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Incomplete Complex Intuitionistic Fuzzy System: Preference Relations, Expert Weight Determination, Group Decision-Making and Their Calculation Algorithms

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Abstract: As is well known, complex intuitionistic fuzzy preference relation can describe the fuzzy characters of things in more detail and comprehensively and is very useful in dealing with decision-making problems that include periodic or recurring phenomena. However, sometimes, a decision-maker may provide incomplete judgments in a complex intuitionistic fuzzy preference relation because of a lack of knowledge, time pressure, and the decision-makers' limited expertise related to the problem domain. In such cases, it would be sensible not to force the expert to express "false" preferences over these objects. Consequently, how to define incomplete complex intuitionistic fuzzy preference relations and to estimate their missing elements in an incomplete complex intuitionistic fuzzy preference relation becomes a necessary step in a decision-making process. In this paper, the concept of incomplete complex intuitionistic fuzzy preference relation is introduced and its properties are discussed. Meanwhile, the multiplicative consistent incomplete complex intuitionistic fuzzy preference relations are defined. Secondly, estimating algorithms are developed to estimate the missing elements in the acceptable incomplete complex intuitionistic fuzzy preference relations. Finally, an expert weight determination algorithm and the group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations are established. The solving process of the algorithms is illustrated by an example, the practicability of the algorithms is verified, the advantages and disadvantages of two group decision-making algorithms are compared and analyzed, and the simulation verification of incomplete complex intuitionistic fuzzy system is carried out by MATLAB software. The framework proposed in this paper effectively generalizes and enriches the previous works and has a good application prospect.

Keywords: complex intuitionistic fuzzy set; incomplete complex intuitionistic fuzzy preference relation; algorithms; group decision-making

MSC: 03B52; 03E72; 28E10

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1. Introduction

Preference relation, as one of the effective decision-making methods, has received great attention from researchers and has been widely applied in many practical decision-making fields, such as forecast projects, economic analysis, management information systems, decision support systems, and so on [1–6]. In the decision-making process, decision-makers provide their preference values over alternatives by comparing them pairwise and then constructing a preference relation. With the different types of decision-making problems, there exist distinct types of preference relations. Xu [7] gave a systematic and exhaustive survey of the existing preference relations, which mainly include multiplicative preference relations, fuzzy preference relations, linguistic preference relations, and intuitionistic preference relations. These four types of preference relations have been systematically investigated over the past decades and applied extensively in a variety of fields, such as medicine, economy, management, and military affairs [8–22].

Note that the above-cited studies of decision-making issues based on preference relations can only deal with one-dimensional decision-making problems. However, many real-world complex problems include two-dimensional data, that is, the properties and periodicity of the parameters associated with the problem. To characterize such two-dimensional information using the above theories, the decision-maker will have to consider two or more fuzzy sets or intuitionistic fuzzy sets, which may increase execution time and the amount of computation required to solve the problem. Consequently, to depict some periodic information in the judgment values, in 2002, Ramot et al. [23] proposed an important concept and called it a complex fuzzy set, where the membership function, instead of being a real-valued function with the range of $[0, 1]$ is replaced by a complex-valued function with codomain unit disc in the complex plane. In 2018, Yazdanbakhsh and Dick [24] made a systematic review of complex fuzzy sets and logic and highlighted several applications of complex fuzzy sets. Furthermore, to incorporate the hesitation degree and the periodicity information into the analysis, Alkouri and Salleh proposed the theory of complex intuitionistic fuzzy sets and discussed several operations of complex intuitionistic fuzzy sets [25,26]. They also defined a distance between two complex intuitionistic fuzzy sets and gave its application in a decision-making problem [27]. Subsequently, Rani and Garg gave a systematic and exhaustive investigation of complex intuitionistic fuzzy set theories [28–34]. In 2021, Rani and Garg [35] utilized complex intuitionistic fuzzy numbers to describe complex intuitionistic fuzzy preference relations and developed individual and group decision-making methods based on them. Meanwhile, some new types of fuzzy sets and their applications have been investigated by many researchers recently [36,37].

The complex intuitionistic fuzzy preference relation as a newly developed tool can describe the fuzzy characters of things in more detail and comprehensively and is very useful in dealing with the vagueness and uncertainty of actual decision-making problems, which include two-dimensional data. Consider that, in some real-life situations, sometimes, a decision-maker may provide incomplete judgments in a complex intuitionistic fuzzy preference relation because of time pressure, lack of knowledge, and the decision maker's limited expertise related to the problem domain. In such cases, it would be sensible not to force the expert to express "false" preferences over these objects, and thus an incomplete complex intuitionistic fuzzy preference relation could be constructed, in which some elements are missing. As a result, how to define incomplete complex intuitionistic fuzzy preference relations and to estimate their missing information becomes a necessary step in a decision-making process. It is a necessity to develop some effective techniques for estimating missing information. Meanwhile, consider that in the practical decision-making problems, since the increasing complexity and uncertainty of the socioeconomic environment, we notice that group decision-making makes decision-making more scientific and democratic. Therefore, it is also necessary to consider group decision-making problems in which there exist multiple incomplete complex intuitionistic fuzzy preference relations. As is well known, the weights of experts reflect the influence of experts on the decision results in the group decision-making problems, which directly determine the feasibility and rationality of the decision results. So in the group decision-making process, how to scientifically determine the weights of experts is also very important. The framework proposed in this paper can describe the fuzzy characteristics of things in more detail and comprehensively. It is very useful in dealing with the vagueness and uncertainty of actual decision-making problems, which include two-dimensional data, that is, the properties and periodicity of the parameters associated with the problem. It also effectively generalizes and enriches the previous works, and has a good application prospect.

In this paper, we will pay attention to these issues. To do that, the remainder of this paper is organized as follows: In Section 2, some basic concepts related to complex fuzzy sets, complex intuitionistic fuzzy sets, and preference relations are reviewed. In Section 3, the concept of incomplete complex intuitionistic fuzzy preference relation is introduced and its properties are discussed. Meanwhile, the multiplicative consistent incomplete complex intuitionistic fuzzy preference relations are defined. Estimating algorithms are developed to estimate the missing elements in the acceptable incomplete complex intuitionistic fuzzy preference relations in Section 4. In Section 5, the expert weight determination algorithm is developed, and then the group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations are established. In Section 6, an example is given to illuminate the solution process of the group decision-making algorithms, verify their practicality and compare and analyze their advantages and disadvantages. Meanwhile, the simulation verification of an incomplete complex intuitionistic fuzzy system is carried out by MATLAB software in this section. The conclusion is given in Section 7.

2. Preliminaries

Some basic concepts related to complex fuzzy sets, complex intuitionistic fuzzy sets, and preference relations are reviewed in this section. Table 1 summarizes the mathematical symbols and their definitions in this paper.

Table 1. Mathematical symbols and definitions.

Mathematical Symbols	Definitions
A	A complex fuzzy set
$\tilde{A}$	A complex intuitionistic fuzzy set
$\tilde{\mathfrak{C}}\mathfrak{E}$	The set of complex intuitionistic fuzzy numbers
$\mathcal{S}$	The score function
$\mathcal{H}$	The accuracy function
$\mathcal{V}$	The discrete set of alternatives
$\mathcal{R}$	The preference relation
$\mathcal{B}$	The intuitionistic preference relation
$\mathfrak{R}$	The complex intuitionistic fuzzy preference relation
$\mathcal{S}$	A scoring matrix
$\bar{\mathcal{S}}$	A mean scoring matrix

Definition 1 ([23])**.** *A complex fuzzy set A over an universe U, is formed by $A = \{\langle x, \mu_A(x)\rangle : x \in U\}$, where the complex-valued membership function $\mu_A(x)$ has the form*

$$\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)} \qquad i = \sqrt{-1},$$

where $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0,1]$.

Definition 2 ([25])**.** *A complex intuitionistic fuzzy set $\tilde{A}$, defined on an universe of discourse U, is characterized by membership and non-membership functions $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$, respectively, that assign any element $x \in U$ a complex-valued grade of both membership and non-membership in $\tilde{A}$.*

By Definition 2, the values of $\mu_{\tilde{A}}(x)$, $\nu_{\tilde{A}}(x)$ and their sum may receive all lying within the unit circle in the complex plane, and are on the form

$$\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x) \cdot e^{i2\pi(\omega_{\mu\tilde{A}}(x))}$$

for membership function in $\tilde{A}$ and

$$\nu_{\tilde{A}}(x) = s_{\tilde{A}}(x) \cdot e^{i2\pi(\omega_{\nu\tilde{A}}(x))}$$

for non-membership function in $\tilde{A}$, where $i = \sqrt{-1}$, each of $r_{\tilde{A}}(x), s_{\tilde{A}}(x), \omega_{\mu\tilde{A}}(x)$ and $\omega_{\nu\tilde{A}}(x)$ are real-valued functions and both belong to the interval $[0,1]$ such that $r_{\tilde{A}}(x) + s_{\tilde{A}}(x) \in [0,1]$ and $\omega_{\mu\tilde{A}}(x) + \omega_{\nu\tilde{A}}(x) \in [0,1]$.

Let $\mathcal{CIF}^{\star}(U)$ be the set of all complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy set $\tilde{A}$ may be represented as the set of ordered pairs

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle : x \in U\},$$

where $\mu_{\tilde{A}}(x) : U \to \{a | a \in C, \ |a| \leq 1\}$, $\nu_{\tilde{A}}(x) : U \to \{a' | a' \in C, \ |a'| \leq 1\}$ and $|\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x)| \leq 1$.

If there is a single element x in U, then for notational convenience, we write complex intuitionistic fuzzy set $\tilde{A}$ over U as $((r_{\tilde{A}}, \omega_{\mu\tilde{A}}), (s_{\tilde{A}}, \omega_{\nu\tilde{A}}))$ and call it complex intuitionistic fuzzy number. The class $\mathfrak{CE}$ denotes the set of all complex intuitionistic fuzzy numbers in $\tilde{A}$.

Definition 3 ([28]). *The score function $\mathcal{S}$ and an accuracy function $\mathcal{H}$ for a complex intuitionistic fuzzy number $\tilde{A}_1 = ((r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}), (s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}))$ are given as*

$$\mathcal{S}(\tilde{A}_1) = r_{\tilde{A}_1} - s_{\tilde{A}_1} + \omega_{\mu\tilde{A}_1} - \omega_{\nu\tilde{A}_1} \tag{1}$$

and

$$\mathcal{H}(\tilde{A}_1) = r_{\tilde{A}_1} + s_{\tilde{A}_1} + \omega_{\mu\tilde{A}_1} + \omega_{\nu\tilde{A}_1}. \tag{2}$$

It is clear that $\mathcal{S}(\tilde{A}_1) \in [-2, 2]$ and $\mathcal{H}(\tilde{A}_1) \in [0, 2]$.

Let $\tilde{A}_1 = ((r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}), (s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}))$ and $\tilde{A}_2 = ((r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_2}))$ be two complex intuitionistic fuzzy numbers. Based on the score function and accuracy function, an ordered relation between $\tilde{A}_1$ and $\tilde{A}_2$ states as

(1) If $\mathcal{S}(\tilde{A}_1) < \mathcal{S}(\tilde{A}_2)$, then $\tilde{A}_1 < \tilde{A}_2$;
(2) If $\mathcal{S}(\tilde{A}_1) > \mathcal{S}(\tilde{A}_2)$, then $\tilde{A}_1 > \tilde{A}_2$;
(3) If $\mathcal{S}(\tilde{A}_1) = \mathcal{S}(\tilde{A}_2)$, then
(i) If $\mathcal{H}(\tilde{A}_1) = \mathcal{H}(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$;
(ii) If $\mathcal{H}(\tilde{A}_1) < \mathcal{H}(\tilde{A}_2)$, then $\tilde{A}_1 < \tilde{A}_2$;
(iii) If $\mathcal{H}(\tilde{A}_1) > \mathcal{H}(\tilde{A}_2)$, then $\tilde{A}_1 > \tilde{A}_2$.

As is well known, the score function of a complex intuitionistic fuzzy number is determined by the difference between the amplitude term of membership degree and non-membership degree and the difference between the phase term of membership degree and non-membership degree, which is defined as the sum of these two parts. In both senses, it expresses the degree of certainty, which corresponds to a positive benefit function in some sense. However, the accuracy function expresses the degree of negation, which corresponds to a cost function in some sense. Therefore, this paper applies the score function first to rank the alternatives. Of course, when the score function values are equal, we will rank the alternatives by calculating and comparing the accuracy function values to choose the best solution.

Definition 4 ([28]). *Suppose $\tilde{A}_1 = ((r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}), (s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}))$ and $\tilde{A}_2 = ((r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_2}))$ be two complex intuitionistic fuzzy numbers, then*
(i) $\tilde{A}_1 \subseteq \tilde{A}_2$ if $r_{\tilde{A}_1} \leq r_{\tilde{A}_2}$, $s_{\tilde{A}_1} \geq s_{\tilde{A}_2}$ and $\omega_{\mu\tilde{A}_1} \leq \omega_{\mu\tilde{A}_2}$, $\omega_{\nu\tilde{A}_1} \geq \omega_{\nu\tilde{A}_2}$;
(ii) $\tilde{A}_1 = \tilde{A}_2$ if and only if $\tilde{A}_1 \subseteq \tilde{A}_2$ and $\tilde{A}_2 \subseteq \tilde{A}_1$;
(iii) $\tilde{A}_1^C = ((s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}), (r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}))$.

