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Computer-Aided Analysis of Solvable Rigid Lie Algebras with a Given Eigenvalue Spectrum

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Abstract: With the help of symbolic computer packages, the study of the cohomological rigidity of real solvable Lie algebras of rank one with a maximal torus of derivations $\mathfrak{t}$ and the eigenvalue spectrum $\operatorname{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$ initiated in a previous work is continued for arbitrary values $k \geq 2$, obtaining new hierarchies of solvable rigid Lie algebras.

Keywords: Lie algebras; solvability; rigid; cohomology; Jacobi scheme

MSC: 17B56, 17B30

1. Introduction

Emerging originally in the context of the deformation theory of complex analytic structures [1,2], the notion of rigidity for Lie algebras has become increasingly important not only for classification purposes, but also for a geometrical comprehension of the variety, $\mathcal{L}^n$, of n-dimensional Lie algebra laws, their irreducible components, and the characteristics of points in $\mathcal{L}^n$ [3–5]. In the mid 1960s, several rigidity criteria based on the Chevalley cohomology were found [6,7], showing that—besides the topological approach to rigidity—a purely algebraic formalism was feasible. Although these approaches are not equivalent, as rigidity does not impose nullity of the Chevalley cohomology [8], this ansatz was crucial for finding large classes of rigid Lie algebras, beyond the well-known, semi-simple, and parabolic Lie algebras [9]. The first examples of rigid algebras with non-trivial cohomology were found in [6]; meanwhile, in [10], certain obstructions for the integrability of cocycles were inspected in detail, in terms of the so-called Rim quadratic map [11], providing a sufficiency criterion to ensure the geometrical rigidity. For the case of solvable Lie algebras, the weight systems of maximal tori of derivations (see, e.g., [12–14]) have been shown to be an effective tool to study the rigidity without recourse to cohomological methods, leading to the first systematic procedures or algorithms to construct rigid Lie algebras [15–17]. In this context, even techniques of non-standard analysis have been shown to be relevant [18,19]. On the other hand, low-dimensional rigid Lie algebras and several hierarchies of solvable rigid Lie algebras in arbitrary dimensions have been classified, taking into account the structural properties of their maximal tori of derivations ([20–23] and references therein); however, this approach only solves partially the rigidity problem, (it is important to mention that this approach has also been used for the analysis of rigid superalgebras [24]). However, this procedure has allowed a better comprehension of the nilpotent Lie algebras in terms of the eigenvalue spectra of their torus. So, in [25], the maximal tori of nilpotent Lie algebras with a maximal nilpotence index, called filiform, were studied in detail. It follows from this study that, among the relevant classes of solvable rigid Lie algebras of rank one, those having a filiform nilradical play a central role, as other classes can be obtained from these by means of special constructions and extensions [26,27]. With the

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introduction of symbolic computer packages, the computation of cohomologies has been largely simplified [28,29], allowing wide classes of cohomologically rigid algebras to be obtained, and the integrability obstructions to be determined with precision, allowing new classes of geometrically rigid Lie algebras to be found [14,30]. The work of R. Carles on the structure theory of rigid Lie algebras and the Jacobi schemes (see [22,31–34] and references therein) offered new approaches to the analysis of geometrical properties of the variety $\mathcal{L}^n$, although their practical computational implementation remains a question that has not yet been solved satisfactorily.

In this work, we follow the analysis begun in [35], analysing a question left open in that work, and concerning the cohomological rigidity of rank-one solvable rigid Lie algebras with a filiform nilradical—an eigenvalue spectrum $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$—for its torus and the additional constraint $C_{2,3}^{k+3} = 0$. Although for low values of $k \leq 4$, various results have been obtained in the literature (see [20,35,36]), the general case, $k \geq 5$, has not been studied in detail due to computational difficulties and the high number of solutions for the quadratic equations determined by the Jacobi conditions. It is found that, for each k and from a certain dimension m_0 onwards, only one cohomologically rigid Lie algebra exists; meanwhile, for dimensions $n < m_0$, several isolated rigid algebras appear. In order to analyse these special solutions, the structure of certain types of factor sequences are studied, allowing researchers to determine different types of rigid algebras according to the specific structural properties of the associated nilradical .

Unless otherwise stated, any Lie algebra in this work is finite-dimensional and defined over the field of real numbers $\mathbb{R}$.

1.1. Generalities

Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{Der}(\mathfrak{g})$ its algebra of derivations, that is, the set of linear maps, $D : \mathfrak{g} \to \mathfrak{g}$, such that the Leibniz condition,

$$D[X,Y] = [D(X),Y] + [X,D(Y)], \quad X,Y \in \mathfrak{g}.$$

is satisfied. For derivations coinciding with the adjoint action, i.e., $D(Y) = \mathrm{ad}(X)(Y) := [X,Y]$ for some $X \in \mathfrak{g}$, we say that the derivation, D, is inner, and is outer otherwise.

Definition 1. *Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus (of derivations) is an Abelian subalgebra $\mathfrak{t}$ of $Der(\mathfrak{g})$, such that its elements are semi-simple.*

The semi-simplicity of a set of outer derivations essentially means that the linear operators, $f \otimes_{\mathbb{R}} \mathrm{Id} \in End(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$, are simultaneously diagonalizable over the field $\mathbb{C}$ of complex numbers. Maximal tori, $\mathfrak{t}$, in the complex case, belongs to the same conjugacy class [37]; meanwhile, over the real field, this class splits into a finite number of classes. The dimension of a maximal torus is easily seen to define a scalar invariant $\mathrm{r}(\mathfrak{g})$ called the rank of the Lie algebra of $\mathfrak{g}$.

From the structure theory of Lie algebras, it is known that any (real or complex) solvable Lie algebra, $\mathfrak{r}$, decomposes as the semi-direct sum of the maximal nilpotent ideal, $\mathfrak{n}$ of $\mathfrak{r}$, called the nilradical, and a linear space, $\mathfrak{t}$, formed by linearly nil-independent outer derivations of $\mathfrak{n}$, as follows:

$$\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}. \tag{1}$$

Further, the following relations hold:

$$\begin{gathered} [\mathfrak{t},\mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{n},\ [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{n}, \\ \dim \mathfrak{n} - \dim[\mathfrak{n},\mathfrak{n}] \geq \dim \mathfrak{t}. \end{gathered} \tag{2}$$

The latter inequality in particular determines an upper bound for the rank of a solvable Lie algebra [37,38].

1.2. Rigid Lie Algebras and Chevalley cohomology

(Real) Lie algebras of a given dimension, n, can be seen as points of an algebraic subset, $\mathcal{L}^n$ of $\mathbb{R}^{n^3}$, as determined by the following quadratic relations:

$$\begin{aligned} C_{ij}^{k}+C_{ji}^{k} &=0, \quad 1\leq i,j,k\leq n \\ C_{ij}^{l}C_{lk}^{m}+C_{jk}^{l}C_{li}^{m}+C_{ki}^{l}C_{lj}^{m} &=0, \quad 1\leq i,j,k,l,m\leq n. \end{aligned} \tag{3}$$

The variety is indeed analytic and, among its salient topological properties, it is locally piecewise connected, a fact that enables us to study the properties of $\mathcal{L}^n$ by means of deformations [39]. The general linear group $GL(n,\mathbb{R})$ induces a natural action on $\mathcal{L}^n$, given by

$$(f\star\mathfrak{g})(X,Y)=f\left(\left[f^{-1}(X),f^{-1}(Y)\right]_{\mathfrak{g}}\right), \quad f\in GL(n,\mathbb{R}),\ X,Y\in\mathfrak{g}. \tag{4}$$

Hence, the orbit $\mathcal{O}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ corresponds to Lie algebras isomorphic to $\mathfrak{g}$. As the isotropy group of any point in an orbit corresponds to the automorphism group $\mathrm{Aut}(\mathfrak{g})$, $\mathcal{O}(\mathfrak{g})$ can be seen as the homogeneous space $GL(n,\mathbb{R})/\mathrm{Aut}(\mathfrak{g})$. Using this fact, the identity $\dim\mathcal{O}(\mathfrak{g})=n^2-\dim\mathrm{Der}(\mathfrak{g})$ follows immediately. As a submanifold of $\mathbb{R}^{n^3}$, we can consider the closure, $\overline{\mathcal{O}}(\mathfrak{g})$, of any orbit with respect to the Euclidean topology.

Definition 2. *A Lie algebra $\mathfrak{g}$ is called rigid if the orbit $\mathcal{O}(\mathfrak{g})$ is an open set of $\mathcal{L}^n$, with respect to the Euclidean topology.*

It can be shown that an equivalent rigidity notion is obtained by considering the natural Zariski topology of algebraic subsets [40]. It is in this context that the relation with the cohomology of Lie algebras emerges naturally [8]. For a generic point, μ, in $\mathcal{L}^n$ (i.e., a point of $\mathcal{L}^n$ thus corresponds to a Lie algebra $\mathfrak{g}$ over a given basis), it was shown in [10] that the following properties hold:

1. The tangent Zariski space to $\mathcal{L}^n$ at the point μ coincides with the space of 2 cocycles $Z^2(\mathfrak{g},\mathfrak{g})$ of the Chevalley cohomology.
2. The tangent space to the orbit $\mathcal{O}(\mathfrak{g})$ coincides with the space of 2 coboundaries $B^2(\mathfrak{g},\mathfrak{g})$ of the Chevalley cohomology.

These properties allow us to analyse the rigidity by means of purely algebraic methods. In fact, only the spaces $H^p(\mathfrak{g},\mathfrak{g})$ for $p\leq 3$ have to be considered, from which a further subdivision of rigid Lie algebras into cohomologically (or algebraically) rigid and geometrically rigid Lie algebras will be deduced. This division actually makes reference to the geometry of singularities in the variety $\mathcal{L}^n$ [5,10]. The space $H^0(\mathfrak{g},\mathfrak{g})$ corresponds to the centre $Z(\mathfrak{g})$ of the Lie algebra, while $H^1(\mathfrak{g},\mathfrak{g})$ is identified with the space of outer derivations [8].

