

Article

Almost Contractions under Binary Relations

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Abstract: After the introduction of almost contraction due to Berinde, the branch of metric fixed point theory has attracted much attention in this direction, and various fixed point results have been proved for almost contractions via different approaches. In this paper, the results on existence and uniqueness of fixed points are proved employing almost contraction conditions in the framework of metric space endowed with binary relation. Finally, some examples are constructed, which substantiate the utility of the newly proved results.

Keywords: almost contractions; φ -self-closedness; $\mathcal{F}$ -closedness

MSC: 47H10; 54H25



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1. Introduction and Preliminaries

Metric fixed point theory is a vital and wide part of nonlinear functional analysis. In fact, metric fixed point theory provides various applications in different fields which include ordinary differential equations, partial differential equations, integral equations, functional equations, matrix equations, random differential equations, variational inequalities, eigen value problems, operator equations, optimization theory, approximation theory, fractal theory, control theory, probability theory, potential theory, electrical heating in the Joule–Thomson effect, fluid flow, chemical equations, steady state temperature distribution, neutron transport theory, Nash equilibria, econometrics, economic theory, epidemics, global analysis, physics, statistics, engineering, computer science, biology, chemistry, and several others. There is an extensive literature on this topic, which includes pure as well as applied aspects. Indeed, the area of metric fixed point theory originated in 1922 with the appearance of the Banach Contraction Principle [1] (abbreviated as: BCP). This core result guarantees the existence and uniqueness of a fixed point under the hypothesis that the ambient space remains a complete metric space, whereas the underlying mapping should be a contraction mapping. Moreover, BCP offers a constructive procedure for approximation of the fixed point. Many authors have extended BCP employing relatively more general contractive conditions. One of the remarkable generalizations is due to Berinde [2], which was first referred to as weak contraction and then almost contraction [3].

Definition 1 ([2]). A function $\mathcal{F}$ (from a metric space $(\mathbb{M}, \varrho)$ into itself) is said to be almost contraction if for some $\delta \in (0, 1)$, for some $L \in [0, \infty)$, and for all $r, t \in \mathbb{M}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta \varrho(r, t) + L\varrho(r, \mathcal{F}t).$$

The symmetry of the metric ϱ concludes that the almost contraction condition is equivalent to the following dual one (c.f. [2]):

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta \varrho(r, t) + L\varrho(t, \mathcal{F}r), \quad \forall r, t \in \mathbb{M}.$$

Theorem 1 ([2]). Suppose that $(\mathbb{M}, \varrho)$ remains a complete metric space while $\mathcal{F}$ is an almost contraction self-mapping on $\mathbb{M}$. Then, $\mathcal{F}$ admits a fixed point.

Theorem 1 subsumes the earlier classical fixed point theorems due to Banach [1], Kannan [4], Chatterjea [5], and Zamfirescu [6]. The class of almost contractions also covers a certain class of quasi-contractions due to Ćirić [7]. Here, it can be highlighted that the almost contraction mapping need not admit a unique fixed point. However, the fixed points of any almost contraction mapping can be computed by the convergence of the sequence of Picard iteration. By slightly modifying the almost contraction condition, Berinde [2] obtained the following uniqueness result:

Theorem 2 ([2]). Suppose that $(\mathbb{M}, \varrho)$ remains a complete metric space while $\mathcal{F}$ is self-mapping on $\mathbb{M}$. If for some $\delta \in (0, 1)$, for some $L \in [0, \infty)$, and for all $r, t \in \mathbb{M}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta \varrho(r, t) + L\varrho(r, \mathcal{F}r),$$

then $\mathcal{F}$ admits a unique fixed point.

For further study on almost contractions, we refer to [8–15]. On the other hand, fixed point theorems in ordered metric space have occupied a significant chapter in metric fixed point theory, especially due to the works of Ran and Reurings [16] and Nieto and Rodríguez-López [17]. The results in ordered metric spaces mainly weaken the contractive condition by employing the fact that the condition needs to hold only for those elements of the space that are related by the underlying partial order. In 2015, Alam and Imdad [18] established yet another novel variant of BCP employing an amorphous (arbitrary) binary relation. In recent years, the fixed point results of Alam and Imdad [18] were further generalized and extended by many authors (e.g., [19–26]).

