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On (k, ψ) -Hilfer Fractional Differential Equations and Inclusions with Mixed (k, ψ) -Derivative and Integral Boundary Conditions

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Abstract: In this paper we study single-valued and multi-valued (k, ψ) -Hilfer-type boundary value problems of fractional order in $(1, 2]$, subject to nonlocal boundary conditions involving (k, ψ) -Hilfer-type derivative and integral operators. The results for single-valued case are established by using Banach and Krasnosel'skiĭ fixed point theorems as well as Leray–Schauder nonlinear alternative. In the multi-valued case, we establish an existence result for the convex valued right-hand side of the inclusion via Leray–Schauder nonlinear alternative for multi-valued maps, while the second one when the right-hand side has non-convex values is obtained by applying Covitz–Nadler fixed point theorem for multi-valued contractions. Numerical examples illustrating the obtained theoretical results are also presented.

Keywords: (k, ψ) -Hilfer fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; existence; uniqueness; fixed point theorems

MSC: 26A33; 34A08; 34A60; 34B10



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1. Introduction

Fractional calculus is concerned with integral and derivative operators of non-integer order and arises in many engineering and scientific fields such as physics, chemistry, mathematical biology, mechanics, and so forth; see the monographs [1–9]. Usually, fractional integral operators are used to define fractional derivative operators. Many kinds of fractional derivative operators such as Riemann–Liouville, Caputo, Hadamard, Katugampola, Hilfer, etc., are proposed in the literature. Certain forms of fractional operators include definitions of other fractional operators. For example, the concept of generalized fractional derivatives and integrals introduced by Katugampola in [10,11] includes Riemann–Liouville and Hadamard fractional derivatives. The Hilfer fractional derivative operator [12] includes Riemann–Liouville and Caputo fractional derivative operators. The ψ -fractional derivative operator [13] unifies Caputo, Caputo–Hadamard and Caputo–Erdélyi–Kober fractional derivative operators. A wide class of fractional operators is covered by the (k, ψ) -Hilfer fractional derivative operator introduced in [14,15].

In [14], the authors, by applying Banach's fixed point theorem, proved the existence of a unique solution for a nonlinear initial value problem involving (k, ψ) -Hilfer-type fractional derivative operator. Tariboon et al. [16] studied the existence and uniqueness of solutions for (k, ψ) -Hilfer fractional differential equations and inclusions with multi-point boundary conditions.

To enrich the literature in this new research topic, which is very limited at the moment, we continue in the present paper the study of boundary value problems involving (k, ψ) -Hilfer-type fractional derivatives in the order of $(1, 2]$, supplemented with nonlocal boundary conditions involving (k, ψ) -Hilfer-type derivative and integral operators of the form

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}u(t) = f(t, u(t)), & t \in (a, b], \\ u(a) = 0, \quad u(b) = \lambda {}^{k,H}D^{p,q;\psi}u(\eta) + \mu {}^k\mathcal{J}^{v,\psi}u(\sigma), \end{cases} \quad (1)$$

where ${}^{k,H}D^{\alpha,\beta;\psi}$ denotes the (k, ψ) -Hilfer-type fractional derivative of order α , $1 < \alpha < 2$ and parameter β , $0 \leq \beta \leq 1$, $k > 0$, $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, ${}^{k,H}D^{p,q;\psi}$ denotes the (k, ψ) -Hilfer-type fractional derivative of order p , $1 < p < 2$ and parameter q , $0 \leq q \leq 1$, $p < \alpha$, ${}^k\mathcal{J}^{v,\psi}$ is the (k, ψ) -Riemann–Liouville fractional integral of order $v > 0$, $\lambda, \mu \in \mathbb{R}$, and $a < \xi, \sigma < b$. Existence and uniqueness will be established, by using Banach's and Krasnosel'skii's fixed point theorems, as well as the Laray–Schauder nonlinear alternative.

We also study the corresponding multi-valued case of the problem (1) given by

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}u(t) \in \mathfrak{F}(t, u(t)), & t \in (a, b], \\ u(a) = 0, \quad u(b) = \lambda {}^{k,H}D^{p,q;\psi}u(\eta) + \mu {}^k\mathcal{J}^{v,\psi}u(\sigma), \end{cases} \quad (2)$$

in which $\mathfrak{F} : [a, b] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map ($\mathcal{P}(\mathbb{R})$ denotes the family of all nonempty subsets of $\mathbb{R}$) while the other parameters are the same as defined in problem (1). The existence results for the problem (2) associated with convex and non-convex cases of the multi-valued map $\mathfrak{F}(t, u(t))$ will be obtained by using the Laray–Schauder nonlinear alternative for multi-valued maps and the Covitz–Nadler fixed point theorem for multi-valued contractions, respectively.

The fixed point theory provides an excellent approach to establish the existence theory for initial and boundary value problems. For some recent publications on this branch of mathematical analysis, we refer the reader to some recent books [17–19].

The content in the rest of paper is arranged as follows. Section 2 contains some necessary definitions and lemmas, while Section 3 is concerned with an auxiliary lemma which enables us to transform the nonlinear (k, ψ) -Hilfer type boundary value problem (1) into an equivalent fixed point problem. The main results for the problems (1) and (2) are presented in Sections 4 and 5, respectively. Finally, in Section 6, numerical examples illustrating the obtained theoretical results are presented.

2. Preliminaries

In this section we introduce some definitions and lemmas that will be used throughout the paper.

Definition 1 ([20]). Let $h \in L^1([a, b], \mathbb{R})$ and $k, \alpha \in \mathbb{R}^+$. Then, the k -Riemann–Liouville fractional derivative of order α of the function h is given by

$${}^k\mathcal{J}_{a+}^{\alpha}h(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-w)^{\frac{\alpha}{k}-1}h(w)dw, \quad (3)$$

where Γ_k is the k -Gamma function for $z \in \mathbb{C}$ with $\Re(z) > 0$ and $k \in \mathbb{R}, k > 0$ which is defined in [21] by

$$\Gamma_k(z) = \int_0^{+\infty} s^{z-1}e^{-\frac{s^k}{k}}ds.$$

The following relations are well known.

$$\Gamma(\theta) = \lim_{k \rightarrow 1} \Gamma_k(\theta), \quad \Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma\left(\frac{\theta}{k}\right) \text{ and } \Gamma_k(\theta + k) = \theta\Gamma_k(\theta).$$

Definition 2 ([22]). Let $h \in L^1([a, b], \mathbb{R})$ and $k, \alpha \in \mathbb{R}^+$. Then the k -Riemann–Liouville fractional derivative of order α of the function h is given by

$${}^{k,RL}D_{a+}^{\alpha} h(t) = \left(k \frac{d}{dt}\right)^n {}^k\mathfrak{J}_{a+}^{nk-\alpha} h(t), \quad n = \left\lceil \frac{\alpha}{k} \right\rceil, \quad (4)$$

where $\left\lceil \frac{\alpha}{k} \right\rceil$ is the ceiling function of $\frac{\alpha}{k}$.

Definition 3 ([2]). Let $h \in L^1([a, b], \mathbb{R})$ and an increasing function $\psi : [a, b] \rightarrow \mathbb{R}$ with $\psi'(t) \neq 0$ for all $t \in [a, b]$. Then the ψ -Riemann–Liouville fractional integral of the function h is given by

$$\mathfrak{J}^{\alpha;\psi} h(t) = \frac{1}{\Gamma_k(\alpha)} \int_a^t \psi'(u) (\psi(t) - \psi(u))^{\alpha-1} h(u) du. \quad (5)$$

Definition 4. Let $n - 1 < \alpha \leq n$, $\psi \in C^n([a, b], \mathbb{R})$ is an increasing function with $\psi'(t) \neq 0$, $t \in [a, b]$, and $h \in C([a, b], \mathbb{R})$.