The notion of deformation in the variety $\mathcal{L}^n$—although expressed analytically in terms of formal series—is essentially a geometric concept relating to paths on $\mathcal{L}^n$ [41–44]. We define a formal deformation $\mathfrak{g}_t$ of a Lie algebra $\mathfrak{g}=(\mathbb{R}^n,\mu)\in\mathcal{L}^n$ through a deformed commutator

$$[X,Y]_{\varepsilon}:=[X,Y]+\psi_m(X,Y)\varepsilon^m,$$

with ε as a real parameter and $\psi_m:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ a skew–symmetric bilinear map. Developing the Jacobi identity up to the quadratic order of ε, we are led to the following identity:

$$\begin{gathered} \left[X_i,[X_j,X_k]_{\varepsilon}\right]_{\varepsilon}+\left[X_k,[X_i,X_j]_{\varepsilon}\right]_{\varepsilon}+[X_j,[X_k,X_i]_{\varepsilon}]_{\varepsilon}= \\ \varepsilon\, d\psi_1(X_i,X_j,X_k)+\varepsilon^2\left(\frac{1}{2}[\psi_1,\psi_1]+d\psi_2\right)(X_i,X_j,X_k)+\mathcal{O}(\varepsilon^3), \end{gathered} \tag{5}$$

with $d\psi_l$ ($l = 1, 2$) and $[\psi_1, \psi_1]$ defined as

$$\begin{aligned} d\psi_l(X_i, X_j, X_k) := [X_i, \psi_l(X_j, X_k)] + [X_k, \psi_l(X_i, X_j)] + [X_j, \psi_l(X_k, X_i)] + \\ + \psi_l(X_i, [X_j, X_k]) + \psi_l(X_k, [X_i, X_j]) + \psi_l(X_j, [X_k, X_i]) \end{aligned} \tag{6}$$

and

$$\frac{1}{2}[\psi_1, \psi_1](X_i, X_j, X_k) := \psi_1(\psi_1(X_i, X_j), X_k) + \psi_1(\psi_1(X_j, X_k), X_i) + \psi_1(\psi_1(X_k, X_i), X_j), \tag{7}$$

respectively. In order to fulfil the Jacobi identity, the constraints

$$d\psi_1(X_i, X_j, X_k) = 0, \tag{8}$$

$$\frac{1}{2}[\psi_1, \psi_1](X_i, X_j, X_k) + d\psi_2(X_i, X_j, X_k) = 0 \tag{9}$$

must be satisfied. Equation (8) shows that ψ_1 is a 2-cocycle in $H^2(\mathfrak{g}, \mathfrak{g})$, while (9) expresses an integrability condition (see, e.g., [8])). We say that a cocycle φ is integrable whenever the linearly deformed commutator deformation

$$[X, Y]_\varepsilon := [X, Y] + \varepsilon\, \varphi(X, Y)$$

satisfies the Jacobi identity and thus defines a Lie algebra. If the space $H^2(\mathfrak{g}, \mathfrak{g})$ reduces to zero, than any deformation of $\mathfrak{g}$ is trivial, implying that the Lie algebra is rigid [6,7,9,41,45]. Algebras such that $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ are called cohomologically rigid, and encompass the class of semi-simple, Borel subalgebra and parabolic Lie algebras, among others. As shown in [6], another type of rigidity can appear for $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$, whenever the cocycles are non-integrable. This constitutes the geometrical rigidity, which is harder to establish, and that often requires techniques from algebraic geometry [5,10,11,31]. We merely indicate that a sufficiency criterion for the geometrical rigidity is given if the Rim map $Sq : H^2(\mathfrak{g}, \mathfrak{g}) \to H^3(\mathfrak{g}, \mathfrak{g})$ is injective, where

$$Sq(\psi) = \frac{1}{2}[\psi, \psi].$$

Within the frame of cohomologically rigid Lie algebras, the Hochschild–Serre factorization plays a central role, as it allows us to restrict the computations to a specific class of cocycles characterized by an invariance property [46]. In the case of solvable real Lie algebras $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$—where $\mathfrak{t}$ is an Abelian algebra of derivations and the adjoint operators $\mathrm{ad}_{\mathfrak{r}} T$ ($T \in \mathfrak{t}$) are diagonal—this decomposition is given by

$$H^p(\mathfrak{r}, \mathfrak{r}) \simeq \sum_{a+b=p} H^a(\mathfrak{t}, \mathbb{R}) \otimes H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}}, \tag{10}$$

where the invariance condition of a b-cochain φ is defined as

$$(T.\varphi)(Z_1, \dots, Z_b) = [T, \varphi(Z_1, \dots, Z_b)] - \sum_{s=1}^{b} \varphi(Z_1, \dots, [T, Z_s], \dots, Z_b). \tag{11}$$

and the space of invariant cocycles is defined as

$$H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = \left\{ [\varphi] \in H^b(\mathfrak{n}, \mathfrak{r}) \mid (T.\varphi) = 0,\ T \in \mathfrak{t} \right\}. \tag{12}$$

Applying Künneth-type formulae, it is straightforward to verify that, for the values $0 \le b \le p$, the condition $H^p(\mathfrak{r}, \mathfrak{r}) = 0$ is equivalent to $H^b(\mathfrak{n}, \mathfrak{r})^{\mathfrak{t}} = 0$.

For Lie algebras that satisfy $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, the cohomology further satisfies the following inequality

$$\sum_{k=0}^{p} (-1)^{p+k} \dim H^k(\mathfrak{g}, \mathfrak{g}) \geq 0,$$

from which the condition $\dim \mathrm{Der}(\mathfrak{g}) \leq \dim \mathfrak{g} + \dim H^2(\mathfrak{g}, \mathfrak{g})$ is easily deduced [31].

In particular, for a rank-one solvable Lie algebra, $\mathfrak{r}$ and $p = 3$, we have

$$\dim H^3(\mathfrak{r}, \mathfrak{r}) - \dim H^2(\mathfrak{r}, \mathfrak{r}) + \dim H^1(\mathfrak{r}, \mathfrak{r}) - \dim Z(\mathfrak{r}) \geq 0$$

It is worth mentioning that it is currently unknown whether there exist rank-one solvable rigid algebras with $Z(\mathfrak{r}) \neq 0$ and $H^1(\mathfrak{r}, \mathfrak{r}) \neq 0$.

2. Rigid Lie Algebras with Fixed Eigenvalue Spectrum

Consider a real solvable Lie algebra, $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$, with a maximal one-dimensional torus of derivations, $\mathfrak{t}$, and a filiform nilradical $\mathfrak{n}$. As the torus acts diagonally on the latter, the action can be described in terms of the eigenvalue spectrum $(\lambda_1, \ldots, \lambda_n)$ of $\mathfrak{t}$ over a given basis $\{X_1, \ldots, X_n\}$ of the nilradical $\mathfrak{n}$. The Jacobi identity implies that for any nonvanishing commutator $[X_i, X_j]$, it corresponds to an eigenvector with eigenvalue $\lambda_i + \lambda_j$. In [35], the notion of saturation of nilradicals was introduced and analysed. In the preceding conditions, we recall that $\mathfrak{n}$ is said to be saturated (with regards to $\mathfrak{t}$) if the commutator $[X_i, X_j]$ is nonzero whenever the sum of the corresponding eigenvalues is an eigenvalue of the torus, i.e., $\lambda_i + \lambda_j \in \mathrm{spec}(\mathfrak{t})$.

In the following, let us suppose that $n \geq k + 4$ and that the eigenvalue spectrum of $\mathfrak{t}$ is given by

$$\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2), \tag{13}$$

and let $\{X_1, \ldots, X_n\}$ be an adapted basis of eigenvectors for the adjoint operator $ad(T)$. From the Jacobi identity, it follows at once that the only nonvanishing commutators in the nilradical $\mathfrak{n}$ are given by

$$\begin{aligned} [X_1, X_j] &= C_{1,j}^{j+1} X_{j+1}, && 2 \leq j \leq n-1 \\ [X_i, X_j] &= C_{i,j}^{i+j+k-2} X_{i+j+k-2}, && 2 \leq i < j;\ i+j+k-2 \leq n. \end{aligned} \tag{14}$$

As we are assuming that the nilradical is filiform, we can suppose without loss of generality that $C_{1,j}^{j+1} = 1$ for $2 \leq j \leq n-1$ [25]. Now, the detailed analysis of the Jacobi identity for different triples (i, j, k) of indices (see [35] for details) provides the following constraints on the structure constants $C_{i,j}^{k}$ of $\mathfrak{n}$:

$$\begin{aligned} (1,i,j):&\quad C_{i,j}^{i+j+k-2} - C_{i+1,j}^{i+j+k-1} - C_{i,j+1}^{i+j+k-1} = 0, \quad 2 \leq i < j \leq n-k-4 \\ (1,i,i+1):&\quad C_{i,i+1}^{2i+k-1} - C_{i,i+2}^{2i+k} = 0, \quad 2 \leq i \leq \left[\frac{n-k}{2}\right], \\ (1,i,i+s):&\quad C_{i,i+s}^{2i+k+s-2} - C_{i,1+i+s}^{2i+k+s-1} - C_{1+i,i+s}^{2i+k+s-1} = 0, \quad 2i+s \leq n-k+1. \end{aligned} \tag{15}$$

These equations determine a linear system that can be solved recursively for the coefficients $C_{i,j}^{i+j+k-2}$ with $|j-i| > 1$ with respect to the set of diagonal entries

$$\Delta = \left\{ C_{2,3}^{k+3}, C_{3,4}^{k+5}, \ldots, C_{\rho,\rho+1}^{2\rho+k-1} \right\}, \quad \rho = \left[\frac{n+1-k}{2}\right] \tag{16}$$

Using these elements as basis for the solution, the remaining structure constants can be obtained as sums

$$C_{i,j}^{i+j+k-2} = \sum_{\ell=2}^{\rho} \lambda_{i,j}^{\ell} C_{\ell,\ell+1}^{2\ell+k-1} \tag{17}$$

for appropriate coefficients $\lambda_{i,s}$ (explicit formulae for these coefficients can be found in [30]). For practical reasons, it is convenient to introduce the parameters, $\alpha_i = C_{i,i+1}^{2i+k-1}$, for $2 \leq i \leq \rho$ and set

$$\Delta = \{\alpha_2, \ldots, \alpha_\rho\} = \left\{C_{2,3}^{k+3}, C_{3,4}^{k+5}, \ldots, C_{\rho,\rho+1}^{2\rho+k-1}\right\}. \tag{18}$$

This formal simplification shows that the nonvanishing brackets of $\mathfrak{n}$ have the form

$$\begin{array}{l} [X_1, X_j] = X_{j+1},\ 1 \leq j \leq n-1, \\ [X_i, X_{i+1}] = \alpha_i X_{2i+k-1},\ 2 \leq i \leq \rho, \\ [X_i, X_{i+s}] = \sum_{\ell=2}^{\rho} \lambda_{i,s}^{\ell} \alpha_\ell X_{2i+k+s-2}, \quad i \geq 2,\ 2i+k+s-2 \leq n. \end{array} \tag{19}$$

The Jacobi identities that correspond to the triples $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$ have still to be evaluated. The result is the quadratic system

$$C_{i,j+\ell+k-2}^{i+j+2k+\ell-4} C_{j,\ell}^{j+\ell+k-2} + C_{i,j}^{i+j+k-2} C_{\ell,i+j+k-2}^{i+j+\ell+2k-4} + C_{i,\ell}^{i+\ell+k-2} C_{j,i+\ell+k-2}^{i+j+\ell+2k-4} = 0, \tag{20}$$

which by virtue of Equation (17) reduces to a quadratic system with respect to the variables $\{\alpha_2, \ldots, \alpha_\rho\}$ of Δ. Any solution of this system determines a nilradical with $\mathfrak{t}$ as torus. For low values of the dimension, the solutions are generically parametrised , so that no rigid algebras will emerge. As a general rule, when n is sufficiently high in comparison with the number of equations, isolated solutions occur. Once such isolated solutions have been detected, their rigidity is analysed either using cohomology or the Jacobi scheme (see [5,20,30,32] for details).