The aim of this paper is to prove the fixed point theorems in the context of metric space equipped with a binary relation employing a relation-preserving almost contractivity condition. These results are indeed relation-theoretic versions of Theorem 1 and Theorem 2. Several examples are furnished in support of the newly proved results, which clearly show the important improvements of the existing results in the literature.

2. Relation-Theoretic Notions

This section is comprised of some definitions which are needed in the subsequent discussion. As usual, $\mathbb{R}$ will stand for the set of real numbers, $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}_0$ for the set of whole numbers. Recall that a binary relation (often called a relation) on a set $\mathbb{M}$ remains a subset $\mathfrak{R}$ of $\mathbb{M}^2$. For simplicity, we sometimes write $r\mathfrak{R}t$ instead of $(r, t) \in \mathfrak{R}$, e.g., in the case of the relations of “less than or equal to” ($\mathfrak{R} := \leq$) and “greater than or equal to” ($\mathfrak{R} := \geq$) on $\mathbb{R}$ is expressed respectively as: $r \leq t$ and $r \geq t$.

Definition 2 ([27]). Given a relation $\mathfrak{R}$ on a set $\mathbb{M}$, the set

$$\mathfrak{R}^{-1} := \{(r, t) \in \mathbb{M}^2 : (t, r) \in \mathfrak{R}\}$$

remains again a relation on $\mathbb{M}$, which is termed as an inverse relation of $\mathfrak{R}$. Moreover, the set

$$\mathfrak{R}^s := \mathfrak{R} \cup \mathfrak{R}^{-1}$$

forms again a relation on $\mathbb{M}$, which is called the symmetric closure of $\mathfrak{R}$. Clearly, $\mathfrak{R}^s$ remains the least symmetric relation on $\mathbb{M}$ among those binary relations that contain $\mathfrak{R}$.

Definition 3 ([18]). Any two elements r and t of a set $\mathbb{M}$ are called $\mathfrak{R}$ -comparative, whereas $\mathfrak{R}$ remains a relation on $\mathbb{M}$ if either $(r, t) \in \mathfrak{R}$ or $(t, r) \in \mathfrak{R}$. Usually, $[r, t] \in \mathfrak{R}$ means that “ r and t are $\mathfrak{R}$ -comparative”.

Proposition 1 ([18]). If $\mathfrak{R}$ remains a relation on $\mathbb{M}$, then

$$(r, t) \in \mathfrak{R}^s \iff [r, t] \in \mathfrak{R}.$$

Definition 4 ([27]). If $\mathfrak{R}$ remains a relation on a set $\mathbb{M}$, then $\mathfrak{R}$ is said to be complete if each pair of elements of $\mathbb{M}$ is $\mathfrak{R}$ -comparative, i.e.,

$$[r, t] \in \mathfrak{R}, \quad \forall r, t \in \mathbb{M}.$$

Definition 5 ([27]). If $\mathbb{E} \subseteq \mathbb{M}$ and $\mathfrak{R}$ remains a relation on $\mathbb{M}$, then the set

$$\mathfrak{R}|_{\mathbb{E}} := \mathfrak{R} \cap \mathbb{E}^2$$

induces a relation on $\mathbb{E}$, which is termed as the restriction of $\mathfrak{R}$ on $\mathbb{E}$.

