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On a New Integral Inequality: Generalizations and Applications

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Abstract: In this paper, we present some generalizations and improvements of a new integral inequality from the 29th IMC in 2022. Some applications of our new results are also provided.

Keywords: IMC 2022; quasi-hyperbolic sine function; exponential function; arithmetic mean-geometric mean (AM-GM) inequality; Cauchy-Schwarz inequality; Lagrange mean value theorem

MSC: 26B05; 26B20; 26D10; 26E60; 47G10

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1. Introduction and Motivation

As we all know, in the history of the research process of inequality theory, many important generalization studies often come from some simple inequalities that have widespread applications. Over the past more than five decades, rapid developments in inequality theory and its applications have contributed greatly to many branches of mathematics, economics, finance, physics, dynamic systems theory, game theory, and so on; for more details, one can refer to [1–4] and the references therein.

The following new integral inequality (here we regard it as a theorem) arose from the 29th International Mathematics Competition for University Students (for short, IMC 2022), which was held in Blagoevgrad, Bulgaria on 1–7 August 2022. For more information (including proofs), please visit the following official website of IMC 2022: https://www.imc-math.org.uk (accessed on 1 August 2022).

Theorem 1. *Let $f : [0,1] \to (0,+\infty)$ be an integrable function such that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Then $\int_0^1 f(x)\,\mathrm{d}x \geq 1$.*

Motivated by the above integral inequality, the following questions arise naturally.

Question 1. Can we establish new real generalizations of Theorem 1?

Question 2. Does Theorem 1 still hold if we replace the codomain $(0,+\infty)$ of f with $(-\infty,+\infty)$?

In this work, our questions will be answered affirmatively. In Section 2, we successfully establish a new real generalization (see Theorem 2 below) of Theorem 1, which is a positive answer to Question 1. In Section 3, we first construct a new simple counterexample to show that Question 2 is not always true. Furthermore, we establish an equivalent theorem (see Theorem 3 below) of Theorem 2. Finally, some applications of our new results are given in Section 4. The new results we present in this paper are novel and developmental.

2. New Results for Question 1

The following result is very crucial for answering Question 1.

Lemma 1. *Let $f(x)$ be an integrable function on $[a,b]$. Then*

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_a^b f(a+b-x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\,\mathrm{d}x \\ &= \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,\mathrm{d}x \\ &= \int_{\frac{a+b}{2}}^b [f(x)+f(a+b-x)]\,\mathrm{d}x.\end{aligned}$$

Proof. By using integration by substitution (see, e.g., [5]), we have

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\xlongequal{(\text{let } t=a+b-x)} \int_b^a f(a+b-t)\,(-\mathrm{d}t) \\ &= \int_a^b f(a+b-t)\,\mathrm{d}t = \int_a^b f(a+b-x)\,\mathrm{d}x.\end{aligned} \tag{1}$$

Hence, we obtain

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \frac{1}{2}\left[\int_a^b f(x)\,\mathrm{d}x + \int_a^b f(a+b-x)\,\mathrm{d}x\right] \\ &= \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\,\mathrm{d}x.\end{aligned} \tag{2}$$

Note that

$$\begin{aligned}\int_{\frac{a+b}{2}}^b [f(x)+f(a+b-x)]\,\mathrm{d}x &\xlongequal{(\text{let } t=a+b-x)} \int_{\frac{a+b}{2}}^a [f(a+b-t)+f(t)]\,(-\mathrm{d}t) \\ &= \int_a^{\frac{a+b}{2}} [f(a+b-t)+f(t)]\,\mathrm{d}t \\ &= \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,\mathrm{d}x,\end{aligned} \tag{3}$$

so it is easy to see that

$$\begin{aligned}&\int_a^b [f(x)+f(a+b-x)]\,\mathrm{d}x \\ &= \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,\mathrm{d}x + \int_{\frac{a+b}{2}}^b [f(x)+f(a+b-x)]\,\mathrm{d}x \\ &= 2\int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\,\mathrm{d}x.\end{aligned}$$

Combining (1) and (2) together with (3), we prove the desired conclusion. □

With the help of Lemma 1, we can establish the following generalization of Theorem 1.

Theorem 2. *Let $f : [a,b] \to (0,+\infty)$ be an integrable function such that*

$$f(x)\cdot f(a+b-x) = c \tag{4}$$

for all $x \in [a,b]$, where $c > 0$ is a constant. Then

$$\int_a^b f(x)\,\mathrm{d}x \ge (b-a)\sqrt{c}. \tag{5}$$

The case of equality holds in (5) *if and only if* $f(x) = f(a+b-x) = \sqrt{c}$ *for all* $x \in [a,b]$.

