbb{M} \text{ with } t \neq g.$$

Definition 4 ([19]). *Let $\mathbb{J}$ be a nonempty closed convex subset of a Banach space $\mathbb{M}$ and $\{g_m\}$ be a bounded sequence in $\mathcal{M}$. For $g \in \mathbb{M}$, we set*

$$r(g, \{g_m\}) = \limsup_{m\to\infty} \|g_m - g\|.$$

The asymptotic radius of $\{g_m\}$ relative to $\mathbb{J}$ is defined by

$$r(\mathbb{J}, \{g_m\}) = \inf\{r(g, \{g_m\}) : g \in \mathbb{J}\}.$$

The asymptotic center of $\{g_m\}$ relative to $\mathbb{J}$ is given as:

$$A(\mathbb{J}, \{g_m\}) = \{g \in \mathbb{J} : r(g, \{g_m\}) = r(\mathbb{J}, \{g_m\})\}.$$

It is known generally that in a uniformly convex Banach $A(\mathbb{J}, \{g_m\})$ consists of only one element.

Definition 5 ([10]). *Let $\mathbb{J}$ be a nonempty closed convex subset of a Banach space $\mathbb{M}$. A mapping $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ is said to be demiclosed with respect to $g \in \mathbb{M}$ if for each sequence $\{g_m\}$ which is weakly convergent to $g \in \mathbb{J}$ and $\{\mathcal{H}g_m\}$ converges strongly to t implies that $\mathcal{H}g = t$.*

Definition 6 ([26]). *A mapping $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ is said to satisfy condition (I) if a nondecreasing function $h : [0, \infty) \to [0, \infty)$ exists with $h(0) = 0$ and for all $c > 0$, then $h(c) > 0$ with $\|g - \mathcal{H}g\| \geq h(d(g, F_{\mathcal{H}}))$ for all $g \in \mathbb{J}$, where $d(g, F_{\mathcal{H}}) = \inf_{g^* \in F_{\mathcal{H}}} \|g - g^*\|$.*

Lemma 1 ([27]). *Let $\mathbb{M}$ be a uniformly convex Banach space and $\{k_m\}$ be any sequence satisfying $0 < g \leq k_m \leq t < 1$ for all $m \geq 1$. Assume that $\{g_m\}$ and $\{t_m\}$ are any sequences of $\mathbb{M}$ such that*

$$\begin{aligned} \limsup_{m\to\infty} \|g_m\| &\leq y, \\ \limsup_{m\to\infty} \|t_m\| &\leq y \text{ and} \\ \limsup_{m\to\infty} \|k_m g_m + (1 - k_m) t_m\| &= y \end{aligned}$$

hold for some $y \geq 0$. Then, $\lim_{m\to\infty} \|g_m - t_m\| = 0$.

Lemma 2 ([10]). *Let $\mathbb{M}$ be a Banach space and $\mathbb{J}$ be a nonempty subset of $\mathbb{M}$. Suppose the mapping $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ is a generalized α-Reich–Suzuki nonexpansive. Then, for all $g, t \in \mathbb{J}$, the following condition holds:*

$$\|g - \mathcal{M}t\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|g - \mathcal{H}g\| + \|g - t\|.$$

3. Convergence Analysis

In this section, we study weak and strong convergence results for fixed points of generalized α-Reich–Suzuki nonexpansive mappings in the setting of uniformly convex Banach spaces.

Theorem 1. *Let $\mathbb{J}$ be a nonempty, closed and convex subset of a uniformly convex Banach space $\mathbb{M}$. If $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ is a generalized α-Reich–Suzuki nonexpansive mapping with $F_H \neq \emptyset$ and $\{g_m\}$ is the A^{**} iterative method defined by (9), then $\lim_{m\to\infty} \|g_m - g^*\|$ exists for each $g^* \in F_H$.*

Proof. Assume $g^* \in F_{\mathcal{H}}$ and $g \in \mathbb{J}$. Since $\mathcal{H}$ is a α-Reich–Suzuki nonexpansive mapping with $F_H \neq \emptyset$, we have

$$\|\mathcal{H}g - \mathcal{H}m^*\| \leq \|g - g^*\|. \tag{10}$$

By (9), we have

$$\begin{aligned} \|p_m - m^*\| &= \|\mathcal{H}((1-a_m)g_m + a_m\mathcal{H}g_m) - g^*\| \\ &\leq \|(1-a_m)g_m + a_m\mathcal{H}g_m - g^*\| \\ &\leq (1-a_m)\|g_m - g^*\| + a_m\|\mathcal{H}g_m - g^*\| \\ &\leq (1-a_m)\|g_m - g^*\| + a_m\|g_m - g^*\| \\ &= \|g_m - g^*\|. \end{aligned} \tag{11}$$

Using (9) and (11), we have

$$\begin{aligned} \|t_m - g^*\| &= \|\mathcal{H}^2 p_m - g^*\| \\ &= \|\mathcal{H}(\mathcal{H}p_m) - g^*\| \\ &\leq \|\mathcal{H}p_m - g^*\| \\ &\leq \|p_m - g^*\| \\ &\leq \|g_m - g^*\|. \end{aligned} \tag{12}$$

Finally, from (9) and (12), we have

$$\begin{aligned} \|g_{m+1} - g^*\| &= \|\mathcal{H}^2 t_m - g^*\| \\ &= \|\mathcal{H}(\mathcal{H}t_m) - g^*\| \\ &\leq \|\mathcal{H}t_m - g^*\| \\ &\leq \|t_m - g^*\| \\ &\leq \|g_m - g^*\|. \end{aligned} \tag{13}$$

This implies that the sequence $\{\|g_m - g^*\|\}$ is nonincreasing and bounded below. Thus, $\lim_{m\to\infty} \|g_m - g^*\|$ exists for each $g^* \in F_H$. □

Theorem 2. *Let $\mathbb{M}$, $\mathbb{J}$, $\mathcal{H}$ and $\{g_m\}$ be as defined in Theorem 1. Then, $F_{\mathcal{H}} \neq \emptyset$ if and only if $\{g_m\}$ is bounded and $\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\|$.*

Proof. From Theorem 1, we know that $\{g_m\}$ is bounded and $\lim_{m\to\infty}$ exists for any $g^* \in F_{\mathcal{H}}$. Assume that

$$\lim_{m\to\infty} \|g_m - m^*\| = y. \tag{14}$$

From (11) and (14), we have

$$\limsup_{m\to\infty} \|p_m - g^*\| \leq \limsup_{m\to\infty} \|g_m - g^*\| = y. \tag{15}$$