Definition 5 ([28]). *Suppose $\tilde{A}_1 = ((r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}), (s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}))$ and $\tilde{A}_2 = ((r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_2}))$ be two complex intuitionistic fuzzy numbers, and $\lambda > 0$, then*
(i) $\tilde{A}_1 \oplus \tilde{A}_2 = ((r_{\tilde{A}_1} + r_{\tilde{A}_2} - r_{\tilde{A}_1} r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_1} + \omega_{\mu\tilde{A}_2} - \omega_{\mu\tilde{A}_1}\omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_1} s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_1}\omega_{\nu\tilde{A}_2}))$;

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = ((r_{\tilde{A}_1} r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_1} \omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_1} + s_{\tilde{A}_2} - s_{\tilde{A}_1} s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_1} + \omega_{\nu\tilde{A}_2} - \omega_{\nu\tilde{A}_1}\omega_{\nu\tilde{A}_2}))$;
(iii) $\lambda\tilde{A}_1 = ((1-(1-r_{\tilde{A}_1})^\lambda, 1-(1-\omega_{\mu\tilde{A}_1})^\lambda), (s^\lambda_{\tilde{A}_1}, \omega^\lambda_{\nu\tilde{A}_1}))$;
(iv) $\tilde{A}_1^\lambda = ((r^\lambda_{\tilde{A}_1}, \omega^\lambda_{\mu\tilde{A}_1}), (1-(1-s_{\tilde{A}_1})^\lambda, 1-(1-\omega_{\nu\tilde{A}_1})^\lambda))$.

Theorem 1. *Let* $\tilde{A}_1 = ((r_{\tilde{A}_1}, \omega_{\mu\tilde{A}_1}), (s_{\tilde{A}_1}, \omega_{\nu\tilde{A}_1}))$ *and* $\tilde{A}_2 = ((r_{\tilde{A}_2}, \omega_{\mu\tilde{A}_2}), (s_{\tilde{A}_2}, \omega_{\nu\tilde{A}_2}))$ *be two complex intuitionistic fuzzy numbers, then* $\tilde{A}_1 \oplus \tilde{A}_2$, $\tilde{A}_1 \otimes \tilde{A}_2$, $\lambda\tilde{A}_1$ *and* $\tilde{A}_1^\lambda$ $(\lambda > 0)$ *are also complex intuitionistic fuzzy numbers and*
(i) $\tilde{A}_1 \oplus \tilde{A}_2 = \tilde{A}_2 \oplus \tilde{A}_1$;
(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = \tilde{A}_2 \otimes \tilde{A}_1$;
(iii) $\lambda(\tilde{A}_1 \oplus \tilde{A}_2) = \lambda\tilde{A}_1 \oplus \lambda\tilde{A}_2,\ \lambda > 0$;
(iv) $(\tilde{A}_1 \otimes \tilde{A}_2)^\lambda = \tilde{A}_1^\lambda \otimes \tilde{A}_2^\lambda,\ \lambda > 0$;
(v) $\lambda_1\tilde{A}_1 \oplus \lambda_2\tilde{A}_1 = (\lambda_1 + \lambda_2)\tilde{A}_1,\ \lambda_1,\ \lambda_2 > 0$;
(vi) $\tilde{A}_1^{\lambda_1} \otimes \tilde{A}_1^{\lambda_2} = \tilde{A}_1^{\lambda_1+\lambda_2},\ \lambda_1,\ \lambda_2 > 0$.

Proof. Obvious from Definition 5. □

Definition 6 ([15]). *For a discrete set of alternatives* $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$, *a preference relation* $\mathcal{R}$ *is characterized by a membership function* $\mu_{\mathcal{R}} : \mathcal{V} \times \mathcal{V} \to \Omega$, *where* Ω *is the domain representing preference degrees.*

Definition 7 ([19]). *An intuitionistic preference relation* $\mathcal{B}$ *on the discrete set of alternatives* $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ *is represented by a matrix* $\mathcal{B} = (b_{ij})_{n\times n} \subseteq \mathcal{V} \times \mathcal{V}$ *with* $b_{ij} = \langle (x_i, x_j), \mu(x_i, x_j), \nu(x_i, x_j)\rangle$ *for all* $i, j = 1, 2, \cdots, n$. *For convenience, let* $b_{ij} = (\mu_{ij}, \nu_{ij})$ *for all* $i, j = 1, 2, \cdots, n$, *where* b_{ij} *is an intuitionistic fuzzy number, composed by the certainty degree* μ_{ij} *to which* x_i *is preferred to* x_j *and the certainty degree* ν_{ij} *to which* x_i *is non-preferred to* x_j, *and* $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ *is interpreted as the uncertainty degree to which* x_i *is preferred to* x_j. *Furthermore,* μ_{ij} *and* ν_{ij} *satisfy the following characteristics*

$$0 \le \mu_{ij} + \nu_{ij} \le 1,\ \mu_{ji} = \nu_{ij},\ \nu_{ji} = \mu_{ij},\ \mu_{ii} = \nu_{ii} = 0.5 \text{ for all } i, j = 1, 2, \cdots, n.$$

Definition 8 ([20]). *An intuitionistic preference relation* $\mathcal{B} = (b_{ij})_{n\times n}$ *with* $b_{ij} = (\mu_{ij}, \nu_{ij})$ $(i, j = 1, 2, \cdots, n)$ *is multiplicative consistent if for any* $i \le k \le j$

$$\mu_{ij} = \begin{cases} 0, & \text{if } (\mu_{ik}, \mu_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj} + (1-\mu_{ik})(1-\mu_{kj})}, & \text{otherwise}, \end{cases} \tag{3}$$

and

$$\nu_{ij} = \begin{cases} 0, & \text{if } (\nu_{ik}, \nu_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{\nu_{ik}\nu_{kj}}{\nu_{ik}\nu_{kj} + (1-\nu_{ik})(1-\nu_{kj})}, & \text{otherwise}. \end{cases} \tag{4}$$

Definition 9 ([19]). *Let* $\mathcal{B}$ *be an intuitionistic preference relation, where* $b_{ij} = (\mu_{ij}, \nu_{ij})$ *for* $i, j = 1, 2, \cdots, n$. *Then* $\mathcal{B}$ *is called an incomplete intuitionistic preference relation, if some of its elements cannot be provided by the decision-makers, which denoted by the unknown variable "−", and the others can be given by the decision-makers, which satisfy*

$$0 \le \mu_{ij} + \nu_{ij} \le 1,\ \mu_{ji} = \nu_{ij},\ \nu_{ji} = \mu_{ij},\ \mu_{ii} = \nu_{ii} = 0.5 \text{ for all } i, j = 1, 2, \cdots, n.$$

Definition 10 ([35]). *A complex intuitionistic fuzzy preference relation* $\mathfrak{R}$ *on the discrete set of alternatives* $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ *is represented by a matrix* $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n} \subseteq \mathcal{V} \times \mathcal{V}$, *where* $\tilde{a}_{ij} = ((r_{ij}, \omega_{\mu_{ij}}), (s_{ij}, \omega_{\nu_{ij}}))$ *is a complex intuitionistic fuzzy number for all* $i, j = 1, 2, \cdots, n$. *Here the amplitude and phase terms corresponding to membership degrees, that is,* r_{ij} *and* $\omega_{\mu_{ij}}$ *explicit the degrees to which the alternative* $\mathcal{V}_i$ *is preferred over the alternative* $\mathcal{V}_j$. *On the other hand, the amplitude and phase terms corresponding to non-membership degrees, that is,* s_{ij} *and* $\omega_{\nu_{ij}}$ *describe* *the extent of non-preference of the alternative $\mathcal{V}_i$ over the alternative $\mathcal{V}_j$. These terms satisfy the following characteristics*

$r_{ij}, s_{ij}, \omega_{\mu_{ij}}, \omega_{\nu_{ij}}, r_{ij} + s_{ij}, \omega_{\mu_{ij}} + \omega_{\nu_{ij}} \in [0,1], r_{ji} = s_{ij}, s_{ji} = r_{ij}, \omega_{\mu_{ji}} = \omega_{\nu_{ij}}, \omega_{\nu_{ji}} = \omega_{\mu_{ij}}, r_{ii} = s_{ii} = \omega_{\mu_{ii}} = \omega_{\nu_{ii}} = 0.5$ for all $i, j = 1, 2, \cdots, n$.

3. Incomplete Complex Intuitionistic Fuzzy Preference Relations

Consider that, sometimes, a decision maker may provide incomplete judgments in a complex intuitionistic fuzzy preference relation because of time pressure, lack of knowledge, and the decision maker's limited expertise related to the problem domain. In this section, we will investigate a novel concept of incomplete complex intuitionistic fuzzy preference relation and discuss its properties. Meanwhile, the concept of multiplicative consistent incomplete complex intuitionistic fuzzy preference relations are given.

Definition 11. *Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be a complex intuitionistic fuzzy preference relation, where $\tilde{a}_{ij} = ((r_{ij}, \omega_{\mu_{ij}}), (s_{ij}, \omega_{\nu_{ij}}))$ for all $i, j = 1, 2, \cdots, n$. Then $\mathfrak{R}$ is called an incomplete complex intuitionistic fuzzy preference relation if some of its elements cannot be given by the decision-makers, which we denote by "$-$", and the others can be provided by the decision maker, which satisfy the following conditions*

$r_{ij}, s_{ij}, \omega_{\mu_{ij}}, \omega_{\nu_{ij}}, r_{ij} + s_{ij}, \omega_{\mu_{ij}} + \omega_{\nu_{ij}} \in [0,1], r_{ji} = s_{ij}, s_{ji} = r_{ij}, \omega_{\mu_{ji}} = \omega_{\nu_{ij}}, \omega_{\nu_{ji}} = \omega_{\mu_{ij}}, r_{ii} = s_{ii} = \omega_{\mu_{ii}} = \omega_{\nu_{ii}} = 0.5$ for all $i, j = 1, 2, \cdots, n$.

Property 1. *Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be an incomplete complex intuitionistic fuzzy preference relation, where $\tilde{a}_{ij} = ((r_{ij}, \omega_{\mu_{ij}}), (s_{ij}, \omega_{\nu_{ij}}))$ for all $i, j = 1, 2, \cdots, n$ and Ω be the set of all the known elements, then*

(i) Triangle condition: If $\tilde{a}_{ij} \subseteq \tilde{a}_{ik} \oplus \tilde{a}_{kj}$ for all $\tilde{a}_{ik}, \tilde{a}_{kj}, \tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies triangle condition. This condition can be explained geometrically as follows: Let the alternatives $\mathcal{V}_i$, $\mathcal{V}_k$ and $\mathcal{V}_j$ represent the triangle vertices and let the lengths of triangle sides be $\tilde{a}_{ik}, \tilde{a}_{kj}$ and $\tilde{a}_{ij}$. Then, the triangle condition states that the sum of lengths of vertices $\mathcal{V}_i, \mathcal{V}_k$ and $\mathcal{V}_k, \mathcal{V}_j$ should be greater than or equal to the lengths of vertices $\mathcal{V}_i, \mathcal{V}_j$.

(ii) Weak transitivity condition: If $((0.5, 0.5), (0.5, 0.5)) \subseteq \tilde{a}_{ik}$, $((0.5, 0.5), (0.5, 0.5)) \subseteq \tilde{a}_{kj}$ then $((0.5, 0.5), (0.5, 0.5)) \subseteq \tilde{a}_{ij}$ for all $\tilde{a}_{ik}, \tilde{a}_{kj}, \tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies weak transitivity condition. This property states that if the alternative $\mathcal{V}_i$ is preferred to $\mathcal{V}_k$ with preference value $\tilde{a}_{ik}$ and the alternative $\mathcal{V}_k$ is preferred to $\mathcal{V}_j$ with preference value $\tilde{a}_{kj}$, then the alternative $\mathcal{V}_i$ is preferred to $\mathcal{V}_j$ with preference value $\tilde{a}_{ij}$.

(iii) Max-min transitivity condition: If $\min\{\tilde{a}_{ik}, \tilde{a}_{kj}\} \subseteq \tilde{a}_{ij}$ for all $\tilde{a}_{ik}, \tilde{a}_{kj}, \tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies max-min transitivity condition. This property can be interpreted as follows: The complex intuitionistic fuzzy value giving the preference value of alternative $\mathcal{V}_i$ over $\mathcal{V}_j$ should be greater than or equal to the minimum complex intuitionistic fuzzy value acquired by comparing alternatives $\mathcal{V}_i$ and $\mathcal{V}_j$ with an intermediate one.