3. Rigid Algebras with Spectrum (13) and $C_{2,3}^{k+3} = 0$

The case with $C_{2,3}^{k+3} \neq 0$, studied in detail in [35], was shown to lead to a small number of possibilities that can be analysed without major computational difficulties, resulting in particular in the existence of some a continuous series of cohomologically rigid Lie algebras. Imposing the constraint $C_{2,3}^{k+3} = 0$, the detailed analysis complicates considerably, leading to a large hierarchy of rigid Lie algebras that satisfy the quadratic Equation (20). In previous work, the cases $k = 2, 3, 4$ have been considered (see [20,35] and references therein). To handle the general case, $k \geq 5$, an algorithmic procedure was proposed in [35] to systematize the determination of such algebras, that we briefly recall there (see [35] for details). The assumptions are the same as above, i.e., $\mathfrak{r}$ is a rank-one Lie algebra with an eigenvalue spectrum of $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$ for the fixed values of k and n.

1. Set $C_{1,j}^{j+1} = 1$ for $2 \leq j \leq n-1$.
2. Start with $q = \rho - 1$.
3. In the main diagonal Δ, set $\alpha_{q+1} = 1$ and $\alpha_\ell = 0$ for $2 \leq \ell \leq q$.
4. If the linear Equations (15) are incompatible, then replace q by $q-1$ and go to step 3.
5. If (15) is satisfied, evaluate the system $\mathcal{S}$ formed by the Jacobi conditions for the triples $\{X_i, X_j, X_\ell\}$ with $2 \leq i < j < \ell$.
6. If the system $\mathcal{S}$ admits no solution, then replace q by $q-1$ and go to step 3.
7. If the system $\mathcal{S}$ admits a solution depending on one or more parameters α_i, then replace q by $q-1$ and go to step 3.
8. If the system $\mathcal{S}$ admits an isolated solution, let $\mathfrak{n}$ be the corresponding nilpotent Lie algebra.
9. Compute $H^2\left(\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}, \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}\right)$.

The algorithm is conclusive for the algebraically rigid case. In the case where the cohomology $H^2(\mathfrak{r}, \mathfrak{r})$ does not vanish, the rigidity can still be inferred if the solution is isolated. Otherwise, in the presence of more than one solution, alternative approaches such as the Jacobi scheme have to be applied [5].

Inserting the constraint $C_{2,3}^{k+3} = 0$ into the system (17), it is easily seen that—among the various linear combinations of structure constants—the following proportions are obtained:

$$C_{2,4}^{k+4} = 0, \quad C_{3,4}^{k+3} = \alpha_3 = C_{2,5}^{k+5}, \quad C_{3,5}^{k+6} = -C_{2,5}^{k+5}, \quad C_{2,6}^{k+6} = 2C_{2,5}^{k+5}. \tag{21}$$

A particularity of this case—that distinguishes it from the saturated case treated in [35]—is that one of the entries in the diagonal, Δ, may be zero, so that additional assumptions must be made. The key observation is that the structure constants $C_{2,p}^{k+p}$ for odd values of $7 \leq p \leq n-k$ are not constrained, so that they be used as auxiliary or alternative parameters for solving the quadratic system (20). For this reason, for certain cases, it is convenient to replace the parameters, α_i, by the corresponding linear combinations of the structure constants, $C_{2,p}^{k+p}$.

Schematically, the commutator table of the nilradical $\mathfrak{n}$ has the following shape (where only the brackets $[X_i, X_j]$ with $j < i$ are shown; non-displayed commutators are either zero or obtained by antisymmetry):

	X_1	X_2	X_3	X_4	$\cdots$	$\cdots$	X_ρ	$X_{\rho+s}$
X_1	0							
X_2	X_3	0						
X_3	X_4	0	0					
X_4	X_5	0	$\alpha_3 X_{k+5}$	0				
X_5	X_6	$C_{2,5}^{k+5} X_{j+k}$	$\vdots$	$\alpha_4 X_{k+7}$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$		
X_j	X_{j+1}	$C_{2,j}^{j+k} X_{j+k}$	$C_{3,j}^{j+k+1} X_{j+k+1}$	$C_{4,j}^{j+k+2} X_{j+k+2}$	$\cdots$	$\cdots$	$\alpha_\rho X_{n-1}$	0
X_{j+1}	X_{j+1}	$C_{2,j+1}^{j+k+1} X_{j+k+1}$	$C_{3,j+1}^{j+k+2} X_{j+k+2}$	$C_{4,j}^{j+k+2} X_{j+k+3}$	$\cdots$	$\cdots$	$\alpha_\rho X_n$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\iddots$	0	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$		0	$\vdots$
X_{n-k-2}	X_{n-k-1}	$C_{2,n-k-2}^{n-2} X_{n-2}$	$C_{3,n-k-2}^{n-1} X_{n-1}$	$C_{4,n-k-2}^{n} X_n$	$\cdots$	$\cdots$	0	0
X_{n-k-1}	X_{n-k}	$C_{2,n-k-1}^{n-1} X_{n-1}$	$C_{3,n-k-1}^{n} X_n$	0	$\cdots$	$\cdots$	0	0
X_{n-k}	X_{n-k+1}	$C_{2,n-k}^{n} X_n$	0	0	$\cdots$	$\cdots$	0	0
X_{n-k+1}	X_{n-k+2}	0	0	0	$\cdots$	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$
X_{n-1}	X_n	0	0	0	$\cdots$	$\cdots$	0	0
X_n	0	0	0	0	$\cdots$	$\cdots$	0	0

where $1 \leq s \leq n - \rho$.

In order to illustrate the algorithmic procedure, we consider the detailed analysis of the eigenvalue spectrum for the value $k = 8$. The computer analysis for the quadratic Jacobi conditions (20) leads to the following results:

1. For $n \leq 26$, we obtain parametrised families with two or more parameters, so that no cohomologically rigid solution is obtained.
2. For $n = 27$, there are seven solutions, but none of the isolated ones has a vanishing cohomology.
3. For $n = 28$, there are six parametrised families and four isolated solutions. Among these, only two have a vanishing cohomology. The main diagonal is given by

$$\begin{aligned} \Delta &= \left\{ 0^2, 1, \frac{24}{11}, 3, \frac{40}{11}, \frac{51}{11}, \frac{84}{11}, \frac{322}{11} \right\}, \\ \Delta &= \left\{ 0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143} \right\} \end{aligned} \tag{22}$$

4. For $n = 29$, there are nine solutions, four of which are isolated, and only two of these have vanishing cohomology. The diagonal is given by

$$\Delta = \left\{0^2, 1, 0^2, \frac{55}{7}, 55, 616, 7535\right\},$$
$$\Delta = \left\{0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}, \frac{180}{143}\right\}$$

5. For $n = 30$, there are five solutions, from which two are isolated with diagonals given, respectively, by

$$\Delta = \left\{0^3, 1, \frac{22}{13}, \frac{25}{13}, \frac{25}{13}, \frac{25}{13}, \frac{28}{13}, \frac{42}{13}\right\},$$
$$\Delta = \left\{0^4, 1, 15, 50, \frac{725}{6}, 285, 846\right\}$$

   Although the associated solvable Lie algebras $\mathfrak{r}$ satisfy in both cases $\dim H^2(\mathfrak{r}, \mathfrak{r}) = 1$, they are geometrically rigid, as can be shown by topological arguments (see, e.g., [30]).

6. For $n = 31$, there are six solutions, from which two are isolated with diagonals given, respectively, by

$$\Delta = \left\{0^3, 1, \frac{22}{13}, \frac{25}{13}, \frac{25}{13}, \frac{25}{13}, \frac{28}{13}, \frac{42}{13}, \frac{150}{13}\right\},$$
$$\Delta = \left\{0^4, 1, 15, 50, \frac{725}{6}, 285, 846, 5880\right\}$$

   As before, $\dim H^2(\mathfrak{r}, \mathfrak{r}) = 1$ for the associated solvable Lie algebras $\mathfrak{r}$. Furthermore, in this case, these Lie algebras are geometrically rigid.

7. For $n = 32$, we find four solutions—one is isolated. It is cohomologically rigid with diagonal

$$\Delta = \left\{0^4, 1, \frac{30}{13}, \frac{45}{43}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}\right\}$$

8. For $n = 33$, we find five solutions—one isolated. It is cohomologically rigid with diagonal

$$\Delta = \left\{0^4, 1, \frac{30}{13}, \frac{45}{43}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}, \frac{990}{13}\right\}.$$

9. For $n \geq 34$, we obtain at most four solutions, three of them parametrised and one isolated with vanishing cohomology. The precise structure of the resulting diagonal and the associated nilradical $\mathfrak{n}_0$ for this series will be described in the following paragraph.