Definition 6 ([18]). By an $\mathfrak{R}$ -preserving sequence, where $\mathfrak{R}$ remains a relation on a set $\mathbb{M}$, we mean the sequence $\{\vartheta_n\} \subset \mathbb{M}$, which satisfies

$$(\vartheta_n, \vartheta_{n+1}) \in \mathfrak{R}, \quad \forall n \in \mathbb{N}_0.$$

Definition 7 ([18]). Given any set $\mathbb{M}$ and a mapping $\mathcal{F} : \mathbb{M} \rightarrow \mathbb{M}$, we say that a relation $\mathfrak{R}$ on $\mathbb{M}$ is $\mathcal{F}$ -closed if for any $r, t \in \mathbb{M}$,

$$(r, t) \in \mathfrak{R} \Rightarrow (\mathcal{F}r, \mathcal{F}t) \in \mathfrak{R}.$$

Proposition 2 ([19]). Suppose that $\mathbb{M}$ is a set so that $\mathfrak{R}$ remains a relation on $\mathbb{M}$ while $\mathcal{F}$ is a function from $\mathbb{M}$ into itself. Then, the $\mathcal{F}$ -closedness of $\mathfrak{R}$ implies the $\mathcal{F}$ -closedness of $\mathfrak{R}^s$.

Proposition 3 ([20]). Suppose that $\mathbb{M}$ is a set so that $\mathfrak{R}$ remains a relation on $\mathbb{M}$ while $\mathcal{F}$ is a function from $\mathbb{M}$ into itself. Then, the $\mathcal{F}$ -closedness of $\mathfrak{R}$ implies the $\mathcal{F}^n$ -closedness of $\mathfrak{R}$, where $n \in \mathbb{N}_0$.

Definition 8 ([19]). A metric space $(\mathbb{M}, \varrho)$ is termed as $\mathfrak{R}$ -complete (whereas $\mathfrak{R}$ remains a relation on $\mathbb{M}$) if each Cauchy $\mathfrak{R}$ -preserving sequence in $\mathbb{M}$ converges.

Obviously, for any metric space completeness implies $\mathfrak{R}$ -completeness whatever the relation $\mathfrak{R}$. In particular, if $\mathfrak{R}$ remains the universal relation, then the concepts of $\mathfrak{R}$ -completeness and usual completeness are equivalent.

Definition 9 ([19]). A function $\mathcal{F}$ (from a metric space $(\mathbb{M}, \varrho)$ into itself) is termed as $\mathfrak{R}$ -continuous (whereas $\mathfrak{R}$ remains a relation on $\mathbb{M}$) at a point $r \in \mathbb{M}$ if

$$\mathcal{F}(\vartheta_n) \xrightarrow{\varrho} \mathcal{F}(r)$$

for any sequence $\{\vartheta_n\}$ which is $\mathfrak{R}$ -preserving and converges to r . Naturally, $\mathfrak{R}$ -continuity of $\mathcal{F}$ means $\mathfrak{R}$ -continuity of $\mathcal{F}$ at every point of $\mathbb{M}$.

Obviously, for any mapping on a metric space endowed with the relation $\mathfrak{R}$, continuity implies $\mathfrak{R}$ -continuity. In particular, if $\mathfrak{R}$ remains the universal relation, then the concepts of $\mathfrak{R}$ -continuity and usual continuity are equivalent.

Definition 10 ([18]). On a metric space $(\mathbb{M}, \varrho)$, an underlying relation $\mathfrak{R}$ is termed as ϱ -self-closed if each sequence $\{\vartheta_n\}$ (which is $\mathfrak{R}$ -preserving and convergent such that $\vartheta_n \xrightarrow{\varrho} r$) has a subsequence $\{\vartheta_{n_k}\}$ whose terms are $\mathfrak{R}$ -comparative with r , i.e.,

$$[\vartheta_{n_k}, r] \in \mathfrak{R}, \quad \forall k \in \mathbb{N}_0.$$

In the sequel, for a metric space $(\mathbb{M}, \varrho)$, a relation $\mathfrak{R}$ and a function $\mathcal{F} : \mathbb{M} \rightarrow \mathbb{M}$, the following notations will be adopted.

