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Article

Comparison of Overlap and Grouping Functions

Songsong Dai

School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China; ssdai@tzc.edu.cn

Abstract: This paper investigates the pointwise comparability of overlap and grouping functions which obtained by Bustince et al.'s and Bedregal et al.'s generator pairs, respectively. Some necessary and sufficient conditions for the comparison of these functions are proved. We also introduce some compositions of these functions and study the order preservation of these compositions.

Keywords: overlap functions; grouping functions; comparison; composition; order preservation

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1. Introduction

Overlap function introduced by Bustince et al. [1] is a particular type of aggregation function [2]. Its dual concept is the grouping function [3]. In recent years, those two concepts have attracted a wide range of interests. For applications, they have been successfully applied to many domains, such as image processing [1,4], classification [5,6] and decision making [7,8]. For theoretical research, general overlap and grouping functions [9,10], N-dimensional overlap functions [11], Archimedean overlap functions [12], general interval-valued overlap functions [13], complex-valued overlap and grouping functions [14,15], quasi-homogeneous overlap functions [16], pseudo-homogeneous overlap and grouping functions [17], overlap functions on bounded lattices [18], overlap and grouping functions on complete lattices [19] have been introduced. Many fuzzy concepts derived from overlap and grouping functions, such as generalized interval-valued OWA operators [20], residual implications [21,22], (G,N)-implications [23], binary relations [24], (IO, O)-fuzzy rough sets [25] and so on.

In the study of overlap and grouping functions, the study of their properties accounts for a large proportion and play an important role. Bustince et al. [1] gave an alternative characterization of overlap functions by their generator pairs. Bedregal et al. [26] gave an alternative characterization of grouping functions in a similar way. Dimuro et al. [27] introduced the additive generators of overlap and grouping functions. Qiao and Hu [28] studied the interval additive generators of interval overlap and grouping functions. They [29] also introduced the multiplicative generators of overlap and grouping functions.

We have already known that there is a partial order between two t-norms T_1 and T_2, i.e., $T_1 \leq T_2$ if $T_1(a,b) \leq T_2(a,b)$ for all $(a,b) \in [0,1]^2$ (see [30], Chapter 6). Klement et al. [31] presented a necessary and sufficient condition for the comparability of continuous Archimedean t-norms. There also exist some pointwise comparison results of fuzzy implications (see [32], Chapter 1). However, comparatively little investigation has been made on the comparability of overlap/grouping functions. Bustince et al. [1] defined the pointwise order of two overlap functions O_1 and O_2, i.e., $O_1 \leq O_2$ if $O_1(a,b) \leq O_2(a,b)$ for all $(a,b) \in [0,1]^2$. Bedregal et al. [26] defined the pointwise order of two grouping functions in a similar way. Dai et al. [33] showed that the meet operation, join operation, convex combination, and $\circledast$-composition of overlap and grouping functions are order preserving. But the research on the pointwise comparability of overlap and grouping functions have not been studied in details. Therefore, in this paper, we study the pointwise comparability of overlap and grouping functions involving Bustince et al. [1] and Bedregal et al. [26] generators. We present some necessary and sufficient conditions for their comparability. We also investigate order preservation of some compositions of overlap and grouping functions.

The paper is organized as follows: In Section 2, we recall the concepts of overlap/grouping functions and their order relationship. In Section 3, we study the pointwise comparability of overlap functions involving Bustince et al. [1] generators. In Section 4, we study the pointwise comparability of grouping functions involving Bedregal et al. [26] generators. In Section 5, we introduce some compositions of overlap/grouping functions and study properties preservation of these compositions. In Section 6, our researches are concluded.

2. Preliminaries

2.1. Overlap and Grouping Functions

In this section, we recall the basic theory of overlap and grouping functions. More details can be found in [1,11,26,28].

Definition 1 ([1]). *A bivariate function $O : [0,1]^2 \to [0,1]$ is a overlap function if, for any $a, b \in [0,1]$, it has the following properties:*

(O1) *O is commutative;*
(O2) *$O(a,b) = 0$ if and only if $ab = 0$;*
(O3) *$O(a,b) = 1$ if and only if $ab = 1$;*
(O4) *O is non-decreasing;*
(O5) *O is continuous.*

Definition 2 ([3]). *A bivariate function $G : [0,1]^2 \to [0,1]$ is a grouping function if, for any $a, b \in [0,1]$, it has the following properties:*

(G1) *G is commutative;*
(G2) *$G(a,b) = 0$ if and only if $a = b = 0$;*
(G3) *$G(a,b) = 1$ if and only if $a = 1$ or $b = 1$.*
(G4) *G is non-decreasing;*
(G5) *G is continuous.*

Denote by $\mathcal{O}$ the set of all overlap functions, and $\mathcal{G}$ the set of all grouping functions.