(iv) Max-max transitivity condition: If $\max\{\tilde{a}_{ik}, \tilde{a}_{kj}\} \subseteq \tilde{a}_{ij}$ for all $\tilde{a}_{ik}, \tilde{a}_{kj}, \tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies max-max transitivity condition. This property can be interpreted as follows: The complex intuitionistic fuzzy value giving the preference value of alternative $\mathcal{V}_i$ over $\mathcal{V}_j$ should be greater than or equal to the maximum complex intuitionistic fuzzy number value acquired by comparing alternatives $\mathcal{V}_i$ and $\mathcal{V}_j$ with an intermediate one.

(v) Restricted Max-min transitivity condition: If $((0.5, 0.5), (0.5, 0.5)) \subseteq \tilde{a}_{ik}, ((0.5, 0.5), (0.5, 0.5)) \subseteq \tilde{a}_{kj} \Rightarrow \min\{\tilde{a}_{ik}, \tilde{a}_{kj}\} \subseteq \tilde{a}_{ij}$ for all $\tilde{a}_{ik}, \tilde{a}_{kj}, \tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies restricted max-min transitivity condition. This property can be interpreted in the following way: When the alternative $\mathcal{V}_i$ is preferred to $\mathcal{V}_k$ with a complex intuitionistic fuzzy value $\tilde{a}_{ik}$, and the alternative $\mathcal{V}_k$ is preferred to $\mathcal{V}_j$ with a complex intuitionistic fuzzy value $\tilde{a}_{kj}$, then $\mathcal{V}_i$ should be preferred to $\mathcal{V}_j$ with at least a complex

intuitionistic fuzzy value $\tilde{a}_{ij}$ equal to the minimum of the above values. With equality only when there is indifference between at least two of the three alternatives.

(vi) Restricted Max-max transitivity condition: If $((0.5,0.5),(0.5,0.5)) \subseteq \tilde{a}_{ik}$, $((0.5,0.5),(0.5,0.5)) \subseteq \tilde{a}_{kj} \Rightarrow \max\{\tilde{a}_{ik},\tilde{a}_{kj}\} \subseteq \tilde{a}_{ij}$ for all $\tilde{a}_{ik},\tilde{a}_{kj},\tilde{a}_{ij} \in \Omega$, then we say that an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$ satisfies restricted max-max transitivity condition. This property can be interpreted in the following way: When the alternative $\mathcal{V}_i$ is preferred to $\mathcal{V}_k$ with a complex intuitionistic fuzzy value $\tilde{a}_{ik}$, and the alternative $\mathcal{V}_k$ are preferred to $\mathcal{V}_j$ with a complex intuitionistic fuzzy value $\tilde{a}_{kj}$, then $\mathcal{V}_i$ should be preferred to $\mathcal{V}_j$ with at least a complex intuitionistic fuzzy value $\tilde{a}_{ij}$ equal to the maximum of the above values. With equality only when there is indifference between at least two of the three alternatives.

Definition 12. *Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be an incomplete complex intuitionistic fuzzy preference relation, then $\mathfrak{R}$ is called a consistent incomplete complex intuitionistic fuzzy preference relation if it satisfies the multiplicative transitivity*

$$\tilde{a}_{ij} = \tilde{a}_{ik} \otimes \tilde{a}_{kj} \quad \text{for all } \tilde{a}_{ij},\tilde{a}_{kj},\tilde{a}_{ik} \in \Omega, \tag{5}$$

where Ω is the set of all the known elements.

Definition 13. *Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be an incomplete complex intuitionistic fuzzy preference relation, if $(i,j) \cap (k,l) \neq \emptyset$. Then the element $\tilde{a}_{ij}$ and $\tilde{a}_{kl}$ are called adjoining. For the unknown element $\tilde{a}_{ij}$, if there exist two adjoining known elements $\tilde{a}_{ik}$ and $\tilde{a}_{kj}$, then $\tilde{a}_{ij}$ is called available. Indeed, the element $\tilde{a}_{ij}$ can be obtained indirectly by using Equation (5), which means that the estimated element $\tilde{a}_{ij}$ should be taken according to the known elements $\tilde{a}_{ik}$ and $\tilde{a}_{kj}$.*

Definition 14. *Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be an incomplete complex intuitionistic fuzzy preference relation if each unknown element can be derived from its adjoining known elements, then $\mathfrak{R}$ is called acceptable; otherwise, $\mathfrak{R}$ is called unacceptable.*

Obviously, for an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$, if $\mathfrak{R}$ is acceptable, then there exists at least one known element (except diagonal elements) in each line or each column of $\mathfrak{R}$,, i.e., there exists at least $(n-1)$ judgments provided by the decision maker (that is to say, each one of the alternatives is compared at least once).

Definition 15. *A complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ with $\tilde{a}_{ij} = ((r_{ij},\omega_{\mu_{ij}}),(s_{ij},\omega_{\nu_{ij}}))$ $(i,j = 1,2,\cdots,n)$ is multiplicative consistent if for any $i \le k \le j$*

$$r_{ij} = \begin{cases} 0, & \text{if } (r_{ik},r_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{r_{ik}r_{kj}}{r_{ik}r_{kj}+(1-r_{ik})(1-r_{kj})}, & \textit{otherwise}, \end{cases} \tag{6}$$

$$s_{ij} = \begin{cases} 0, & \text{if } (s_{ik},s_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{s_{ik}s_{kj}}{s_{ik}s_{kj}+(1-s_{ik})(1-s_{kj})}, & \textit{otherwise}, \end{cases} \tag{7}$$

$$\omega_{\mu_{ij}} = \begin{cases} 0, & \text{if } (\omega_{\mu_{ik}},\omega_{\mu_{kj}}) \in \{(0,1),(1,0)\}, \\ \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}}+(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})}, & \textit{otherwise}, \end{cases} \tag{8}$$

and

$$\omega_{\nu_{ij}} = \begin{cases} 0, & \text{if } (\omega_{\nu_{ik}},\omega_{\nu_{kj}}) \in \{(0,1),(1,0)\}, \\ \frac{\omega_{\nu_{ik}}\omega_{\nu_{kj}}}{\omega_{\nu_{ik}}\omega_{\nu_{kj}}+(1-\omega_{\nu_{ik}})(1-\omega_{\nu_{kj}})}, & \textit{otherwise}. \end{cases} \tag{9}$$

Remark 1. *Since the function $f(x) = \frac{x}{x+b}$ is a monotone increasing function with $x, b > 0$, then in Equations (6)–(9), it is clear to see that $r_{ij}, s_{ij}, \omega_{\mu_{ij}}, \omega_{\nu_{ij}} \in [0,1]$ and*

$$\begin{aligned}
&r_{ij} + s_{ij} \\
&= \frac{r_{ik}r_{kj}}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})} + \frac{s_{ik}s_{kj}}{s_{ik}s_{kj} + (1-s_{ik})(1-s_{kj})} \\
&\leq \frac{r_{ik}r_{kj}}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})} + \frac{(1-r_{ik})(1-r_{kj})}{(1-r_{ik})(1-r_{kj}) + (1-s_{ik})(1-s_{kj})} \\
&\leq \frac{r_{ik}r_{kj}}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})} + \frac{(1-r_{ik})(1-r_{kj})}{(1-r_{ik})(1-r_{kj}) + r_{ik}r_{kj}} \\
&= \frac{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})} = 1,
\end{aligned}$$

$$\begin{aligned}
&\omega_{\mu_{ij}} + \omega_{\nu_{ij}} \\
&= \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})} + \frac{\omega_{\nu_{ik}}\omega_{\nu_{kj}}}{\omega_{\nu_{ik}}\omega_{\nu_{kj}} + (1-\omega_{\nu_{ik}})(1-\omega_{\nu_{kj}})} \\
&\leq \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})} + \frac{(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})}{(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj})} + (1-\omega_{\nu_{ik}})(1-\omega_{\nu_{kj}})} \\
&\leq \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})} + \frac{(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})}{(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj})} + \omega_{\mu_{ik}}\omega_{\mu_{kj}}} \\
&= \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})}{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})} = 1,
\end{aligned}$$

that is $r_{ij} + s_{ij} \leq 1$ and $\omega_{\mu_{ij}} + \omega_{\nu_{ij}} \leq 1$. In particular, we have

$$r_{ij} + s_{ij} \begin{cases} = 1, & \text{if } r_{ik} + s_{ik} = 1 \text{ and } r_{kj} + s_{kj} = 1, \\ < 1, & \text{otherwise}, \end{cases}$$

and

$$\omega_{\mu_{ij}} + \omega_{\nu_{ij}} \begin{cases} = 1, & \text{if } \omega_{\mu_{ik}} + \omega_{\nu_{ik}} = 1 \text{ and } \omega_{\mu_{kj}} + \omega_{\nu_{kj}} = 1, \\ < 1, & \text{otherwise}. \end{cases}$$

Theorem 2. *Any complex intuitionistic preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{2\times 2}$ is multiplicative consistent.*

Proof. Since

$$\frac{r_{11}r_{12}}{r_{11}r_{12} + (1-r_{11})(1-r_{12})} = \frac{0.5r_{12}}{0.5r_{12} + 0.5(1-r_{12})} = r_{12},$$

$$\frac{s_{11}s_{12}}{r_{11}s_{12} + (1-s_{11})(1-s_{12})} = \frac{0.5s_{12}}{0.5s_{12} + 0.5(1-s_{12})} = s_{12},$$

$$\frac{\omega_{\mu_{11}}\omega_{\mu_{12}}}{\omega_{\mu_{11}}\omega_{\mu_{12}} + (1-\omega_{\mu_{11}})(1-\omega_{\mu_{12}})} = \frac{0.5\omega_{\mu_{12}}}{0.5\omega_{\mu_{12}} + 0.5)(1-\omega_{\mu_{12}})} = \omega_{\mu_{12}},$$

and

$$\frac{\omega_{\nu_{11}}\omega_{\nu_{12}}}{\omega_{\nu_{11}}\omega_{\nu_{12}} + (1-\omega_{\nu_{11}})(1-\omega_{\nu_{12}})} = \frac{0.5\omega_{\nu_{12}}}{0.5\omega_{\nu_{12}} + 0.5)(1-\omega_{\nu_{12}})} = \omega_{\nu_{12}},$$

which satisfy Equations (6)–(9). In addition, in the case where $r_{12} = 0, \omega_{\mu_{12}} = 0$ or $s_{12} = 0, \omega_{\nu_{12}} = 0$, Equations (6)–(9) also hold. □

Example 1. *Let $\mathfrak{R}_1$, $\mathfrak{R}_2$ and $\mathfrak{R}_3$ be three complex intuitionistic fuzzy preference relations, where*

$$\mathfrak{R}_1 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) \\ ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) \\ ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

$$\mathfrak{R}_2 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.2),(0.5,0.5)) & ((0.5,0.2),(0.2,0.4)) \\ ((0.5,0.5),(0.4,0.2)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.5),(0.2,0.4)) \\ ((0.2,0.4),(0.5,0.2)) & ((0.2,0.4),(0.6,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

$$\mathfrak{R}_3 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.6,0.4),(0.3,0.5)) & ((0.7,0.5),(0.2,0.4)) \\ ((0.3,0.5),(0.6,0.4)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.4),(0.4,0.3)) \\ ((0.2,0.4),(0.7,0.5)) & ((0.4,0.3),(0.5,0.4)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

It is clear to see from Equations (6)–(9) that both $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are multiplicative consistent, complex intuitionistic preference relations. However, in $\mathfrak{R}_3$, we have

$$r_{13} = \frac{r_{12}r_{23}}{r_{12}r_{23} + (1-r_{12})(1-r_{23})} = \frac{0.6 \times 0.5}{0.6 \times 0.5 + (1-0.6)(1-0.5)} = 0.6,$$

$$s_{13} = \frac{s_{12}s_{23}}{s_{12}s_{23} + (1-s_{12})(1-s_{23})} = \frac{0.3 \times 0.4}{0.3 \times 0.4 + (1-0.3)(1-0.4)} = 0.2222,$$

$$\omega_{\mu_{13}} = \frac{\omega_{\mu_{12}}\omega_{\mu_{23}}}{\omega_{\mu_{12}}\omega_{\mu_{23}} + (1-\omega_{\mu_{12}})(1-\omega_{\mu_{23}})} = \frac{0.4 \times 0.4}{0.4 \times 0.4 + (1-0.4)(1-0.4)} = 0.3077,$$

and

$$\omega_{\nu_{13}} = \frac{\omega_{\nu_{12}}\omega_{\nu_{23}}}{\omega_{\nu_{12}}\omega_{\nu_{23}} + (1-\omega_{\nu_{12}})(1-\omega_{\nu_{23}})} = \frac{0.5 \times 0.3}{0.5 \times 0.3 + (1-0.5)(1-0.3)} = 0.3,$$

obviously, $((0.7,0.5),(0.2,0.4)) \neq ((0.6,0.3077),(0.2222,0.3))$. Hence, $\mathfrak{R}_3$ is not a multiplicative consistent, complex intuitionistic preference relation.