(a) The ψ -Riemann–Liouville fractional derivative of the function h of order α is given in [2] as

$${}^{RL}D^{\alpha;\psi} h(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \mathfrak{J}_{a+}^{n-\alpha;\psi} h(t). \quad (6)$$

(b) The ψ -Caputo fractional derivative of the function h of order α is defined in [13] as

$${}^CD^{\alpha;\psi} h(t) = \mathfrak{J}_{a+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n h(t). \quad (7)$$

(c) The ψ -Hilfer fractional derivative of the function $h \in C([a, b], \mathbb{R})$ of order $\alpha \in (n - 1, n]$ and type $\beta \in [0, 1]$ is defined in [23] as

$${}^HD^{\alpha,\beta;\psi} h(t) = \mathfrak{J}_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \mathfrak{J}_{a+}^{(1-\beta)(n-\alpha);\psi} h(t). \quad (8)$$

Definition 5 ([24]). Let $\psi : [a, b] \rightarrow \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. Then the (k, ψ) -Riemann–Liouville fractional integral of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of a function $h \in L^1([a, b], \mathbb{R})$ is given by

$${}^k\mathfrak{J}_{a+}^{\alpha;\psi} h(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \psi'(u) (\psi(t) - \psi(u))^{\frac{\alpha}{k}-1} h(u) du, \quad k > 0. \quad (9)$$

Definition 6 ([14]). Let $\alpha, k \in \mathbb{R}^+ = (0, +\infty)$, $\beta \in [0, 1]$, $\psi \in C^n([a, b], \mathbb{R})$ is an increasing function with $\psi'(t) \neq 0$, $t \in [a, b]$ and $h \in C^n([a, b], \mathbb{R})$. Then the (k, ψ) -Hilfer fractional derivative of the function h of order α and type β , is defined by

$${}^{k,H}D^{\alpha,\beta;\psi} h(t) = {}^k\mathfrak{J}_{a+}^{\beta(nk-\alpha);\psi} \left(\frac{k}{\psi'(t)} \frac{d}{dt}\right)^n {}^k\mathfrak{J}_{a+}^{(1-\beta)(nk-\alpha);\psi} h(t), \quad n = \left\lceil \frac{\alpha}{k} \right\rceil. \quad (10)$$

Remark 1. Observe that the (k, ψ) -Hilfer-type fractional derivative can be expressed in terms of (k, ψ) -Riemann–Liouville fractional derivative as

$$\begin{aligned} {}^{k,H}D^{\alpha,\beta;\psi} h(t) &= {}^k\mathfrak{J}_{a+}^{\theta_k-\alpha;\psi} \left(\frac{k}{\psi'(t)} \frac{d}{dt}\right)^n {}^k\mathfrak{J}_{a+}^{nk-\theta_k;\psi} h(t) \\ &= {}^k\mathfrak{J}_{a+}^{\theta_k-\alpha;\psi} \left({}^{k,RL}D^{\theta_k;\psi} h\right)(t), \end{aligned}$$

where $\theta_k = \alpha + \beta(nk - \alpha)$, $\beta(nk - \alpha) = \theta_k - \alpha$ and $(1 - \beta)(nk - \alpha) = nk - \theta_k$, $\beta \in [0, 1]$. Note that $n - 1 < \frac{\theta_k}{k} \leq n$ when $n - 1 < \frac{\alpha}{k} \leq n$.

We recall now some useful lemmas.

Lemma 1 ([14]). Assume that $\mathfrak{h} \in C^n([a, b], \mathbb{R})$ and ${}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h} \in C^n([a, b], \mathbb{R})$ with $\mu, k \in \mathbb{R}^+ = (0, +\infty)$ and $n = \left\lceil \frac{\mu}{k} \right\rceil$. Then

$${}^k\mathfrak{J}^{\mu;\psi}\left({}^{k,RL}D^{\mu;\psi}\mathfrak{h}(t)\right) = \mathfrak{h}(t) - \sum_{j=1}^n \frac{(\psi(t) - \psi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(\frac{k}{\psi'(t)} \frac{d}{dt} \right)^{n-j} {}^k\mathfrak{J}_{a+}^{nk-\mu;\psi}\mathfrak{h}(t) \right]_{z=a}.$$

Lemma 2 ([14]). Let $\theta_k = \alpha + \beta(k - \alpha)$ with $\alpha, k \in \mathbb{R}^+ = (0, +\infty)$, $\alpha < k$, $\beta \in [0, 1]$. Then

$${}^k\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}D^{\theta_k;\psi}\mathfrak{h}\right)(t) = {}^k\mathfrak{J}^{\alpha;\psi}\left({}^{k,H}D^{\alpha;\beta;\psi}\mathfrak{h}\right)(t), \quad \mathfrak{h} \in C^n([a, b], \mathbb{R}).$$

Lemma 3 ([14]). Let $\zeta, k \in \mathbb{R}^+ = (0, +\infty)$ and $\eta \in \mathbb{R}$ such that $\frac{\eta}{k} > -1$. Then

$$(i). \quad {}^k\mathcal{I}^{\zeta,\psi}(\psi(t) - \psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k + \zeta)} (\psi(t) - \psi(a))^{\frac{\eta+\zeta}{k}}.$$

$$(ii). \quad {}^k\mathcal{D}^{\zeta,\psi}(\psi(t) - \psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k - \zeta)} (\psi(t) - \psi(a))^{\frac{\eta-\zeta}{k}}.$$

3. An Auxiliary Result

In this section, an auxiliary result dealing with the linear variant the problem (1) is presented.

Lemma 4. Let $g \in C(a, b) \cap L^1(a, b)$ (see [25,26]) and

$$\begin{aligned} \Omega : &= \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} - \lambda \Gamma_k(\theta_k) \frac{(\psi(\eta) - \psi(a))^{\frac{\theta_k-p}{k}-1}}{\Gamma_k(\theta_k - p)} \\ &\quad - \mu \Gamma_k(\theta_k) \frac{(\psi(\sigma) - \psi(a))^{\frac{\theta_k+v}{k}-1}}{\Gamma_k(\theta_k + v)} \neq 0. \end{aligned} \quad (11)$$

Then, the function $u \in C^2([a, b], \mathbb{R})$ is a solution of the (k, ψ) -Hilfer-type nonlocal fractional boundary value problem

$$\begin{cases} {}^{k,H}D^{\alpha;\beta;\psi}u(t) = g(t), & t \in (a, b], k > 0, 1 < \alpha \leq 2, \beta \in [0, 1], \\ u(a) = 0, & u(b) = \lambda {}^{k,H}D^{p,q;\psi}u(\eta) + \mu {}^k\mathfrak{J}^{v,\psi}u(\sigma), \end{cases} \quad (12)$$

if and only if

$$u(t) = {}^k\mathfrak{J}^{\alpha;\psi}g(t) + \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi}g(\eta) + \mu {}^k\mathfrak{J}^{v+\alpha;\psi}g(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi}g(b) \right], \quad (13)$$

where $\theta_k = \alpha + \beta(2k - \alpha)$.