Let O be an overlap function, the dual grouping function of O is defined as $G_O(a,b) = 1 - O(1-a, 1-b)$.

Example 1 ([1,26]). *The following are typical examples of overlap and grouping functions, where $p > 0$,*

- $O_{nm}(a,b) = \min(a,b)\max(a^2, b^2)$;
- $O_p(a,b) = a^p b^p$;
- $O_{mp}(a,b) = \min(a^p, b^p)$;
- $O_{Mp}(a,b) = 1 - \max((1-a)^p, (1-b)^p)$;
- $O_{DB}(a,b) = \begin{cases} \frac{2ab}{a+b}, & if\ a+b \neq 0, \\ 0, & if\ a+b = 0. \end{cases}$
- $G_{nm}(a,b) = 1 - \min(1-a, 1-b)\max((1-a)^2, (1-b)^2)$;
- $G_p(a,b) = 1 - (1-a)^p(1-b)^p$;
- $G_{mp}(a,b) = 1 - \min((1-a)^p, (1-b)^p)$;
- $G_{Mp}(a,b) = \max(a^p, b^p)$;
- $G_{DB}(a,b) = \begin{cases} \frac{a+b-2ab}{2-a-b}, & if\ a \neq 1\ or\ b \neq 1, \\ 1, & if\ a = b = 1. \end{cases}$

2.2. Orders of Overlap and Grouping Functions

Bustince et al. [1] and Bedregal et al. [26] introduced the following partial order for overlap and grouping functions, respectively.

Definition 3 ([1,26])**.** *Let $f_1, f_2 \in \mathcal{O}$ (or both $f_1, f_2 \in \mathcal{G}$),*

(i) *we say that f_1 is weaker than f_2, denote $f_1 \preceq f_2$, if $f_1(a,b) \leq f_2(a,b)$ holds for all $(a,b) \in [0,1]^2$.*
(ii) *we write $f_1 \prec f_2$ if $f_1 \preceq f_2$ and $f_1 \neq f_2$.*

Proposition 1. *Let O_1 and O_2 be two overlap functions, if $O_1 \preceq O_2$, then $G_{O_2} \preceq G_{O_1}$, where G_{O_1} and G_{O_2} are the dual grouping functions of O_1 and O_2, respectively.*

Proof. First $O_1 \preceq O_2$ means $O_1(a,b) \leq O_2(a,b)$ holds for all $(a,b) \in [0,1]^2$. Then $O_1(1-a, 1-b) \leq O_2(1-a, 1-b)$ holds for all $(a,b) \in [0,1]^2$.

Afterwards we have $1 - O_1(1-a, 1-b) \geq 1 - O_2(1-a, 1-b)$ holds for all $(a,b) \in [0,1]^2$. Thus $G_{O_2} \preceq G_{O_1}$, i.e., $G_{O_2}(a,b) \leq G_{O_1}(a,b)$ holds for all $(a,b) \in [0,1]^2$. □

Example 2. *Consider the overlap and grouping functions in Example 1, we have*

- $O_{nm} \preceq O_{mp}$, *where* $0 < p \leq 1$;
- $O_{mp} \preceq O_{nm}$, *where* $p \geq 3$;
- $O_p \preceq O_{mp}$;
- $O_p \preceq O_{DB}$, *where* $p \geq 1$;
- $G_{mp} \preceq G_{nm}$, *where* $0 < p \leq 1$;
- $G_{nm} \preceq G_{mp}$, *where* $p \geq 3$;
- $G_{mp} \preceq G_p$;
- $G_{DB} \preceq G_p$, *where* $p \geq 1$.

Remark 1. *$\preceq$ is a partial order, but not a linear order. For example, consider the O_{mp} with $p = 2$ and O_{nm}, $O_{mp}(a,b) = \min(a^2, b^2)$ and O_{nm} are incomparable since $O_{mp}(1, \frac{1}{2}) = \frac{1}{4} < O_{nm}(1, \frac{1}{2}) = \frac{1}{2}$ and $O_{mp}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} > O_{nm}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$.*

3. Comparison of Overlap Functions

Bustince et al. [1] gave an alternative characterization of overlap functions.