Using (10) and (14), we have

$$\limsup_{m\to\infty} \|\mathcal{H}g_m - g^*\| \leq \limsup_{m\to\infty} \|g_m - g^*\| = y. \tag{16}$$

By (9), we have

$$\begin{aligned} \|g_{m+1} - g^*\| &= \|\mathcal{H}^2 t_m - g^*\| \\ &= \|\mathcal{H}(\mathcal{H}t_m) - g^*\| \\ &\le \|\mathcal{H}t_m - g^*\| \\ &\le \|t_m - g^*\| \\ &= \|\mathcal{H}^2 p_m - g^*\| \\ &= \|\mathcal{H}(\mathcal{H}p_m) - g^*\| \\ &\le \|\mathcal{H}p_m - g^*\| \\ &\le \|p_m - g^*\|. \end{aligned} \tag{17}$$

Therefore,

$$y \le \liminf_{m\to\infty} \|p_m - g^*\|. \tag{18}$$

By (15) and (18), we have

$$\begin{aligned} y &= \lim_{m\to\infty} \|p_m - g^*\| \\ &= \lim_{m\to\infty} \|\mathcal{H}((1-a_m)g_m + a_m\mathcal{H}g_m) - g^*\| \\ &\le \lim_{m\to\infty} \|(1-a_m)g_m + a_m\mathcal{H}g_m - g^*\| \\ &= \lim_{m\to\infty} \|(1-a_m)(g_m - g^*) + a_m(\mathcal{H}g_m - g^*)\| \\ &= \lim_{m\to\infty} ((1-a_m)\|g_m - g^*\| + a_m\|\mathcal{H}g_m - g^*\|) \\ &\le \lim_{m\to\infty} ((1-a_m)\|g_m - g^*\| + a_m\|g_m - g^*\|) \\ &\le y. \end{aligned} \tag{19}$$

Hence,

$$\lim_{m\to\infty} \|(1-a_m)(g_m - g^*) + a_m(\mathcal{H}g_m - g^*)\| = y. \tag{20}$$

Using (14), (16), (20) and Lemma 1, we obtain

$$\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0. \tag{21}$$

Conversely, if $\{g_m\}$ is bounded and $\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0$. Let $g^* \in A(\mathbb{J}, \{g_m\})$, then, by Lemma 2, we have

$$\begin{aligned} r(\mathcal{H}g^*, \{g_m\}) &= \limsup_{m\to\infty} \|g_m - \mathcal{H}g^*\| \\ &\le \left(\frac{3+\varpi}{1-\varpi}\right) \limsup_{m\to\infty} \|g_m - \mathcal{H}g_m\| + \limsup_{m\to\infty} \|\mathcal{H}g_m - g^*\| \\ &= \limsup_{m\to\infty} \|g_m - g^*\| \\ &= r(g^*, \{g_m\}). \end{aligned}$$

This shows that $\mathcal{H}g^* \in A(\mathbb{J}, \{g_m\})$. Since the Banach space $\mathbb{M}$ is uniformly convex, we know that $A(\mathbb{J}, \{g_m\})$ is a set with one element and it follows that $\mathcal{H}g^* = g^*$. Hence, $F_{\mathcal{H}} \neq \emptyset$. □

Theorem 3. *Let $\mathbb{M}$, $\mathbb{J}$, $\mathcal{H}$ and $\{g_m\}$ be as defined in Theorem 1 such that $F_{\mathcal{H}} \neq \emptyset$. Suppose that $\mathbb{M}$ posses the Opial's condition, then the A^{**} iteration method $\{g_m\}$ converges weakly to a point in $F_{\mathcal{H}}$.*

Proof. For $F_{\mathcal{H}} \neq \emptyset$, we have established in Theorems 1 and 2 that $\lim_{m\to\infty} \|g_m - g^*\|$ exists and $\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0$. Now, we will prove that it is not possible for $\{g_m\}$ to have two weak subsequential limits in $F_{\mathcal{H}}$. Suppose that q and w are two weak subsequential limits of $\{g_{m_i}\}$ and $\{g_{m_j}\}$, respectively. Then, by Theorem 2, we know that $(I - \mathcal{H})$ is demiclosed at 0—this implies that $(I - \mathcal{H})q = 0$. Thus, $\mathcal{H}q = q$. By a similar argument, we can show that $\mathcal{H}w = w$. Next, we show uniqueness. Assume $q \neq w$, then, from Opial's property,

$$\begin{aligned}
\lim_{m\to\infty} \|g_m - q\| &= \lim_{m_i\to\infty} \|g_{m_i} - q\| < \lim_{m_i\to\infty} \|g_{m_i} - w\| = \lim_{m\to\infty} \|g_m - w\| \\
&= \lim_{m_j\to\infty} \|g_{m_j} - w\| < \lim_{m_j\to\infty} \|g_{m_j} - q\| = \lim_{m\to\infty} \|g_m - q\|,
\end{aligned}$$

which is a contradiction, so $q = w$. Hence, $\{g_m\}$ converges weakly to $q \in F_{\mathcal{H}}$. □

We now prove some convergence theorems.

Theorem 4. *Let $\mathbb{M}$, $\mathbb{J}$, $\mathcal{H}$ and $\{g_m\}$ be as defined in Theorem 1 such that $F_{\mathcal{H}} \neq \emptyset$. Then, the iteration method $\{g_m\}$ converges to a point in $F_{\mathcal{H}}$ if and only if $\liminf_{m\to\infty} d(g_m, F_{\mathcal{H}}) = 0$, where $d(g_m, F_{\mathcal{H}}) = \inf\{\|g_m - g^*\| : g^* \in F_{\mathcal{H}}\}$.*

Proof. The necessity is trivial.