Proof. We use two methods to show (5).
<u>*Method 1*</u>. By (4) and using the arithmetic-mean–geometric-mean (AM-GM) inequality, we have

$$f(x) + f(a+b-x) \geq 2\sqrt{f(x)f(a+b-x)} = 2\sqrt{c} \quad \text{for any } x \in [a,b]. \tag{6}$$

By (6) and applying Lemma 1, we arrive at

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\,\mathrm{d}x \geq \int_a^{\frac{a+b}{2}} 2\sqrt{c}\,\mathrm{d}x = (b-a)\sqrt{c}.$$

Obviously, the equality holds in (5) if and only if the equality holds in (6) and if and only if $f(x) = f(a+b-x) = \sqrt{c}$ for all $x \in [a,b]$.
<u>*Method 2*</u>. By applying Lemma 1 and using (4), it follows that

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x = \int_a^b \frac{c}{f(x)}\,\mathrm{d}x.$$

Applying Cauchy–Schwarz inequality, we obtain

$$\left(\int_a^b f(x)\,\mathrm{d}x\right)^2 = \int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b \frac{c}{f(x)}\,\mathrm{d}x \geq \left(\int_a^b \sqrt{c}\,\mathrm{d}x\right)^2 = (b-a)^2 c.$$

This proves the inequality (5). Clearly, the equality holds in (5) if and only if $f(x) = f(a+b-x) = \sqrt{c}$ for all $x \in [a,b]$. □

Remark 1. *By taking* $a = 0$ *and* $b = c = 1$ *in Theorem 2, we can prove Theorem 1.*

Remark 2. *There are many functions satisfying condition (4), as in Theorem 2, such as*

(i) $f(x) = \sqrt{c},\ x \in [a,b]$, *where* $c > 0$ *is a constant;*
(ii) $f(x) = \alpha^x,\ x \in [a,b]$, *where* $\alpha > 0$ *with* $\alpha \neq 1$;
(iii) $f(x) = \dfrac{x}{a+b-x},\ x \in [a,b]$;
(iv) $f(x) = \dfrac{a+b-x}{x},\ x \in [a,b]$;
(v) $f(x) = \begin{cases} (a+b-x)^2+1, & x \in [a, \frac{a+b}{2}), \\ 1, & x = \frac{a+b}{2}, \\ \frac{1}{x^2+1}, & x \in (\frac{a+b}{2}, b]. \end{cases}$

3. New Results for Question 2

In this section, we first provide a simple counterexample to show that Question 2 is not always true if we replace the codomain $(0, +\infty)$ of f with $(-\infty, +\infty)$.

Example 1. *Let* $f : [0,1] \to (-\infty, +\infty)$ *be defined by*

$$f(x) = \begin{cases} 1, & x \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1], \\ -1, & x \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

Then, f *is an integrable function on* $[0,1]$ *but not continuous on* $[0,1]$. *Clearly,* $f(x)$ *satisfies* $f(x) \cdot f(1-x) = 1$ *for all* $x \in [0,1]$. *However, it is easy to see that*

$$\int_0^1 f(x)\,\mathrm{d}x = 0 < 1.$$

By applying Theorem 2, we obtain the following result.

Theorem 3. *Let $g : [a, b] \to (-\infty, +\infty)$ be an integrable function such that*

$$g(x) \cdot g(a + b - x) = \lambda \tag{7}$$

for all $x \in [a, b]$, where λ is a nonzero constant. Then

$$\int_a^b |g(x)| \, \mathrm{d}x \geq (b - a)\sqrt{|\lambda|}. \tag{8}$$

The case of equality holds in (8) if and only if $|g(x)| = |g(a + b - x)| = \sqrt{|\lambda|}$ for all $x \in [a, b]$.