Definition 16. *An incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ with $\tilde{a}_{ij} = ((r_{ij},\omega_{\mu_{ij}}),(s_{ij},\omega_{\nu_{ij}}))$ $(i,j = 1,2,\cdots,n)$ is multiplicative consistent if for all $\tilde{a}_{ik} = ((r_{ik},\omega_{\mu_{ik}}),(s_{ik},\omega_{\nu_{ik}})), \tilde{a}_{kj} = ((r_{kj},\omega_{\mu_{kj}}),(s_{kj},\omega_{\nu_{kj}})) \in \Omega$ and $i \le k \le j$*

$$r_{ij} = \begin{cases} 0, & \text{if } (r_{ik},r_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{r_{ik}r_{kj}}{r_{ik}r_{kj}+(1-r_{ik})(1-r_{kj})}, & \textit{otherwise}, \end{cases} \tag{10}$$

$$s_{ij} = \begin{cases} 0, & \text{if } (s_{ik},s_{kj}) \in \{(0,1),(1,0)\}, \\ \frac{s_{ik}s_{kj}}{s_{ik}s_{kj}+(1-s_{ik})(1-s_{kj})}, & \textit{otherwise}, \end{cases} \tag{11}$$

$$\omega_{\mu_{ij}} = \begin{cases} 0, & \text{if } (\omega_{\mu_{ik}},\omega_{\mu_{kj}}) \in \{(0,1),(1,0)\}, \\ \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}}+(1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})}, & \textit{otherwise}, \end{cases} \tag{12}$$

and

$$\omega_{\nu_{ij}} = \begin{cases} 0 & \text{if } (\omega_{\nu_{ik}},\omega_{\nu_{kj}}) \in \{(0,1),(1,0)\} \\ \frac{\omega_{\nu_{ik}}\omega_{\nu_{kj}}}{\omega_{\nu_{ik}}\omega_{\nu_{kj}}+(1-\omega_{\nu_{ik}})(1-\omega_{\nu_{kj}})} & \textit{otherwise}. \end{cases} \tag{13}$$

where Ω is the set of all the known elements of $\mathfrak{R}$.

Example 2. *Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two incomplete complex intuitionistic fuzzy preference relations, where*

$$\mathfrak{R}_1 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.5),(0.5,0.5)) & ((0.5,0.5),(0.4,0.5)) & - \\ ((0.5,0.5),(0.4,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.5),(0.4,0.5)) & - \\ ((0.4,0.5),(0.5,0.5)) & ((0.4,0.5),(0.6,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.4,0.5),(0.6,0.5)) \\ - & - & ((0.6,0.5),(0.4,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

$$\mathfrak{R}_2 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.5),(0.2,0.5)) & - & ((0.6,0.5),(0.3,0.5)) \\ ((0.2,0.5),(0.4,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.3,0.5),(0.1,0.5)) & ((0.2,0.4),(0.8,0.5)) \\ - & ((0.1,0.5),(0.3,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.5),(0.4,0.5)) \\ ((0.3,0.5),(0.6,0.5)) & ((0.8,0.5),(0.2,0.4)) & ((0.4,0.5),(0.6,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

and "$-$" denotes the unknown element.

It is clear to see from Equations (10)–(13) that $\mathfrak{R}_1$ is a multiplicative consistent incomplete complex intuitionistic preference relation. However, in $\mathfrak{R}_2$, we have

$$r_{24} = \frac{r_{23}r_{34}}{r_{23}r_{34} + (1 - r_{23})(1 - r_{34})} = \frac{0.3 \times 0.6}{0.3 \times 0.6 + (1 - 0.3)(1 - 0.6)} = 0.3913,$$

$$s_{24} = \frac{s_{23}s_{34}}{s_{23}s_{34} + (1 - s_{23})(1 - s_{34})} = \frac{0.1 \times 0.4}{0.1 \times 0.4 + (1 - 0.1)(1 - 0.4)} = 0.0690,$$

$$\omega_{\mu_{24}} = \frac{\omega_{\mu_{23}}\omega_{\mu_{34}}}{\omega_{\mu_{23}}\omega_{\mu_{34}} + (1 - \omega_{\mu_{23}})(1 - \omega_{\mu_{34}})} = \frac{0.5 \times 0.5}{0.5 \times 0.5 + (1 - 0.5)(1 - 0.5)} = 0.5,$$

and

$$\omega_{\nu_{24}} = \frac{\omega_{\nu_{23}}\omega_{\nu_{34}}}{\omega_{\nu_{23}}\omega_{\nu_{34}} + (1 - \omega_{\nu_{23}})(1 - \omega_{\nu_{34}})} = \frac{0.5 \times 0.5}{0.5 \times 0.5 + (1 - 0.5)(1 - 0.5)} = 0.5,$$

obviously, $((0.2,0.4),(0.8,0.5)) \neq ((0.3913,0.5),(0.0690,0.5))$. Hence, $\mathfrak{R}_2$ is not a multiplicative consistent incomplete complex intuitionistic preference relation.

4. Estimation Algorithms for the Acceptable Incomplete Complex Intuitionistic Fuzzy Preference Relations

4.1. The Estimation Algorithm with the Least Judgments

In the following, we shall develop a straightforward algorithm (i.e., Algorithm 1) for estimating the missing elements of an acceptable incomplete complex intuitionistic fuzzy preference relation with the least judgments based on Definition 16 (i.e., there are only $n - 1$ known off-diagonal elements).

Algorithm 1: The estimation algorithm with the least judgments.

Step 1: For a decision-making problem, let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ be a discrete set of alternatives. A decision-maker only compares $n - 1$ pairs of objects, $(\mathcal{V}_i, \mathcal{V}_j)(i = 1, 2, \cdots, n-1,\ j = i+1)$, on the set $\mathcal{V}$ and provides his/her judgements, each of which is expressed as an complex intuitionistic fuzzy number, all the judgements are contained in an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$.

Step 2: Utilize Definition 16 to estimate all the missing elements in $\mathfrak{R}$ using the known elements, and thus get a multiplicative consistent, complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}$ of $\mathfrak{R}$.

Step 3: End.

Example 3. *Consider a decision-making problem with a discrete alternatives set $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$. An expert compares three pairs of objects $(\mathcal{V}_1, \mathcal{V}_2), (\mathcal{V}_2, \mathcal{V}_3), (\mathcal{V}_3, \mathcal{V}_4)$ and provides the preference values over these pairs of objects respectively as follows*

$$\tilde{a}_{12} = ((0.4,0.2),(0.2,0.4)), \tilde{a}_{23} = ((0.6,0.4),(0.3,0.5)), \tilde{a}_{34} = ((0.8,0.5),(0.1,0.5)).$$

Thus, by Definition 11, we can construct an incomplete complex intuitionistic fuzzy preference relation as follows

$$\mathfrak{R} = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.2),(0.2,0.4)) & - & - \\ ((0.2,0.4),(0.4,0.2)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.4),(0.3,0.5)) & - \\ - & ((0.3,0.5),(0.6,0.4)) & ((0.5,0.5),(0.5,0.5)) & ((0.8,0.5),(0.1,0.5)) \\ - & - & ((0.1,0.5),(0.8,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

where "$-$" denotes the unknown element.

Based on the known elements in $\mathfrak{R}$, we can utilize Equations (10)–(13) to estimate all the missing elements as follows

$$r_{13} = \frac{r_{12}r_{23}}{r_{12}r_{23} + (1-r_{12})(1-r_{23})} = \frac{0.4 \times 0.6}{0.4 \times 0.6 + (1-0.4)(1-0.6)} = 0.5,$$

$$s_{13} = \frac{s_{12}s_{23}}{s_{12}s_{23} + (1-s_{12})(1-s_{23})} = \frac{0.2 \times 0.3}{0.2 \times 0.3 + (1-0.2)(1-0.3)} = 0.0968,$$

$$\omega_{\mu_{13}} = \frac{\omega_{\mu_{12}}\omega_{\mu_{23}}}{\omega_{\mu_{12}}\omega_{\mu_{23}} + (1-\omega_{\mu_{12}})(1-\omega_{\mu_{23}})} = \frac{0.2 \times 0.4}{0.2 \times 0.4 + (1-0.2)(1-0.4)} = 0.1429,$$

and

$$\omega_{\nu_{13}} = \frac{\omega_{\nu_{12}}\omega_{\nu_{23}}}{\omega_{\nu_{12}}\omega_{\nu_{23}} + (1-\omega_{\nu_{12}})(1-\omega_{\nu_{23}})} = \frac{0.4 \times 0.5}{0.4 \times 0.5 + (1-0.4)(1-0.5)} = 0.4.$$

That is,

$$\tilde{a}_{13} = ((r_{13}, \omega_{\mu_{13}}), (s_{13}, \omega_{\nu_{13}})) = ((0.5, 0.1429), (0.0968, 0.4)),$$

and

$$\tilde{a}_{31} = ((r_{31}, \omega_{\mu_{31}}), (s_{31}, \omega_{\nu_{31}})) = ((0.0968, 0.4), (0.5, 0.1429)).$$

In a similar way, we can obtain the other missing elements as follows

$$\tilde{a}_{14} = ((0.8, 0.1429), (0.0118, 0.4)),\ \tilde{a}_{41} = ((0.0118, 0.4), (0.8, 0.1429)),$$

and

$$\tilde{a}_{24} = ((0.8571, 0.5), (0.0455, 0.5)),\ \tilde{a}_{42} = ((0.0455, 0.5), (0.8571, 0.5)).$$

Based on all these elements, we can construct a multiplicative consistent, complete complex intuitionistic fuzzy preference relation as follows

$$\bar{\mathfrak{R}} = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.5),(0.2,0.5)) & ((0.5,0.1429),(0.0968,0.4))) & ((0.8,0.1429),(0.0118,0.4)) \\ ((0.2,0.5),(0.4,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.5),(0.3,0.5)) & ((0.8571,0.5),(0.0455,0.5)) \\ ((0.0968,0.4),(0.5,0.1429)) & ((0.3,0.5),(0.6,0.5)) & ((0.5,0.5),(0.5,0.5)) & ((0.8,0.5),(0.1,0.5)) \\ ((0.0118,0.4),(0.8,0.1429)) & ((0.0455,0.5),(0.8571,0.5)) & ((0.1,0.5),(0.8,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

4.2. The Estimation Algorithm with More Known Judgments

Now we consider an acceptable incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ in a more general form, that is, there may exist other known elements in $\mathfrak{R}$. In this case, for all $\tilde{a}_{ik} = ((r_{ik}, \omega_{\mu_{ik}}), (s_{ik}, \omega_{\nu_{ik}})), \tilde{a}_{kj} = ((r_{kj}, \omega_{\mu_{kj}}), (s_{kj}, \omega_{\nu_{kj}})) \in \Omega$ and $i, k, j = 1, 2, \cdots, n$, where Ω is the set of all the known elements in $\mathfrak{R}$, each missing element $\tilde{a}_{ij}$ in $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ can be estimated as $\bar{\tilde{a}}_{ij} = ((\bar{r}_{ij}, \bar{\omega}_{\mu_{ij}}), (\bar{s}_{ij}, \bar{\omega}_{\nu_{ij}}))$.

Let $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ be an acceptable incomplete complex intuitionistic fuzzy preference relation, then, based on Equation (5), each unknown element $\tilde{a}_{ij}$ can be estimated indirectly by using

$$\bar{\tilde{a}}_{ij} = (\bigotimes_{k \in Z_{ij}} (\tilde{a}_{ik} \otimes \tilde{a}_{kj}))^{\frac{1}{z_{ij}}} \tag{14}$$

where $Z_{ij} = \{k|\tilde{a}_{ik}, \tilde{a}_{kj} \in \Omega\}$, z_{ij} is the number of the elements in Z_{ij}. Then we construct a complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}} = (\bar{\tilde{a}}_{ij})_{n\times n}$, where

$$\bar{\tilde{a}}_{ij} = \begin{cases} \tilde{a}_{ij}, & \text{if } \tilde{a}_{ij} \in \Omega, \\ \bar{\tilde{a}}_{ij}, & \text{if } \tilde{a}_{ij} \notin \Omega. \end{cases} \tag{15}$$

Clearly, an unknown element $\tilde{a}_{ij}$ can be estimated if there exists at least one k so that the element $\tilde{a}_{ik}$ and $\tilde{a}_{kj}$ are known. The complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}$ contains both the direct complex intuitionistic fuzzy preference information given by the decision maker and the indirect complex intuitionistic preference information derived from the known complex intuitionistic preference information. The estimation algorithm with more known judgments is shown in Algorithm 2.

Algorithm 2: The estimation algorithm with more known judgments (I).

Step 1: For a decision-making problem, let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ be a discrete set of alternatives. A decision-maker provides their complex intuitionistic fuzzy preference number for each pair of alternatives and constructs an acceptable incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$.
Step 2: Utilize Equation (15) to construct the complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}} = (\bar{\tilde{a}}_{ij})_{n\times n}$ of $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$.
Step 3: End.