As follows from this analysis, for given values of k and n, there appear nilradicals $\mathfrak{n}$ with a diagonal, Δ, having quite a different structure. Comparing for instance the two nilradicals $\mathfrak{n}_1$ and $\mathfrak{n}_2$, associated with the diagonals in (22), the commutator table has the following form (as before, the non-displayed commutators are either zero or obtained by antisymmetry):

$\mathfrak{n}_1$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_3	0								
X_3	X_4	0	0							
X_4	X_5	0	0	0						
X_5	X_6	0	0	X_{15}	0					
X_6	X_7	0	$-X_{15}$	X_{16}	$\frac{24}{11}X_{17}$	0				
X_7	X_8	X_{15}	$-2X_{16}$	$\frac{-13}{11}X_{17}$	$\frac{24}{11}X_{18}$	$3X_{19}$	0			
X_8	X_9	$3X_{16}$	$-\frac{9}{11}X_{17}$	$-\frac{37}{11}X_{18}$	$\frac{-9}{11}X_{19}$	$3X_{20}$	$\frac{40}{11}X_{21}$	0		
X_9	X_{10}	$\frac{42}{11}X_{17}$	$\frac{28}{11}X_{18}$	$\frac{-28}{11}X_{19}$	$\frac{-42}{11}X_{20}$	$\frac{-7}{11}X_{21}$	$\frac{40}{11}X_{22}$	$\frac{51}{11}X_{23}$	0	
X_{10}	X_{11}	$\frac{14}{11}X_{18}$	$\frac{56}{11}X_{19}$	$\frac{14}{11}X_{20}$	$\frac{-35}{11}X_{21}$	$\frac{-47}{11}X_{22}$	$-X_{23}$	$\frac{51}{11}X_{24}$	$\frac{84}{11}X_{25}$	0
X_{11}	X_{12}	$\frac{-42}{11}X_{19}$	$\frac{42}{11}X_{20}$	$\frac{49}{11}X_{21}$	$\frac{12}{11}X_{22}$	$\frac{-36}{11}X_{23}$	$\frac{-62}{11}X_{24}$	$-3X_{25}$	$\frac{84}{11}X_{26}$	$\frac{322}{11}X_{27}$
X_{12}	X_{13}	$\frac{-84}{11}X_{20}$	$\frac{-7}{11}X_{21}$	$\frac{37}{11}X_{22}$	$\frac{48}{11}X_{23}$	$\frac{26}{11}X_{24}$	$\frac{-25}{11}X_{25}$	$\frac{-117}{11}X_{26}$	$\frac{-238}{11}X_{27}$	$\frac{322}{11}X_{28}$
X_{13}	X_{14}	$-7X_{21}$	$-4X_{22}$	$-X_{23}$	$2X_{24}$	$5X_{25}$	$8X_{26}$	$11X_{27}$	$\frac{-560}{11}X_{28}$	0
X_{14}	X_{15}	$-3X_{22}$	$-3X_{23}$	$-3X_{24}$	$-3X_{25}$	$-3X_{26}$	$-3X_{27}$	$\frac{681}{11}X_{28}$	0	0
X_{15}	X_{16}	0	0	0	0	0	$\frac{-714}{11}X_{28}$	0	0	0
X_{16}	X_{17}	0	0	0	0	$\frac{714}{11}X_{28}$	0	0	0	0
X_{17}	X_{18}	0	0	0	$\frac{-714}{11}X_{28}$	0	0	0	0	0
X_{18}	X_{19}	0	0	$\frac{714}{11}X_{28}$	0	0	0	0	0	0
X_{19}	X_{20}	0	$\frac{-714}{11}X_{28}$	0	0	0	0	0	0	0
X_{20}	X_{21}	$\frac{714}{11}X_{28}$	0	0	0	0	0	0	0	0
X_{21}	X_{22}	0								
X_{22}	X_{23}	0								
X_{23}	X_{24}	0								
X_{24}	X_{25}	0								
X_{25}	X_{26}	0								
X_{26}	X_{27}	0								
X_{27}	X_{28}	0								
X_{28}	0	0								

$\mathfrak{n}_2$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_2	X_3	0								
X_3	X_4	0	0							
X_4	X_5	0	0	0						
X_5	X_6	0	0	X_{15}	0					
X_6	X_7	0	$-X_{15}$	X_{16}	$\frac{15}{13}X_{17}$	0				
X_7	X_8	X_{15}	$-2X_{16}$	$\frac{-2}{13}X_{17}$	$\frac{15}{13}X_{18}$	$\frac{135}{143}X_{19}$	0			
X_8	X_9	$3X_{16}$	$\frac{-24}{13}X_{17}$	$\frac{-17}{13}X_{18}$	$\frac{30}{143}X_{19}$	$\frac{135}{143}X_{20}$	$\frac{100}{143}X_{21}$	0		
X_9	X_{10}	$\frac{63}{13}X_{17}$	$\frac{-7}{13}X_{18}$	$\frac{-217}{143}X_{19}$	$\frac{105}{143}X_{20}$	$\frac{35}{143}X_{21}$	$\frac{100}{143}X_{22}$	$\frac{75}{143}X_{23}$	0	
X_{10}	X_{11}	$\frac{70}{13}X_{18}$	$\frac{140}{143}X_{19}$	$\frac{-112}{143}X_{20}$	$\frac{-140}{153}X_{21}$	$\frac{-5}{11}X_{22}$	$\frac{25}{143}X_{23}$	$\frac{75}{143}X_{24}$	$\frac{63}{143}X_{25}$	0
X_{11}	X_{12}	$\frac{630}{143}X_{19}$	$\frac{252}{143}X_{20}$	$\frac{28}{143}X_{21}$	$\frac{-75}{143}X_{22}$	$\frac{-90}{143}X_{23}$	$\frac{-50}{143}X_{24}$	$\frac{12}{143}X_{25}$	$\frac{63}{143}X_{26}$	$\frac{70}{143}X_{27}$
X_{12}	X_{13}	$\frac{378}{143}X_{20}$	$\frac{224}{143}X_{21}$	$\frac{103}{143}X_{22}$	$\frac{15}{143}X_{23}$	$\frac{-40}{143}X_{24}$	$\frac{-62}{143}X_{25}$	$\frac{-51}{143}X_{26}$	$\frac{-7}{143}X_{27}$	$\frac{70}{143}X_{28}$
X_{13}	X_{14}	$\frac{14}{13}X_{21}$	$\frac{11}{13}X_{22}$	$\frac{8}{13}X_{22}$	$\frac{5}{13}X_{23}$	$\frac{2}{13}X_{25}$	$\frac{-1}{13}X_{26}$	$\frac{-4}{13}X_{27}$	$\frac{-7}{13}X_{28}$	0
X_{14}	X_{15}	$\frac{3}{13}X_{22}$	$\frac{3}{13}X_{23}$	$\frac{3}{13}X_{24}$	$\frac{3}{13}X_{25}$	$\frac{3}{13}X_{26}$	$\frac{3}{13}X_{27}$	$\frac{3}{13}X_{28}$	0	0
X_{15}	X_{16}	0	0	0	0	0	0	0	0	0
X_{16}	X_{17}									
X_{17}	X_{18}									
X_{18}	X_{19}									
X_{19}	X_{20}									
X_{20}	X_{21}									
X_{21}	X_{22}									
X_{22}	X_{23}									
X_{23}	X_{24}									
X_{24}	X_{25}									
X_{25}	X_{26}									
X_{26}	X_{27}									
X_{27}	X_{28}									
X_{28}	0									

We observe that the main difference between the nilradicals—that are far from being saturated—lies in the fact that, from the second column onwards, we either obtain a number of consecutive nonvanishing commutators or a sequence with some zeroes. This suggests to consider the values of the second column as an additional criterion to properly separate the isolated solutions to Equation (20). We will inspect the precise description of the cohomologically rigid solutions of Equation (20) according to this principle.

3.1. Algebras with $C^n_{2,n-k} = 0$

As follows from Equation (14), the last index—for which the commutator, $[X_2, X_j]$, is defined—is $j = n - k$. For computational reasons, we analyse the solutions of system (20) in dependence of the value $C^n_{2,n-k}$. Imposing that the latter is zero, for any $k \geq 3$ there are isolated solutions, and for $k \geq 8$, the minimal dimension for which such a solution appears is given by $n \geq 3k + 4$. With this constraint, there is always an isolated solution for the Jacobi Equation (20), leading to a new continuous family of cohomologically rigid Lie algebras with a resulting nilradical, $\mathfrak{n}_0$, of dimension $n = 3k + 2s + 4$ or $n = 3k + 2s + 5$ $(s \geq -1)$, depending on the parity of the dimension. In order to describe the generating functions of the diagonal (16) properly, it is convenient to separate the analysis according to the parity of both the dimension, n, and the index, k.

3.1.1. k Even, n Odd

We define the auxiliar parameter, t, given by $t = \frac{n-1}{2}$. In this case, the first $t - \frac{3k}{2}$ entries of the main diagonal are zero, and we normalize the first nonvanishing value. Following this normalized value, the k remaining entries in the diagonal are different from zero, as follows:

$$\Delta = \{\overbrace{0,0,\ldots,0}^{t-\frac{3k}{2}},1,\phi_1(k,n),\phi_2(k,n),\ldots,\phi_{k-1}(k,n)\}, \tag{23}$$

where the generating functions are defined as

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!}\prod_{i=0}^{\ell-1}\frac{(1+2i-3k+n)(n-1-2i-k)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-1. \tag{24}$$

3.1.2. k Even, n Even

Here, $t = \frac{n}{2}$. As before, the first $t - \frac{3k}{2}$ entries of Δ are zero, so that

$$\Delta = \{\overbrace{0,0,\ldots,0}^{t-\frac{3k}{2}},1,\phi_1(k,n),\phi_2(k,n),\ldots,\phi_{k-2}(k,n)\}. \tag{25}$$

The generating functions are given by

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!}\prod_{i=0}^{\ell-1}\frac{(1+2i-3k+n)(n-2i-k)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-2. \tag{26}$$

3.1.3. k Odd, n Odd

The auxiliary parameter is given by $t = \frac{n-1}{2}$, and the diagonal, Δ, is given by

$$\Delta = \{\overbrace{0,0,\ldots,0}^{t-\frac{3k-1}{2}},1,\phi_1(k,n),\phi_2(k,n),\ldots,\phi_{k-2}(k,n)\}, \tag{27}$$

with generating functions

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!}\prod_{i=0}^{\ell-1}\frac{(n-2i-k)(2+2i-3k+n)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-2. \tag{28}$$

3.1.4. k Odd, n Even

The auxiliary parameter is given by $t = \frac{n-2}{2}$, and the diagonal, Δ, is given by

$$\Delta = \{\overbrace{0,0,\ldots,0}^{t-\frac{3k-1}{2}},1,\phi_1(k,n),\phi_2(k,n),\ldots,\phi_{k-1}(k,n)\}, \tag{29}$$

with generating function

$$\phi_\ell(k,n) = \frac{1}{2^\ell \ell!}\prod_{i=0}^{\ell-1}\frac{(n-2i-k-1)(1+2i-3k+n)}{4(2k-2\ell+2i-1)}, \quad 1 \leq \ell \leq k-1. \tag{30}$$

As a particularity of this case, we observe that, although the isolated solutions for $k = 8$ and dimensions $n = 30, 31$ turn out to be rigid, they are not cohomologically rigid, as in both cases the identity $\dim H^2\left(\mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}, \mathfrak{t}\overrightarrow{\oplus}\mathfrak{n}\right) = 1$ holds. This anomaly has already been observed for other series of cohomologically rigid algebras (see [20,22,35] and references therein), and is directly related to the number of quadratic Jacobi equations to be solved.