Proof. Due to (7), we know that $g(x) \neq 0$ for all $x \in [a, b]$. So we can define $f : [a, b] \to (0, +\infty)$ by

$$f(x) = |g(x)| \quad \text{for } x \in [a, b].$$

Since g is integrable, f is integrable. From (7) again, we obtain

$$f(x) \cdot f(a + b - x) = |g(x) \cdot g(a + b - x)| = |\lambda| \quad \text{for all } x \in [a, b].$$

Let $c = |\lambda|$. Then $c > 0$. Hence, all conditions in Theorem 2 are satisfied. By Theorem 2, we obtain

$$\int_a^b |g(x)| \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x \geq (b - a)\sqrt{c} = (b - a)\sqrt{|\lambda|},$$

and the equality holds in (8) if and only if $|g(x)| = |g(a + b - x)| = \sqrt{|\lambda|}$. The proof is completed. □

Remark 3. *We applied Theorem 2 to show Theorem 3. It is obvious that Theorem 2 is a special case of Theorem 3. Therefore, we can conclude that Theorems 2 and 3 are indeed equivalent.*

Taking advantage of Theorem 3, we easily obtain the following results.

Corollary 1. *Let $g : [0, 1] \to (-\infty, +\infty)$ be an integrable function such that*

$$g(x) \cdot g(1 - x) = \lambda$$

for all $x \in [0, 1]$, where λ is a nonzero constant. Then

$$\int_0^1 |g(x)| \mathrm{d}x \geq \sqrt{|\lambda|}. \tag{9}$$

The case of equality holds in (9) if and only if $|g(x)| = |g(1 - x)| = \sqrt{|\lambda|}$ for all $x \in [0, 1]$.

Proof. Taking $a = 0$ and $b = 1$ in Theorem 3, then the desired result is obtained. □

Corollary 2. *Let $m > 0$. Suppose that $g : [-m, m] \to (-\infty, +\infty)$ is an integrable function such that*

$$g(x) \cdot g(-x) = 1$$

for all $x \in [-m, m]$. Then

$$\int_{-m}^m |g(x)| \mathrm{d}x \geq 2m. \tag{10}$$

The case of equality holds in (10) if and only if $|g(x)| = |g(-x)| = 1$ for all $x \in [-m, m]$.

Proof. Take $a = -m$, $b = m$, and $\lambda = 1$ in Theorem 3, then the desired conclusion is proved. □

As a consequence of Theorem 3, we obtain the following theorem.

Theorem 4. *Let $h : (-\infty, +\infty) \to (-\infty, +\infty)$ be a function satisfying $\int_{-\infty}^{+\infty} h(x)\,\mathrm{d}x < \infty$. Suppose that there exist $a, b \in (-\infty, +\infty)$ with $a < b$ such that*

$$h(x) \cdot h(a+b-x) = \lambda \tag{11}$$

for all $x \in [a, b]$, where λ is a nonzero constant. Then, for any $u, v \in (-\infty, +\infty)$ with $u \leq a$ and $b \leq v$, we have

$$\int_u^v |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|},$$

$$\int_u^{+\infty} |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|},$$

$$\int_{-\infty}^{v} |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|}$$

and

$$\int_{-\infty}^{+\infty} |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|}.$$

Proof. Define $g : [a, b] \to (-\infty, +\infty)$ by

$$g(x) = h(x) \quad \text{for } x \in [a, b].$$

Since $\int_{-\infty}^{+\infty} h(x)\,\mathrm{d}x < \infty$, $\int_a^b h(x)\,\mathrm{d}x < \infty$. Hence, h is integrable on $[a, b]$. It follows that g, $|g|$, and $|h|$ are integrable on $[a, b]$ and

$$\int_a^b |g(x)|\,\mathrm{d}x = \int_a^b |h(x)|\,\mathrm{d}x.$$

By (11), we obtain

$$g(x) \cdot g(a+b-x) = h(x) \cdot h(a+b-x) = \lambda \quad \text{for all } x \in [a, b].$$

Hence all conditions in Theorem 3 are satisfied. By utilizing Theorem 3, we obtain

$$\int_u^v |h(x)|\,\mathrm{d}x \geq \int_a^b |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|},$$

$$\int_u^{+\infty} |h(x)|\,\mathrm{d}x \geq \int_a^b |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|},$$

$$\int_{-\infty}^{v} |h(x)|\,\mathrm{d}x \geq \int_a^b |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|}$$

and

$$\int_{-\infty}^{+\infty} |h(x)|\,\mathrm{d}x \geq \int_a^b |h(x)|\,\mathrm{d}x \geq (b-a)\sqrt{|\lambda|}.$$

The proof is completed. □

4. Some Applications

In this section, we first establish the following new useful inequalities, which improve the known inequalities for exponential functions.