Theorem 1 ([1])**.** *The bivariate function $O_{fg} : [0,1]^2 \to [0,1]$ is an overlap function if and only if*

$$O_{fg}(a,b) = \frac{f(a,b)}{f(a,b) + g(a,b)} \tag{1}$$

for some $f, g : [0,1]^2 \to [0,1]$ satisfying the following conditions

(F1) *f and g are symmetric;*
(F2) *f is non decreasing and g is non increasing;*
(F3) *$f(a,b) = 0$ if and only if $ab = 0$;*
(F4) *$g(a,b) = 0$ if and only if $ab = 1$;*
(F5) *f and g are continuous functions.*

For any overlap function O_{fg} characterized by Equation (1), (f, g) is said to be the generator pair of O_{fg}.

We give the following necessary and sufficient condition for the comparison of overlap functions characterized by different generator pairs.

Theorem 2. *Let $O_{f_1g_1}$ and $O_{f_2g_2}$ be two overlap functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $O_{f_1g_1} \preceq O_{f_2g_2}$ if and only if $f_1g_2 \leq f_2g_1$, i.e., for all $a, b \in [0,1]$,*

$$f_1(a,b)g_2(a,b) \leq f_2(a,b)g_1(a,b). \tag{2}$$

Proof. ($\Rightarrow$) From the definition of the generator pair in Equation (1), if $O_{f_1g_1} \preceq O_{f_2g_2}$, then for all $a, b \in [0,1]$, by Definition 3,

$$\frac{f_1(a,b)}{f_1(a,b)+g_1(a,b)} \leq \frac{f_2(a,b)}{f_2(a,b)+g_2(a,b)}.$$

From Theorem 1 (F3) and (F4), we have $f_1(a,b)+g_1(a,b) > 0$ and $f_2(a,b)+g_2(a,b) > 0$ for all $a, b \in [0,1]$, and we have

$$f_1(a,b)[f_2(a,b)+g_2(a,b)] \leq f_2(a,b)[f_1(a,b)+g_1(a,b)].$$

Then for all $a, b \in [0,1]$, it holds that

$$f_1(a,b)g_2(a,b) \leq f_2(a,b)g_1(a,b)$$

Thus $f_1g_2 \leq f_2g_1$.
($\Leftarrow$) If $f_1g_2 \leq f_2g_1$, i.e., for all $a, b \in [0,1]$, it holds that

$$f_1(a,b)g_2(a,b) \leq f_2(a,b)g_1(a,b).$$

By adding $f_1(a,b)f_2(a,b)$ in both sides of this inequality, we obtain

$$f_1(a,b)f_2(a,b) + f_1(a,b)g_2(a,b) \leq f_1(a,b)f_2(a,b) + f_2(a,b)g_1(a,b),$$

i.e.,

$$f_1(a,b)[f_2(a,b)+g_2(a,b)] \leq f_2(a,b)[f_1(a,b)+g_1(a,b)].$$

From $f_1(a,b)+g_1(a,b) > 0$ and $f_2(a,b)+g_2(a,b) > 0$ for all $a, b \in [0,1]$, one has that $[f_2(a,b)+g_2(a,b)][f_1(a,b)+g_1(a,b)] > 0$ for all $a, b \in [0,1]$.
Then by dividing both sides of the equation by $[f_2(a,b)+g_2(a,b)][f_1(a,b)+g_1(a,b)]$, we get for all $a, b \in [0,1]$

$$\frac{f_1(a,b)}{f_1(a,b)+g_1(a,b)} \leq \frac{f_2(a,b)}{f_2(a,b)+g_2(a,b)}.$$

Thus by Definition 3, $O_{f_1g_1} \preceq O_{f_2g_2}$. $\square$

Corollary 1. *Let $O_{f_1g_1}$ and $O_{f_2g_2}$ be two overlap functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $O_{f_1g_1} \preceq O_{f_2g_2}$ if and only if $\frac{f_1}{f_2} \leq \frac{g_1}{g_2}$, i.e., for all $(a,b) \in (0,1]^2 \backslash \{(1,1)\}$,*

$$\frac{f_1(a,b)}{f_2(a,b)} \leq \frac{g_1(a,b)}{g_2(a,b)}. \tag{3}$$

Corollary 2. *Let O_{f_1g} and O_{f_2g} be two overlap functions with generator pair (f_1, g) and (f_2, g), respectively. If $f_1 \leq f_2$, i.e., $f_1(a,b) \leq f_2(a,b)$ for all $a, b \in [0,1]$. Then $O_{f_1g} \preceq O_{f_2g}$.*