From these special dimension onwards, the solutions tend to stabilize, and cohomologically rigid Lie algebras are obtained.

In Table 1, we enumerate the lowest-dimensional solutions for the continuous series for $k \geq 5$ and $n \leq 33$. (The values $k = 3, 4$ have already been considered in [35]).

Table 1. Cohomologically rigid Lie algebras with $C^n_{2,n-k} = 0$.

n	k	Δ	n	k	Δ
21	5	$\left(0^3, 1, \frac{16}{7}, 4, 8\right)$	29	5	$\left(0^7, 1, \frac{48}{7}, \frac{1188}{35}, \frac{1320}{7}\right)$
22	5	$\left(0^3, 1, \frac{16}{7}, 4, 8, 35\right)$	29	6	$\left(0^5, 1, \frac{11}{3}, \frac{55}{6}, 22, 66, 462\right)$
23	5	$\left(0^4, 1, \frac{45}{14}, \frac{54}{7}, 21\right)$	29	7	$(0^4, 1, \frac{5}{2}, \frac{25}{6}, \frac{25}{4}, 10, 21)$
24	5	$\left(0^4, 1, \frac{45}{14}, \frac{54}{7}, 21, 126\right)$	29	8	$\left(0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}, \frac{180}{143}\right)$
24	6	$(0^3, 1, 2, \frac{20}{7}, 4, 7)$	30	5	$\left(0^7, 1, \frac{48}{7}, \frac{1188}{35}, \frac{1320}{7}, \frac{16335}{7}\right)$
25	5	$(0^5, 1, \frac{30}{7}, \frac{27}{2}, 48)$	30	6	$\left(0^6, 1, \frac{14}{2}, \frac{44}{3}, 44, 165\right)$
25	6	$(0^3, 1, 2, \frac{20}{7}.4, 7, 28)$	30	7	$(0^4, 1, \frac{5}{2}, \frac{25}{6}, \frac{25}{4}, 10, 21, 105)$
25	7	$\left(0^2, 1, \frac{27}{22}, \frac{12}{11}, \frac{10}{11}, \frac{9}{11}, \frac{21}{22}\right)$	31	5	$\left(0^8, 1, \frac{117}{14}, \frac{351}{7}, \frac{4719}{14}\right)$
26	5	$(0^5, 1, \frac{30}{7}, \frac{27}{2}, 48, 378)$	31	6	$\left(0^6, 1, \frac{14}{2}, \frac{44}{3}, 44, 165, 1452\right)$
26	6	$(0^4, 1, \frac{25}{9}, \frac{75}{14}, 10, \frac{70}{13})$	31	7	$\left(0^5, 1, \frac{36}{11}, 7, \frac{40}{3}, 27, 72\right)$
26	7	$\left(0^2, 1, \frac{27}{22}, \frac{12}{11}, \frac{10}{11}, \frac{9}{11}, \frac{21}{22}, \frac{28}{11}\right)$	32	5	$\left(0^8, 1, \frac{117}{14}, \frac{351}{7}, \frac{4719}{14}, \frac{70785}{14}\right)$
27	5	$\left(0^6, 1, \frac{11}{2}, 22, 99\right)$	32	6	$\left(0^7, 1, \frac{52}{9}, \frac{156}{7}, \frac{572}{7}, \frac{7865}{21}\right)$
27	6	$(0^4, 1, \frac{25}{9}, \frac{75}{14}, 10, \frac{70}{13}, 126)$	32	7	$\left(0^5, 1, \frac{36}{11}, 7, \frac{40}{3}, 27, 72, 462\right)$
27	7	$(0^3, 1, \frac{20}{11}, \frac{25}{11}, \frac{200}{77}, \frac{35}{11}, \frac{56}{11})$	32	8	$\left(0^4, 1, \frac{30}{13}, \frac{45}{13}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}\right)$
28	5	$\left(0^6, 1, \frac{11}{2}, 22, 99, 990\right)$	33	5	$\left(0^9, 1, 10, \frac{143}{2}, 572\right)$
28	6	$\left(0^5, 1, \frac{11}{3}, \frac{55}{6}, 22, 66\right)$	33	6	$\left(0^8, 1, 7, \frac{65}{2}, 143, \frac{1573}{2}\right)$
28	7	$\left(0^3, 1, \frac{20}{11}, \frac{25}{11}, \frac{200}{77}, \frac{35}{11}, \frac{56}{11}, \frac{210}{11}\right)$	33	7	$\left(0^6, 1, \frac{91}{22}, \frac{364}{33}, 26, 65, \frac{429}{2}\right)$
28	8	$\left(0^2, 1, \frac{15}{13}, \frac{135}{143}, \frac{100}{143}, \frac{75}{143}, \frac{63}{143}, \frac{70}{143}\right)$	33	8	$\left(0^4, 1, \frac{30}{13}, \frac{45}{13}, \frac{175}{39}, \frac{75}{13}, \frac{108}{13}, \frac{210}{13}, \frac{990}{13}\right)$

For any k, there exists an integer, $m(k)$, such that, for $n \geq m(k)$, the only isolated solution for equations (20) is given by $\mathfrak{n}_0$, with the remaining depending on two or more parameters. For values $k \geq 5$, $m(k)$ is given by (the corresponding value for $k = 2$ was given in [22], for $k = 3$ in [20] and for $k = 4$ in [35])

$$m(k) = \left\{ \begin{array}{cl} 21 & k = 5 \\ 4k & k \;\; \text{even} \\ 4k - 1 & k \;\; \text{odd} \end{array} \right. \tag{31}$$

In each case, several other types of (cohomologically or geometrically) rigid algebras appear in dimensions lower than $m(k)$, the analysis of which requires the imposition of additional constraints. The characteristic fact of the series determined by $\mathfrak{n}_0$ is that it is the only cohomologically rigid solution for which the condition $C^n_{2,n-k} = 0$ is satisfied.

Concerning the commutators of the resulting nilradical, $\mathfrak{n}_0$, the structure constants have the following structure:

$$\left\{ \begin{array}{ll} C_{1,j}^{j+1} = 1 & 2 \leq j \leq n-1 \\ C_{2,j}^{k+j} \neq 0 & 2s+7 \leq j \leq k+s+6 \\ C_{i,j}^{i+j+k-2} \neq 0 & s+9 \leq i+j \leq n-k-2 \\ C_{i,j}^{i+j+k-2} = 0 & 2 \leq i < j,\ j > k+2s+6 \end{array} \right. \tag{32}$$

where, in particular

$$C_{2,k+2s+6}^{2k+2s+6} = C_{3,k+2s+6}^{2k+2s+7} = \cdots = C_{n-2k-2s-5,k+2s+6}^{n-1} = C_{n-2k-2s-4,k+2s+6}^{n}. \tag{33}$$

All the remaining values for the constants are recovered using Equation (17).

4. Algebras with $C_{2,n-k}^{n} \neq 0$

Nilradicals satisfying this condition can only appear for dimensions $n < m(k)$. If the commutator $[X_2, X_{n-k}]$ does not vanish, the distinction of the solutions is computationally simplified if we consider the following factor sequence (whenever the denominator is nonzero):

$$\frac{C_{2,i}^{k+i}}{C_{2,i+1}^{k+i+1}}, \quad 4 \leq i \leq n-k-1. \tag{34}$$

A relevant structural difference with respect to the previous case is that the diagonal (16) may have one or more zeroes after the first nonvanishing entry, implying that we do not obtain continuous series of algebras, but only solutions that exist in a certain range of dimensions, as expected. For the same values of k and n, several cohomologically rigid Lie algebras may exist, implying that a generic generating function that describes this type of solution cannot be found in general. It must also be observed that not all values of k provide cohomologically rigid solutions. The computational analysis of the case $C_{2,n-k}^{n} \neq 0$ shows that it depends essentially on the rest class of k, mod 4. We thus introduce the auxiliary parameter, p, such that $k = 4p$, $k = 4p+1$, $k = 4p+2$, and $k = 4p+3$. Within the range of $k \leq 30$ and $n \leq 200$, the analysis of solutions for the constraint $C_{2,n-k}^{n} \neq 0$ leads to three main subcases.

4.1. *Subcase 1:* $C_{2,n-k-1}^{n-1} \neq 0$, $C_{s+r+6,s+r+7}^{2s+2r+k+11} = 0$, $0 \leq r \leq p-1$

For values $k \leq 7$, there is only one solution. For values $k \geq 8$, isolated solutions of this type appear with the diagonal (16) adopting the following form:

$$\Delta(\mathfrak{g}) = \{\overbrace{0,\ldots,0}^{s+3}, 1, \overbrace{0,\ldots,0}^{p}, C_{s+p+6,s+p+7}^{2s+2p+k+11}, \ldots, C_{3p+s+6,3p+s+7}^{6p+2s+k+11}\} \tag{35}$$

for even values of n, while

$$\Delta(\mathfrak{g}) = \{\overbrace{0,\ldots,0}^{s+3}, 1, \overbrace{0,\ldots,0}^{p}, C_{s+p+6,s+p+7}^{2s+2p+k+11}, \ldots, C_{3p+s+5,3p+s+6}^{6p+2s+k+9}\} \tag{36}$$

if n is an odd number. In both cases, the $3+s$ first entries as well as the p zeroes following the normalized entry are a consequence of the unique solution of the linear system of equations

$$\frac{C_{2,i+s}^{k+i+s}}{C_{2,i+s+1}^{k+i+s+1}} = \frac{i-8-2s}{i-5-s}, \tag{37}$$

where $9+2s \leq i \leq 2p+2s+10$. Taking into account the possible values of p and the parity of the dimension of the nilradical $\mathfrak{n}$, the following six possibilities are given:

1. For odd values of n and $k = 4p$, there exists an $n + 1 = (10 + 2s + 10)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \le s \le 2p - 4$.
2. For odd values of n and $k = 4p + 1$, there exists an $n + 1 = (10 + 2s + 12)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \le s \le 2p - 3$.
3. For odd values of n and either $k = 4p + 2$ or $k = 4p + 3$, there do not exist cohomologically rigid solutions.
4. For even values of n and either $k = 4p$ or $k = 4p + 3$, there do not exist cohomologically rigid solutions.
5. For even values of n and $k = 4p + 1$, there exists an $n + 1 = (10 + 2s + 13)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \le s \le 2p - 3$.
6. For even values of n and $k = 4p + 2$, there exists an $n + 1 = (10 + 2s + 15)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \le s \le 2p - 2$.