Theorem 5. *Let $a > 0$. Then, the following hold.*

(i) *If $0 < a < 1$, then $a^x < 1 + x a^{\frac{x}{2}} \ln a$ for all $x > 0$.*
(ii) *If $a = 1$, then $a^x = 1 + x a^{\frac{x}{2}} \ln a = 1$ for all $x > 0$.*

(iii) *If $a > 1$, then $a^x > 1 + xa^{\frac{x}{2}} \ln a$ for all $x > 0$.*
In particular, we have

$$e^x > xe^{\frac{x}{2}} + 1 > x + 1 \text{ for all } x > 0.$$

Proof. Given $x > 0$. Let $f(y) = a^y$ for $y \in [0, x]$. Then f is integrable on $[0, x]$, and

$$f(y) \cdot f(x - y) = a^x \quad \text{for all } y \in [0, x].$$

Hence, by applying Theorem 2, we have

$$\frac{1}{\ln a}(a^x - 1) = \int_0^x a^y \, dy \geq xa^{\frac{x}{2}}. \tag{12}$$

(i) If $0 < a < 1$, then $\ln a < 0$. Note that $f(y) = f(x - y) = a^{\frac{x}{2}}$ holds for $y = \frac{x}{2}$. So the equality does not hold in (12). From (12), we obtain

$$a^x < 1 + xa^{\frac{x}{2}} \ln a \quad \text{for all } x > 0.$$

(ii) Clearly, if $a = 1$, then $a^x = 1 = 1 + xa^{\frac{x}{2}} \ln a$ for all $x > 0$.
(iii) If $a > 1$, then $\ln a > 0$. Since $f(y) = f(x - y) = a^{\frac{x}{2}}$ holds for $y = \frac{x}{2}$, the equality does not hold in (12). Hence, using (12) again, we obtain

$$a^x > 1 + xa^{\frac{x}{2}} \ln a \quad \text{for all } x > 0.$$

In particular, by taking $a := e$, we have

$$e^x > xe^{\frac{x}{2}} + 1 > x + 1 \quad \text{for all } x > 0.$$

The proof is completed. □

Next, we provide a new simple proof of the following important fundamental inequality for hyperbolic sine functions by applying Theorem 2, Theorem 3, or their corollaries.

Theorem 6. $\sinh x > x$ *for all* $x > 0$.

Proof. Given $x > 0$. Let $f(y) = e^y$ for $y \in [-x, x]$. Then f is integrable on $[-x, x]$ and

$$f(y) \cdot f(-y) = e^0 = 1 \quad \text{for all } y \in [-x, x].$$

By applying Theorem 2 (or Theorem 3 or Corollary 2), we obtain

$$e^x - e^{-x} = \int_{-x}^x e^y \, dy \geq 2x.$$

Since $e^x \neq e^{-x}$ for $x \neq 0$, we obtain

$$\sinh x = \frac{e^x - e^{-x}}{2} > x.$$

The proof is completed. □

In this paper, we introduce the concept of quasi-hyperbolic sine function.

Definition 1. *A function q-$\sinh : (0, +\infty) \times (-\infty, +\infty) \to (-\infty, +\infty)$ is said to be a quasi-hyperbolic sine function if*

$$q\text{-}\sinh(a, x) = \frac{a^x - a^{-x}}{2} \quad \text{for } a > 0 \text{ and } x \in (-\infty, +\infty).$$

Remark 4. *In [6], Nantomah, Okpoti, and Nasiru defined generalized hyperbolic sine function using*

$$\sinh_a x = \frac{a^x - a^{-x}}{2} \quad \textit{for } a > 1 \textit{ and } x \in (-\infty, +\infty).$$

It is obvious that a hyperbolic sine function is a generalized hyperbolic sine function, and a generalized hyperbolic sine function is a quasi-hyperbolic sine function, but the converse is not true.

We now give the following new inequalities for quasi-hyperbolic sine functions.

Theorem 7. *Let $a > 0$. Then, the following hold.*

(i) If $0 < a < 1$, then $q\text{-}\sinh(a, x) < x \ln a$ for all $x > 0$.
(ii) If $a = 1$, then $q\text{-}\sinh(a, x) = 0$ for all $x > 0$.
(iii) If $a > 1$, then $q\text{-}\sinh(a, x) = \sinh_a x > x \ln a$ for all $x > 0$.