Now, we prove the converse. Suppose $\liminf_{m\to\infty} d(g_m, F_{\mathcal{H}}) = 0$ and $g^* \in F_{\mathcal{H}}$. From Theorem 1, $\lim_{m\to\infty} \|g_m - g^*\|$ exists, for each $g^* F_{\mathcal{H}}$. It suffices to show that $\{g_m\}$ is Cauchy in $\mathbb{J}$. Since $\lim_{m\to\infty} d(g_m, F_{\mathcal{H}}) = 0$, then for $\epsilon > 0$, there exists $l_0 \in \mathbb{N}$ such that for all $m \geq l_0$,

$$\begin{aligned}
d(g_m, F_{\mathcal{H}}) &< \frac{\epsilon}{2} \\
\inf\{\|g_m - g^*\| : g^* \in F_{\mathcal{H}}\} &< \frac{\epsilon}{2}.
\end{aligned}$$

Particularly, $\inf\{\|g_{l_0} - g^*\| : g^* \in F_{\mathcal{H}}\} < \frac{\epsilon}{2}$. Thus, there exists $g^* \in F_{\mathcal{H}}$ such that

$$\|g_{l_0} - g^*\| < \frac{\epsilon}{2}.$$

Let $l, m \geq l_0$—we obtain

$$\begin{aligned}
\|g_{m+l} - g_m\| &\leq \|g_{m+l} - g^*\| + \|g_m - g^*\| \\
&\leq \|g_{l_0} - g^*\| + \|g_{l_0} - g^*\| \\
&= 2\|g_{l_0} - g^*\| < \epsilon.
\end{aligned}$$

This shows that $\{g_m\}$ is Cauchy in $\mathbb{J}$. By the closeness $\mathbb{J}$, there must be a point $v \in \mathbb{J}$ satisfying $\lim_{m\to\infty} g_m = v$. Now, $\lim_{m\to\infty} d(g_m, F_{\mathcal{H}}) = 0$ implies that $d(v, F_{\mathcal{H}}) = 0$, that is $v \in F_{\mathcal{H}}$. □

Theorem 5. *Let $\mathbb{M}$, $\mathcal{H}$ and $\{g_m\}$ be as defined in Theorem 1 such that $F_{\mathcal{H}} \neq \emptyset$. If $\mathbb{J}$ is a nonempty convex compact subset of $\mathbb{M}$, then the A^{**} iteration method $\{g_m\}$converges strongly to a point in $F_{\mathcal{H}}$.*

Proof. By Theorem 2, we have $\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0$. Since $\mathbb{J}$ is compact, then we know that $\{g_m\}$ has a strong convergent subsequence $\{g_{m_i}\}$ with a strong limit k. Using Lemma 2, we have

$$\|g_{m_i} - \mathcal{H}k\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|g_{m_i} - \mathcal{H}g_{m_i}\| + \|g_{m_i} - k\|.$$

As $i \to \infty$, we have $g_{m_i} \to \mathcal{H}k$. Therefore, $\mathcal{H}k = k$, i.e., $k \in F_{\mathcal{H}}$. By Theorem 1, $\lim_{m\to\infty} \|g_m - k\|$ exists. This shows that k is a strong limit for $\{g_m\}$. $\square$

Theorem 6. *Let $\mathbb{M}$, $\mathcal{H}$ and $\{g_m\}$ be as defined in Theorem 1 such that $F_{\mathcal{H}} \neq \emptyset$. If $\mathbb{J}$ is a nonempty convex compact subset of $\mathbb{M}$, then the A^{**} iteration method $\{g_m\}$converges strongly to a fixed point in $F_{\mathcal{H}}$.*

Proof. By Theorem 2, it is established that

$$\lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0. \tag{22}$$

From Definition 6 and (22), we obtain

$$0 \leq \lim_{m\to\infty} f(d(g_m, F_{\mathcal{H}})) \leq \lim_{m\to\infty} \|g_m - \mathcal{H}g_m\| = 0 \;\Rightarrow\; f(d(g_m, F_{\mathcal{H}})) = 0.$$

Since $h : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $h(0) = 0$ and $h(c) > 0$, for all $c > 0$, we obtain

$$\lim_{m\to\infty} d(g_m, F_{\mathcal{H}}) = 0.$$

By the application of Theorem 4, the conclusion follows. $\square$

In this section, we provide an example of a mapping which is generalized α-Reich–Suzuki nonexpnsive, but does not satisfy condition (C). With the provided example, we will conduct an experiment to demonstrate the efficiency of our new iterative method (2) over some existing iterative methods.

Example 1. *Let $\mathbb{M} = \mathbb{R}$ with the usual norm and $\mathbb{J} = [7, 10]$. Define $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ by*

$$\mathcal{H}g = \begin{cases} \frac{g+56}{8}, & \text{if } g < 10, \\ 7, & \text{if } g = 10, \end{cases}$$

for all $g \in \mathbb{J}$.

(1). Let $g = 9$ and $t = 10$, then we have

$$\frac{1}{2}\|g - \mathcal{H}g\| = \frac{7}{8} < 1 = \|g - t\|.$$

However,

$$\|\mathcal{H}g - \mathcal{H}h\| = \frac{9}{8} > 1 = \|g - h\|.$$

Therefore, $\mathcal{H}$ does not satisfy condition (C).

(2). We will now show that $\mathcal{H}$ is a generalized α-Reich–Suzuki nonexpansive mappings with $\alpha = \frac{1}{3}$. We consider the following cases:

***Case I:** Let $g, h < 10$, then*

$$\begin{aligned}
\mathbb{W}(g,t) &= \alpha\|g - \mathcal{H}g\| + \alpha\|t - \mathcal{H}t\| + (1 - 2\alpha)\|g - t\| \\
&= \frac{1}{3}\left|g - \left(\frac{g+56}{8}\right)\right| + \frac{1}{3}\left|t - \left(\frac{t+56}{8}\right)\right| + \frac{1}{3}|g - t| \\
&= \frac{1}{3}\left|\frac{7g - 56}{8}\right| + \frac{1}{3}\left|\frac{7t - 56}{8}\right| + \frac{1}{3}|g - t| \\
&\geq \frac{1}{8}|g - t| = \|\mathcal{H}g - \mathcal{H}t\|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{T}(g,t) &= \alpha\|g-\mathcal{H}t\| + \alpha\|t-\mathcal{H}g\| + (1-2\alpha)\|g-t\| \\
&= \frac{1}{3}\left|g-\left(\frac{t+56}{8}\right)\right| + \frac{1}{3}\left|t-\left(\frac{g+56}{8}\right)\right| + \frac{1}{3}|g-h| \\
&\geq \frac{1}{8}|g-t| = \|\mathcal{H}g-\mathcal{H}t\|.
\end{aligned}$$