Proof. Assume that u is a solution of the (k, ψ) -Hilfer-type nonlocal fractional boundary value problem (12). Operating on both sides of equation in (12), the fractional integral ${}^k\mathfrak{J}^{\alpha;\psi}$ and using Lemmas 2 and 1, we obtain

$$\begin{aligned} {}^k\mathfrak{J}^{\alpha;\psi}\left({}^{k,H}D^{\alpha;\beta;\psi}u\right)(t) &= {}^k\mathfrak{J}^{\theta_k;\psi}\left({}^{k,RL}D^{\theta_k;\psi}u\right)(t) \\ &= u(t) - \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} \left[\left(\frac{k}{\psi'(t)} \frac{d}{dt} \right) {}^k\mathfrak{J}^{2k-\theta_k;\psi}u(t) \right]_{w=a} \\ &\quad - \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k - k)} \left[{}^k\mathfrak{J}^{2k-\theta_k;\psi}u(t) \right]_{w=a}. \end{aligned}$$

Consequently

$$u(t) = {}^k\mathfrak{J}^{\alpha;\psi}g(t) + c_0 \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Gamma_k(\theta_k)} + c_1 \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-2}}{\Gamma_k(\theta_k - k)}, \quad (14)$$

where

$$c_0 = \left[\left(\frac{k}{\psi'(t)} \frac{d}{dt} \right) {}^k\mathfrak{J}^{2k-\theta_k;\psi}u(t) \right]_{w=a}, \quad c_1 = \left[{}^k\mathfrak{J}^{2k-\theta_k;\psi}u(t) \right]_{w=a}.$$

By the condition $u(a) = 0$, we find that $c_1 = 0$ as $\frac{\theta_k}{k} - 2 < 0$ by Remark 1. By using Lemma 3, we obtain

$${}^kD^{p,q;\psi}(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1} = \frac{\Gamma_k(\theta_k)}{\Gamma_k(\theta_k - p)}(\psi(t) - \psi(a))^{\frac{\theta_k-p}{k}-1}. \quad (15)$$

and

$${}^k\mathfrak{J}^{v,\psi}(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1} = \frac{\Gamma_k(\theta_k)}{\Gamma_k(\theta_k + v)}(\psi(t) - \psi(a))^{\frac{\theta_k+v}{k}-1}. \quad (16)$$

From (15), (16) and the boundary condition: $u(b) = \lambda u(\zeta) + \mu {}^k\mathfrak{J}^{v,\psi}u(\sigma)$, we obtain

$$c_0 = \frac{1}{\Omega} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi}g(\eta) + \mu {}^k\mathfrak{J}^{v+\alpha,\psi}g(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi}g(b) \right].$$

Replacing c_0 and c_1 in (14) by their above values, we obtain the solution (13). The converse can be proved easily by direct computation. This finishes the proof. $\square$

4. The Single Valued Problem

Let us begin this section by defining the solution of problem (1).

Definition 7. A function $u \in C^2([a, b], \mathbb{R})$ possessing the (k, ψ) -Hilfer fractional derivative in the sense of Definition 6 is said to be a solution of the (k, ψ) -Hilfer-type nonlocal fractional boundary value problem (1) if it satisfies the equations ${}^{k,H}D^{\alpha,\beta;\psi}u(t) = f(t, u(t))$, $t \in (a, b]$ with $f \in C([a, b], \mathbb{R})$ and the boundary conditions $u(a) = 0$, $u(b) = \lambda {}^{k,H}D^{p,q;\psi}u(\eta) + \mu {}^k\mathfrak{J}^{v,\psi}u(\sigma)$.

In view of Lemma 4, we define an operator $\mathcal{A} : C([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$ by

$$\begin{aligned} (\mathcal{A}u)(t) &= \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega\Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi}f(\eta, u(\eta)) + \mu {}^k\mathfrak{J}^{v+\alpha,\psi}f(\sigma, u(\sigma)) \right. \\ &\quad \left. - {}^k\mathfrak{J}^{\alpha;\psi}f(b, u(b)) \right] + {}^k\mathfrak{J}^{\alpha;\psi}f(t, u(t)), \quad t \in [a, b], \end{aligned} \quad (17)$$

where $C([a, b], \mathbb{R})$ denotes the Banach space of all continuous real valued functions defined on $[a, b]$ equipped with the sup-norm $\|u\| = \sup_{t \in [a, b]} |u(t)|$.

Observe that the solutions of the (k, ψ) -Hilfer-type nonlocal fractional boundary value problem (1) must be sought among the fixed points of $\mathcal{A}$.

For computational convenience we put:

$$\begin{aligned} \mathbb{G} &= \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \\ &\quad \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right]. \end{aligned} \quad (18)$$

4.1. Existence of a Unique Solution

In this subsection, we make use of Banach's fixed point theorem [27] to prove the existence of a unique solution to the problem (1).

Theorem 1. *Let the following condition hold:*

$$(H_1) \quad |\mathfrak{f}(t, u) - \mathfrak{f}(t, y)| \leq \mathfrak{L}|u - y|, \quad \mathfrak{L} > 0 \text{ for each } t \in [a, b] \text{ and } u, y \in \mathbb{R}.$$

Then there exists a unique solution to the (k, ψ) -Hilfer-type nonlocal fractional boundary value problem (1) on $[a, b]$, provided that

$$\mathfrak{L}\mathbb{G} < 1, \quad (19)$$

where $\mathbb{G}$ is defined by (18).

Proof. We verify that the operator $\mathcal{A}$ defined in (17) satisfies the hypothesis of Banach's fixed point theorem. Letting $\sup_{t \in [a, b]} |\mathfrak{f}(t, 0)| = \mathbb{M} < +\infty$, we define $B_r = \{u \in C([a, b], \mathbb{R}) : \|u\| \leq r\}$ with

$$r \geq \frac{\mathbb{M}\mathbb{G}}{1 - \mathfrak{L}\mathbb{G}}. \quad (20)$$

We will first show that $\mathcal{A}B_r \subset B_r$. By the assumptions (H_1) , we have

$$\begin{aligned} |\mathfrak{f}(t, u(t))| &\leq |\mathfrak{f}(t, u(t)) - \mathfrak{f}(t, 0)| + |\mathfrak{f}(t, 0)| \\ &\leq \mathfrak{L}|u(t)| + \mathbb{M} \leq \mathfrak{L}\|u\| + \mathbb{M} \leq \mathfrak{L}r + \mathbb{M}. \end{aligned}$$

Then, for any $u \in B_r$, we obtain

$$\begin{aligned} |(\mathcal{A}u)(t)| &\leq \sup_{t \in [a, b]} \left\{ \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathfrak{J}^{\alpha-p; \psi} |\mathfrak{f}(\eta, u(\eta))| + |\mu| {}^k\mathfrak{J}^{v+\alpha; \psi} |\mathfrak{f}(\sigma, u(\sigma))| \right. \right. \\ &\quad \left. \left. + {}^k\mathfrak{J}^{\alpha; \psi} |\mathfrak{f}(b, u(b))| \right] + {}^k\mathfrak{J}^{\alpha; \psi} |\mathfrak{f}(t, u(t))| \right\} \\ &\leq {}^k\mathfrak{J}^{\alpha; \psi} (|\mathfrak{f}(t, u(t)) - \mathfrak{f}(t, 0)| + |\mathfrak{f}(t, 0)|) \\ &\quad + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Omega|\Gamma_k(\theta_k)} \left(|\lambda| {}^k\mathfrak{J}^{\alpha-p; \psi} (|\mathfrak{f}(\eta, u(\eta)) - \mathfrak{f}(\eta, 0)| + |\mathfrak{f}(\eta, 0)|) \right. \\ &\quad \left. + |\mu| {}^k\mathfrak{J}^{v+\alpha; \psi} (|\mathfrak{f}(\sigma, u(\sigma)) - \mathfrak{f}(\sigma, 0)| + |\mathfrak{f}(\sigma, 0)|) \right. \\ &\quad \left. + {}^k\mathfrak{J}^{\alpha; \psi} (|\mathfrak{f}(b, u(b)) - \mathfrak{f}(b, 0)| + |\mathfrak{f}(b, 0)|) \right) \\ &\leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ &\quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} (\mathfrak{L}\|u\| + \mathbb{M}) \\ &\leq (\mathfrak{L}r + \mathbb{M})\mathbb{G} \leq r. \end{aligned}$$

Hence, $\|\mathcal{A}u\| \leq r$, which means that $\mathcal{A}B_r \subset B_r$ as $u \in B_r$ is an arbitrary element.