Table 2 enumerates the diagonals, Δ, associated with the lowest-dimensional cohomologically rigid solutions:

Table 2. Lowest-dimensional solutions for Subcase 1.

$\dim \mathfrak{n}$	k	Δ
24	6	$\{0,0,0,1,0,0,-\frac{36}{7},-24,-189\}$
29	8	$\{0,0,0,1,0,0,\frac{55}{7},55,616,7535\}$
31	9	$\left\{0,0,0,1,0,0,\frac{100}{21},\frac{45}{2},96,\frac{2650}{7}\right\}$
32	9	$\left\{0,0,0,1,0,0,\frac{100}{21},\frac{45}{2},96,\frac{2650}{7},\frac{20340}{7}\right\}$
33	9	$\{0,0,0,0,1,0,0,\frac{65}{6},65,351,3900\}$
34	9	$\{0,0,0,0,1,0,0,\frac{65}{6},65,351,3900,\frac{124605}{2}\}$
34	10	$\{0,0,0,1,0,0,\frac{13}{4},\frac{143}{12},\frac{65}{2},78,\frac{871}{4}\}$
36	10	$\{0,0,0,0,1,0,0,\frac{245}{33},35,126,525,\frac{7421}{3}\}$
38	10	$\{0,0,0,0,0,1,0,0,\frac{490}{33},\frac{945}{11},378,1925,22275\}$
39	12	$\{0,0,0,1,0,0,0,-\frac{675}{187},-\frac{302}{17},-\frac{1075}{17},-\frac{3375}{17},-\frac{12345}{17},-\frac{100380}{17}\}$
41	12	$\{0,0,0,0,1,0,0,0,-\frac{980}{99},-\frac{672}{11},-280,-\frac{4400}{3},-\frac{79450}{9},-\frac{1005760}{9}\}$
43	12	$\{0,0,0,0,0,1,0,0,0,-\frac{3332}{143},-\frac{1904}{11},-952,-5984,-85085,-2167218\}$

4.2. Subcase 2: $C^{n-1}_{2,n-k-1} \neq 0$ and $C^{n-2}_{2,n-k-2} \neq 0$

Solutions subjected to these additional constraints appear for $p \ge 3$, i.e., $k \ge 12$. The diagonal, Δ, has its $3 + s$ first entries equal to zero as in (35) and (36), but the remaining nonvanishing entries have no easily recognizable pattern, and no zeroes have to appear. The structure constants of the nilradical are characterized by the following factor sequence:

$$\frac{C^{k+i+s}_{2,i+s}}{C^{k+i+s+1}_{2,i+s+1}} = \frac{i-7-2s}{i+2p-2s-j}, \quad 2p+2 \le i \le 4p+6, \tag{38}$$

and where j is constrained by p and s depending on the parity of n and the value of k. More precisely:

1. For odd values of n and $k = 4p$, there exists an $n + 1 = (10p + 2s + 10)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \le s \le p - 4$ and $8 \le j \le 2p - 2s + 1$.
2. For even values of n and $k = 4p$, there do not exist cohomologically rigid solutions.

3. For odd values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+12)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \leq s \leq p-4$ and $8 \leq j \leq 2p-2s+1$.
4. For even values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+13)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \leq s \leq p-3$ and $7 \leq j \leq 2p-2s+1$.
5. For odd values of n and $k = 4p+2$, there do not exist cohomologically rigid solutions.
6. For even values of n and $k = 4p+2$, there exists an $n+1 = (10p+2s+15)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $0 \leq s \leq p-3$ and $7 \leq j \leq 2p-2s+1$.
7. For $k = 4p+3$, there do not exist cohomologically rigid solutions, regardless of the parity on n.

Table 3 presents the lowest-dimensional cohomologically rigid solutions corresponding to this subcase:

Table 3. Lowest-dimensional solutions for Subcase 2.

dim $\mathfrak{n}$	k	$\boldsymbol{\Delta}$
40	13	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, -\frac{1998557}{194446}, -\frac{2985595}{97223}, -\frac{6456225}{97223}, -\frac{13282053}{97223}, -\frac{45764301}{194446}, -\frac{43441288}{97223}, -\frac{335042213}{194446}\right\}$
40	13	$\left\{0^2, 1, -1, \frac{1}{5}, 0, -\frac{9317}{2470}, -\frac{27161}{2470}, -\frac{25917}{1235}, -\frac{31782}{1235}, -\frac{58629}{2470}, -\frac{16027}{2470}, \frac{471163}{2470}\right\}$
40	13	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, -\frac{2537}{102}, -\frac{11032}{153}, -\frac{24085}{153}, -\frac{5863}{17}, -\frac{32923}{34}, -\frac{401005}{153}, -\frac{4202891}{306}\right\}$
42	13	$\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{715}{34}, -\frac{1378}{17}, -\frac{26853}{119}, -\frac{73947}{119}, -\frac{520113}{238}, -\frac{2895530}{357}, \frac{-14343043}{238}\}$
42	14	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, -\frac{20119}{969}, -\frac{153853}{2907}, -\frac{282385}{2907}, -\frac{52987}{323}, -\frac{93731}{323}, -\frac{9053}{19}, -\frac{2622697}{2907}\right\}$
42	14	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, -\frac{4745}{578}, -\frac{6136}{289}, -\frac{76121}{2023}, -\frac{115959}{2023}, -\frac{283371}{4046}, -\frac{158285}{2023}, -\frac{353951}{4046}\right\}$
42	14	$\left\{0^2, 1, -1, \frac{1}{5}, 0, -\frac{1001}{380}, -\frac{275977}{45220}, -\frac{90023}{11305}, -\frac{71571}{22610}, \frac{76443}{9044}, \frac{1425457}{45220}, \frac{33407}{340}\right\}$
44	14	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{87035}{5168}, -\frac{73463}{1292}, -\frac{1196673}{9044}, -\frac{2524737}{9044}, -\frac{22311201}{36176}, -\frac{24418735}{18088}, -\frac{43441288}{97223}, -\frac{335042213}{194446}\right\}$
47	16	$\left\{0^2, 1, -5, 7, -4, 1, \frac{21917}{323}, \frac{816326}{3553}, \frac{1923088}{3553}, \frac{377858}{323}, \frac{885456}{323}, \frac{2778490}{323}, \frac{217347689}{7106}, \frac{1588345421}{7106}\right\}$
47	16	$\left\{0^2, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, \frac{27981499}{1069776}, \frac{24341191}{267444}, \frac{57588575}{267444}, \frac{122804099}{267444}, \frac{1093212263}{1069776}, \frac{133386825}{59432}, \frac{2101240531}{356592}, \frac{4052486529}{118864}\right\}$
47	16	$\left\{0^2, 1, -2, 1, -\frac{1}{7}, 0, \frac{748839}{81719}, \frac{2620562}{81719}, \frac{6026233}{81719}, \frac{1076861}{7429}, \frac{1819180}{7429}, \frac{232695697}{572033}, \frac{65841958}{81719}, \frac{267268274}{81719}\right\}$
47	16	$\left\{0^2, 1, -1, \frac{1}{5}, 0^2, \frac{1183}{437}, \frac{15413}{1748}, \frac{74447}{4370}, \frac{36155}{1748}, \frac{24915}{1748}, -\frac{6655}{437}, -\frac{62465}{437}, -\frac{6514053}{4370}\right\}$

4.3. *Subcase 3:* $\boldsymbol{C^{n-1}_{2,n-k-1} \neq 0}$, $\boldsymbol{C^{n-2}_{2,n-k-2} \neq 0}$ *and* $\boldsymbol{C^{n-3}_{2,n-k-3} \neq 0}$

In this subcase, the solutions appear for $p \geq 3$, thus $k \geq 13$. The diagonal, Δ, has its $3+s$ first entries are zero as before, the remaining entries being nonzero. The relevant factor sequence that determines the structure constants of the nilradical is given by

$$\frac{C^{k+i+s}_{2,i+s}}{C^{k+i+s+1}_{2,i+s+1}} = \frac{i-8-2s}{i+2p-2s-j}, \quad 2p+2 \leq i \leq 4p+6, \tag{39}$$

with j constrained by s and p. The analysis provides the following cases:

1. For odd values of n and $k = 4p$, there exists an $n+1 = (10p+2s+10)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $-1 \leq s \leq p-4$ and $10 \leq j \leq 2p-2s+1$.
2. For even values of n and $k = 4p$, there do not exist cohomologically rigid solutions.
3. For odd values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+12)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $-1 \leq s \leq p-4$ and $10 \leq j \leq 2p-2s+1$.
4. For even values of n and $k = 4p+1$, there exists an $n+1 = (10p+2s+13)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $-1 \leq s \leq p-3$ and $9 \leq j \leq 2p-2s+1$.
5. For odd values of n and $k = 4p+2$, there do not exist cohomologically rigid solutions.
6. For even values of n and $k = 4p+2$, there exists an $n+1 = (10p+2s+15)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $-1 \leq s \leq p-3$ and $9 \leq j \leq 2p-2s+1$.
7. For $k = 4p+3$, there do not exist cohomologically rigid solutions, regardless of the parity on n.

The lowest dimensional solution of this case are given in Table 4.