Case II: *Let $g < 10$ and $t = 10$, we have*

$$\begin{aligned}
\mathbb{W}(g,t) &= \alpha\|g-\mathcal{H}g\| + \alpha\|t-\mathcal{H}t\| + (1-2\alpha)\|g-t\| \\
&= \frac{1}{3}\left|g-\left(\frac{g+56}{8}\right)\right| + \frac{1}{3}|10-7| + \frac{1}{3}|g-10| \\
&= \frac{1}{3}\left|\frac{7g-56}{8}\right| + 1 + \frac{1}{3}|g-10| \\
&\geq \frac{1}{8}|g| = \|\mathcal{H}g-\mathcal{H}t\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{T}(g,t) &= \alpha\|g-\mathcal{H}t\| + \alpha\|t-\mathcal{H}g\| + (1-2\alpha)\|g-t\| \\
&= \frac{1}{3}|g-7| + \frac{1}{3}\left|10-\left(\frac{g+56}{8}\right)\right| + \frac{1}{3}|g-10| \\
&= \frac{1}{3}|g-7| + \frac{1}{3}\left|\frac{24-g}{8}\right| + \frac{1}{3}|g-10| \\
&\geq 1 + \frac{1}{3}\left|\frac{24-g}{8}\right| \\
&\geq \frac{1}{8}|g| = \|\mathcal{H}g-\mathcal{H}t\|.
\end{aligned}$$

Case III: *Let $g = 10$ and $h < 10$—we obtain*

$$\begin{aligned}
\mathbb{W}(g,t) &= \alpha\|g-\mathcal{H}g\| + \alpha\|t-\mathcal{H}t\| + (1-2\alpha)\|g-t\| \\
&= \frac{1}{3}|10-7| + \frac{1}{3}\left|t-\left(\frac{t+56}{8}\right)\right| + \frac{1}{3}|10-t| \\
&= 1 + \left|\frac{7t-56}{7}\right| + \frac{1}{3}|10-t| \\
&\geq \frac{1}{8}|t| = \|\mathcal{H}g-\mathcal{H}t\|.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\mathbb{T}(g,t) &= \alpha\|g-\mathcal{H}t\| + \alpha\|t-\mathcal{H}g\| + (1-2\alpha)\|g-t\| \\
&= \frac{1}{3}\left|10-\left(\frac{t+56}{8}\right)\right| + \frac{1}{3}|t-7| + \frac{1}{3}|10-t| \\
&= \frac{1}{3}\left|\frac{24-t}{8}\right| + \frac{1}{3}|t-7| + \frac{1}{3}|10-t| \\
&\geq \frac{1}{3}\left|\frac{24-t}{8}\right| + 1 \\
&\geq \frac{1}{8}|t| = \|\mathcal{H}g-\mathcal{H}t\|.
\end{aligned}$$

Case IV: *Let $g = t = 10$, we obtain*

$$\mathbb{W}(g,t) = \alpha\|g-\mathcal{H}g\| + \alpha\|t-\mathcal{H}t\| + (1-2\alpha)\|g-t\| \geq 0 = \|\mathcal{H}g-\mathcal{H}t\|.$$

Also,

$$\mathbb{T}(g,t) = \alpha\|g - \mathcal{H}t\| + \alpha\|t - \mathcal{H}g\| + (1-2\alpha)\|g - t\| \geq 0 = \|\mathcal{H}g - \mathcal{H}t\|.$$

In all the cases above, it clear that $\|\mathcal{H}g - \mathcal{H}t\| \leq \max\{\mathbb{W}(g,h), \mathbb{T}(g,h)\}$ *for* $\alpha = \frac{1}{3}$. *Hence,* $\mathcal{H}$ *is a generalized α-Reich–Suzuki nonexpansive mapping with fixed point* $g^* = 8$.

If we choose starting $g_1 = 7$ *and control sequences* $a_m = b_m = c_m = \frac{10}{11}$, *then the following Tables 1 and 2 and Figures 1 and 2 show that the* A^{**} *iteration method (9) converges faster to 8 than the Mann (1), Ishikiwa (2), Noor (3), Abbas (4), S (5), Thakur (6), M (7) and F (8) iteration methods.*

Table 1. Comparison of speed of convergence of the A^{**} iteration method with some known iteration methods.

g_m	**Mann**	**Ishikawa**	**S**	**M**	A^{**}
g_1	7.0000000000	7.0000000000	7.0000000000	7.0000000000	7.0000000000
g_2	7.7954545455	7.8858471074	7.9653925620	7.9968039773	7.9999937578
g_3	7.9581611570	7.9869691171	7.9988023252	7.9999897854	8.0000000000
g_4	7.9914420548	7.9985124870	7.9999585515	7.9999999674	8.0000000000
g_5	7.9982495112	7.9998301961	7.9999985656	7.9999999999	8.0000000000
g_6	7.9996419455	7.9999806164	7.9999999504	8.0000000000	8.0000000000
g_7	7.9999267616	7.9999977873	7.9999999983	8.0000000000	8.0000000000
g_8	7.9999850194	7.9999997474	7.9999999999	8.0000000000	8.0000000000
g_9	7.9999969358	7.9999999712	8.0000000000	8.0000000000	8.0000000000
g_{10}	7.9999993732	7.9999999967	8.0000000000	8.0000000000	8.0000000000
g_{11}	7.9999998718	7.9999999996	8.0000000000	8.0000000000	8.0000000000
g_{12}	7.9999999738	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{13}	7.9999999946	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{14}	7.9999999989	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{15}	7.9999999998	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{16}	8.0000000000	8.0000000000	8.0000000000	8.0000000000	8.0000000000

Table 2. Comparison of speed of convergence of the A^{**} iteration method with some known iteration methods.

g_m	**Noor**	**Abbas**	**Thakur**	**F**	A^{**}
g_1	7.0000000000	7.0000000000	7.0000000000	7.0000000000	7.0000000000
g_2	7.8961189895	7.9763629320	7.9956740702	7.9996004972	7.9999937578
g_3	7.9892087357	7.9994412890	7.9999812863	7.9999998404	8.0000000000
g_4	7.9988789926	7.9999867937	7.9999999190	7.9999999999	8.0000000000
g_5	7.9998835486	7.9999996878	7.9999999996	8.0000000000	8.0000000000
g_6	7.9999879029	7.9999999926	8.0000000000	8.0000000000	8.0000000000
g_7	7.9999987433	7.9999999998	8.0000000000	8.0000000000	8.0000000000
g_8	7.9999998695	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_9	7.9999999864	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{10}	7.9999999986	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{11}	7.9999999999	8.0000000000	8.0000000000	8.0000000000	8.0000000000
g_{12}	8.0000000000	8.0000000000	8.0000000000	8.0000000000	8.0000000000