- (i) $F(\mathcal{F}) :=$ the collection of the fixed points of $\mathcal{F}$,
- (ii) $\mathbb{M}(\mathcal{F}, \mathfrak{R}) := \{r \in \mathbb{M} : (r, \mathcal{F}r) \in \mathfrak{R}\}$.

3. Main Results

In the following lines, a fixed point theorem for almost contraction employing a binary relation is presented.

Theorem 3. Let $(\mathbb{M}, \varrho)$ be a metric space equipped with a binary relation $\mathfrak{R}$. Let $\mathcal{F}$ be a self-mapping on $\mathbb{M}$. Moreover, assume that

- (i) $(\mathbb{M}, \varrho)$ is $\mathfrak{R}$ -complete,
- (ii) $\mathfrak{R}$ is $\mathcal{F}$ -closed,
- (iii) $\mathcal{F}$ is $\mathfrak{R}$ -continuous or $\mathfrak{R}$ is ϱ -self-closed,
- (iv) $\mathbb{M}(\mathcal{F}, \mathfrak{R}) \neq \emptyset$,
- (v) for some $\delta \in (0, 1)$ and $L \in [0, \infty)$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta \varrho(r, t) + L\varrho(t, \mathcal{F}r).$$

Then, $\mathcal{F}$ admits a fixed point.

Proof. In view of assumption (iv), we can choose $\vartheta_0 \in \mathbb{M}(\mathcal{F}, \mathfrak{R})$. We define a sequence $\{\vartheta_n\}$ of Picard iteration based at the initial point ϑ_0 , so that

$$\vartheta_n = \mathcal{F}^n(\vartheta_0) = \mathcal{F}(\vartheta_{n-1}), \quad \forall n \in \mathbb{N}_0. \quad (1)$$

As $(\vartheta_0, \mathcal{F}\vartheta_0) \in \mathfrak{R}$, the $\mathcal{F}$ -closedness of $\mathfrak{R}$ together with Proposition 3 yields that

$$(\mathcal{F}^n \vartheta_0, \mathcal{F}^{n+1} \vartheta_0) \in \mathfrak{R}$$

so that

$$(\vartheta_n, \vartheta_{n+1}) \in \mathfrak{R}, \quad \forall n \in \mathbb{N}_0. \quad (2)$$

It follows that the sequence $\{\vartheta_n\}$ is $\mathfrak{R}$ -preserving.

We denote $\varrho_n := \varrho(\vartheta_{n+1}, \vartheta_n)$. Applying the contractivity condition (v) and using (1) and (2), we obtain for all $n \in \mathbb{N}_0$ that

$$\begin{aligned} \varrho_n &= \varrho(\vartheta_n, \vartheta_{n+1}) = \varrho(\mathcal{F}\vartheta_{n-1}, \mathcal{F}\vartheta_n) \\ &\leq \delta \varrho(\vartheta_{n-1}, \vartheta_n) + L\varrho(\vartheta_n, \mathcal{F}\vartheta_{n-1}), \end{aligned}$$

which in view of (1) reduces to

$$\varrho_n \leq \delta \varrho_{n-1}. \quad (3)$$

By induction, (3) reduces to

$$\varrho_n \leq \delta \varrho_{n-1} \leq \delta^2 \varrho_{n-2} \leq \cdots \leq \delta^n \varrho_0$$

so that

$$\varrho_n \leq \delta^n \varrho_0, \quad \forall n \in \mathbb{N}. \quad (4)$$

For $n < m$, using (4), we obtain

$$\begin{aligned} \varrho(\vartheta_n, \vartheta_m) &\leq \varrho(\vartheta_n, \vartheta_{n+1}) + \varrho(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots + \varrho(\vartheta_{m-1}, \vartheta_m) \\ &\leq (\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})\varrho_0 \\ &= \delta^n(1 + \delta + \delta^2 + \cdots + \delta^{m-n-1})\varrho_0 \\ &< \frac{\delta^n}{1 - \delta}\varrho_0 \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty. \end{aligned}$$