In the second step, it will be established that $\mathcal{A}$ is a contraction. For $u, y \in C([a, b], \mathbb{R})$ and $t \in [a, b]$, we obtain

$$\begin{aligned} & |(\mathcal{A}u)(t) - (\mathcal{A}y)(t)| \\ & \leq {}^k\mathfrak{J}^{\alpha;\psi} |\mathfrak{f}(t, u(t)) - \mathfrak{f}(t, y(t))| \\ & \quad + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left(|\lambda| {}^k\mathfrak{J}^{\alpha-p;\psi} |\mathfrak{f}(\eta, u(\eta)) - \mathfrak{f}(\eta, y(\eta))| \right. \\ & \quad + |\mu| {}^k\mathfrak{J}^{\alpha+v;\psi} |\mathfrak{f}(\sigma, u(\sigma)) - \mathfrak{f}(\sigma, y(\sigma))| \\ & \quad \left. + {}^k\mathfrak{J}^{\alpha;\psi} (|\mathfrak{f}(b, u(b)) - \mathfrak{f}(b, y(b))|) \right) \\ & \leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ & \quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} \mathfrak{L} \|u - y\| \\ & = \mathfrak{L}\mathbb{G} \|u - y\|, \end{aligned}$$

which, on taking the norm for $t \in [a, b]$, yields $\|\mathcal{A}u - \mathcal{A}y\| \leq \mathfrak{L}\mathbb{G} \|u - y\|$. Since $\mathfrak{L}\mathbb{G} < 1$, therefore $\mathcal{A}$ is a contraction. As the hypothesis of the Banach's fixed point theorem is verified, we conclude that the operator $\mathcal{A}$ has a unique fixed point, which is indeed a unique solution of the problem (1). This finishes the proof. $\square$

4.2. Existence Results

Here we present two existence results for the problem (1) by applying Krasnosel'skiĭ's fixed point theorem [28] and nonlinear alternative of Leray–Schauder type [29].

Theorem 2. Suppose that (H_1) and the following condition hold:

$(H_2) \forall (t, u) \in [a, b] \times \mathbb{R}$, there exists $\theta \in C([a, b], \mathbb{R}^+)$ such that $|\mathfrak{f}(t, u)| \leq \theta(t)$.

Then, if $\mathbb{G}_1 \mathfrak{L} < 1$, where

$$\begin{aligned} \mathbb{G}_1 : &= \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} \right. \\ & \quad \left. + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right], \end{aligned} \quad (21)$$

the problem (1) has at least one solution on $[a, b]$.

Proof. Set $\sup_{t \in [a, b]} \theta(t) = \|\theta\|$ and consider the ball $B_\rho = \{u \in C([a, b], \mathbb{R}) : \|u\| \leq \rho\}$, with $\rho \geq \|\theta\| \mathbb{G}$. Introduce the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on B_ρ to $\mathbb{R}$ as

$$\mathcal{A}_1 u(t) = {}^k\mathfrak{J}^{\alpha;\psi} \mathfrak{f}(t, u(t)), \quad t \in [a, b],$$

$$\begin{aligned} (\mathcal{A}_2 u)(t) &= \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega\Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi} \mathfrak{f}(\eta, u(\eta)) + \mu {}^k\mathfrak{J}^{\alpha+v;\psi} \mathfrak{f}(\sigma, u(\sigma)) \right. \\ & \quad \left. - {}^k\mathfrak{J}^{\alpha;\psi} \mathfrak{f}(b, u(b)) \right], \quad t \in [a, b]. \end{aligned}$$

For any $u, y \in B_\rho$, we have

$$\begin{aligned} & |(\mathcal{A}_1 u)(t) + (\mathcal{A}_2 y)(t)| \\ & \leq \sup_{t \in [a, b]} \left\{ \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| {}^k \mathfrak{J}^{\alpha-p; \psi} |\mathfrak{f}(\eta, u(\eta))| + |\mu| {}^k \mathfrak{J}^{v+\alpha, \psi} |\mathfrak{f}(\sigma, y(\sigma))| \right. \right. \\ & \quad \left. \left. + {}^k \mathfrak{J}^{\alpha; \psi} |\mathfrak{f}(b, y(b))| \right] + {}^k \mathfrak{J}^{\alpha; \psi} |\mathfrak{f}(t, u(t))| \right\} \\ & \leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k} - 1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ & \quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} \|\theta\| \\ & = \mathbb{G} \|\theta\| \leq \rho. \end{aligned}$$

Therefore $\|(\mathcal{A}_1 u) + (\mathcal{A}_2 y)\| \leq \rho$, which shows that $\mathcal{A}_1 u + \mathcal{A}_2 y \in B_\rho$. It is easy to show, by using (21), that the operator $\mathcal{A}_2$ is a contraction.

Note that continuity of $\mathfrak{f}$ implies the continuity of the operator $\mathcal{A}_1$. Furthermore, the operator $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1 u\| \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \|\theta\|.$$

Now we establish that the operator $\mathcal{A}_1$ is compact. For $t_1 < t_2$ ($t_1, t_2 \in [a, b]$), we have

$$\begin{aligned} & |(\mathcal{A}_1 u)(t_2) - (\mathcal{A}_1 u)(t_1)| \\ & \leq \frac{1}{\Gamma_k(\alpha)} \left| \int_a^{t_1} \psi'(s) [(\psi(t_2) - \psi(s))^{\frac{\alpha}{k} - 1} - (\psi(t_1) - \psi(s))^{\frac{\alpha}{k} - 1}] \mathfrak{f}(s, u(s)) ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} \psi'(s) (\psi(t_2) - \psi(s))^{\frac{\alpha}{k} - 1} \mathfrak{f}(s, u(s)) ds \right| \\ & \leq \frac{\|\theta\|}{\Gamma_k(\alpha + k)} [2(\psi(t_2) - \psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2) - \psi(a))^{\frac{\alpha}{k}} - (\psi(t_1) - \psi(a))^{\frac{\alpha}{k}}|] \\ & \longrightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0, \end{aligned}$$

independently of u . Hence, the operator $\mathcal{A}_1$ is equicontinuous, and consequently, it follows by the Arzelà–Ascoli theorem that the operator $\mathcal{A}_1$ is completely continuous. Thus, the conclusion of Krasnosel’skiĭ’s fixed point theorem applies, and the problem (1) has at least one solution on $[a, b]$. The proof is completed. $\square$

Theorem 3. Suppose that the following conditions hold:

- (H₃) $\exists$ a continuous, nondecreasing function $\chi : [0, +\infty) \rightarrow (0, +\infty)$ and a function $\gamma \in C([a, b], \mathbb{R}^+)$ such that, $\forall (t, u) \in [a, b] \times \mathbb{R}$, $|\mathfrak{f}(t, u)| \leq \gamma(t)\chi(|u|)$;
(H₄) $\exists$ a constant $\mathcal{K} > 0$ such that

$$\frac{\mathcal{K}}{\chi(\mathcal{K})\|\gamma\|\mathbb{G}} > 1.$$

Then there exists at least one solution for the problem (1) on $[a, b]$.