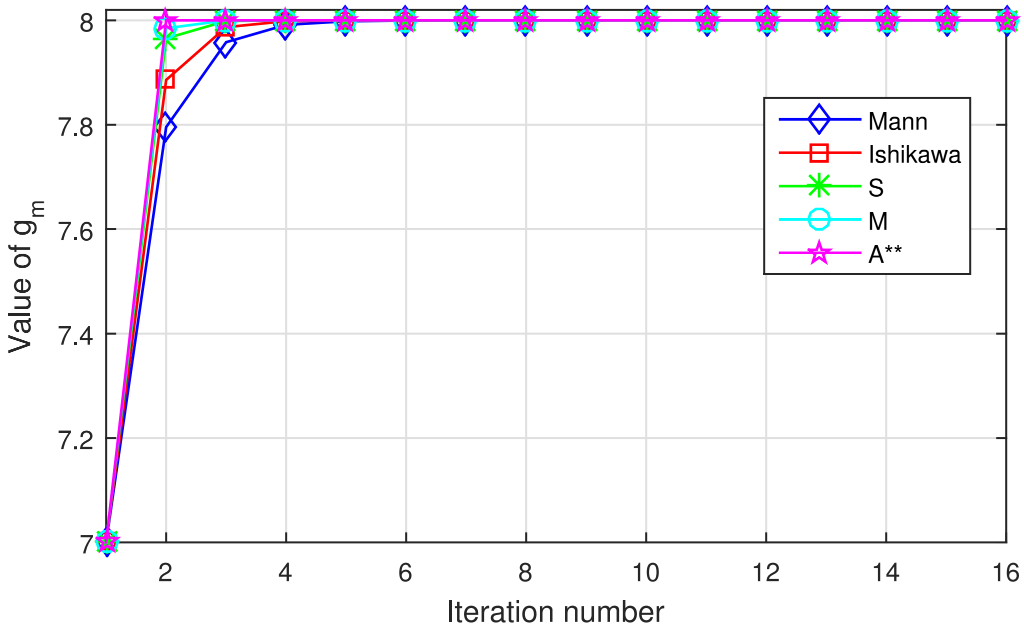

Figure 1. Graph corresponding to Table 1.

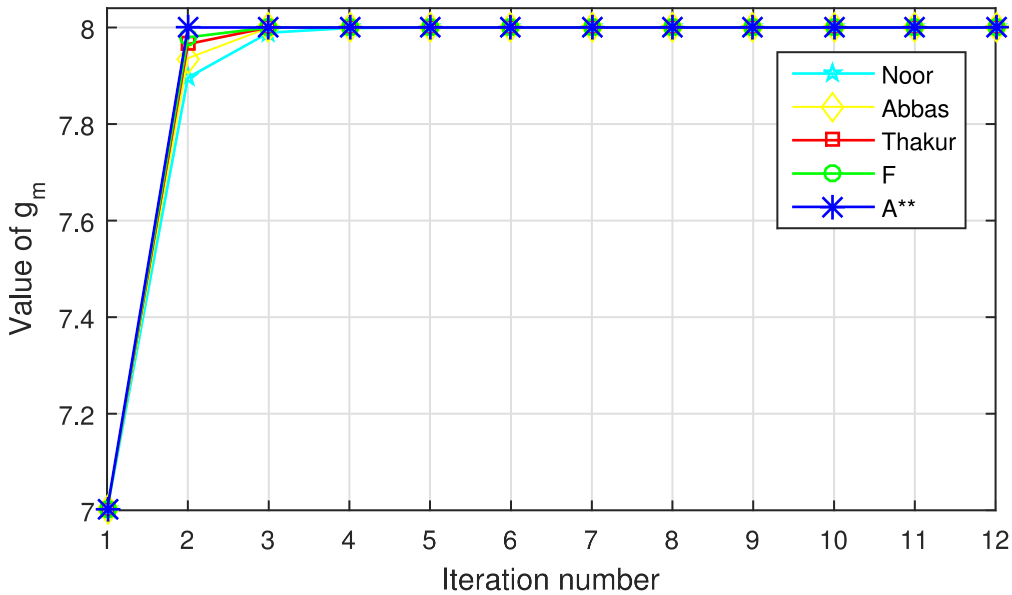

Figure 2. Graph corresponding to Table 2.

4. w^2-Stability Result

In this section, we prove the convergence and weak w^2-stability results of A^{**} iteration method (9) with respect to almost contraction mappings. Before that, we will recall the following definitions that will serve as an important tool in achieving our results.

Definition 7 ([28])**.** *A mapping $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ is said to be an almost contraction if there exists $\gamma \in [0, 1)$ and some constant $L \geq 0$, such that*

$$\|\mathcal{H}g - \mathcal{H}t\| \leq \gamma\|g - t\| + L\|g - \mathcal{H}g\|, \;\; \forall\, g, t \in \mathbb{J}. \tag{23}$$

Definition 8 ([29])**.** *Two sequences $\{g_m\}$ and $\{t_m\}$ are said to be equivalent if*

$$\|g_m - t_m\| \to 0, \;\; as \;\; m \to \infty.$$

Definition 9 ([30]). *Let $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ be a self-map and for arbitrary $g_1 \in \mathbb{J}$, let $\{g_m\}$ be the iterative algorithm defined by*

$$g_{m+1} = f(\mathcal{H}, g_m), \ m \geq 1. \tag{24}$$

Assume that $g_m \to g^*$ as $m \to \infty$, for all $g^* \in F_{\mathcal{H}}$ and for any sequence $\{w_m\} \subset \mathbb{J}$ which is equivalent to $\{g_m\}$, we have

$$\lim_{m\to\infty} \epsilon_m = \lim_{m\to\infty} \|w_{m+1} - f(\mathcal{H}, w_m)\| = 0 \implies \lim_{m\to\infty} w_m = g^*,$$

then we say that the iterative algorithm (10) is weak w^2-stable with respect to $\mathcal{H}$.