It follows that $\{\vartheta_n\}$ is a Cauchy sequence. Hence, $\{\vartheta_n\}$ is an $\mathfrak{R}$ -preserving Cauchy sequence. Due to the $\mathfrak{R}$ -completeness of $(\mathbb{M}, \varrho)$, one can find $r \in \mathbb{M}$ satisfying

$$\vartheta_n \xrightarrow{\varrho} r.$$

Now, we use assumption (iii) to show that r is a fixed point of $\mathcal{F}$. Suppose that $\mathcal{F}$ is $\mathfrak{R}$ -continuous. Since $\{\vartheta_n\}$ is $\mathfrak{R}$ -preserving with $\vartheta_n \xrightarrow{\varrho} r$, therefore, the $\mathfrak{R}$ -continuity of $\mathcal{F}$ asserts that $\vartheta_{n+1} = \mathcal{F}(\vartheta_n) \xrightarrow{\varrho} \mathcal{F}(r)$. Using the uniqueness of the limit, we obtain $\mathcal{F}(r) = r$ so that r is a fixed point of $\mathcal{F}$. Alternately, assume that $\mathfrak{R}$ is ϱ -self-closed. Since $\{\vartheta_n\}$ is an $\mathfrak{R}$ -preserving sequence such that $\vartheta_n \xrightarrow{\varrho} r$, therefore, by the ϱ -self-closedness of $\mathfrak{R}$, one can find a subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$ with $[\vartheta_{n_k}, r] \in \mathfrak{R}$ for all $k \in \mathbb{N}_0$. Obviously, we have

$$\vartheta_{n_k} \xrightarrow{\varrho} r. \quad (5)$$

On using $[\vartheta_{n_k}, r] \in \mathfrak{R}$, symmetry of ϱ , assumption (e), and (5), we have

$$\begin{aligned} \varrho(\vartheta_{n_k+1}, \mathcal{F}r) &= \varrho(\mathcal{F}\vartheta_{n_k}, \mathcal{F}r) \\ &\leq \delta\varrho(\vartheta_{n_k}, r) + L\varrho(r, \mathcal{F}\vartheta_{n_k}) \\ &= \delta\varrho(\vartheta_{n_k}, r) + L\varrho(r, \vartheta_{n_k+1}) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

so that $\vartheta_{n_k+1} \xrightarrow{\varrho} \mathcal{F}(r)$. By the uniqueness of limit, we obtain $\mathcal{F}(r) = r$. Thus, r remains a fixed point of $\mathcal{F}$. $\square$

Now, a result on uniqueness of fixed points is proved by making a slight modification in almost contractivity conditions in addition to using an additional hypothesis.

Theorem 4. Under the hypotheses (i)–(iv) of Theorem 3, if the following assumptions hold:

(vi) for some $\delta \in (0, 1)$ and $L \in [0, \infty)$ and for all $r, t \in \mathbb{M}$ with $(r, t) \in \mathfrak{R}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta\varrho(r, t) + L\varrho(r, \mathcal{F}r),$$

(vii) $\mathfrak{R}|_{\mathcal{F}(\mathbb{M})}$ is complete,

then, $\mathcal{F}$ has a unique fixed point.

Proof. Proceeding along the lines similar to the proof of Theorem 3, we can find that $F(\mathcal{F}) \neq \emptyset$. We take $r_1, r_2 \in F(\mathcal{F})$ so that

$$\mathcal{F}(r_1) = r_1 \text{ and } \mathcal{F}(r_2) = r_2. \quad (6)$$

Clearly, $r_1, r_2 \in \mathcal{F}(\mathbb{M})$. Hence, by assumption (vii), we obtain

$$[r_1, r_2] \in \mathfrak{R}. \quad (7)$$

Making use of (6), (7), and the contractivity condition (vi), we obtain

$$\varrho(r_1, r_2) = \varrho(\mathcal{F}r_1, \mathcal{F}r_2) \leq \delta\varrho(r_1, r_2) + L\varrho(r_1, \mathcal{F}r_1). \quad (8)$$