Proof. Consider the operator $\mathcal{A}$ is defined by (17). We will split the proof into several steps. It will be shown in the first step that the operator $\mathcal{A}$ maps bounded sets into bounded set in $C([a, b], \mathbb{R})$. For $r > 0$, let $B_r = \{u \in C([a, b], \mathbb{R}) : \|u\| \leq r\}$. Then, for $t \in [a, b]$, we obtain

$$\begin{aligned} |(\mathcal{A}u)(t)| &\leq \sup_{t \in [a, b]} \left\{ \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathcal{J}^{\alpha-p;\psi} |\mathfrak{f}(\eta, u(\eta))| + |\mu| {}^k\mathcal{J}^{v+\alpha;\psi} |\mathfrak{f}(\sigma, u(\sigma))| \right. \right. \\ &\quad \left. \left. + {}^k\mathcal{J}^{\alpha;\psi} |\mathfrak{f}(b, u(b))| \right] + {}^k\mathcal{J}^{\alpha;\psi} |\mathfrak{f}(t, u(t))| \right\} \\ &\leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ &\quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} \|\gamma\| \chi(\|u\|), \end{aligned}$$

which implies that

$$\|\mathcal{A}x\| \leq \chi(r) \|\gamma\| \mathbb{G}.$$

Now, we establish that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C([a, b], \mathbb{R})$. Consider $u \in B_r$ and $t_1, t_2 \in [a, b]$ such that $t_1 < t_2$. Then we obtain

$$\begin{aligned} &|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)| \\ &\leq \frac{1}{\Gamma_k(\alpha)} \left| \int_a^{t_1} \psi'(s) [(\psi(t_2) - \psi(s))^{\frac{\alpha}{k}-1} - (\psi(t_1) - \psi(s))^{\frac{\alpha}{k}-1}] \mathfrak{f}(s, u(s)) ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \psi'(s) (\psi(t_2) - \psi(s))^{\frac{\alpha}{k}-1} \mathfrak{f}(s, u(s)) ds \right| \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathcal{J}^{\alpha-p;\psi} |\mathfrak{f}(\eta, u(\eta))| \right. \\ &\quad \left. + |\mu| {}^k\mathcal{J}^{v+\alpha;\psi} |\mathfrak{f}(\sigma, u(\sigma))| + {}^k\mathcal{J}^{\alpha;\psi} |\mathfrak{f}(b, u(b))| \right] \\ &\leq \frac{\|\gamma\| \chi(r)}{\Gamma_k(\alpha + k)} [2(\psi(t_2) - \psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2) - \psi(a))^{\frac{\alpha}{k}} - (\psi(t_1) - \psi(a))^{\frac{\alpha}{k}}|], \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega|\Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \\ &\quad \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \|\gamma\| \chi(r) \\ &\longrightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0, \end{aligned}$$

independently of $u \in B_r$. Thus, an immediate consequence of the Arzelà–Ascoli theorem implies that the operator $\mathcal{A} : C([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$ is completely continuous.

In the final step, we establish the boundedness of the set of all solutions to the equation $u = \omega \mathcal{A}u$ for $\omega \in (0, 1)$.

As in the first step, we obtain

$$|u(t)| \leq \chi(\|u\|) \|\gamma\| \mathbb{G},$$

or

$$\frac{\|u\|}{\chi(\|u\|) \|\gamma\| \mathbb{G}} \leq 1.$$

By (H_4) , we can find $\mathcal{K}$ such that $\|u\| \neq \mathcal{K}$. Consider the set

$$U = \{u \in C([a, b], \mathbb{R}) : \|u\| < \mathcal{K}\}.$$

Observe that the operator $\mathcal{A} : \overline{U} \rightarrow C([a, b], \mathbb{R})$ is continuous and completely continuous, where $\overline{U}$ denotes the closure of U . By the given choice of U , we cannot find any $u \in \partial U$ (∂U denotes the boundary of U) such that $u = \omega \mathcal{A}u$ for some $\omega \in (0, 1)$ in view of the assumption (H_4) . In consequence, we deduce by the nonlinear alternative of Leray–Schauder type the operator $\mathcal{A}$ has a fixed point $u \in \overline{U}$, which is a solution of the problem (1). This finishes the proof. $\square$

5. The Multivalued Problem

Definition 8. A function $u \in C^2([a, b], \mathbb{R})$ possessing the (k, ψ) -Hilfer fractional derivative in the sense of Definition 6 is said to be a solution of the (k, ψ) -Hilfer-type nonlocal fractional multi-valued boundary value problem (2) if there exists a function $f \in L^1([a, b], \mathbb{R})$ with $f(t) \in \mathfrak{F}(t, u)$ for a.e. $t \in [a, b]$ such that u satisfies the differential equation ${}^{k,H}D^{\alpha,\beta;\psi}u(t) = f(t)$ on $[a, b]$ and the boundary conditions $u(a) = 0$, $u(b) = \lambda {}^{k,H}D^{p,q;\psi}u(\eta) + \mu {}^{k}\mathfrak{J}^{v,\psi}u(\sigma)$.

For each $u \in C([a, b], \mathbb{R})$, we define the set of selections of $\mathfrak{F}$ as

$$S_{\mathfrak{F},u} := \{f \in L^1([a, b], \mathbb{R}) : f(t) \in \mathfrak{F}(t, u(t)) \text{ on } [a, b]\}.$$

Our first result for the multi-valued problem (2) is concerned with the case when the multi-valued map $\mathfrak{F}$ has convex values, and relies on the nonlinear alternative of Leray–Schauder type for multi-valued maps [29].

Theorem 4. Suppose that:

- (G₁) $\mathfrak{F} : [a, b] \times \mathbb{R} \rightarrow \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1 -Carathéodory, where $\mathcal{P}_{cp,c}(\mathbb{R}) = \{\mathfrak{R} \in \mathcal{P}(\mathbb{R}) : \mathfrak{R} \text{ is compact and convex}\}$;
- (G₂) $\exists$ a continuous nondecreasing function $z : [0, +\infty) \rightarrow (0, +\infty)$ and a positive continuous real valued function q such that, $\forall (t, u) \in [a, b] \times \mathbb{R}$,

$$\|\mathfrak{F}(t, u)\|_{\mathcal{P}} := \sup\{|f| : f \in \mathfrak{F}(t, u)\} \leq q(t)z(\|u\|);$$

- (G₃) $\exists$ a constant $\mathfrak{M} > 0$ such that

$$\frac{\mathfrak{M}}{\|q\|z(\mathfrak{M})\mathbb{G}} > 1,$$

where $\mathbb{G}$ is defined by (18).

Then the multi-valued problem (2) has at least one solution on $[a, b]$.

Proof. Define an operator $\mathcal{F} : C([a, b], \mathbb{R}) \rightarrow \mathcal{P}(C([a, b], \mathbb{R}))$ as

$$\mathcal{F}(u) = \left\{ \begin{array}{l} h \in C([a, b], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^{k}\mathfrak{J}^{\alpha-p;\psi}f(\eta) + \mu {}^{k}\mathfrak{J}^{\alpha+v;\psi}f(\sigma) - {}^{k}\mathfrak{J}^{\alpha;\psi}f(b) \right] \\ + {}^{k}\mathfrak{J}^{\alpha;\psi}f(t), \end{array} \right. \end{array} \right\}$$

for $t \in [a, b]$ and $f \in S_{\mathfrak{F},u}$. Observe that the fixed points of the operator $\mathcal{F}$ are solutions to the multi-valued problem (2).

We will split the proof in different steps.

Step 1. $\mathcal{F}(u)$ is convex for each $u \in C([a, b], \mathbb{R})$.

Since $S_{\mathfrak{F},u}$ is convex, this step is obvious.

Step 2. Bounded sets are mapped into bounded sets in $C([a, b], \mathbb{R})$ by $\mathcal{F}$.