Corollary 3. *Let O_{fg_1} and O_{fg_2} be two overlap functions with generator pair (f, g_1) and (f, g_2), respectively. If $g_1 \leq g_2$, i.e., $g_1(a,b) \leq g_2(a,b)$ for all $a, b \in [0,1]$. Then $O_{fg_2} \preceq O_{fg_1}$.*

Example 3. *Consider the following functions* $f_1, g_1, f_2, g_2 : [0,1]^2 \to [0,1]$, *defined by*

$$f_1(a,b) = \sqrt{ab}, \tag{4}$$
$$f_2(a,b) = a^2b^2, \tag{5}$$
$$g_1(a,b) = 1 - ab, \tag{6}$$
$$g_2(a,b) = \max(1-a, 1-b). \tag{7}$$

Obviously, they satify the conditions of Theorem 1. We also have $f_2 \leq f_1$ *and* $g_2 \leq g_1$. *Then it holds that* $f_2g_2 \leq f_1g_1$.

From Theorem 2, we obtain $O_{f_2g_1} \preceq O_{f_1g_2}$, *i.e.,*

$$\frac{a^2b^2}{a^2b^2 + 1 - ab} \leq \frac{\sqrt{ab}}{\sqrt{ab} + \max(1-a, 1-b)}$$

for all $a, b \in [0,1]$.

Moreover,

$$\frac{f_1(a,b)}{f_2(a,b)} = \frac{\sqrt{ab}}{a^2b^2} = \frac{1}{a^{3/2}b^{3/2}}$$

and

$$\frac{g_1(a,b)}{g_2(a,b)} = \frac{1-ab}{\max(1-a,1-b)}$$

are incomparable since

$$\frac{f_1(0.9,0.9)}{f_2(0.9,0.9)} = \frac{1}{0.9^3} \approx 1.372 < \frac{g_1(0.9,0.9)}{g_2(0.9,0.9)} = 1.9$$

and

$$\frac{f_1(0.1,0.1)}{f_2(0.1,0.1)} = \frac{1}{0.1^3} = 1000 > \frac{g_1(0.1,0.1)}{g_2(0.1,0.1)} = 1.1.$$

Then $O_{f_1g_1}(a,b) = \frac{\sqrt{ab}}{\sqrt{ab}+1-ab}$ *and* $O_{f_2g_2}(a,b) = \frac{a^2b^2}{a^2b^2+\max(1-a,1-b)}$ *are incomparable because of Corollary 1.*

4. Comparison of Grouping Functions

Bedregal et al. [26] gave an alternative characterization of grouping functions.

Theorem 3 ([26])**.** *The bivariate function* $G_{fg} : [0,1]^2 \to [0,1]$ *is a grouping function if and only if*

$$G_{fg}(a,b) = 1 - \frac{f(a,b)}{f(a,b) + g(a,b)} \tag{8}$$

for some $f, g : [0,1]^2 \to [0,1]$ *satisfying the following conditions*

(T1) f *and* g *are symmetric;*
(T2) f *is non increasing and* g *is non decreasing;*
(T3) $f(a,b) = 0$ *if and only if* $a = 1$ *or* $b = 1$;
(T4) $g(a,b) = 0$ *if and only if* $a = b = 0$;
(T5) f *and* g *are continuous functions.*

For any grouping function G_{fg} characterized by Equation (8), (f, g) is said to be the generator pair of G_{fg}.

We give the following necessary and sufficient condition for the comparison of grouping functions characterized by different generator pairs.