Example 4. *Consider a decision making problem with a discrete alternatives set $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$, an expert provides his/her complex intuitionistic fuzzy preference value for each pair of alternatives, and constructs an acceptable incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{3\times 3}$ as follows*

$$\mathfrak{R} = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & - & ((0.7,0.5),(0.2,0.4)) \\ - & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.4),(0.4,0.3)) \\ ((0.2,0.4),(0.7,0.5)) & ((0.4,0.3),(0.5,0.4)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

where "$-$" denotes the unknown element.

Utilize Equation (15) to construct the complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}} = (\bar{\tilde{a}}_{ij})_{3\times 3}$ of $\mathfrak{R} = (\tilde{a}_{ij})_{3\times 3}$

$$\bar{\mathfrak{R}} = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.28,0.15),(0.69,0.64)) & ((0.7,0.5),(0.2,0.4)) \\ ((0.69,0.64),(0.28,0.15)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.4),(0.4,0.3)) \\ ((0.2,0.4),(0.7,0.5)) & ((0.4,0.3),(0.5,0.4)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

Next, we still consider an acceptable incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ in a more general form as above. Then we shall provide another algorithm (i.e., Algorithm 3) to estimate the missing elements $\tilde{a}_{ij}$ in $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$. For all $\tilde{a}_{ik} = ((r_{ik}, \omega_{\mu_{ik}}), (s_{ik}, \omega_{\nu_{ik}})), \tilde{a}_{kj} = ((r_{kj}, \omega_{\mu_{kj}}), (s_{kj}, \omega_{\nu_{kj}})) \in \Omega$ and $i \le k \le j$, where Ω is the set of all the known elements in $\mathfrak{R}$, then each missing element $\tilde{a}_{ij}$ in $\mathfrak{R} = (\tilde{a}_{ij})_{n\times n}$ can be estimated as $\bar{\tilde{a}}'_{ij} = ((\bar{r}'_{ij}, \bar{\omega}'_{\mu_{ij}}), (\bar{s}'_{ij}, \bar{\omega}'_{\nu_{ij}}))$, where

$$\bar{r}'_{ij} = \frac{1}{z_{ij}} \sum_{k\in Z_{ij}} \frac{r_{ik}r_{kj}}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})},$$

$$\bar{s}'_{ij} = \frac{1}{z_{ij}} \sum_{k\in Z_{ij}} \frac{s_{ik}s_{kj}}{s_{ik}s_{kj} + (1-s_{ik})(1-s_{kj})},$$

$$\bar{\omega}'_{\mu_{ij}} = \frac{1}{z_{ij}} \sum_{k\in Z_{ij}} \frac{\omega_{\mu_{ik}}\omega_{\mu_{kj}}}{\omega_{\mu_{ik}}\omega_{\mu_{kj}} + (1-\omega_{\mu_{ik}})(1-\omega_{\mu_{kj}})},$$

and

$$\bar{\omega}'_{\nu_{ij}} = \frac{1}{z_{ij}} \sum_{k \in Z_{ij}} \frac{\omega_{\nu_{ik}} \omega_{\nu_{kj}}}{\omega_{\nu_{ik}} \omega_{\nu_{kj}} + (1 - \omega_{\nu_{ik}})(1 - \omega_{\nu_{kj}})},$$

where $Z_{ij} = \{k | \tilde{a}_{ik}, \tilde{a}_{kj} \in \Omega\}$, z_{ij} is the number of the elements in Z_{ij}. Then we get a complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}' = (\bar{\tilde{a}}'_{ij})_{n \times n}$, where

$$\bar{\tilde{a}}'_{ij} = \begin{cases} \tilde{a}_{ij}, & \text{if } \tilde{a}_{ij} \in \Omega, \\ \bar{\tilde{a}}'_{ij}, & \text{if } \tilde{a}_{ij} \notin \Omega. \end{cases} \tag{16}$$

Algorithm 3: The estimation algorithm with more known judgments (II).

Step 1: It is similar to ***Step 1*** of Algorithm 2.
Step 2: Utilize Equation (16) to construct the complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}' = (\bar{\tilde{a}}'_{ij})_{n \times n}$ of $\mathfrak{R} = (\tilde{a}_{ij})_{n \times n}$.
Step 3: End.

Example 5. *Consider an incomplete complex intuitionistic fuzzy preference relation*

$$\mathfrak{R} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.4,0.3), \\ (0.2,0.4) \end{pmatrix} & \begin{pmatrix} (0.6,0.5), \\ (0.3,0.1) \end{pmatrix} & - & \begin{pmatrix} (0.6,0.3), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.2,0.4), \\ (0.4,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.5), \\ (0.1,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.3), \\ (0.8,0.1) \end{pmatrix} & - \\ \begin{pmatrix} (0.3,0.1), \\ (0.6,0.5) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.3,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.4,0.7) \end{pmatrix} & \begin{pmatrix} (0.7,0.2), \\ (0.2,0.5) \end{pmatrix} \\ - & \begin{pmatrix} (0.8,0.1), \\ (0.2,0.3) \end{pmatrix} & \begin{pmatrix} (0.4,0.7), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.4), \\ (0.2,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1,0.5), \\ (0.6,0.3) \end{pmatrix} & - & \begin{pmatrix} (0.2,0.5), \\ (0.7,0.2) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.6,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

where "−" denotes the unknown element.

Based on the known elements in $\mathfrak{R}$, we can utilize Equation (16) to estimate all the missing elements and construct a complete complex intuitionistic fuzzy preference relation as follows

$$\bar{\mathfrak{R}}' = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.4,0.3), \\ (0.2,0.4) \end{pmatrix} & \begin{pmatrix} (0.6,0.5), \\ (0.3,0.1) \end{pmatrix} & \begin{pmatrix} (0.4176,0.2276), \\ (0.3611,0.1374) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.2,0.4), \\ (0.4,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.5), \\ (0.1,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.3), \\ (0.8,0.1) \end{pmatrix} & \begin{pmatrix} (0.3864,0.2111), \\ (0.2635,0.3) \end{pmatrix} \\ \begin{pmatrix} (0.3,0.1), \\ (0.6,0.5) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.3,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.4,0.7) \end{pmatrix} & \begin{pmatrix} (0.7,0.2), \\ (0.2,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.3611,0.1374), \\ (0.4176,0.2276) \end{pmatrix} & \begin{pmatrix} (0.8,0.1), \\ (0.2,0.3) \end{pmatrix} & \begin{pmatrix} (0.4,0.7), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.4), \\ (0.2,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1,0.5), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.2635,0.3), \\ (0.3864,0.2111) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.7,0.2) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.6,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

5. Group Decision-Making Algorithms Based on Incomplete Complex Intuitionistic Fuzzy Preference Relations

Consider that in the practical decision-making problems, since the increasing complexity and uncertainty of the socioeconomic environment, we notice that group decision-making makes decision-making more scientific and democratic. Therefore, it is necessary to consider group decision-making problems in which there exist multiple incomplete complex intuitionistic fuzzy preference relations. As is well known, the weight of experts reflects the influence of experts on the decision results in the group decision-making problems, which directly determines the feasibility and rationality of the decision results.

Based on the analysis above, in this section, we will first develop an expert weight determination algorithm for the group decision-making problems, and then develop group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations.

First of all, the expert weight determination algorithm is developed and shown in Algorithm 4.

Algorithm 4: The expert weight determination algorithm.

Step 1: Utilize Equation (1) to calculate scoring matrix $\mathcal{S}^{(k)} = (\mathcal{S}(\tilde{a}_{ij}^{(k)}))_{n\times n}$ of $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}(k = 1, 2, \cdots, m)$.

Step 2: Utilize the mean of the score function values $b_{ij} = \frac{1}{m}\sum_{k=1}^{m} \mathcal{S}(\tilde{a}_{ij}^{(k)})(i, j = 1, \ 2, \cdots, n)$ to construct the mean scoring matrix $\bar{\mathcal{S}} = (b_{ij})_{n\times n}$ of the scoring matrix $\mathcal{S}^{(k)} = (\mathcal{S}(\tilde{a}_{ij}^{(k)}))_{n\times n}$.

Step 3: Utilize the mean deviation analysis formula $\mathcal{A}_k = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mid \mathcal{S}(\tilde{a}_{ij}^{(k)}) \ - b_{ij} \mid$ to calculate the weight of the decision-maker, where $w_k = \frac{1-\mathcal{A}_k}{\sum_{k=1}^{m}(1-\mathcal{A}_k)}$, $k = 1, 2, \cdots, m$, $i, j = 1, 2, \cdots, n$, $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$.

Consider a group decision-making problem, let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ be a discrete set of alternatives and $D = \{d_1, d_2, \cdots, d_m\}$ be the set of decision-makers. The decision-maker $d_k \in D$ provides his/her complex intuitionistic fuzzy preference value for each pair of alternatives and constructs a complex intuitionistic fuzzy preference relation $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}$, where $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots, m$. Note that if $\mathfrak{R}^{(k)}$ is an acceptable complete complex intuitionistic fuzzy preference relation, then go to the algorithm. However, if $\mathfrak{R}^{(k)}$ is an acceptable incomplete complex intuitionistic fuzzy preference relation, then we can utilize Algorithms 1–3 proposed above to construct a new acceptable complete complex intuitionistic fuzzy preference relation.

Example 6 is used to illustrate the expert weight determination algorithm.

Example 6. *Suppose that a committee of three experts $D = \{d_1, d_2, d_3\}$ is set up to assess the alternatives $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$. Each expert d_k conduct pairwise comparisons among alternatives and gave their judgement values in an acceptable complete complex intuitionistic fuzzy preference relation $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{3\times 3}$, where $\tilde{a}_{ij}^{(k)} = ((r_{ij}^{(k)}, \omega_{\mu_{ij}}^{(k)}), (s_{ij}^{(k)}, \omega_{\nu_{ij}}^{(k)}))$ expresses the preference degree of alternative $\mathcal{V}_i$ over $\mathcal{V}_j$ for each $i, j = 1, 2, 3$ and $k = 1, 2, 3$.*

$$\mathfrak{R}^1 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.3),(0.5,0.5)) & ((0.8,0.6),(0.1,0.3)) \\ ((0.5,0.5),(0.4,0.3)) & ((0.5,0.5),(0.5,0.5)) & ((0.4,0.3),(0.5,0.6)) \\ ((0.1,0.3),(0.8,0.6)) & ((0.5,0.6),(0.4,0.3)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

$$\mathfrak{R}^2 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.4,0.2),(0.5,0.5)) & ((0.5,0.2),(0.2,0.4)) \\ ((0.5,0.5),(0.4,0.2)) & ((0.5,0.5),(0.5,0.5)) & ((0.6,0.5),(0.2,0.4)) \\ ((0.2,0.4),(0.5,0.2)) & ((0.2,0.4),(0.6,0.5)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

$$\mathfrak{R}^3 = \begin{bmatrix} ((0.5,0.5),(0.5,0.5)) & ((0.6,0.4),(0.3,0.5)) & ((0.7,0.5),(0.2,0.4)) \\ ((0.3,0.5),(0.6,0.4)) & ((0.5,0.5),(0.5,0.5)) & ((0.5,0.4),(0.4,0.3)) \\ ((0.2,0.4),(0.7,0.5)) & ((0.4,0.3),(0.5,0.4)) & ((0.5,0.5),(0.5,0.5)) \end{bmatrix}$$

Step 1: *Utilize Equation (1) to calculate scoring matrix $\mathcal{S}^{(k)} = (\mathcal{S}(\tilde{a}_{ij}^{(k)}))_{3\times 3}$ of $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{3\times 3}(k = 1, 2, 3)$ as follows*

$$\mathcal{S}^{(1)} = \begin{bmatrix} 0 & -0.3 & 1 \\ 0.3 & 0 & -0.4 \\ -1 & 0.4 & 0 \end{bmatrix}$$

$$\mathcal{S}^{(2)} = \begin{bmatrix} 0 & -0.4 & 0.1 \\ 0.4 & 0 & 0.5 \\ -0.1 & -0.5 & 0 \end{bmatrix}$$

$$\mathcal{S}^{(3)} = \begin{bmatrix} 0 & 0.2 & 0.6 \\ -0.2 & 0 & 0.2 \\ -0.6 & -0.2 & 0 \end{bmatrix}$$

Step 2: *Utilize the mean of the score function values* $b_{ij} = \frac{1}{3}\sum_{k=1}^{3} \mathcal{S}(\tilde{a}_{ij}^{(k)})(i, j = 1, 2, 3)$ *to construct the mean scoring matrix* $\bar{\mathcal{S}} = (b_{ij})_{3\times 3}$ *of the scoring matrix* $\mathcal{S}^{(k)} = (\mathcal{S}(\tilde{a}_{ij}^{(k)}))_{3\times 3}$ *as follows*

$$\bar{\mathcal{S}} = \begin{bmatrix} 0 & -0.17 & 0.57 \\ 0.17 & 0 & 0.1 \\ -0.57 & -0.1 & 0 \end{bmatrix}$$

Step 3: *Utilize the mean deviation analysis formula* $\mathcal{A}_k = \frac{1}{3^2}\sum_{i=1}^{3}\sum_{j=1}^{3} \mid \mathcal{S}(\tilde{a}_{ij}^{(k)}) - b_{ij} \mid$ *to calculate the weight of the decision-maker as follows*

$$w = (0.3986, 0.4134, 0.1880)^T,$$

where $w_k = \frac{1-\mathcal{A}_k}{\sum_{k=1}^{3}(1-\mathcal{A}_k)}$, $k = 1, 2, 3$, $i, j = 1, 2, 3$, $w_k \geq 0$ *and* $\sum_{k=1}^{3} w_k = 1$.