Table 4. Lowest-dimensional solutions with $n \leq 50$ for Subcase 3.

dim $\mathfrak{n}$	$\boldsymbol{k}$	$\boldsymbol{\Delta}$
40	13	$\left\{0,0,1,-\frac{9}{2},3,-\frac{1}{2},-\frac{891}{34},-\frac{5313}{68},-\frac{2933}{17},-\frac{12933}{34},-\frac{36459}{34},-\frac{245773}{68},-\frac{879219}{34}\right\}$
42	14	$\left\{0^2,1,-\frac{9}{2},3,-\frac{1}{2},-\frac{4433}{204},-\frac{35035}{612},-\frac{962}{9},-\frac{18713}{102},-\frac{22703}{68},-\frac{44737}{68},-\frac{1012121}{612}\right\}$
47	16	$\left\{0^2,1,-8,10,-4,\frac{1}{2},\frac{55055}{646},\frac{95277}{323},\frac{226686}{323},\frac{985611}{646},\frac{2318591}{646},\frac{7295717}{646},\frac{30283877}{646},\frac{140233457}{323}\right\}$
47	16	$\left\{0^2,1,-\frac{9}{2},3,-\frac{1}{2},0,\frac{188097}{7429},\frac{671671}{7429},\frac{3228417}{14858},\frac{3492414}{7429},\frac{7953275}{7429},\frac{39831759}{14858},\frac{128959077}{14858},\frac{487959836}{7429}\right\}$
49	17	$\left\{0^2,1,-8,10,-4,\frac{1}{2},\frac{801944}{11305},\frac{358917}{1615},\frac{39561}{85},\frac{3845109}{4522},\frac{17478197}{11305},\frac{70572931}{22610},\frac{164397571}{22610},\frac{510384407}{22610}\right\}$
49	17	$\left\{0^2,1,-\frac{9}{2},3,-\frac{1}{2},0,\frac{4719}{230},\frac{144508}{2185},\frac{304281}{2185},\frac{109971}{437},\frac{964304}{2185},\frac{1738341}{2185},\frac{28170363}{17480},\frac{9591842}{2185}\right\}$ $\frac{241548541}{874},\frac{1393530272}{437}\}$
50	17	$\left\{0^2,1,-\frac{25}{2},25,-\frac{35}{2},5,\frac{3212359}{14858},\frac{4851210}{7429},\frac{10021665}{7429},\frac{18255630}{7429},\frac{33456995}{7429},\frac{69751682}{7429},\frac{200439395}{7429},\frac{794583244}{7429},\frac{7199857252}{7429}\right\}$
52	17	$\left\{0^2,1,-\frac{9}{2},3,-\frac{1}{2},0,\frac{4719}{230},\frac{144508}{2185},\frac{304281}{2185},\frac{109971}{437},\frac{964304}{2185},\frac{1738341}{2185},\frac{28170363}{17480},\frac{9591842}{2185},\frac{50647275}{1748}\right\}$

4.4. Subcase 4: $C^{n}_{2,n-k} \neq 0$ and $C^{n-1}_{2,n-k-1} = 0$

Structurally, this subcase differs from the previous ones. Isolated solutions to Equation (20) exist for $p \geq 2$ (hence $k = 8$), implying that $n = 28$ is the lowest dimension for which Lie algebras of this type appear. As before, the first $3+s$ entries of the diagonal, Δ, are zero, and again, after the first normalized entry there appear additional zeroes. The structure constants of the nilradical satisfy the relations

$$C^{2k+2s+8}_{2,k+2s+8} = C^{2k+2s+i+6}_{i,k+2s+8}, \quad 3 \leq i \leq n-2k-2s-7. \tag{40}$$

Depending on the rest class of k, the following possibilities are given:

1. For $k = 4p$, there exists an $n + 1 = (10p + 2s + 9)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $2p - 6 \leq s \leq 2p - 4$.
2. For $k = 4p + 1$, there exists an $n + 1 = (10p + 2s + 10)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for $2p - 4 \leq s \leq 2p - 2$.
3. For $k = 4p + 2$, there exists an $n + 1 = (10p + 2s + 13)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $2p - 3 \leq s \leq 2p - 1$.
4. For $k = 4p + 3$, there exists an $n + 1 = (10p + 2s + 16)$-dimensional cohomologically rigid Lie algebra, $\mathfrak{r}$, for any $2p - 5 \leq s \leq 2p - 3$.

In Table 5, the lowest dimensional solutions corresponding to this case are given.

Table 5. Lowest-dimensional solutions for Subcase 4.

$\dim \mathfrak{n}$	k	Δ
28	8	$\left\{0^2, 1, \frac{24}{11}, 3, \frac{40}{11}, \frac{51}{11}, \frac{84}{11}, \frac{322}{11}\right\}$
29	9	$\left\{0^2, 1, -2, 1, \frac{85}{7}, 37, 112, 714\right\}$
31	9	$\left\{0^3, 1, -1, 0, \frac{475}{42}, 50, 206, 1764\right\}$
33	9	$\left\{0^4, 1, 0, 0, \frac{65}{6}, 65, 351, 3900\right\}$
34	10	$\{0^3, 1, -2, 1, 17, 55, 148, 474, 3462\}$
36	10	$\left\{0^4, 1, -1, 0, \frac{518}{33}, \frac{209}{3}, 243, 993, 9240\right\}$
38	10	$\left\{0^5, 1, 0, 0, \frac{490}{33}, \frac{945}{11}, 378, 1925, 22275\right\}$
35	11	$\left\{0^2, 1, \frac{52}{25}, -\frac{2}{25}, -4, -\frac{17}{2}, -14, -\frac{588}{25}, -\frac{1272}{25}, -\frac{519}{2}\right\}$
37	11	$\left\{0^3, 1, \frac{49}{15}, 3, -1, -\frac{161}{18}, -23, -\frac{267}{5}, -154, -\frac{2079}{2}\right\}$
38	12	$\left\{0^2, 1, \frac{13}{17}, -\frac{26}{17}, -\frac{72}{17}, -\frac{13}{2}, -\frac{287}{34}, -\frac{184}{17}, -\frac{264}{17}, -\frac{1023}{34}, -\frac{4807}{34}\right\}$
40	12	$\left\{0^3, 1, \frac{410}{221}, \frac{32}{221}, -4, -\frac{4333}{442}, -\frac{3850}{221}, -\frac{6450}{221}, -\frac{11886}{221}, -\frac{58971}{442}, -\frac{178673}{221}\right\}$
42	12	$\left\{0^4, 1, \frac{7400}{2499}, 3, -1, -\frac{526}{51}, -\frac{468}{17}, -\frac{7305}{119}, -\frac{51733}{357}, -\frac{378576}{833}, -\frac{414128}{119}\right\}$

5. Cohomological Rigidity of the Families

The proof of the rigidity of the previous solutions follows by the application of the Hochschild–Serre factorization. Let $\mathfrak{r} = \mathfrak{t} \overrightarrow{\oplus} \mathfrak{n}$ be the rank-one solvable Lie algebra with nilradical $\mathfrak{n}$, possessing the diagonal Δ.

For an arbitrary 1-cochain $f \in C^1(\mathfrak{r}, \mathfrak{r})$

$$f(X_i) = \sum_{i=1}^{n} a_i^j X_j + b_i T, \;\; 1 \leq i \leq n, \tag{41}$$

that is $\mathfrak{t}$-invariant, we necessarily have $b_i = 0$. The image by the coboundary operator is given by

$$\begin{aligned} df(X_1, X_i) &= \left(a_1^1 + a_i^j - a_{i+1}^{i+1}\right) X_{i+1}, \;\; 2 \leq i \leq n-1, \\ df(X_i, X_j) &= \left(a_i^i + a_j^j - a_{i+j+k-2}^{i+j+k-2}\right) C_{i,j}^{i+j+k-2} X_{i+j+k-2}, \end{aligned}$$

where $2 \leq i \leq j + n + k - 2, \; i + j \leq n - k + 2$. From this, we extract a basis of $Z^1(\mathfrak{n}, \mathfrak{t})^{\mathfrak{t}}$:

$$f(X_1) = X_1, \; f(X_j) = (j + k - 2)X_j, \; 2 \leq j \leq n \tag{42}$$

corresponding to the linear operator $ad(T)$; hence, $Z^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}} = B^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}}$ and $H^1(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}} = 0$. It follows that

$$\dim B^2(\mathfrak{n},\mathfrak{t})^{\mathfrak{t}} = n-1. \tag{43}$$

Now, let $\varphi \in C^2(\mathfrak{n},\mathfrak{n})$ be a generic 2-cochain defined by

$$\varphi(X_i, X_j) = \sum_1^n \alpha_{i,j}^r C_r + \beta_{i,j} T, \tag{44}$$

Applying the coboundary operator leads to the following identities (k and s are fixed):

$$\begin{aligned}
\mathrm{d}\varphi(X_1, X_i, X_j) =& \Big(\alpha_{i,j}^{i+j+k-2} - \alpha_{1,j}^{j+k-1} C_{i,j+1}^{+j+k-1} - \alpha_{1,i}^{i+1} C_{i+1,j}^{i+j+k-1} \\
& -\alpha_{i+1,j}^{i+j+k-1} - \alpha_{i,j+1}^{i+j+k+1} - \alpha_{1,i+j+k-2}^{i+j+k-1} C_{i,j}^{i+j+k-2}\Big) X_{i+j+k-1}, \\
\mathrm{d}\varphi(X_i, X_j, X_l) =& \Big(\alpha_{j,l}^{j+l+k-2} C_{i,j+l+k-2}^{+j+2k+l-4} - \alpha_{i,l}^{i+l+k-2} C_{j,i+l+k-2}^{i+j+2k+l-4} \\
& +\alpha_{i,j}^{i+j+k-2} C_{l,i+j+k-2}^{i+j+2k+l-4} - \alpha_{l,i+j+k-2}^{i+j+l+2k-4} C_{i,j}^{+j+k-2} \\
& -\alpha_{j,i+l+k-2}^{i+j+l+2k-4} C_{i,l}^{+l+k-2} + \alpha_{i,j+l+k-2}^{i+j+l+2k-4} C_{j,l}^{j+l+k-2}\Big) X_{i+j+l+2k-4}
\end{aligned} \tag{45}$$

We distinguish the choice of fundamental parameters of the preceding system according to the various constraints used in the analysis of the different cases:

1. For the nilradicals belonging to the subcase $C_{2,n-k}^n = 0$, the generic form of a cocycle is given by

$$\begin{aligned}
\varphi(X_i, X_j) =& \Bigg((-1)^s \alpha_{2,7+2s}^{7+k+2s} + \sum_{t=0}^{i-3} \Big(-\alpha_{1,2+t}^{3+t} + \alpha_{1,9-i+2s+t}^{10-i+2s+t} \Big) \\
& + \sum_{t=0}^{i+j-2s-10} \Big(\alpha_{1,7+k+2s+t}^{8+k+t+2s} - \alpha_{1,9-i+2s+t}^{10-i+2s+t} \Big) \Bigg) C_{i,j}^{i+j+k-2} X_{i,j}^{i+j+k-2},
\end{aligned} \tag{46}$$

so that a basis of 2-cocycles is given by the cocycle classes of

$$\alpha_{2,7+2s}^{7+k+2s}; \quad \alpha_{1,j}^{j+1}, \quad 2 \le j \le n-1, \tag{47}$$