Theorem 4. *Let $G_{f_1g_1}$ and $G_{f_2g_2}$ be two grouping functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $G_{f_1g_1} \preceq G_{f_2g_2}$ if and only if $f_2g_1 \leq f_1g_2$, i.e., for all $a, b \in [0,1]$,*

$$f_2(a,b)g_1(a,b) \leq f_1(a,b)g_2(a,b). \tag{9}$$

Proof. ($\Rightarrow$) From the definition of the generator pair in Equation (8), if $G_{f_1g_1} \preceq G_{f_2g_2}$, then for all $a, b \in [0,1]$, by Definition 3,

$$1 - \frac{f_1(a,b)}{f_1(a,b) + g_1(a,b)} \leq 1 - \frac{f_2(a,b)}{f_2(a,b) + g_2(a,b)}.$$

This is

$$\frac{f_2(a,b)}{f_2(a,b) + g_2(a,b)} \leq \frac{f_1(a,b)}{f_1(a,b) + g_1(a,b)}.$$

From Theorem 3 (T3) and (T4), we have $f_1(a,b) + g_1(a,b) > 0$ and $f_2(a,b) + g_2(a,b) > 0$ for all $a, b \in [0,1]$, and we have

$$f_2(a,b)[f_1(a,b) + g_1(a,b)] \leq f_1(a,b)[f_2(a,b) + g_2(a,b)].$$

Then for all $a, b \in [0,1]$, it holds that

$$f_2(a,b)g_1(a,b) \leq f_1(a,b)g_2(a,b).$$

Thus $f_2g_1 \leq f_1g_2$.
($\Leftarrow$) If $f_2g_1 \leq f_1g_2$, i.e., for all $a, b \in [0,1]$, it holds that

$$f_2(a,b)g_1(a,b) \leq f_1(a,b)g_2(a,b).$$

By adding $f_1(a,b)f_2(a,b)$ in both sides of this inequality, we obtain

$$f_1(a,b)f_2(a,b) + f_2(a,b)g_1(a,b) \leq f_1(a,b)f_2(a,b) + f_1(a,b)g_2(a,b),$$

i.e.,

$$f_2(a,b)[f_1(a,b) + g_1(a,b)] \leq f_1(a,b)[f_2(a,b) + g_2(a,b)].$$

From $f_1(a,b) + g_1(a,b) > 0$ and $f_2(a,b) + g_2(a,b) > 0$ for all $a, b \in [0,1]$, one has that $[f_2(a,b) + g_2(a,b)][f_1(a,b) + g_1(a,b)] > 0$ for all $a, b \in [0,1]$.

Then by dividing both sides of the equation by $[f_2(a,b) + g_2(a,b)][f_1(a,b) + g_1(a,b)]$, we get for all $a, b \in [0,1]$

$$\frac{f_2(a,b)}{f_2(a,b) + g_2(a,b)} \leq \frac{f_1(a,b)}{f_1(a,b) + g_1(a,b)}.$$

So we have, for all $a, b \in [0,1]$

$$1 - \frac{f_1(a,b)}{f_1(a,b) + g_1(a,b)} \leq 1 - \frac{f_2(a,b)}{f_2(a,b) + g_2(a,b)}.$$

Thus by Definition 3, $G_{f_1g_1} \preceq G_{f_2g_2}$. □

Corollary 4. *Let $G_{f_1g_1}$ and $G_{f_2g_2}$ be two grouping functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $G_{f_1g_1} \preceq G_{f_2g_2}$ if and only if $\frac{g_1}{g_2} \leq \frac{f_1}{f_2}$, i.e., for all $(a,b) \in [0,1)^2 \backslash \{(0,0)\}$,*

$$\frac{g_1(a,b)}{g_2(a,b)} \leq \frac{f_1(a,b)}{f_2(a,b)}. \tag{10}$$

Corollary 5. *Let G_{f_1g} and G_{f_2g} be two grouping functions with generator pair (f_1, g) and (f_2, g), respectively. If $f_1 \leq f_2$, then $G_{f_2g} \preceq G_{f_1g}$.*

Corollary 6. *Let G_{fg_1} and G_{fg_2} be two grouping functions with generator pair (f, g_1) and (f, g_2), respectively. If $g_1 \leq g_2$, then $G_{fg_1} \preceq G_{fg_2}$.*

Example 4. *Consider the following functions $f_1, g_1, f_2, g_2 : [0,1]^2 \to [0,1]$, defined by*

$$f_1(a,b) = (1-a)(1-b), \tag{11}$$

$$f_2(a,b) = (1-a^2)(1-b^2), \tag{12}$$

$$g_1(a,b) = \frac{a+b}{2}, \tag{13}$$

$$g_2(a,b) = \min(a,b). \tag{14}$$

Obviously, they satify the conditions of Theorem 3. We also have $f_1 \leq f_2$ and $g_2 \leq g_1$. Then it holds that $f_1g_2 \leq f_2g_1$.