Finally, we will develop two group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations, which are shown in Algorithms 5 and 6.

Algorithm 5: Group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations (I).

Step 1: Utilize Algorithm 2 to construct the complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}^{(k)} = (\bar{\tilde{a}}_{ij}^{(k)})_{n\times n}$ of $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}(k = 1, 2, \cdots, m)$.
Step 2: Utilize the complex intuitionistic fuzzy arithmetic averaging operator $\tilde{b}_i^{(k)} = \frac{1}{n}\bigoplus_{j=1}^{n} \bar{\tilde{a}}_{ij}^{(k)}$, $i = 1, 2, \cdots, n$ to aggregate all $\bar{\tilde{a}}_{ij}^{(k)}(j = 1, 2, \cdots, n)$ corresponding to the alternative $\mathcal{V}_i$, and then get the averaged complex intuitionistic fuzzy value $\tilde{b}_i^{(k)}$ of the alternative $\mathcal{V}_i$ over all the other alternatives.
Step 3: Calculate the weight vector $w = (w_1, w_2, \cdots, w_m)^T$ of decision-makers using Algorithm 4.
Step 4: Utilize the complex intuitionistic fuzzy weighted arithmetic averaging operator $\tilde{b}_i = \bigoplus_{k=1}^{m} w_k \tilde{b}_i^{(k)}(i = 1, 2, \cdots, n)$ to aggregate all $\tilde{b}_i^{(k)}(k = 1, 2, \cdots, m)$ corresponding to m decision-makers into a collective complex intuitionistic fuzzy value $\tilde{b}_i$ of the alternative $\mathcal{V}_i$ over all the other alternatives.
Step 5: Rank all the $\tilde{b}_i(i = 1, 2, \cdots, n)$ by means of the score Function (1) and the accuracy Function (2), and then rank all the alternatives $\mathcal{V}_i(i = 1, 2, \cdots, n)$ and select the best one in accordance with the values of $\tilde{b}_i$.

Consider a group decision-making problem, let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_n\}$ be a discrete set of alternatives and $D = \{d_1, d_2, \cdots, d_m\}$ be the set of decision-makers. The decision-maker $d_k \in D$ provides his/her complex intuitionistic fuzzy preference value for each pair of alternatives, and constructs an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}$, where $\tilde{a}_{ij}^{(k)} = ((r_{ij}^{(k)}, \omega_{\mu_{ij}}^{(k)}), (s_{ij}^{(k)}, \omega_{\nu_{ij}}^{(k)})), r_{ij}^{(k)}, s_{ij}^{(k)}, \omega_{\mu_{ij}}^{(k)}, \omega_{\nu_{ij}}^{(k)}, r_{ij}^{(k)} + s_{ij}^{(k)}, \omega_{\mu_{ij}}^{(k)} + \omega_{\nu_{ij}}^{(k)} \in [0, 1], r_{ji}^{(k)} = s_{ij}^{(k)}, s_{ji}^{(k)} = r_{ij}^{(k)}, \omega_{\mu_{ji}}^{(k)} = \omega_{\nu_{ij}}^{(k)}, \omega_{\nu_{ji}}^{(k)} = \omega_{\mu_{ij}}^{(k)}, r_{ii}^{(k)} = s_{ii}^{(k)} = \omega_{\mu_{ii}}^{(k)} = \omega_{\nu_{ii}}^{(k)} = 0.5$ for all $i, j = 1, 2, \cdots, n$ and $\tilde{a}_{ij}^{(k)} \in \Omega$. If $\mathfrak{R}^{(k)}$ is an acceptable incomplete complex intuitionistic fuzzy preference relation, then go to the algorithm. However, if $\mathfrak{R}^{(k)}$ is an unacceptable incomplete complex intuitionistic fuzzy preference relation (this kind of situation is not very common in real problems), then the decision-maker needs to construct a new incomplete complex intuitionistic fuzzy preference relation, and follow this procedure until it is acceptable.

Algorithm 6: Group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relations (II).

Step 1: Utilize Algorithm 3 to construct the complete complex intuitionistic fuzzy preference relation $\bar{\mathfrak{R}}'^{(k)} = (\bar{\tilde{a}}_{ij}'^{(k)})_{n\times n}$ of $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times n}(k = 1, 2, \cdots, m)$.

Step 2: Calculate the weight vector $w' = (w_1', w_2', \cdots, w_m')^T$ of decision-makers using Algorithm 4.

Step 3: Utilize the complex intuitionistic fuzzy weighted averaging operator to aggregate all individual complete complex intuitionistic fuzzy preference relations $\bar{\mathfrak{R}}'^{(k)}$ $(k = 1, 2, \cdots, m)$ together with the experts' weights w_k' $(k = 1, 2, \cdots, m)$ into the collective complete complex intuitionistic fuzzy preference relation $\mathfrak{R}' = (\tilde{a}_{ij}')_{n\times n}$ with $\tilde{a}_{ij}' = ((r_{ij}', \omega_{\mu_{ij}}'), (s_{ij}', \omega_{\nu_{ij}}'))$, where $r_{ij}' = \sum_{k=1}^{m} w_k' \bar{r}_{ij}'^{(k)}, \omega_{\mu_{ij}}' = \sum_{k=1}^{m} w_k' \bar{\omega}_{\mu_{ij}}'^{(k)}, s_{ij}' = \sum_{k=1}^{m} w_k' \bar{s}_{ij}'^{(k)}, \omega_{\nu_{ij}}' = \sum_{k=1}^{m} w_k' \bar{\omega}_{\nu_{ij}}'^{(k)},\ \ i, j = 1, 2, \cdots, n.$

Step 4: Utilize the complex intuitionistic fuzzy averaging operator $r_i = \frac{1}{n}\sum_{j=1}^{n} r_{ij}'$, $\omega_{\mu_i} = \frac{1}{n}\sum_{j=1}^{n} \omega_{\mu_{ij}}', s_i = \frac{1}{n}\sum_{j=1}^{n} s_{ij}', \omega_{\nu_i} = \frac{1}{n}\sum_{j=1}^{n} \omega_{\nu_{ij}}'$ to aggregate all $\tilde{a}_{ij}'$ corresponding to m decision-makers into a collective complex intuitionistic fuzzy value $\tilde{b}_i' = ((r_i, \omega_{\mu_i}), (s_i, \omega_{\nu_i}))$ of the alternative $\mathcal{V}_i$ over all the other alternatives.

Step 5: Rank all the $\tilde{b}_i'(i = 1, 2, \cdots, n)$ by means of the score Function (1) and the accuracy Function (2), and then rank all the alternatives $\mathcal{V}_i(i = 1, 2, \cdots, n)$ and select the best one in accordance with the values of $\tilde{b}_i'$.

Remark 2. *In an incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}$, when the phase terms of membership and non-membership of element $\tilde{a}_{ij}$ are equal to zero (i.e., $\omega_{\mu_{ij}} = 0$ and $\omega_{\nu_{ij}} = 0$), that is, the periodic change of uncertain information is not considered, then the incomplete complex intuitionistic fuzzy preference relation effectively reduces to an incomplete intuitionistic preference relation. Meanwhile, the framework proposed in this paper is consistent with the conclusion of the classical incomplete intuitionistic preference relation [19]. Therefore, it has more obvious advantages than the classical intuitionistic fuzzy evaluation when considering the periodic conditions of uncertain information change.*

Remark 3. *Since the complex intuitionistic fuzzy number is composed of four quantities, i.e., amplitude terms and phase terms of both membership degree and non-membership degree. Therefore, the algorithms proposed in this paper have a large amount of computation. We mainly use MATLAB software for numerical calculation.*

6. Illustrative Example

In this section, we will utilize a practical example to illustrate the solution processes of the group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relation, verify their practicality, compare and analyse the advantages and disadvantages of two group decision-making algorithms. Finally, the simulation verification of complex intuitionistic fuzzy system proposed in this paper is carried out by MATLAB software.

Example 7. *Since 2019, the COVID-19 pandemic has been raging. In response to the call of the Ministry of Education to "suspend classes and continue learning", the online teaching models have rapidly spread due to its advantages such as distance teaching and abundant teaching resources. An online teaching experience consists of computer-assisted teaching media, various computer tools and software. Therefore, there is an intense need to choose the online teaching platform carefully by keeping in view the various factors, such as cost, media characteristics, ease of use, flexibility, and so forth.*

Consider that an educational institute is interested in ranking five online teaching platform, namely, $\mathcal{V}_1$: *Cisco Webex,* $\mathcal{V}_2$: *Zoom,* $\mathcal{V}_3$: *Google classroom,* $\mathcal{V}_4$: *Tencent conference and* $\mathcal{V}_5$: *Coursera.*

Suppose that a committee of four experts $D = \{d_1, d_2, d_3, d_4\}$ *is set up to assess these alternatives* $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5\}$. *On the basis of their influence on student's academic achievement and the time taken by them for this influence, each expert* d_k *conduct pairwise comparisons among alternatives and gave their judgement values in an acceptable incomplete complex intuitionistic fuzzy preference relation* $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{5\times 5}$, *where* $\tilde{a}_{ij}^{(k)} = ((r_{ij}^{(k)}, \omega_{\mu_{ij}}^{(k)}), (s_{ij}^{(k)}, \omega_{\nu_{ij}}^{(k)}))$ *expresses the preference degree of alternative* $\mathcal{V}_i$ *over* $\mathcal{V}_j$ *for each* $i, j = 1, 2, 3, 4, 5$ *and* $k = 1, 2, 3, 4$.

The acceptable incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ *are given by decision-makers as follows*

$$\mathfrak{R}^{(1)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.6,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.5,0.1)\end{pmatrix} & - & \begin{pmatrix}(0.7,0.3),\\(0.3,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.6,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.1,0.1)\end{pmatrix} & - \\ \begin{pmatrix}(0.5,0.1),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.7,0.3),\\(0.2,0.7)\end{pmatrix} & \begin{pmatrix}(0.5,0.2),\\(0.2,0.5)\end{pmatrix} \\ - & \begin{pmatrix}(0.1,0.1),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.7,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.8,0.4),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.5),\\(0.7,0.3)\end{pmatrix} & - & \begin{pmatrix}(0.2,0.5),\\(0.5,0.2)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.8,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\mathfrak{R}^{(2)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.5,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.4,0.1)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.1,0.5)\end{pmatrix} & - \\ \begin{pmatrix}(0.5,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.7,0.5),\\(0.2,0.5)\end{pmatrix} & - & \begin{pmatrix}(0.6,0.3),\\(0.2,0.1)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.1),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.7,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.2,0.7)\end{pmatrix} & \begin{pmatrix}(0.8,0.2),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.5),\\(0.8,0.3)\end{pmatrix} & - & \begin{pmatrix}(0.2,0.7),\\(0.8,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} \\ - & \begin{pmatrix}(0.2,0.1),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.8,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\mathfrak{R}^{(3)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.5,0.4)\end{pmatrix} & - & \begin{pmatrix}(0.4,0.5),\\(0.4,0.1)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.3),\\(0.1,0.1)\end{pmatrix} & - \\ - & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.3,0.7)\end{pmatrix} & \begin{pmatrix}(0.7,0.2),\\(0.3,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.1),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.1,0.1),\\(0.9,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.4),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.5),\\(0.8,0.3)\end{pmatrix} & - & \begin{pmatrix}(0.3,0.5),\\(0.7,0.2)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.9,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\mathfrak{R}^{(4)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.3),\\(0.6,0.4)\end{pmatrix} & - & - & - \\ \begin{pmatrix}(0.6,0.4),\\(0.4,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & - & - \\ - & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.2,0.7)\end{pmatrix} & - \\ - & - & \begin{pmatrix}(0.2,0.7),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.4),\\(0.1,0.5)\end{pmatrix} \\ - & - & - & \begin{pmatrix}(0.1,0.5),\\(0.9,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

where "−" denotes the unknown variable.

Firstly, we use Algorithm 5 to prioritize the online course platform, which involves the following steps.