Now, we present our main results in this section as follows:

Theorem 7. *Let $\mathbb{J}$ be a nonempty closed convex subset of a Banach space $\mathbb{M}$ and $\mathcal{H} : \mathbb{J} \to \mathbb{J}$ be an almost contraction mapping. If $\{g_m\}$ is the sequence defined by (2), then $\{g_m\}$ converges to a unique fixed point of $\mathcal{H}$. Moreover, $\{g_m\}$ is weakly w^2-stable with respect to $\mathcal{H}$.*

Proof. First, we show that $\lim_{m\to\infty} g_m = g^* \in F_{\mathcal{H}}$. Using (9), we obtain

$$\begin{aligned}
\|p_m - m^*\| &= \|\mathcal{H}((1-a_m)g_m + a_m\mathcal{H}g_m) - g^*\| \\
&\leq \gamma\|(1-a_m)g_m + a_m\mathcal{H}g_m - g^*\| \\
&\leq \gamma((1-a_m)\|g_m - g^*\| + a_m\|\mathcal{H}g_m - g^*\|) \\
&\leq \gamma((1-a_m)\|g_m - g^*\| + a_m\gamma\|g_m - g^*\|) \\
&= \gamma(1-(1-\gamma)a_m)\|g_m - g^*\|.
\end{aligned} \tag{25}$$

Using (9) and (25), we obtain

$$\begin{aligned}
\|t_m - g^*\| &= \|\mathcal{H}^2 p_m - g^*\| \\
&= \|\mathcal{H}(\mathcal{H}p_m) - g^*\| \\
&\leq \gamma\|\mathcal{H}p_m - g^*\| \\
&\leq \gamma^2\|p_m - g^*\| \\
&\leq \gamma^3(1-(1-\gamma)a_m)\|g_m - g^*\|.
\end{aligned} \tag{26}$$

Finally, by (9) and (26), we obtain

$$\begin{aligned}
\|g_{m+1} - g^*\| &= \|\mathcal{H}^2 t_m - g^*\| \\
&= \|\mathcal{H}(\mathcal{H}t_m) - g^*\| \\
&\leq \gamma\|\mathcal{H}t_m - g^*\| \\
&\leq \gamma^2\|t_m - g^*\| \\
&\leq \gamma^5(1-(1-\gamma)a_m)\|g_m - g^*\|.
\end{aligned} \tag{27}$$

Since $0 \leq \gamma < 1$ and $0 < a_m < 1$, we have $1-(1-\gamma)a_m < 1$. Therefore, (27) becomes

$$\|g_{m+1} - g^*\| \leq \gamma^5\|g_m - g^*\|. \tag{28}$$

Inductively, we obtain

$$\|g_{m+1} - g^*\| \leq \gamma^{5(m+1)}\|g_0 - g^*\|.$$

Since $0 \leq \gamma < 1$, it follows that $\lim_{\gamma\to\infty} u_\gamma = g^*$.

Now, we will prove that $\{g_m\}$ is weak w^2-stable with respect to $\mathcal{H}$.

We assume that the sequence $\{w_m\} \in \mathbb{J}$ is equivalent to sequence $\{g_m\}$ defined by (9). Let the sequence $\epsilon_m \in \mathbb{R}^+$ be defined by

$$\begin{cases} w_1 \in \mathbb{J}, \\ q_m = \mathcal{H}((1-a_m)w_m + a_m\mathcal{H}w_m), \\ h_m = \mathcal{H}^2 q_m, \\ \epsilon_m = \|w_{m+1} - \mathcal{H}^2 h_m\|, \end{cases} \quad m \in \mathbb{N}, \tag{29}$$

where $\{a_m\}$ is a sequences in (0,1).

Suppose $\lim_{m\to\infty} \epsilon_m = 0$. From (9) and (29), we set $u_m = (1-a_m)g_m + a_m\mathcal{H}g_m$ and $v_m = (1-a_m)w_m + a_m\mathcal{H}w_m$. Noting that $0 \le \gamma < 1$ and $0 < a_m < 1$, implies $1-(1-\gamma)a_m < 1$, then we have

$$\begin{aligned} \|u_m - v_m\| &= \|(1-a_m)g_m + a_m\mathcal{H}g_m - ((1-a_m)w_m + a_m\mathcal{H}w_m)\| \\ &\le (1-a_m)\|g_m - w_m\| + a_m\|\mathcal{H}g_m - \mathcal{H}w_m\| \\ &\le (1-a_m)\|g_m - w_m\| + a_m\gamma\|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\| \\ &= (1-(1-\gamma)a_m)\|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\| \\ &\le \|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\|. \end{aligned} \tag{30}$$

Using (9), (29) and (30), we obtain

$$\begin{aligned} \|p_m - q_m\| &= \|\mathcal{H}u_m - \mathcal{H}v_m\| \\ &\le \gamma\|u_m - v_m\| + L\|u_m - \mathcal{H}u_m\| \\ &\le \gamma[\|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\|] + L\|u_m - \mathcal{H}u_m\| \\ &\le \|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\| + L\|u_m - \mathcal{H}u_m\|. \end{aligned} \tag{31}$$

By (9), (29) and (31), we have

$$\begin{aligned} \|t_m - h_m\| &= \|\mathcal{H}^2 p_m - \mathcal{H}^2 q_m\| \\ &= \|\mathcal{H}(\mathcal{H}p_m) - \mathcal{H}(\mathcal{H}q_m)\| \\ &\le \gamma\|\mathcal{H}p_m - \mathcal{H}q_m\| + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| \\ &\le \|\mathcal{H}p_m - \mathcal{H}q_m\| + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| \\ &\le \gamma\|p_m - q_m\| + L\|p_m - \mathcal{H}p_m\| + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| \\ &\le \|p_m - q_m\| + L\|p_m - \mathcal{H}p_m\| + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| \\ &\le \|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\| + L\|u_m - \mathcal{H}u_m\| + L\|p_m - \mathcal{H}p_m\| \\ &\quad + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\|. \end{aligned} \tag{32}$$