2. For the remaining cases with $C_{2,n-k}^n \neq 0$, the cocycles adopt the form

$$\begin{aligned}
\varphi(X_i, X_j) =& \Bigg((-1)^{s+1} \alpha_{2,9+2s}^{9+k+2s} + \sum_{t=0}^{i-3} \Big(-\alpha_{1,2+t}^{3+t} + \alpha_{1,11-i+2s+t}^{12-i+2s+t} \Big) \\
& + \sum_{t=0}^{i+j-2s-12} \Big(\alpha_{1,9+k+2s+t}^{10+k+t+2s} - \alpha_{1,11-i+2s+t}^{12-i+2s+t} \Big) \Bigg) C_{i,j}^{i+j+k-2} X_{i,j}^{i+j+k-2}
\end{aligned} \tag{48}$$

with basis

$$\alpha_{2,9+2s}^{9+k+2s}; \quad \alpha_{1,j}^{j+1}, \quad 2 \le j \le n-1, \tag{49}$$

In both cases $\dim Z^2(\mathfrak{n},\mathfrak{r})^t = n-1$, thus from (43) it follows that

$$\dim H^2(\mathfrak{n},\mathfrak{r})^t = 0, \tag{50}$$

showing the cohomological rigidity of these algebras.

Other Isolated Cohomologically Rigid Solutions

Besides the three main types described above, for some values low of k and n, there appear cohomological rigid Lie algebras whose nilradical does not satisfy any of the factor sequences (37)–(40). This indicates that, from a certain dimension onwards, for a fixed value of k, new factor sequences have to be introduced, in order to obtain new types of solutions. Due to this fact, a full classification is probably not possible, albeit for fixed dimensions, a complete table of the various possible factor sequences can be obtained, eventually allowing an extrapolation to higher dimensions. Another conceivable strategy that has been partially used in this work for high values of k is to set a certain number of the parameters α_j in the diagonal (18) equal to zero—i.e., imposing additional constraints on the dimension of $C^1(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{n}]$—and inspect the minimal dimension for which a solution to Equation (20), depending on only one parameter, exists. The cohomology of such isolated solutions has then be computed, allowing us to derive cohomologically rigid algebras whose nilradical does not belong to the main types described above. We enumerate the solutions for dimensions $n \leq 56$ found by this procedure in Table 6.

Table 6. Other isolated cohomologically rigid solutions.

$\dim \mathfrak{n}$	k	Δ
39	12	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{55}{2}, -\frac{878}{7}, -\frac{3117}{7}, -\frac{12891}{7}, -\frac{241989}{14}, -\frac{1439933}{7}\right\}$
40	13	$\left\{0^2, 1, -5, 7, -4, -\frac{20681}{374}, -\frac{5215}{34}, -\frac{5585}{17}, -\frac{12102}{17}, -\frac{67467}{34}, -\frac{446677}{34}, -\frac{39588077}{374}\right\}$
41	13	$\left\{0^3, 1, -2, 1, -\frac{1}{7}, -\frac{715}{34}, -\frac{1378}{17}, -\frac{26853}{119}, -\frac{73947}{119}, -\frac{520113}{238}, -\frac{2895530}{357}\right\}$
42	13	$\left\{0^3, 1, -\frac{10}{3}, 3, -1, -\frac{4627}{102}, -\frac{8554}{51}, -\frac{7785}{17}, -\frac{63569}{51}, -\frac{147873}{34}, -\frac{615329}{17}, -\frac{42938467}{102}\right\}$
42	14	$\left\{0^2, 1, -5, 7, -4, -\frac{30649}{646}, -\frac{829535}{7106}, -\frac{68915}{323}, -\frac{118338}{323}, -\frac{26169}{38}, -\frac{1121263}{646}, -\frac{32265271}{7106}\right\}$
44	14	$\left\{0^3, 1, -\frac{10}{3}, 3, -1, -\frac{72667}{1938}, -\frac{39494}{323}, -\frac{91287}{323}, -\frac{584903}{969}, -\frac{906257}{646}, -\frac{1413269}{323}, -\frac{30078347}{1938}\right\}$
49	16	$\left\{0^3, 1, -5, 7, -4, 1, \frac{2408}{19}, \frac{110670}{209}, \frac{316738}{209}, \frac{75114}{19}, \frac{212069}{19}, \frac{803374}{19}, \frac{7495085}{19}, \frac{1274187707}{209}\right\}$
51	16	$\left\{0^4, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, 104, 540, \frac{16852}{9}, \frac{17567}{3}, 19734, \frac{802945}{9}, \frac{2989675}{3}, 19951542\right\}$
52	17	$\left\{0^3, 1, -7, 14, -12, 5, \frac{313796}{1311}, \frac{4962456}{5681}, \frac{945708}{437}, \frac{6169163}{1311}, \frac{4490915}{437}, \frac{11168586}{437}, \frac{115303838}{1311}, \frac{332435922}{437}, \frac{5046314533}{437}\right\}$
53	17	$\left\{0^4, 1, -\frac{10}{3}, 3, -1, \frac{1}{9}, \frac{5746}{69}, \frac{26860}{69}, \frac{81532}{69}, \frac{640123}{207}, \frac{1668238}{207}, \frac{4934215}{207}, \frac{20201675}{207}, \frac{220088839}{414}\right\}$
54	17	$\left\{0^4, 1, -5, 7, -4, 1, \frac{4269}{23}, \frac{19380}{23}, \frac{57936}{23}, \frac{150163}{23}, \frac{388713}{23}, \frac{1143987}{23}, \frac{4665700}{23}, \frac{48039654}{23}, \frac{942321832}{23}, -\frac{3885024}{23}, -\frac{43575246}{23}, -\frac{105666980678}{2783}\right\}$
54	18	$\left\{0^3, 1, -7, 14, -12, 5, \frac{453389}{2185}, \frac{1523136}{2185}, \frac{385968}{247}, \frac{16988433}{5681}, \frac{2402785}{437}, \frac{23268102}{2185}, \frac{52825188}{2185}, \frac{33833332}{437}, \frac{13043524332}{39767}\right\}$
56	18	$\left\{0^4, 1, -5, 7, -4, 1, \frac{17991}{115}, \frac{14994}{23}, \frac{40392}{23}, \frac{91683}{23}, \frac{198653}{23}, \frac{2260843}{115}, \frac{1206608}{23}, \frac{4553926}{23}, \frac{47612189}{46}\right\}$

6. Conclusions

Using symbolic computer packages, a computational analysis of rank-one solvable Lie algebras with a filiform nilradical and an eigenvalue spectrum for the torus $\mathfrak{t}$, given by $\mathrm{spec}(\mathfrak{t}) = (1, k, k+1, \ldots, n+k-2)$, and subjected to the additional constraint $C_{2,3}^{k+3} = 0$, has been carried out. This was performed according to the algorithmic procedure proposed in [35], and extends the results of that work to arbitrary values of k. It turns out that this analysis is far more complicated than that corresponding to the case of $C_{2,3}^{k+3} \neq 0$, leading to a continuous series of cohomologically rigid Lie algebras, as well as to an increasing number of algebras that only appear for certain values of k and the dimension, n, of the nilradical. A relevant albeit expected fact is that for the constraints, $C_{2,3}^{k+3} = C_{2,n-k}^{n} = 0$, only one rigid Lie algebra appears from a certain dimension onwards, the remaining solutions to the quadratic equations being parametrised families that can be deformed non-trivially.

In particular, for dimensions $n > m(k)$, no geometrically rigid subjected to the conditions $C_{2,3}^{k+3} = C_{2,n-k}^{n} = 0$ exist. This case can thus be considered as completely classified. On the contrary, for $C_{2,n-k}^{n} \neq 0$, we only obtain discrete solutions, i.e, rigid algebras that appear merely for certain values of k and n and only within a certain range of dimensions. For given values of k and n, several structurally different Lie algebras appear, which makes the search of generating functions a hopeless task, at least for the general description. A full classification of this case is probably impossible, as for increasing values of k new types of special solutions that do not follow any of the patterns determined by the factor sequences of the type (37)–(40) appear, implying that new factor sequences have to be defined. Albeit the number of such sequences is surely infinite, the analysis could be refined, analysing values of the constants α_j, β_j for which the rational functions

$$\frac{C_{2,i+s}^{i+k+s}}{C_{2,i+s+1}^{i+k+s+1}} = \frac{\alpha_1 i + \alpha_2 s + \alpha_3}{\beta_1 i + \beta_2 s + \beta_3},\ s \geq 0 \tag{51}$$

provide isolated solutions to the quadratic Jacobi conditions (20). It is still unclear whether all admissible factor sequences of this type can be described globally in terms of some kind of generating function.

We restricted the preceding discussion to the generic shape of cohomologically rigid Lie algebras. In particular, we have not considered the nonrational (and even purely complex) solutions to the Jacobi equations, which are also known to exist and provide rigid Lie algebras [20]. On the other hand, as expected, there are also various (families of) rigid algebras with varying cohomology, the dimension of which depends on the parameters k and s. As follows from the computational analysis, all these geometrically rigid solutions are subjected to the following two constraints in common:

$$C_{2,3}^{k+3} = 0, \quad C_{2,n-k}^{n} \neq 0. \tag{52}$$

However, for the algebras having nonzero cohomology the quadratic Rim map (see, e.g., [8,10] and references therein) is not conclusive, so that the rigidity has to be established by other means, while this is cumbersome but realizable for low values of k, the increasing number of isolated solutions with nonvanishing cohomology makes a direct approach quite impractical for values $k > 5$. For these cases, the best method is to analyse the nilpotence conditions in the scheme defined by the Jacobi conditions [5]. However, even this ansatz is not devoid of technical difficulties, as the analysis of nilpotent elements is computationally far from being an easy task. The authors are currently developing an algorithmic method in the language MATHEMATICA® in order to systematize the analysis of geometrically rigid Lie algebras corresponding to the eigenvalue spectrum (13), and decide whether the deformation parameters are subjected to some nilpotence condition. We hope to report on some advances in this context in future work.

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