From Theorem 4, we obtain $G_{f_2g_2} \preceq G_{f_1g_1}$, i.e.,

$$1 - \frac{(1-a^2)(1-b^2)}{(1-a^2)(1-b^2) + \min(a,b)} \leq 1 - \frac{(1-a)(1-b)}{(1-a)(1-b) + \frac{a+b}{2}}$$

for all $a, b \in [0,1]$.

Moreover, for all $(a,b) \in [0,1)^2 \backslash \{(0,0)\}$

$$\frac{f_1(a,b)}{f_2(a,b)} = \frac{(1-a)(1-b)}{(1-a^2)(1-b^2)} = \frac{1}{(1+a)(1+b)}$$

and

$$\frac{g_2(a,b)}{g_1(a,b)} = \frac{\min(a,b)}{\frac{a+b}{2}} = \frac{2\min(a,b)}{a+b}$$

are incomparable since

$$\frac{f_1(0.25,0.25)}{f_2(0.25,0.25)} = \frac{1}{1.25^2} = 0.64 < \frac{g_2(0.25,0.25)}{g_1(0.25,0.25)} = 1$$

and

$$\frac{f_1(0.1,0.9)}{f_2(0.1,0.9)} = \frac{1}{1.1 * 1.9} \approx 0.4785 > \frac{g_1(0.1,0.9)}{g_2(0.1,0.9)} = 0.2.$$

Then $G_{f_1g_2}(a,b) = 1 - \frac{(1-a)(1-b)}{(1-a)(1-b)+\min(a,b)}$ and $G_{f_2g_1}(a,b) = 1 - \frac{(1-a^2)(1-b^2)}{(1-a^2)(1-b^2)+\frac{a+b}{2}}$ are incomparable because of Corollary 4.

5. Order Preservation of Some Compositions of Overlap and Grouping Functions

In this section, we consider the following problem.

Problem 1. *Whether we have*

$$H_{f_1g_1} \preceq H_{f_2g_2}, H_{f_3g_3} \preceq H_{f_4g_4} \Rightarrow H_{(f_1 \circ_1 f_3)(g_1 \circ_1 g_3)} \preceq H_{(f_2 \circ_2 f_4)(g_2 \circ_2 g_4)} \tag{15}$$

for some operations $\circ_1$ and $\circ_2$ of bivariate functions, where $H_{f_ig_i}$, with $i = 1, ..., 4$ are all overlap functions or all grouping functions?

Let h_1 and h_2 be two bivariate functions, their meet, join and product operations are defined as

$$(h_1 \vee h_2)(a,b) = \max\big(h_1(a,b), h_2(a,b)\big), \tag{16}$$
$$(h_1 \wedge h_2)(a,b) = \min\big(h_1(a,b), h_2(a,b)\big), \tag{17}$$
$$(h_1 \times h_2)(a,b) = h_1(a,b)h_2(a,b), \tag{18}$$

for all $(a,b) \in [0,1]^2$.

First, we prove the closures of the proposed compositions of overlap (or grouping) functions $H_{(f_1 \circ f_2)(g_1 \circ g_2)}$, where $\circ \in \{\vee, \wedge, \times\}$.

Lemma 1. *If $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$ satisfy the following conditions (F1)-(F5) of Theorem 1. Then $(f_1 \vee f_2, g_1 \vee g_2)$, $(f_1 \wedge f_2, g_1 \wedge g_2)$ and $(f_1 \times f_2, g_1 \times g_2)$ also satisfy these conditions.*

Proof. The cases for **(F1)**, **(F2)**, and **(F5)** are straightforward.

(F3) ($\Rightarrow$) If $(f_1 \vee f_2)(a,b) = \max\big(f_1(a,b), f_2(a,b)\big) = 0$. Then, $f_1(a,b) = f_2(a,b) = 0$, thus $ab = 0$.

If $(f_1 \wedge f_2)(a,b) = \min\big(f_1(a,b), f_2(a,b)\big) = 0$. Case I, $f_1(a,b) = 0$ then $ab = 0$. Case II, $f_2(a,b) = 0$ then $ab = 0$.