Using (9), (29) and (32), we have

$$\begin{aligned} \|w_{m+1} - g^*\| &\le \|w_{m+1} - g_{m+1}\| + \|g_{m+1} - g^*\| \\ &\le \|w_{m+1} - \mathcal{H}^2 h_m\| + \|\mathcal{H}^2 h_m - g_{m+1}\| + \|g_{m+1} - g^*\| \\ &= \epsilon_m + \|\mathcal{H}(\mathcal{H}t_m) - \mathcal{H}(\mathcal{H}h_m)\| + \|g_{m+1} - g^*\| \\ &= \epsilon_m + \gamma\|\mathcal{H}t_m - \mathcal{H}h_m\| + L\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| + \|g_{m+1} - g^*\| \\ &\le \epsilon_m + \|\mathcal{H}t_m - \mathcal{H}h_m\| + L\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| + \|g_{m+1} - g^*\| \\ &\le \epsilon_m + \gamma\|t_m - h_m\| + L\|t_m - \mathcal{H}t_m\| + L\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| + \|g_{m+1} - g^*\| \\ &\le \epsilon_m + \|t_m - h_m\| + L\|t_m - \mathcal{H}t_m\| + L\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| + \|g_{m+1} - g^*\| \\ &\le \epsilon_m + \|g_m - w_m\| + a_m L\|g_m - \mathcal{H}g_m\| + L\|u_m - \mathcal{H}u_m\| + L\|p_m - \mathcal{H}p_m\| \\ &\quad + L\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| + L\|t_m - \mathcal{H}t_m\| \\ &\quad + L\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| + \|g_{m+1} - g^*\|. \end{aligned} \tag{33}$$

As shown above, $\lim_{m\to\infty} \|g_m - g^*\| = 0$. Thus, observe that

$$\begin{aligned}
\|u_m - \mathcal{H}u_m\| &\leq \|u_m - g^*\| + \|g^* - \mathcal{H}u_m\| \\
&\leq \|u_m - g^*\| + \gamma\|g^* - u_m\| \\
&= (1+\gamma)\|u_m - g^*\| \\
&= (1+\gamma)\|(1-a_m)g_m + a_m\mathcal{H}g_m - g^*\| \\
&\leq (1+\gamma)[(1-a_m)\|g_m - g^*\| + a_m\|\mathcal{H}g_m - g^*\|] \\
&\leq (1+\gamma)[(1-a_m)\|g_m - g^*\| + a_m\gamma\|g_m - g^*\|] \\
&= (1+\gamma)(1-(1-\gamma)a_m)\|g_m - g^*\| \\
&\leq (1+\gamma)\|g_m - g^*\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Again, by (11), we have

$$\begin{aligned}
\|\mathcal{H}p_m - \mathcal{H}(\mathcal{H}p_m)\| &\leq \|\mathcal{H}p_m - g^*\| + \|g^* - \mathcal{H}(\mathcal{H}p_m)\| \\
&\leq \gamma\|p_m - g^*\| + \gamma\|g^* - \mathcal{H}p_m\| \\
&\leq \gamma\|p_m - g^*\| + \gamma^2\|g^* - p_m\| \\
&= \gamma(1+\gamma)\|p_m - g^*\| \to 0, \text{as } m \to \infty.
\end{aligned}$$

Similarly, we can use same approach as above to show that

$$\|\mathcal{H}t_m - \mathcal{H}(\mathcal{H}t_m)\| \to 0 \text{ as } m \to \infty.$$

Additionally, observe that

$$\begin{aligned}
\|g_m - \mathcal{H}g_m\| &\leq \|g_m - g^*\| + \|g^* - \mathcal{H}g_m\| \\
&\leq \|g_m - g^*\| + \gamma\|g^* - g_m\| \\
&= (1+\gamma)\|g_m - g^*\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Using the same argument above, we can show that

$$\|p_m - \mathcal{H}p_m\| = \|t_m - \mathcal{H}t_m\| \to 0 \text{ as } m \to \infty.$$

Since $\lim_{m\to\infty} \|g_m - g^*\| = 0$, then we know that $\lim_{m\to\infty} \|g_{m+1} - g^*\| = 0$. Now, from the equivalence of $\{g_m\}$ and $\{w_m\}$, we also know that $\lim_{m\to\infty} \|g_m - w_m\| = 0$.
Thus, taking the limit of both sides of (33), we have

$$\lim_{m\to\infty} \|w_m - g^*\| = 0.$$

This implies that A^{**} iterative method (9) is weakly w^2-stable with respect to $\mathcal{H}$. $\square$

5. Application to Fractional Volterra–Fredholm Integro-Differential Equations

Fractional differential equations remain a significant tool for modeling several problems in various field of engineering and applied sciences. It is well known that fraction models are more reliable than the classical models. In fact, fractional differential equations can be applied in certain fields such as economics, physics, blood flow phenomena, image processing, aerodynamics, and so on (see [31] and the references therein). Many applications of the fractional calculus can be found in the literature [32]. Different analytical and numerical methods have been used for solving nonlinear fractional differential equations [33]. Most of the physical processes are modeled by nonlinear fractional order differential equations. Solving nonlinear fractional differential equations by analytical methods is very difficult [34]. In this article, we will use our iterative method (9) to solve a nonlinear fractional order differential equation.

Definition 10. *The fractional derivative of $f(u)$ in the sense of Caputo is defined by*

$$ {}^{C}\mathcal{D}_u^k f(u) = \frac{1}{\Gamma(k-m)} \int_b^u f^{(m)}(v)(u-v)^{n-k-1} dv, \quad (m-1 < k < m), $$

where k is the order of the derivation which could be real or complex number with $\Re k > 0$.

In this article, we consider the following nonlinear fractional Volterra–Fredholm integro-differential equation:

$$ {}^{C}\mathcal{D}^k g(u) = c(u)g(u) + r(u) + \int_0^u M_1(u,s)K_1(g(s))ds + \int_0^1 M_2(u,s)K_2(g(s))ds, \tag{34} $$

with initial condition

$$ z^{(i)}(0) = \eta_i, \; i = 0, 1, 2, \cdots, m-1. \tag{35} $$

where ${}^{C}\mathcal{D}^k$ is the Caputo fractional derivative, $m-1 < k \leq m$ and $m \in \mathbb{N}$, $g : \mathbb{G} \to \mathbb{R}$, where $\mathbb{G} = [0,1]$ is the unknown continuous function, $d : \mathbb{G} \to \mathbb{R}$ and $M_i : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$, are continuous functions. $K_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are Lipschitz continuous functions.