Step 1. *Utilize Algorithm 2 to construct the complete complex intuitionistic fuzzy preference relations* $\bar{\mathfrak{R}}^{(k)} = (\bar{\tilde{a}}_{ij}^{(k)})_{5\times 5}$ *of* $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{5\times 5}(k = 1,2,3,4)$ *as follows*

$$\bar{\mathfrak{R}}^{(1)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.6,0.4)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.5,0.1)\end{pmatrix} & \begin{pmatrix}(0.1518,0.1260),\\(0.7277,0.6478)\end{pmatrix} & \begin{pmatrix}(0.7,0.3),\\(0.3,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.6,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.1,0.1)\end{pmatrix} & \begin{pmatrix}(0.2723,0.1119),\\(0.4246,0.6597)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.1),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.7,0.3),\\(0.2,0.7)\end{pmatrix} & \begin{pmatrix}(0.5,0.2),\\(0.2,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.7277,0.6478),\\(0.1518,0.1260)\end{pmatrix} & \begin{pmatrix}(0.1,0.1),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.7,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.8,0.4),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.3,0.5),\\(0.7,0.3)\end{pmatrix} & \begin{pmatrix}(0.4246,0.6597),\\(0.2723,0.1119)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.5,0.2)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.8,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}^{(2)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.5,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.4,0.1)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.1,0.5)\end{pmatrix} & \begin{pmatrix}(0.2586,0.1032),\\(0.5115,0.6070)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.7,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.3775,0.1392),\\(0.3911,0.6953)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.2,0.1)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.1),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.7,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.2,0.7)\end{pmatrix} & \begin{pmatrix}(0.8,0.2),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.5),\\(0.8,0.3)\end{pmatrix} & \begin{pmatrix}(0.3911,0.6953),\\(0.3775,0.1392)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.8,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.4),\\(0.4,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.5112,0.6070),\\(0.2586,0.1032)\end{pmatrix} & \begin{pmatrix}(0.2,0.1),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.8,0.2)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.3,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}^{(3)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.3,0.3),\\(0.5,0.4)\end{pmatrix} & \begin{pmatrix}(0.1193,0.1992),\\(0.7313,0.5772)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.4,0.1)\end{pmatrix} & \begin{pmatrix}(0.8,0.3),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.5,0.4),\\(0.3,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.3),\\(0.1,0.1)\end{pmatrix} & \begin{pmatrix}(0.3567,0.1119),\\(0.4016,0.6597)\end{pmatrix} \\ \begin{pmatrix}(0.7313,0.5772),\\(0.1193,0.1992)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.3,0.7)\end{pmatrix} & \begin{pmatrix}(0.7,0.2),\\(0.3,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.4,0.1),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.1,0.1),\\(0.9,0.3)\end{pmatrix} & \begin{pmatrix}(0.3,0.7),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.4),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.1,0.5),\\(0.8,0.3)\end{pmatrix} & \begin{pmatrix}(0.4016,0.6597),\\(0.3567,0.1119)\end{pmatrix} & \begin{pmatrix}(0.3,0.5),\\(0.7,0.2)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.9,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}^{(4)} = \begin{bmatrix} \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.4,0.3),\\(0.6,0.4)\end{pmatrix} & \begin{pmatrix}(0.0800,0.1500),\\(0.7600,0.700)\end{pmatrix} & \begin{pmatrix}(0.0480,0.0450),\\(0.8080,0.9100)\end{pmatrix} & \begin{pmatrix}(0.0432,0.0180),\\(0.8272,0.9550)\end{pmatrix} \\ \begin{pmatrix}(0.6,0.4),\\(0.4,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.2,0.5),\\(0.4,0.5)\end{pmatrix} & \begin{pmatrix}(0.0592,0.0520),\\(0.7648,0.9030)\end{pmatrix} & \begin{pmatrix}(0.0374,0.0122),\\(0.8520,0.9613)\end{pmatrix} \\ \begin{pmatrix}(0.7600,0.700),\\(0.0800,0.1500)\end{pmatrix} & \begin{pmatrix}(0.4,0.5),\\(0.2,0.5)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.6,0.3),\\(0.2,0.7)\end{pmatrix} & \begin{pmatrix}(0.0642,0.0208),\\(0.7613,0.9536)\end{pmatrix} \\ \begin{pmatrix}(0.8080,0.9100),\\((0.0480,0.0450)\end{pmatrix} & \begin{pmatrix}(0.7648,0.9030),\\(0.0592,0.0520)\end{pmatrix} & \begin{pmatrix}(0.2,0.7),\\(0.6,0.3)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} & \begin{pmatrix}(0.9,0.4),\\(0.1,0.5)\end{pmatrix} \\ \begin{pmatrix}(0.8272,0.9550),\\(0.0432,0.0180)\end{pmatrix} & \begin{pmatrix}(0.8520,0.9613),\\(0.0374,0.0122)\end{pmatrix} & \begin{pmatrix}(0.7613,0.9536),\\(0.0642,0.0208)\end{pmatrix} & \begin{pmatrix}(0.1,0.5),\\(0.9,0.4)\end{pmatrix} & \begin{pmatrix}(0.5,0.5),\\(0.5,0.5)\end{pmatrix} \end{bmatrix}$$

Step 2. *Utilize the complex intuitionistic fuzzy arithmetic averaging operator*

$$\tilde{b}_i^{(k)} = \frac{1}{5}\bigoplus_{j=1}^{5}\bar{\tilde{a}}_{ij}^{(k)}, \quad i = 1,2,3,4,5$$

to aggregate all $\bar{\tilde{a}}_{ij}^{(k)}(j = 1,2,3,4,5)$ *corresponding to the alternative* $\mathcal{V}_i$*, and then get the averaged complex intuitionistic fuzzy value* $\tilde{b}_i^{(k)}$ *of the alternative* $\mathcal{V}_i$ *over all the other alternatives as follows*

$$\tilde{b}_1^{(1)} = ((0.4434, 0.3604), (0.5047, 0.4816)), \tilde{b}_2^{(1)} = ((0.4585, 0.3778), (0.3028, 0.3458)),$$

$$\tilde{b}_3^{(1)} = ((0.5318, 0.3392), (0.2759, 0.5348)), \tilde{b}_4^{(1)} = ((0.5445, 0.5090), (0.3167, 0.3094)),$$

$$\tilde{b}_5^{(1)} = ((0.3204, 0.5370), (0.5203, 0.2664)), \tilde{b}_1^{(2)} = ((0.5182, 0.3571), (0.3842, 0.3603)),$$

$$\tilde{b}_2^{(2)} = ((0.5489, 0.3817), (0.2979, 0.3495)), \tilde{b}_3^{(2)} = ((0.6051, 0.3392), (0.3227, 0.5348)),$$

$$\tilde{b}_4^{(2)} = ((0.3126, 0.5760), (0.5455, 0.3156)), \tilde{b}_5^{(2)} = ((0.3622, 0.4641), (0.4508, 0.2622)),$$

$$\tilde{b}_1^{(3)} = ((0.4829, 0.3715), (0.3740, 0.3567)), \tilde{b}_2^{(3)} = ((0.5813, 0.3778), (0.2995, 0.3499)),$$

$$\tilde{b}_3^{(3)} = ((0.6045, 0.4319), (0.2548, 0.4449)), \tilde{b}_4^{(3)} = ((0.5478, 0.4077), (0.4043, 4076)),$$

$$\tilde{b}_5^{(3)} = ((0.2987, 0.5370), (0.6176, 0.2664)), \tilde{b}_1^{(4)} = ((0.2413, 0.2253), (0.6864, 0.6562)),$$

$$\tilde{b}_2^{(4)} = ((0.3205, 0.3247), (0.5539, 0.5791)), \tilde{b}_3^{(4)} = ((0.5146, 0.4477), (0.2613, 0.4783)),$$

$$\tilde{b}_4^{(4)} = ((0.7173, 0.7606), (0.1535, 0.1774)), \tilde{b}_5^{(4)} = ((0.6926, 0.8849), (0.1361, 0.0620)).$$

Step 3. *Utilize Algorithm 4 to calculate the weight vector of decision-makers as follows*

$$w = (0.2833, 0.3039, 0.3243, 0.0885)^T.$$

Step 4. *Utilize the complex intuitionistic fuzzy weighted arithmetic averaging operator*

$$\tilde{b}_i = \bigoplus_{k=1}^{4} w_k \tilde{b}_i^{(k)}, \quad i = 1, 2, 3, 4, 5$$

to aggregate all $\tilde{b}_i^{(k)} (k = 1, 2, 3, 4)$ *corresponding to four decision-makers into a collective complex intuitionistic fuzzy value* $\tilde{b}_i$ *of the alternative* $\mathcal{V}_i$ *over all the other alternatives as follows*

$$\tilde{b}_1 = ((0.4654, 0.3521), (0.4331, 0.4111)), \tilde{b}_2 = ((0.5192, 0.3744), (0.3167, 0.3645)),$$

$$\tilde{b}_3 = ((0.5777, 0.3807), (0.2806, 0.4988)), \tilde{b}_4 = ((0.5063, 0.5316), (0.3793, 0.3215)),$$

$$\tilde{b}_5 = ((0.3722, 0.5721), (0.4677, 0.2331)).$$

Step 5. *Rank all the* $\tilde{b}_i (i = 1, 2, 3, 4, 5)$ *by means of the score Function (1) and the accuracy Function (2), and then rank all the alternatives* $\mathcal{V}_i (i = 1, 2, 3, 4, 5)$ *and select the best one in accordance with the values of* $\tilde{b}_i (i = 1, 2, 3)$. *Since*

$$\mathcal{S}(\tilde{b}_1) = -0.0267, \mathcal{S}(\tilde{b}_2) = 0.2124, \mathcal{S}(\tilde{b}_3) = 0.1790, \mathcal{S}(\tilde{b}_4) = 0.3371, \mathcal{S}(\tilde{b}_5) = 0.2345,$$

hence, the ranking order of alternatives is

$$\mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_1.$$

Finally, we use Algorithm 6 to prioritize the online course platform, which involves the following steps.

Step 1. *Utilize Algorithm 3 to construct the complete complex intuitionistic fuzzy preference relations* $\bar{\mathfrak{R}}'^{(k)} = (\bar{\tilde{a}}_{ij}'^{(k)})_{5\times 5}$ *of* $\mathfrak{R}^{(k)} = (\tilde{a}_{ij}^{(k)})_{5\times 5} (k = 1, 2, 3, 4)$ *as follows*

$$\bar{\mathfrak{R}}'^{(1)} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.3), \\ (0.6,0.4) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.5,0.1) \end{pmatrix} & \begin{pmatrix} (0.5,0.2276), \\ (0.1714,0.1374) \end{pmatrix} & \begin{pmatrix} (0.3,0.7), \\ (0.3,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.6,0.4), \\ (0.3,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.4,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.1,0.1) \end{pmatrix} & \begin{pmatrix} (0.5286,0.2111), \\ (0.0775,0.3) \end{pmatrix} \\ \begin{pmatrix} (0.5,0.1), \\ (0.4,0.5) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.2,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.7,0.3), \\ (0.2,0.7) \end{pmatrix} & \begin{pmatrix} (0.5,0.2), \\ (0.2,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1714,0.1374), \\ (0.5,0.2276) \end{pmatrix} & \begin{pmatrix} (0.1,0.1), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.2,0.7), \\ (0.7,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.8,0.4), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.3,0.5), \\ (0.7,0.3) \end{pmatrix} & \begin{pmatrix} (0.0775,0.3), \\ (0.5286,0.2111) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.5,0.2) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.8,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}'^{(2)} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.3), \\ (0.5,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.4,0.1) \end{pmatrix} & \begin{pmatrix} (0.8,0.3), \\ (0.1,0.5) \end{pmatrix} & \begin{pmatrix} (0.6076,0.1925), \\ (0.1126,0.2230) \end{pmatrix} \\ \begin{pmatrix} (0.5,0.4), \\ (0.3,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.7,0.5), \\ (0.2,0.5) \end{pmatrix} & \begin{pmatrix} (0.9032,0.3)), \\ (0.0588,0.7) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.2,0.1) \end{pmatrix} \\ \begin{pmatrix} (0.4,0.1), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.7,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.8,0.3), \\ (0.2,0.7) \end{pmatrix} & \begin{pmatrix} (0.8,0.2), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1,0.5), \\ (0.8,0.3) \end{pmatrix} & \begin{pmatrix} (0.0588,0.7), \\ (0.9032,0.3) \end{pmatrix} & \begin{pmatrix} (0.2,0.7), \\ (0.8,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.4), \\ (0.4,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1126,0.2230), \\ (0.6076,0.1925) \end{pmatrix} & \begin{pmatrix} (0.2,0.1), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.8,0.2) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.3,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}'^{(3)} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3,0.3), \\ (0.5,0.4) \end{pmatrix} & \begin{pmatrix} (0.0968,0.3), \\ (0.4,0.4) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.4,0.1) \end{pmatrix} & \begin{pmatrix} (0.8,0.3), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.5,0.4), \\ (0.3,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.4,0.5) \end{pmatrix} & \begin{pmatrix} (0.9,0.3)), \\ (0.1,0.1) \end{pmatrix} & \begin{pmatrix} (0.6781,0.2111), \\ (0.1172,0.3) \end{pmatrix} \\ \begin{pmatrix} (0.4,0.4), \\ (0.0968,0.3) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.2,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.3,0.7) \end{pmatrix} & \begin{pmatrix} (0.7,0.2), \\ (0.3,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.4,0.1), \\ (0.4,0.5) \end{pmatrix} & \begin{pmatrix} (0.1,0.1), \\ (0.9,0.3) \end{pmatrix} & \begin{pmatrix} (0.3,0.7), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.9,0.4), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1,0.5), \\ (0.8,0.3) \end{pmatrix} & \begin{pmatrix} (0.1172,0.3), \\ (0.6781,0.2111) \end{pmatrix} & \begin{pmatrix} (0.3,0.5), \\ (0.7,0.2) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.9,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