Let $B_r = \{u \in C([a, b], \mathbb{R}) : \|u\| \leq r\}$, $r > 0$. Then, for each $h \in \mathcal{F}(u)$, $u \in B_r$, there exists $f \in S_{\mathcal{F}, x}$ such that

$$h(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathcal{J}^{\alpha-p; \psi} f(\eta) + \mu {}^k\mathcal{J}^{\alpha+v; \psi} f(\sigma) - {}^k\mathcal{J}^{\alpha; \psi} f(b) \right] + {}^k\mathcal{J}^{\alpha; \psi} f(t).$$

Then, for $t \in [a, b]$, we have

$$\begin{aligned} |h(t)| &\leq \sup_{t \in [a, b]} \left\{ \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathcal{J}^{\alpha-p; \psi} |f(\eta)| + |\mu| {}^k\mathcal{J}^{\alpha+v; \psi} |f(\sigma)| + {}^k\mathcal{J}^{\alpha; \psi} |f(b)| \right] \right. \\ &\quad \left. + {}^k\mathcal{J}^{\alpha; \psi} |f(t)| \right\} \\ &\leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ &\quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} \|q\| z(\|u\|), \end{aligned}$$

which implies that $\|h\| \leq z(r) \|q\| \mathbb{G}$.

Step 3. Bounded sets are mapped into equicontinuous sets of $C([a, b], \mathbb{R})$ by $\mathcal{F}$.

Consider $t_1, t_2 \in [a, b]$, $t_1 < t_2$ and $u \in B_r$. Then, for each $h \in \mathcal{F}(u)$, we have

$$\begin{aligned} &|h(t_2) - h(t_1)| \\ &\leq \frac{1}{\Gamma_k(\alpha)} \left| \int_a^{t_1} \psi'(s) [(\psi(t_2) - \psi(s))^{\frac{\alpha}{k}-1} - (\psi(t_1) - \psi(s))^{\frac{\alpha}{k}-1}] f(s) ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \psi'(s) (\psi(t_2) - \psi(s))^{\frac{\alpha}{k}-1} f(s) ds \right| \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathcal{J}^{\alpha-p; \psi} |f(\eta)| \right. \\ &\quad \left. + |\mu| {}^k\mathcal{J}^{\alpha+v; \psi} |f(\sigma)| + {}^k\mathcal{J}^{\alpha; \psi} |f(b)| \right] \\ &\leq \frac{\|q\| z(r)}{\Gamma_k(\alpha + k)} [2(\psi(t_2) - \psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2) - \psi(a))^{\frac{\alpha}{k}} - (\psi(t_1) - \psi(a))^{\frac{\alpha}{k}}|] \\ &\quad + \frac{(\psi(t_2) - \psi(a))^{\frac{\theta_k}{k}-1} - (\psi(t_1) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \\ &\quad \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \|q\| z(r) \\ &\longrightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0, \end{aligned}$$

independently of $u \in B_r$. Hence it follows by the Arzelà–Ascoli theorem that $\mathcal{F} : C([a, b], \mathbb{R}) \rightarrow \mathcal{P}(C([a, b], \mathbb{R}))$ is completely continuous.

To prove that $\mathcal{F}$ is upper semi-continuous multivalued mapping, it is enough to prove that the $\mathcal{F}$ has a closed graph, by Proposition 1.2 of [30].

Step 4. $\mathcal{F}$ has a closed graph.

Assume that $u_n \rightarrow u_*$, $h_n \in \mathcal{F}(u_n)$ and $h_n \rightarrow h_*$. Then we must to show that $h_* \in \mathcal{F}(u_*)$. Since $h_n \in \mathcal{F}(u_n)$, there exists $v_n \in S_{\mathcal{F}, u_n}$ such that for each $t \in [a, b]$,

$$h_n(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k \mathcal{J}^{\alpha-p; \psi} f_n(\eta) + \mu {}^k \mathcal{J}^{\alpha+v; \psi} f_n(\sigma) - {}^k \mathcal{J}^{\alpha; \psi} f_n(b) \right] + {}^k \mathcal{J}^{\alpha; \psi} f_n(t).$$

We must show that there exists $v_* \in S_{\mathcal{F}, u_*}$ such that for each $t \in [a, b]$,

$$h_*(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k \mathcal{J}^{\alpha-p; \psi} f_*(\eta) + \mu {}^k \mathcal{J}^{\alpha+v; \psi} f_*(\sigma) - {}^k \mathcal{J}^{\alpha; \psi} f_*(b) \right] + {}^k \mathcal{J}^{\alpha; \psi} f_*(t).$$

Consider the linear operator $\Theta : L^1([a, b], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$ given by

$$v \mapsto \Theta(v)(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k \mathcal{J}^{\alpha-p; \psi} f(\eta) + \mu {}^k \mathcal{J}^{\alpha+v; \psi} f(\sigma) - {}^k \mathcal{J}^{\alpha; \psi} f(b) \right] + {}^k \mathcal{J}^{\alpha; \psi} f(t).$$

Observe that $\|h_n(t) - h_*(t)\| \rightarrow 0$, as $n \rightarrow +\infty$. From a result due to Lazota–Opial [31], we deduce that $\Theta \circ S_{\mathcal{F}}$ is a closed graph operator, and moreover we have $h_n(t) \in \Theta(S_{\mathcal{F}, u_n})$. We have, since $u_n \rightarrow u_*$,

$$h_*(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k \mathcal{J}^{\alpha-p; \psi} f_*(\eta) + \mu {}^k \mathcal{J}^{\alpha+v; \psi} f_*(\sigma) - {}^k \mathcal{J}^{\alpha; \psi} f_*(b) \right] + {}^k \mathcal{J}^{\alpha; \psi} f_*(t),$$

for some $v_* \in S_{\mathcal{F}, u_*}$.

Step 5. An open set $\mathcal{U} \subseteq C([a, b], \mathbb{R})$ exists with $u \notin v\mathcal{F}(u)$ for any $v \in (0, 1)$ and all $u \in \partial \mathcal{U}$.

Consider $v \in (0, 1)$ and $u \in v\mathcal{F}(u)$. Then there exists $f \in L^1([a, b], \mathbb{R})$ with $f \in S_{\mathcal{F}, u}$ such that, for $t \in [a, b]$, we have

$$u(t) = v \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k \mathcal{J}^{\alpha-p; \psi} f(\eta) + \mu {}^k \mathcal{J}^{\alpha+v; \psi} f(\sigma) - {}^k \mathcal{J}^{\alpha; \psi} f(b) \right] + v {}^k \mathcal{J}^{\alpha; \psi} f(t).$$

As in second step, we obtain

$$|u(t)| \leq \|q\|z(\|u\|)\mathbb{G}.$$

Consequently

$$\|u\| \leq \|q\|z(\|u\|)\mathbb{G},$$

or

$$\frac{\|u\|}{\|q\|z(\|u\|)\mathbb{G}} \leq 1.$$

By (H_3) , we can find $\mathfrak{M}$ satisfying $\|u\| \neq \mathfrak{M}$. Consider the set

$$\mathcal{U} = \{u \in C([a, b], \mathbb{R}) : \|u\| < \mathfrak{M}\}.$$

From the preceding arguments, $\mathcal{F} : \overline{\mathcal{U}} \rightarrow \mathcal{P}(C([a, b], \mathbb{R}))$ is a compact and upper semi-continuous multivalued map with convex closed values. By definition $\mathcal{U}$, there does not exist any $u \in \partial \mathcal{U}$ such that $u \in v\mathcal{F}(u)$ for some $v \in (0, 1)$. Hence, it follows by the nonlinear alternative of Leray–Schauder type for multi-valued maps [29] that $\mathcal{F}$ has a fixed point $u \in \overline{\mathcal{U}}$, which is indeed a solution of the multi-valued problem (2). The proof is complete. $\square$

Now we consider the case when F is not necessarily convex valued and show that there exists a solution to the problem (2) with the aid of a fixed point theorem for multivalued contractive maps due to Covitz and Nadler [32].