Suppose the following hypotheses are performed:

(H_1) Two constants L_{K_1} and L_{K_2} exist such that for any $g_1, g_2 \in C(\mathbb{G}, \mathbb{R})$ we have

$$ |K_1(g_1(u)) - K_1(g_1(u))| \leq L_{K_1}|g_1 - g_2| $$

and

$$ |K_2(g_1(u)) - K_2(g_1(u))| \leq L_{K_2}|g_1 - g_2|. $$

(H_2) Two functions $M_1^*, M_2^* \in C(\mathbb{D}, \mathbb{R}^+)$ exist, the set of all positive functions is continuous on $\mathbb{D} = \{(u,s) \in \mathbb{R} \times \mathbb{R} : 0 \leq s \leq u \leq 1\}$ such that

$$ M_1^* = \sup_{u,s\in[0,1]} \int_0^u |M_1(u,s)|ds < \infty, \quad M_2^* = \sup_{u,s\in[0,1]} \int_0^u |M_2(u,s)|ds < \infty. $$

(H_3) The functions $c, r : \mathbb{G} \to \mathbb{R}$ are continuous.

(H_4)

$$ \left(\frac{\|c\|_\infty + M_1^* L_{K_1} + M_2^* L_{K_2}}{\Gamma(k+1)} \right) < 1. $$

A function $g^* \in (\mathbb{G}, \mathbb{R})$ is said to be a solution of the initial value problem if it satisfies (34) and (35). For $g_0(u) \in (\mathbb{G}, \mathbb{R})$, finding the solution of (34) and (35) is equivalent to finding the solution of the following integral equation [35]:

$$ \begin{aligned} g(u) \quad = \quad & g_0 + \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} c(t) g(t) dt + \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} r(t) dt \\ & + \left(\int_0^t M_1(t,\tau) K_1(g(\tau)) d\tau + \int_0^1 M_2(t,\tau) K_2(g(\tau)) d\tau \right) dt, \end{aligned} \tag{36} $$

for each $u \in \mathbb{J}$, and $g_0 = \sum_{n=0}^{m-1} g^n(0^+) \frac{u^n}{n!}$. Under the Hypotheses (H_1)–(H_4), Hamoud et al. [35] prove that the problem (34) and (35) has a unique solution.

In the following Theorem, we approximate the solution of (34) and (35) via A^{**} iteration method (9).

Theorem 8. *Let $\mathbb{M} = (\mathbb{G}, \mathbb{R})$ be a Banach space with the Chebyshev norm $\|f - h\|_\infty = \max_{y \in \mathbb{G}} |f(y) - h(y)|$. Let $\{g_m\}$ be the iteration method (9) for the operator $\mathcal{H} : \mathbb{M} \to \mathbb{M}$ which is define by*

$$\begin{aligned} \mathcal{H}g(u) &= g_0 + \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} c(t) g(t) dt + \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} r(t) dt \\ &\quad + \left(\int_0^t M_1(t,\tau) K_1(g(\tau)) d\tau + \int_0^1 M_2(t,\tau) K_2(g(\tau)) d\tau \right) dt. \end{aligned} \tag{37}$$

If Hypotheses (H_1)–(H_4) are fulfilled, then the problem (34) and (35) has a unique solution $g^* \in (\mathbb{G}, \mathbb{R})$ and the A^{**} iteration method converges to g^*.

Proof. The existence of the unique solution g^* follows from [35]. If $g^* \in (\mathbb{G}, \mathbb{R})$ is a fixed point of $\mathcal{H}$, then g^* is a solution of (34) and (35). Now, we show that the A^{**} iteration method converges to g^*. First, we will prove that the operator $\mathcal{H}$, which is defined by (37), is an almost contraction.

Using the Hypotheses (H_1)–(H_4), we have

$$\begin{aligned} |\mathcal{H}g(u) - \mathcal{H}g^*(u)| &\leq \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} |c(t)||g(t) - g^*(t)| \\ &\quad + \frac{1}{\Gamma(k)} \int_0^u (u-t)^{k-1} \left\{ \begin{array}{l} \int_0^t |M_1(t,\tau)| K_1(g(\tau)) - K_1(g^*(\tau))| d\tau \\ + \int_0^t |M_2(t,\tau)| K_2(g(\tau)) - K_2(g^*(\tau))| d\tau \end{array} \right\} dt \\ &\leq \left(\frac{\|c\|_\infty}{\Gamma(k+1)} + \frac{M_1^* L_{K_1}}{\Gamma(k+1)} + \frac{+M_2^* L_{K_2}}{\Gamma(k+1)} \right) |g(u) - g^*(u)| \\ &= \left(\frac{\|c\|_\infty + M_1^* L_{K_1} + M_2^* L_{K_2}}{\Gamma(k+1)} \right) |g(u) - g^*(u)|. \end{aligned}$$

Therefore,

$$\|\mathcal{H}g - \mathcal{H}g^*\| \leq \left(\frac{\|c\|_\infty + M_1^* L_{K_1} + M_2^* L_{K_2}}{\Gamma(k+1)} \right) \|g - g^*\|. \tag{38}$$

By Hypothesis (H_4), we have $\left(\frac{\|c\|_\infty + M_1^* L_{K_1} + M_2^* L_{K_2}}{\Gamma(k+1)} \right) < 1$. If we take $\gamma = \left(\frac{\|c\|_\infty + M_1^* L_{K_1} + M_2^* L_{K_2}}{\Gamma(k+1)} \right)$. Then, for any $L \geq 0$, (38) can be written as

$$\|\mathcal{H}g - \mathcal{H}g^*\| \leq \gamma \|g - g^*\| + L\|g - \mathcal{H}g\|. \tag{39}$$

This implies that $\mathcal{H}$ is an almost contraction mapping. Therefore, by Theorem 7, the A^{**} iteration method $\{g_m\}$ defined by (9) converges strongly to the unique solution of the problem (34) and (35). □

6. Conclusions

In this article, we have introduced a new iteration method called the A^{**} iteration method (9). We have proven several weak and strong convergence results of our new method involving the fixed points of generalized α-nonexpansive mappings. A novel numerical example has been used to show that our new iteration method enjoys a better speed of convergence than several existing iteration methods. Additionally, the convergence results of the A^{**} iteration method are also established for almost contraction mappings. We have shown that the A^{**} iteration method is weak w^2-stable with respect to almost contraction mappings. As an application of our main results, we approximated the solution of fractional Volterra–Fredholm integro-differential equation.

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