Using (6) again, we obtain

$$\varrho(r_1, \mathcal{F}r_1) = \varrho(r_1, r_1) = 0.$$

Therefore, Equation (8) reduces to

$$\varrho(r_1, r_2) \leq \delta\varrho(r_1, r_2)$$

so that

$$(1 - \delta)\varrho(r_1, r_2) \leq 0.$$

However, due to $1 - \delta > 0$, the above inequality yields $\varrho(r_1, r_2) = 0$, i.e., $r_1 = r_2$. This concludes that the fixed point of $\mathcal{F}$ is unique. $\square$

Remark 1. Under the restriction $L = 0$, Theorem 4 reduces to the classical relation-theoretic contraction principle [18]. Therefore, our main results (i.e., Theorems 3 and 4) extend and improve the results of Alam and Imdad [18].

4. Examples

In this section, we furnish examples, which demonstrate the importance of Theorem 3.

Example 1. Take $\mathbb{M} = [0, 1]$ with usual metric $\varrho(r, t) = |r - t|$. Define a binary relation $\mathfrak{R} = \mathbb{R} \times \mathbb{Q}$. Then, $(\mathbb{M}, \varrho)$ is an $\mathfrak{R}$ -complete metric space. Let $\mathcal{F}$ be the identity mapping on $\mathbb{M}$. Then, $\mathcal{F}$ is $\mathfrak{R}$ -continuous and $\mathfrak{R}$ is $\mathcal{F}$ -closed. Now, for any arbitrary $\delta \in [0, 1)$, $L \geq 1 - \delta$ and for all $r, t \in \mathbb{M}$ satisfying $(r, t) \in \mathfrak{R}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) = \varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta\varrho(r, t) + L\varrho(t, \mathcal{F}r),$$

i.e., $\mathcal{F}$ satisfies assumption (e) of Theorem 3. Thus, all the conditions of Theorem 3 are satisfied. Consequently, $\mathcal{F}$ has a fixed point in $\mathbb{M}$. Indeed, here, $F(\mathcal{F}) = [0, 1]$.

Example 2. Take $\mathbb{M} = [0, 1]$ with usual metric $\varrho(r, t) = |r - t|$. Define a binary relation $\mathfrak{R} = \geq$ (natural ordering). Then, $(\mathbb{M}, \varrho)$ is an $\mathfrak{R}$ -complete metric space. Let $\mathcal{F}$ be a mapping on $\mathbb{M}$ defined by

$$\mathcal{F}(r) = \begin{cases} 0, & \text{if } r = 1 \\ 2/3, & \text{otherwise.} \end{cases}$$

Then, $\mathcal{F}$ is $\mathfrak{R}$ -continuous and $\mathfrak{R}$ is $\mathcal{F}$ -closed. Now, for any $\delta \geq 2/3$, $L \geq \delta$ and for all $r, t \in \mathbb{M}$ satisfying $(r, t) \in \mathfrak{R}$, we have

$$\varrho(\mathcal{F}r, \mathcal{F}t) = \varrho(\mathcal{F}r, \mathcal{F}t) \leq \delta\varrho(r, t) + L\varrho(t, \mathcal{F}r),$$

i.e., $\mathcal{F}$ satisfies assumption (e) of Theorem 3. Thus, all the conditions of Theorem 3 are satisfied. Consequently, $\mathcal{F}$ has a fixed point in $\mathbb{M}$. Indeed, here, $\mathcal{F}$ has a fixed point, namely: $r = 2/3$.