Theorem 5. Suppose that:

- (A₁) $\mathfrak{F} : [a, b] \times \mathbb{R} \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is such that $\mathfrak{f}(\cdot, u) : [a, b] \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$, where $\mathcal{P}_{cp}(\mathbb{R}) = \{\mathfrak{K} \in \mathcal{P}(\mathbb{R}) : \mathfrak{K} \text{ is compact}\}$;
(A₂) $\exists$ a function $m \in C([a, b], \mathbb{R}^+)$ such that

$$H_d(\mathfrak{F}(t, u), \mathfrak{F}(t, \bar{u})) \leq m(t)|u - \bar{u}|,$$

with $d(0, \mathfrak{F}(t, 0)) \leq m(t)$ for almost all $t \in [a, b]$ and $u, \bar{u} \in \mathbb{R}$.

Then the problem (2) has at least one solution on $[a, b]$, provided that

$$\delta := \mathbb{G}\|m\| < 1, \quad (22)$$

where $\mathbb{G}$ is given by (18).

Proof. Observe that the set $S_{\mathfrak{F}, u}$ is nonempty for each $u \in C([a, b], \mathbb{R})$, by the assumption (A₁). Thus, by Theorem III.6 [33], $\mathfrak{F}$ has a measurable selection. Now for each $u \in C([a, b], \mathbb{R})$, it will be shown that $\mathcal{F}(u) \in \mathcal{P}_{cl}(C([a, b], \mathbb{R}))$, where $\mathcal{P}_{cl}(\mathbb{R}) = \{\mathfrak{K} \in \mathcal{P}(\mathbb{R}) : \mathfrak{K} \text{ is closed}\}$. Assume that $\{u_n\}_{n \geq 0} \in \mathcal{F}(u)$ with $u_n \rightarrow u$ ($n \rightarrow +\infty$) in $C([a, b], \mathbb{R})$. Then we have $u \in C([a, b], \mathbb{R})$ and $v_n \in S_{\mathfrak{F}, u_n}$ such that, for each $t \in [a, b]$,

$$u_n(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi} f_n(\eta) + \mu {}^k\mathfrak{J}^{\alpha+v;\psi} f_n(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_n(b) \right] + {}^k\mathfrak{J}^{\alpha;\psi} f_n(t).$$

Since $\mathfrak{F}$ has compact values, one can pass onto a subsequence (if necessary) to get v_n converges to v in $L^1([a, b], \mathbb{R})$. In consequence, $v \in S_{\mathfrak{F}, x}$ and we have

$$\begin{aligned} u_n(t) \rightarrow u(t) &= \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi} f(\eta) + \mu {}^k\mathfrak{J}^{\alpha+v;\psi} f(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f(b) \right] \\ &\quad + {}^k\mathfrak{J}^{\alpha;\psi} f(t), \text{ for each } t \in [a, b]. \end{aligned}$$

Therefore, $u \in \mathcal{F}(u)$.

Finally we show that

$$H_d(\mathcal{F}(u), \mathcal{F}(\bar{u})) \leq \delta \|u - \bar{u}\|, \quad \delta < 1, \text{ for each } u, \bar{u} \in C^2([a, b], \mathbb{R}).$$

Assume that $u, \bar{u} \in C^2([a, b], \mathbb{R})$ and $h_1 \in \mathcal{F}(u)$. Then we can find $v_1(t) \in \mathfrak{F}(t, u(t))$ satisfying

$$h_1(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathfrak{J}^{\alpha-p;\psi} f_1(\eta) + \mu {}^k\mathfrak{J}^{\alpha+v;\psi} f_1(\sigma) - {}^k\mathfrak{J}^{\alpha;\psi} f_1(b) \right] + {}^k\mathfrak{J}^{\alpha;\psi} f_1(t), \text{ for each } t \in [a, b].$$

By (A₂), we have

$$H_d(\mathfrak{F}(t, u), \mathfrak{F}(t, \bar{u})) \leq m(t)|u(t) - \bar{u}(t)|.$$

So, there exists an element $v \in \mathfrak{F}(t, \bar{x}(t))$ such that

$$|v_1(t) - v| \leq m(t)|u(t) - \bar{u}(t)|, \quad t \in [a, b].$$

Define $V : [a, b] \rightarrow \mathcal{P}(\mathbb{R})$ by

$$V(t) = \{v \in \mathbb{R} : |v_1(t) - v| \leq m(t)|u(t) - \bar{u}(t)|\}.$$

By Proposition III.4 [33], the multivalued operator $V(t) \cap \mathfrak{F}(t, \bar{u}(t))$ is measurable, and thus we can find a measurable selection $v_2(t)$ for V . So $v_2(t) \in \mathfrak{F}(t, \bar{u}(t))$ and $|v_1(t) - v_2(t)| \leq m(t)|u(t) - \bar{u}(t)|$, for each $t \in [a, b]$. Let us define

$$h_2(t) = \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[\lambda {}^k\mathcal{J}^{\alpha-p;\psi} f_2(\eta) + \mu {}^k\mathcal{J}^{\alpha+v;\psi} f_2(\sigma) - {}^k\mathcal{J}^{\alpha;\psi} f_2(b) \right] + {}^k\mathcal{J}^{\alpha;\psi} f_2(t),$$

for each $t \in [a, b]$. Consequently, we have

$$\begin{aligned} & |h_1(t) - h_2(t)| \\ & \leq \frac{(\psi(t) - \psi(a))^{\frac{\theta_k}{k}-1}}{\Omega \Gamma_k(\theta_k)} \left[|\lambda| {}^k\mathcal{J}^{\alpha-p;\psi} (|f_1(s) - f_2(s)|)(\eta) \right. \\ & \quad \left. + |\mu| {}^k\mathcal{J}^{\alpha+v;\psi} (|f_1(s) - f_2(s)|)(\sigma) + {}^k\mathcal{J}^{\alpha;\psi} (|f_1(s) - f_2(s)|)(b) \right] \\ & \quad + {}^k\mathcal{J}^{\alpha;\psi} (|f_1(s) - f_2(s)|)(t) \\ & \leq \left\{ \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\theta_k}{k}-1}}{|\Omega| \Gamma_k(\theta_k)} \left[|\lambda| \frac{(\psi(\eta) - \psi(a))^{\frac{\alpha-p}{k}}}{\Gamma_k(\alpha - p + k)} \right. \right. \\ & \quad \left. \left. + |\mu| \frac{(\psi(\sigma) - \psi(a))^{\frac{\alpha+v}{k}}}{\Gamma_k(\alpha + v + k)} + \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \right] \right\} \|m\| \|u - \bar{u}\| \\ & = \mathbb{G} \|m\| \|u - \bar{u}\|, \end{aligned}$$

which yields

$$\|h_1 - h_2\| \leq \mathbb{G} \|m\| \|u - \bar{u}\|.$$

Similarly, switching the roles of u and $\bar{u}$, we can obtain

$$H_d(\mathcal{F}(u), \mathcal{F}(\bar{u})) \leq \mathbb{G} \|m\| \|u - \bar{u}\|.$$

Hence, $\mathcal{F}$ is a contraction and we deduce by Covitz and Nadler fixed point theorem [32] that $\mathcal{F}$ has a fixed point u , which is indeed a solution of the multi-valued problem (2). This ends the proof. $\square$

6. Examples

In this section, numerical examples illustrating the applicability of our theoretical results are presented.