$$\bar{\mathfrak{R}}'^{(4)} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.4,0.3), \\ (0.6,0.4) \end{pmatrix} & \begin{pmatrix} (0.1429,0.3), \\ (0.5,0.4) \end{pmatrix} & \begin{pmatrix} (0.2001,0.1552), \\ (0.2,0.6087) \end{pmatrix} & \begin{pmatrix} (0.6924,0.1091), \\ (0.027,0.6087) \end{pmatrix} \\ \begin{pmatrix} (0.6,0.4), \\ (0.4,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.2,0.5), \\ (0.4,0.5) \end{pmatrix} & \begin{pmatrix} (0.2727,0.3)), \\ (0.1429,0.7) \end{pmatrix} & \begin{pmatrix} (0.7714,0.2222), \\ (0.0182,0.7) \end{pmatrix} \\ \begin{pmatrix} (0.5,0.4), \\ (0.1429,0.3) \end{pmatrix} & \begin{pmatrix} (0.4,0.5), \\ (0.2,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6,0.3), \\ (0.2,0.7) \end{pmatrix} & \begin{pmatrix} (0.931,0.2222), \\ (0.027,0.7) \end{pmatrix} \\ \begin{pmatrix} (0.2,0.6087), \\ (0.2001,0.1552) \end{pmatrix} & \begin{pmatrix} (0.1429,0.7), \\ (0.2727,0.3) \end{pmatrix} & \begin{pmatrix} (0.2,0.7), \\ (0.6,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.9,0.4), \\ (0.1,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.027,0.6087), \\ (0.6924,0.1091) \end{pmatrix} & \begin{pmatrix} (0.0182,0.7), \\ (0.7714,0.2222) \end{pmatrix} & \begin{pmatrix} (0.027,0.7), \\ (0.931,0.2222) \end{pmatrix} & \begin{pmatrix} (0.1,0.5), \\ (0.9,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

Step 2. *Utilize Algorithm 4 to calculate the weight vector of decision-makers as follows*

$$w^{'} = (0.2693, 0.2334, 0.2587, 0.2386)^{T}.$$

Step 3. *Utilize the complex intuitionistic fuzzy weighted averaging operator to aggregate all individual complete complex intuitionistic fuzzy preference relations $\bar{\mathfrak{R}}^{'(k)}$ $(k = 1, 2, 3, 4)$ together with the experts' weights w_k $(k = 1, 2, 3, 4)$ into the collective complete complex intuitionistic fuzzy preference $\mathfrak{R}^{'} = (\tilde{a}^{'}_{ij})_{5\times 5}$ as follows*

$$\mathfrak{R}^{'} = \begin{bmatrix} \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3239,0.3), \\ (0.5507,0.4) \end{pmatrix} & \begin{pmatrix} (0.2836,0.4005), \\ (0.4508,0.2492) \end{pmatrix} & \begin{pmatrix} (0.4726,0.2977), \\ (0.2207,0.3248) \end{pmatrix} & \begin{pmatrix} (0.5948,0.3371), \\ (0.1394,0.4613) \end{pmatrix} \\ \begin{pmatrix} (0.5507,0.4), \\ (0.3239,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.3167,0.5), \\ (0.3533,0.5) \end{pmatrix} & \begin{pmatrix} (0.6703,0.3)), \\ (0.1006,0.3832) \end{pmatrix} & \begin{pmatrix} (0.6419,0.2345), \\ (0.1022,0.3488) \end{pmatrix} \\ \begin{pmatrix} (0.4508,0.2492), \\ (0.2836,0.4005) \end{pmatrix} & \begin{pmatrix} (0.3533,0.5), \\ (0.3167,0.5) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.6736,0.3), \\ (0.2259,0.7) \end{pmatrix} & \begin{pmatrix} (0.7246,0.2053), \\ (0.1613,0.5477) \end{pmatrix} \\ \begin{pmatrix} (0.2207,0.3248), \\ (0.4726,0.2977) \end{pmatrix} & \begin{pmatrix} (0.1006,0.3832), \\ (0.6703,0.3) \end{pmatrix} & \begin{pmatrix} (0.2259,0.7), \\ (0.6736,0.3) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} & \begin{pmatrix} (0.7330,0.4), \\ (0.1700,0.5) \end{pmatrix} \\ \begin{pmatrix} (0.1394,0.4613), \\ (0.5948,0.3371) \end{pmatrix} & \begin{pmatrix} (0.1022,0.3488), \\ (0.6419,0.2345) \end{pmatrix} & \begin{pmatrix} (0.1613,0.5477), \\ (0.7246,0.2053) \end{pmatrix} & \begin{pmatrix} (0.1700,0.5), \\ (0.7330,0.4) \end{pmatrix} & \begin{pmatrix} (0.5,0.5), \\ (0.5,0.5) \end{pmatrix} \end{bmatrix}$$

Step 4. *Utilize the complex intuitionistic fuzzy averaging operator*

$$r_i = \frac{1}{5}\sum_{j=1}^{5} r^{'}_{ij}, \omega_{\mu_i} = \frac{1}{5}\sum_{j=1}^{5} \omega^{'}_{\mu_{ij}}, s_i = \frac{1}{5}\sum_{j=1}^{5} s^{'}_{ij}, \omega_{\nu_i} = \frac{1}{5}\sum_{j=1}^{5} \omega^{'}_{\nu_{ij}}, i = 1, 2, 3, 4, 5$$

to aggregate all $\tilde{a}^{'}_{ij}$ corresponding to four decision-makers into a collective complex intuitionistic fuzzy value $\tilde{b}^{'}_{i} = ((r_i, \omega_{\mu_i}), (s_i, \omega_{\nu_i}))$ of the alternative $\mathcal{V}_i$ over all the other alternatives as follows

$$\tilde{b}^{'}_{1} = ((0.4350, 0.3671), (0.3723, 0.3871)), \tilde{b}^{'}_{2} = ((0.5359, 0.3869), (0.2760, 0.4064)),$$

$$\tilde{b}^{'}_{3} = ((0.5405, 0.3509), (0.2975, 0.5296)), \tilde{b}^{'}_{4} = ((0.3560, 0.4616), (0.4973, 0.3795)),$$

$$\tilde{b}^{'}_{5} = ((0.2146, 0.4716), (0.6389, 0.3354)).$$

Step 5. *Rank all the $\tilde{b}^{'}_{i}(i = 1, 2, 3, 4, 5)$ by means of the score Function (4) and the accuracy Function (5), and then rank all the alternatives $\mathcal{V}_i(i = 1, 2, 3, 4, 5)$ and select the best one in accordance with the values of $\tilde{b}^{'}_{i}(i = 1, 2, 3, 4, 5)$. Since*

$$\mathcal{S}(\tilde{b}_1) = 0.0427, \mathcal{S}(\tilde{b}_2) = 0.2404, \mathcal{S}(\tilde{b}_3) = 0.0643, \mathcal{S}(\tilde{b}_4) = -0.0592, \mathcal{S}(\tilde{b}_5) = -0.2881,$$

hence, the ranking order of alternatives is

$$\mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5.$$

From the numerical results above, it can be clearly seen that the rankings of the locations $\mathcal{V}_i(i = 1, 2, 3, 4, 5)$ obtained by Algorithms 5 and 6 differ significantly. By Algorithm 6, $\mathcal{V}_2$ is ranked the first, then $\mathcal{V}_3$, $\mathcal{V}_1$ and $\mathcal{V}_4$, respectively, and then $\mathcal{V}_5$ the last. However, by Algorithm 5, $\mathcal{V}_2$ drops from the first to the third, $\mathcal{V}_3$ from the second to the forth, and $\mathcal{V}_1$ from the third to the last, while $\mathcal{V}_4$ moves up from the fourth to the first, and $\mathcal{V}_5$ from the last to the second (which changes the most among all the locations). Table 2 reflects these changes.

Table 2. Differences in the results obtained by the Algorithm 5 and Algorith.

	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$
Algorithm 5	−0.0267	0.2124	0.1790	0.3371	0.2435
Algorithm 6	0.0427	0.2404	0.0643	−0.0592	−0.2881
Differences	−0.0694	−0.0280	0.1147	0.3963	0.5316

We also observe that the distinct changes in the rankings of the locations are mainly caused by the substantial differences among the estimated elements in the incomplete complex intuitionistic fuzzy preference relation $\mathfrak{R}^{(4)}$ when using Algorithms 5 and 6. In the process of estimating missing elements, the estimation Formula (15) in the Algorithm 5 results in the decrease of the membership degrees and the increase of the non-membership degrees of complex intuitionistic fuzzy preference values sharply, which produce distortion of the estimated information, but the estimation Formula (16) in Algorithm 6, which is motivated by the multiplicative transitivity of traditional complex intuitionistic fuzzy preference relations, can overcome this issue by making the estimated results more intuitive and reasonable.

Compared with Algorithm 5, Algorithm 6 has the following advantages.

(i) The estimation formulas (10)–(13) used in Algorithm 6 are natural extensions of multiplicative transitivity Formulas (3) and (4) of traditional intuitionistic fuzzy preference relations. The estimation formula used in Algorithm 5, however, sometimes may produce distortion of the estimated information.

(ii) The complex intuitionistic fuzzy weighted averaging operator and the complex intuitionistic fuzzy averaging operator are consistent with the aggregation operations on the ordinary intuitionistic fuzzy sets. These two aggregation operators are also monotone with respect to the total order based on the scores and accuracy degrees. But the aggregation operators used in Algorithm 5 are only monotone with respect to the partial order. As a result, Algorithm 6 is more intuitive and reasonable and thus has good application prospects.

Based on the analysis above, we will take the calculation result of the Algorithm 6 as the decision-making result, for example 7, that is, the ranking order of alternatives is $\mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5$.

Remark 4. *In this paper, the simulation verification of an incomplete complex intuitionistic fuzzy system is carried out by MATLAB software with the help of Example 7. According to Definition 5, complex intuitionistic fuzzy numbers are aggregated by creating numerical formulas in MATLAB software. Meanwhile, the best alternative is selected using the score function value and accuracy function value. Simulation results show that the algorithm proposed in this paper is accurate and effective, and is suitable for group decision-making problems that contain periodic changes of uncertain information.*

7. Conclusions

In this paper, firstly, we introduced the concept of incomplete complex intuitionistic fuzzy preference relation and discussed its properties. Meanwhile, the concept of multiplicative consistent incomplete complex intuitionistic fuzzy preference relations are defined and three estimation algorithms are developed to estimate the missing elements in the acceptable incomplete complex intuitionistic fuzzy preference relations. Finally, the group decision-making algorithms based on incomplete complex intuitionistic fuzzy preference relation are established and the solving process of the algorithms is illustrated by an example, the practicability of the algorithms is verified, and the advantages and disadvantages of group decision-making algorithms are compared and analyzed. As a result, the resulting put forward in this paper is more intuitive and reasonable and thus has good application prospects.

We note that the estimation algorithms (i.e., Algorithms 1–3) for the acceptable incomplete complex intuitionistic fuzzy preference relations are used to translate incomplete complex intuitionistic fuzzy preference relations to complete complex intuitionistic fuzzy preference relations, and the expert weight determination algorithm (i.e., Algorithm 4) is used to calculate the expert weights. Finally, group decision-making algorithms (i.e., Algorithms 5 and 6) based on incomplete complex intuitionistic fuzzy reference relations are used to solve group decision-making problems and select the best alternatives. Although both Algorithms 5 and 6 are group decision-making algorithms, there are advantages of Algorithm 6 compared with Algorithm 5, that is, (i) the estimation formulas used in Algorithm 6 are natural extensions of multiplicative transitivity formulas of traditional intuitionistic fuzzy preference relations, which can avoid distorting the complex intuitionistic fuzzy preference information in the estimation process; (ii) The complex intuitionistic fuzzy weighted averaging operator and the complex intuitionistic fuzzy averaging operator are consistent with the aggregation operations on the ordinary intuitionistic fuzzy sets. These two aggregation operators are also monotone with respect to the total order based on the scores and accuracy degrees. However, the aggregation operators used in Algorithm 5 are only monotone with respect to the partial order. As a result, Algorithm 6 is more intuitive and reasonable and thus has good application prospects. We have used a practical example of an educational institute selecting an online course platform to compare and analyze Algorithms 5 and 6 and verify the practicality and superiority of Algorithm 6. Finally, note that complex intuitionistic fuzzy number itself is more complex, so the algorithms proposed in this paper have a large amount of computation. We mainly use MATLAB software for calculation. Meanwhile, the simulation verification of incomplete complex intuitionistic fuzzy system is carried out by MATLAB software with the help of Example 7. Simulation results show that the algorithm proposed in this paper is accurate and effective, and is suitable for group decision making problems that contain periodic changes of uncertain information.

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