Proof. Given $x > 0$. Let $f(y) = a^y$ for $y \in [-x, x]$. Thus, f is integrable on $[-x, x]$ and

$$f(y) \cdot f(-y) = a^0 = 1 \quad \text{for all } y \in [-x, x].$$

By applying Theorem 2 (or Theorem 3 or Corollary 2), we obtain

$$\frac{1}{\ln a}\left(a^x - a^{-x}\right) = \int_{-x}^{x} a^y \,\mathrm{d}y \geq 2x. \tag{13}$$

(i) If $0 < a < 1$, then $\ln a < 0$. Since $a^x \neq a^{-x}$ for $x \neq 0$, the equality does not hold in (13). Hence (13) yields

$$q\text{-}\sinh(a, x) = \frac{a^x - a^{-x}}{2} < x \ln a.$$

(ii) Clearly, $q\text{-}\sinh(1, x) = \frac{1^x - 1^{-x}}{2} = 0$ for all $x > 0$.
(iii) If $a > 1$, then $\ln a > 0$. Since $a^x \neq a^{-x}$ for $x \neq 0$, the equality does not hold in (13). So, from (13) again, we obtain

$$q\text{-}\sinh(a, x) = \sinh_a x = \frac{a^x - a^{-x}}{2} > x \ln a.$$

The proof is completed. □

Remark 5. *Theorem 6 is a special case of Theorem 7 (iii).*

Theorem 8. *Let $a < b$. Then there exists $\xi \in \left(\frac{a+b}{2}, b\right)$ such that*

$$\mathrm{e}^{\xi} = \frac{\mathrm{e}^b - \mathrm{e}^a}{b - a}.$$

Proof. From the Lagrange mean value theorem or integral mean value theorem, it is easy to see that there exists $\xi \in (a, b)$ such that

$$\mathrm{e}^{\xi} = \frac{\mathrm{e}^b - \mathrm{e}^a}{b - a}. \tag{14}$$

We now claim that $\xi \in \left(\frac{a+b}{2}, b\right)$. Let $f(x) = \mathrm{e}^x$ for $x \in [a, b]$. Then f is integrable on $[a, b]$ and

$$f(x) \cdot f(a + b - x) = \mathrm{e}^{a+b} := c \quad \text{for all } x \in [a, b].$$

Note that $f(x) = f(a+b-x) = \sqrt{c}$ holds for $x = \frac{a+b}{2}$. Accordingly, by applying Theorem 2, we obtain

$$e^b - e^a = \int_a^b e^x \,dx > (b-a)\sqrt{c} = (b-a)e^{\frac{a+b}{2}}, \tag{15}$$

which follows immediately from (14) and (15) that $\xi > \frac{a+b}{2}$. Therefore, $\xi \in \left(\frac{a+b}{2}, b\right)$. $\square$

Theorem 9. *Let* $0 < a < b$*. Then there exists* $\xi \in \left(a, \frac{a+b}{2}\right)$ *such that*

$$b - a = \xi(\ln b - \ln a).$$

Proof. Making full use of the Lagrange mean value theorem, we can find $\xi \in (a, b)$, such that

$$\frac{1}{\xi} = \frac{\ln b - \ln a}{b - a}. \tag{16}$$

We now speculate that $\xi \in \left(a, \frac{a+b}{2}\right)$. To this end, put $f(x) = \dfrac{x}{a+b-x}$ for $x \in [a, b]$. Thus f is integrable on $[a, b]$ and

$$f(x) \cdot f(a+b-x) = 1 \quad \text{for all } x \in [a, b].$$

Note that $f(x) = f(a+b-x) = 1$ holds for $x = \frac{a+b}{2}$. So, by utilizing Theorem 2, we obtain

$$(a+b)(\ln b - \ln a) - (b-a) = \int_a^b \frac{x}{a+b-x} \,dx > b - a. \tag{17}$$

Combining (16) and (17), we obtain $\xi < \frac{a+b}{2}$. Therefore, we show $\xi \in \left(a, \frac{a+b}{2}\right)$. $\square$

5. Conclusions

In this paper, we study two questions for Theorem 1 as follows:

Question 1. Can we establish new real generalizations of Theorem 1?

Question 2. Does Theorem 1 still hold if we replace the codomain $(0, +\infty)$ of f with $(-\infty, +\infty)$?

We establish Theorem 2, which is a new real generalization of Theorem 1, and a positive answer to Question 1. A new simple counterexample is given to verify that Question 2 is not always true. Furthermore, we prove Theorem 3, which is equivalent to Theorem 2, and show some applications of our new results. In summary, our new results are original, novel, and developmental in the literature. We hope that our new results can be applied to nonlinear analysis, mathematical physics, and related fields in the future.

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