Example 1. Consider the following (k, ψ) -Hilfer-type fractional differential equations and inclusions with mixed (k, ψ) -derivative and integral boundary conditions of the form

$$\begin{cases} {}^{\frac{3}{2},H}D^{\frac{7}{4},\frac{1}{4},\frac{t+1}{t+2}} u(t) = f(t, u(t)), & \frac{1}{5} < t < \frac{8}{5}, \\ u\left(\frac{1}{5}\right) = 0, & u\left(\frac{8}{5}\right) = \frac{2}{31} {}^{\frac{3}{2},H}D^{\frac{3}{4},\frac{1}{2},\frac{t+1}{t+2}} u\left(\frac{3}{5}\right) + \frac{3}{51} {}^{\frac{3}{2}}\mathcal{J}^{\frac{15}{16},\frac{t+1}{t+2}} u\left(\frac{6}{5}\right). \end{cases} \quad (23)$$

Here, we choose $k = 3/2$, $\alpha = 7/4$, $\beta = 1/4$, $\psi(t) = (t+1)/(t+2)$, $a = 1/5$, $b = 8/5$, $\lambda = 2/31$, $p = 3/4$, $q = 1/2$, $\eta = 3/5$, $\mu = 3/51$, $v = 15/16$, $\sigma = 6/5$. The computational yields $\theta_{\frac{3}{2}} = 33/16$, $\Omega \approx 0.3485295041$, $\mathbb{G} \approx 0.2007468250$, $\mathbb{G}_1 \approx 0.1245096496$.

(i) Let the nonlinear unbounded Lipschitzian function $f : [1/5, 8/5] \times \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(t, u) = \left(\frac{u^2 + 4|u|}{1 + |u|} \right) \sin^2 \pi t + \frac{1}{3} t^2 + \frac{1}{4} t + \frac{1}{5}. \quad (24)$$

Now, we see that f satisfies the Lipschitzian condition as

$$|f(t, u_1) - f(t, u_2)| \leq 4|u_1 - u_2|,$$

for each $u_1, u_2 \in \mathbb{R}$ and $t \in [1/5, 8/5]$ with $\mathfrak{L} = 4$. Therefore, we obtain $\mathfrak{L}\mathbb{G} \approx 0.802987300 < 1$, which means that the condition (19) is satisfied. Hence, by Theorem 1, the (k, ψ) -Hilfer-type fractional differential equation with mixed (k, ψ) -derivative and integral boundary conditions (23) with the function f given by (24), has a unique solution on the interval $[1/5, 8/5]$.

(ii) Suppose that a nonlinear bounded Lipschitzian function $f : [1/5, 8/5] \times \mathbb{R} \rightarrow \mathbb{R}$ is

$$f(t, u) = \left(\frac{8|u|}{1 + |u|} \right) \cos^4 \pi t + 3t + 2, \quad (25)$$

which is bounded as

$$|f(t, u)| \leq 8 \cos^4 \pi t + 3t + 2 := \theta(t),$$

for all $t \in [1/5, 8/5]$. Observe that f satisfies the Lipschitz condition: $|f(t, u_1) - f(t, u_2)| \leq 8|u_1 - u_2|$, with Lipschitz constant $\mathfrak{L} = 8$. Hence, we obtain $\mathfrak{L}\mathfrak{G}_1 \approx 0.9960771968 < 1$. Then, the result in Theorem 2 yields that the (k, ψ) -Hilfer-type boundary value problem (23) with f presented by (25), has at least one solution on $[1/5, 8/5]$. As $\mathfrak{L}\mathbb{G} \approx 1.605974600 > 1$, the uniqueness result (Theorem 1) does not apply in this situation.

(iii) Assume that

$$f(t, u) = \theta(t)(Ag_1(u) + B), \quad (26)$$

where $\theta : [1/5, 8/5] \rightarrow \mathbb{R}^+$, $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ with $|g_1(u)| \leq |u|$, for example, $g_1(u) = u \sin^8 u$. In addition, $0 \leq A < 1/(\|\theta\|\mathbb{G})$ and $B > 0$. Then we have

$$|f(t, u)| \leq \|\theta\|(A|u| + B).$$

Writing $\chi(u) = A|u| + B$, there exists a constant $\mathfrak{K}$ satisfying condition (H_4) in Theorem 3:

$$\mathfrak{K} > \frac{B\|\theta\|\mathbb{G}}{1 - A\|\theta\|\mathbb{G}}.$$

Therefore, by applying Theorem 3, we deduce that the boundary value problem of (k, ψ) -Hilfer-type fractional differential equation (23) with f given in (26) has at least one solution on $[1/5, 8/5]$.

(iv) Suppose that

$$f(t, u) = \theta(t)(Ag_2(u) + B), \quad (27)$$

where $\theta : [1/5, 8/5] \rightarrow \mathbb{R}^+$, $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ with $|g_2(u)| \leq u^2$, for example, $g_2(u) = u^{18}/(1 + u^{16})$. In addition, two positive constants A and B are such that $AB < 1/(4\|\theta\|^2\mathbb{G}^2)$. Then, we have

$$|f(t, u)| \leq \|\theta\|(Au^2 + B).$$

Setting a function $\chi(u) = Au^2 + B$, there exists a constant

$$\mathfrak{K} \in \left(\frac{1 - \sqrt{1 - 4AB\|\theta\|^2\mathbb{G}^2}}{2A\|\theta\|\mathbb{G}}, \frac{1 + \sqrt{1 - 4AB\|\theta\|^2\mathbb{G}^2}}{2A\|\theta\|\mathbb{G}} \right),$$

satisfying condition (H_4) in Theorem 3. Therefore, all conditions of Theorem 3 are satisfied. Hence (k, ψ) -Hilfer-type nonlocal fractional boundary value problem (23), with f given in (27) has at least one solution on $[1/5, 8/5]$.

(v) Suppose that the first equation of (23) is presented by

$${}^{3/2}_H D^{7/4, 1/4}_{t+} u(t) \in \mathfrak{F}(t, u(t)), \quad \frac{1}{5} < t < \frac{8}{5}, \quad (28)$$

where

$$\mathfrak{F}(t, u) = \left[0, \frac{1}{5t+1} \left(\frac{u^2 + 6|u|}{1+|u|} + \frac{1}{4} \sin^2 t + \frac{3}{4} e^{-|5t-1|} \right) \right].$$

Obviously $\mathfrak{F}(t, u)$ is a measurable set. Moreover,

$$H_d(\mathfrak{F}(t, u), \mathfrak{F}(t, \bar{u})) \leq \frac{6}{5t+1} |u - \bar{u}|.$$

Let us choose $m(t) = 6/(5t+1)$ and observe that $d(0, \mathfrak{F}(t, 0)) \leq 1/(5t+1) < 6/(5t+1) = m(t)$ for almost all $t \in [1/8, 5/8]$. Since $\delta = \mathbb{G}\|m\| \approx 0.6022404750 < 1$, the (k, ψ) -Hilfer-type nonlocal fractional inclusion (28) with mixed (k, ψ) -derivative and integral boundary conditions as in (23), has at least one solution on $[1/5, 8/5]$.

7. Conclusions

This research is devoted to the analysis of single-valued and multi-valued (k, ψ) -Hilfer-type nonlocal fractional boundary value problems involving (k, ψ) -Hilfer fractional derivative and integral operators in boundary conditions. We established existence and uniqueness results for the single-valued case after transforming the given problem into a fixed point problem, with the help of Banach contraction mapping principle, Krasnosel'skiĭ fixed point theorem and the Leray–Schauder nonlinear alternative. Two existence results for the multi-valued problem are obtained by applying Leray–Schauder nonlinear alternative for multivalued maps and Covitz–Nadler fixed point theorem for contractive multivalued maps, respectively, for the cases of convex and non-convex multivalued map involved in the problem. All the obtained theoretical results are well-illustrated by numerical examples. The results are new and enrich the new research area on (k, ψ) -Hilfer nonlocal fractional boundary value problems in the order of $(1, 2]$. In a future study, we plan to extend the results of this paper to cover (k, ψ) -Hilfer nonlocal fractional systems of order in $(1, 2]$.

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