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References

- Banach, S. Sur les operations dans les ensembles abstraits et leur application aux quations intgrales. *Fund. Math.* **1922**, *3*, 133–181. [\[CrossRef\]](#)
- Berinde, V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **2004**, *9*, 43–53.
- Berinde, V.; Păcurar, M. Fixed points and continuity of almost contractions. *Fixed Point Theory* **2008**, *9*, 23–34.
- Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
- Chatterjea, S.K. Fixed point theorem. *C. R. Acad. Bulg. Sci.* **1972**, *25*, 727–730. [\[CrossRef\]](#)
- Zamfirescu, T. Fix point theorems in metric spaces. *Arch. Math.* **1972**, *23*, 292–298. [\[CrossRef\]](#)
- Ćirić, L.B. A generalization of Banach’s contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [\[CrossRef\]](#)
- Berinde, V.; Takens, F. *Iterative Approximation of Fixed Points*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 1912.
- Berinde, M.; Berinde, V. On a general class of multi-valued weakly Picard mappings. *J. Math. Anal. Appl.* **2007**, *326*, 772–782. [\[CrossRef\]](#)
- Berinde, V. General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces. *Carpathian J. Math.* **2008**, *24*, 10–19.
- Alghamdi, M.A.; Berinde, V.; Shahzad, N. Fixed points of non-self almost contractions. *Carpathian J. Math.* **2014**, *30*, 7–14. [\[CrossRef\]](#)
- Altun, I.; Acar, Ö. Fixed point theorems for weak contractions in the sense of Berinde on partial metric spaces. *Topol. Appl.* **2012**, *159*, 2642–2648. [\[CrossRef\]](#)
- Berinde, V.; Vetro, F. Common fixed points of mappings satisfying implicit contractive conditions. *Fixed Point Theory Appl.* **2012**, *2012*, 105. [\[CrossRef\]](#)
- Babu, G.V.R.; Sandhy, M.L.; Kameshwari, M.V.R. A note on a fixed point theorem of Berinde on weak contractions. *Carpathian J. Math.* **2008**, *24*, 8–12.
- Berinde, V. Stability of Picard iteration for contractive mappings satisfying an implicit relation. *Carpathian J. Math.* **2011**, *27*, 13–23. [\[CrossRef\]](#)
- Ran, A.C.M.; Reurings, M.C.B. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [\[CrossRef\]](#)
- Nieto, J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Relation-theoretic metrical coincidence theorems. *Filomat* **2017**, *31*, 4421–4439. [\[CrossRef\]](#)
- Alam, A.; Imdad, M. Nonlinear contractions in metric spaces under locally T -transitive binary relations. *Fixed Point Theory* **2018**, *19*, 13–24. [\[CrossRef\]](#)
- Alam, A.; Arif, M.; Imdad, M. Metrical fixed point theorems under locally finitely T -transitive binary relations using comparison functions. *Miskolc Math. Notes* **2019**, *20*, 59–73. [\[CrossRef\]](#)
- Alam, A.; Imdad, M.; Arif, M. Observations on relation-theoretic coincidence theorems under Boyd—Wong type nonlinear contractions. *Fixed Point Theory Appl.* **2019**, *2019*, 6. [\[CrossRef\]](#)
- Arif, M.; Khan, I.A.; Imdad, M.; Alam, A. Employing locally finitely T -transitive binary relations to prove coincidence theorems for nonlinear contractions. *J. Funct. Spaces* **2020**, *2020*, 6574695. [\[CrossRef\]](#)
- Arif, M.; Imdad, M.; Alam, A. Fixed point theorems under locally T -transitive binary relations employing Matkowski contractions. *Miskolc Math. Notes* **2022**, *23*, 71–83. [\[CrossRef\]](#)
- Sk, F.; Khan, F.A.; Khan, Q.H.; Alam, A. Relation-preserving generalized nonlinear contractions and related fixed point theorems. *AIMS Math.* **2022**, *7*, 6634–6649. [\[CrossRef\]](#)
- Khan, F.A.; Sk, F.; Alshehri, M.G.; Khan, Q.H.; Alam, A. Relational Meir-Keeler contractions and common fixed point theorems. *Jour. Func. Spaces* **2022**, *2022*, 3550923. [\[CrossRef\]](#)
- Lipschutz, S. *Schaum’s Outlines of Theory and Problems of Set Theory and Related Topics*; McGraw-Hill: New York, NY, USA, 1964.