If $(f_1 \times f_2)(a,b) = f_1(a,b)f_2(a,b) = 0$. Case I, $f_1(a,b) = 0$ then $ab = 0$. Case II, $f_2(a,b) = 0$ then $ab = 0$.

($\Leftarrow$) is straightforward.

(F4) ($\Rightarrow$) If $(g_1 \vee g_2)(a,b) = \max\big(g_1(a,b), g_2(a,b)\big) = 0$. Then $g_1(a,b) = g_2(a,b) = 0$, thus $ab = 1$.

If $(g_1 \wedge g_2)(a,b) = \min\big(g_1(a,b), g_2(a,b)\big) = 0$. Case I, $g_1(a,b) = 0$ then $ab = 1$. Case II, $g_2(a,b) = 0$ then $ab = 1$.

If $(g_1 \times g_2)(a,b) = g_1(a,b)g_2(a,b) = 0$. Case I, $g_1(a,b) = 0$ then $ab = 1$. Case II, $g_2(a,b) = 0$ then $ab = 1$.

($\Leftarrow$) is straightforward. □

Corollary 7. *If $O_{f_1 g_1}$ and $O_{f_2 g_2}$ be two overlap ~~function~~ functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $O_{(f_1 \circ f_2)(g_1 \circ g_2)}$ is an overlap function, where $\circ \in \{\vee, \wedge, \times\}$*

Lemma 2. *If $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$ satisfy the following conditions (T1)-(T5) of Theorem 3. Then $(f_1 \vee f_2, g_1 \vee g_2)$, $(f_1 \wedge f_2, g_1 \wedge g_2)$ and $(f_1 \times f_2, g_1 \times g_2)$ also satisfy these conditions.*

Proof. The cases for **(T1)**, **(T2)**, and **(T5)** are straightforward.

(T3) ($\Rightarrow$) If $(f_1 \vee f_2)(a,b) = \max\big(f_1(a,b), f_2(a,b)\big) = 0$. Then, $f_1(a,b) = f_2(a,b) = 0$, thus $a = 1$ or $b = 1$.

If $(f_1 \wedge f_2)(a,b) = \min\big(f_1(a,b), f_2(a,b)\big) = 0$. Case I, $f_1(a,b) = 0$ then $a = 1$ or $b = 1$. Case II, $f_2(a,b) = 0$ then $a = 1$ or $b = 1$.

If $(f_1 \times f_2)(a,b) = f_1(a,b)f_2(a,b) = 0$. Case I, $f_1(a,b) = 0$ then $a = 1$ or $b = 1$. Case II, $f_2(a,b) = 0$ then $a = 1$ or $b = 1$.

($\Leftarrow$) is straightforward.

(T4) ($\Rightarrow$) If $(g_1 \vee g_2)(a,b) = \max\big(g_1(a,b), g_2(a,b)\big) = 0$. Then $g_1(a,b) = g_2(a,b) = 0$, thus $a = b = 0$.

If $(g_1 \wedge g_2)(a,b) = \min\big(g_1(a,b), g_2(a,b)\big) = 0$. Case I, $g_1(a,b) = 0$ then $a = b = 0$. Case II, $g_2(a,b) = 0$ then $a = b = 0$.

If $(g_1 \times g_2)(a,b) = g_1(a,b)g_2(a,b) = 0$. Case I, $g_1(a,b) = 0$ then $a = b = 0$. Case II, $g_2(a,b) = 0$ then $a = b = 0$.

($\Leftarrow$) is straightforward. □

Corollary 8. *If $G_{f_1g_1}$ and $G_{f_2g_2}$ be two grouping functions with generator pair $f_1, g_1 : [0,1]^2 \to [0,1]$ and $f_2, g_2 : [0,1]^2 \to [0,1]$, respectively. Then $G_{(f_1 \circ f_2)(g_1 \circ g_2)}$ is a grouping function, where $\circ \in \{\vee, \wedge, \times\}$*

Lemma 3. *Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in [0,1]$. If $a_1b_2 \leq a_2b_1$ and $a_3b_4 \leq a_4b_3$. Then*

(1) $(a_1a_3)(b_2b_4) \leq (a_2a_4)(b_1b_3)$.
(2) $(a_1 \wedge a_3)(b_2 \wedge b_4) \leq (a_2 \vee a_4)(b_1 \vee b_3)$.

Theorem 5. *Let $O_{f_1g_1}$, $O_{f_2g_2}$, $O_{f_3g_3}$ and $O_{f_4g_4}$ be four overlap functions with generator pair (f_1, g_1), $(f_2, g_2$, (f_3, g_3) and (f_4, g_4), respectively. If $O_{f_1g_1} \preceq O_{f_2g_2}$ and $O_{f_3g_3} \preceq O_{f_4g_4}$, then*

(1) $O_{(f_1 \times f_3)(g_1 \times g_3)} \preceq O_{(f_2 \times f_4)(g_2 \times g_4)}$;
(2) $O_{(f_1 \wedge f_3)(g_1 \wedge g_3)} \preceq O_{(f_2 \vee f_4)(g_2 \vee g_4)}$.

Proof. From $O_{f_1g_1} \preceq O_{f_2g_2}$ and $O_{f_3g_3} \preceq O_{f_4g_4}$, by Theorem 1, it hold that $f_1g_2 \leq f_2g_1$ and $f_3g_4 \leq f_4g_3$. By Lemma 3(1) and (2), we have $(f_1 \times f_3)(g_2 \times g_4) \leq (f_2 \times f_4)(g_1 \times g_3)$ and $(f_1 \wedge f_3)(g_2 \wedge g_4) \leq (f_2 \vee f_4)(g_1 \vee g_3)$, respectively. Thus we obtain $O_{(f_1 \times f_3)(g_1 \times g_3)} \preceq O_{(f_2 \times f_4)(g_2 \times g_4)}$ and $O_{(f_1 \wedge f_3)(g_1 \wedge g_3)} \preceq O_{(f_2 \vee f_4)(g_2 \vee g_4)}$. □

Theorem 6. *Let $G_{f_1g_1}$, $G_{f_2g_2}$, $G_{f_3g_3}$ and $G_{f_4g_4}$ be four grouping functions with generator pair (f_1, g_1), $(f_2, g_2$, (f_3, g_3) and (f_4, g_4), respectively. If $G_{f_1g_1} \preceq G_{f_2g_2}$ and $G_{f_3g_3} \preceq G_{f_4g_4}$, then*

(1) $G_{(f_1 \times f_3)(g_1 \times g_3)} \preceq G_{(f_2 \times f_4)(g_4 \times g_4)}$;
(2) $G_{(f_1 \vee f_3)(g_1 \wedge g_3)} \preceq G_{(f_2 \wedge f_4)(g_2 \vee g_4)}$.

Proof. From $G_{f_1g_1} \preceq G_{f_2g_2}$ and $G_{f_3g_3} \preceq G_{f_4g_4}$, by Theorem 4, it hold that $f_2g_1 \leq f_1g_2$ and $f_4g_3 \leq f_3g_4$. By Lemma 3(1) and (2), we have $(f_2 \times f_4)(g_1 \times g_3) \leq (f_1 \times f_3)(g_2 \times g_4)$ and $(f_2 \wedge f_4)(g_1 \wedge g_3) \leq (f_1 \vee f_3)(g_2 \vee g_4)$, respectively. Thus we obtain $G_{(f_1 \times f_3)(g_1 \times g_3)} \preceq G_{(f_2 \times f_4)(g_2 \times g_4)}$ and $G_{(f_1 \vee f_3)(g_1 \wedge g_3)} \preceq G_{(f_2 \wedge f_4)(g_2 \vee g_4)}$. □

6. Conclusions

This paper studies the pointwise comparability of overlap and grouping functions, respectively. We give some necessary and sufficient conditions for the comparison of overlap functions characterized by Bustince et al. generator pairs [1] and grouping functions characterized by Bedregal et al. generator pairs [26]. We present some more general results on order preservation with respect to some compositions of overlap and grouping functions.

In this paper, we only focus on overlap and grouping functions characterized by Bustince et al. [1] and Bedregal et al. [26] generators, respectively. Naturally, a more detailed discussion of other generators of overlap and grouping functions, such as additive generators proposed by Dimuro et al. [27] and multiplicative generators proposed by Qiao and Hu [29], will be both necessary and interesting.

It was observed that there are various compositions of overlap and grouping functions. Obviously, the results presented in this paper are particular cases of order preservation with respect to some compositions of overlap and grouping functions. Therefore, it will be meaningful to further discuss order preservation of other compositions.

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