Interval Fejér-Type Inequalities for Left and Right-λ-Preinvex Functions in Interval-Valued Settings

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Abstract: For left and right λ-preinvex interval-valued functions (left and right λ-preinvex IVFs) in interval-valued Riemann operator settings, we create Hermite–Hadamard (*H-H*) type inequalities in the current study. Additionally, we create Hermite–Hadamard–Fejér (*H-H*-Fejér)-type inequalities for preinvex functions of the left and right interval-valued type under some mild conditions. Moreover, some exceptional new and classical cases are also obtained. Some useful examples are also presented to prove the validity of the results.

Keywords: left and right λ-preinvex interval-valued function; interval Riemann integral; Hermite–Hadamard-type inequality; Hermite–Hadamard–Fejér-type inequality

1. Introduction

One of the most well-known inequalities in the theory of convex functions with a geometrical meaning and a wide range of applications is the traditional Hermite–Hadamard inequality. This disparity might be seen as a more sophisticated use of convexity. Recent years have seen resurgence in interest in the Hermite–Hadamard inequality for convex functions, leading to the study of several noteworthy improvements and extensions [1,2].

It is generally recognized how important set-valued analysis research is both theoretically and in terms of practical applications. Control theory and dynamical games have been the driving forces behind several developments in set-valued analysis. Since the beginning of the 1960s, mathematical programming and optimal control theory have been the driving forces behind these fields. A specific example is interval analysis, which was developed in an effort to address the interval uncertainty that frequently emerges in mathematical or computer models of some deterministic real-world processes, see [3–7]. In recent years, certain important inequalities for interval valued functions, including Hermite–Hadamard- and Ostrowski-type inequalities, have also been developed. By utilizing Hukuhara derivatives for interval valued functions, Chalco-Cano et al. developed Ostrowski-type inequalities for interval valued functions in [8,9]. The inequalities of Minkowski and Beckenbach for interval valued functions were established by Roman-Flores et al. in [10]. We direct readers to [11–16] for further results that are related to generalization of convex and interval-valued convex functions

As additional references, Zhao et al. [17] introduced the idea of interval-valued coordinated convex functions; An et al. [18] introduced interval (h1, h2) convex functions; Nwaeze et al. [19] proved H-H inequality for n-polynomial convex interval-valued functions; and Tariboon et al. [20], Kalsoom et al. [21] and Ali et al. [22] refined this idea

using quantum calculus. Recently, this concept was also generalized to convex fuzzy interval-valued functions by Khan et al. [23]. Interval-valued analysis has also been used in optimization in fuzzy environments [24–30].

We construct some new mappings in relation to Hermite–Hadamard-type inequalities and show new Hermite–Hadamard–Fejér-type inequalities that do actually give refinement inequalities in order to be motivated by the investigations undertaken in [14–16,28]. Some special cases which can be vied as applications of our main results are also discussed. Some non-trivial examples are also presented to discuss the validity of our main findings.

2. Preliminaries

We begin by recalling the basic notations and definitions. We define interval as $[\mathfrak{B}_*, \mathfrak{B}^*] = \{\imath \in \mathbb{R} : \mathfrak{B}_* \leq \imath \leq \mathfrak{B}^* \text{ and } \mathfrak{B}_*, \mathfrak{B}^* \in \mathbb{R} \}$, where $\mathfrak{B}_* \leq \mathfrak{B}^*$.

We write len $[\mathfrak{B}_*, \mathfrak{B}^*] = \mathfrak{B}^* - \mathfrak{B}_*$. If len $[\mathfrak{B}_*, \mathfrak{B}^*] = 0$, then $[\mathfrak{B}_*, \mathfrak{B}^*]$ is named as degenerate. In this article, all intervals will be non-degenerate intervals. The collection of all closed and bounded intervals of $\mathbb{R}$ is denoted and defined as $\mathcal{K}_C = \{[\mathfrak{B}_*, \mathfrak{B}^*] : \mathfrak{B}_*, \mathfrak{B}^* \in \mathbb{R} \text{ and } \mathfrak{B}_* \leq \mathfrak{B}^*\}$. If $\mathfrak{B}_* \geq 0$, then $[\mathfrak{B}_*, \mathfrak{B}^*]$ is named as a positive interval. The set of all positive intervals is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\mathfrak{B}_*, \mathfrak{B}^*] : [\mathfrak{B}_*, \mathfrak{B}^*] \in \mathcal{K}_C \text{ and } \mathfrak{B}_* \geq 0\}$.

We will now look at some of the properties of intervals using arithmetic operations. Let $[\mathfrak{B}_*, \mathfrak{B}^*]$, $[\mathfrak{G}_*, \mathfrak{G}^*] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$; then, we have

$$[\mathfrak{B}_*, \mathfrak{B}^*] + [\mathfrak{G}_*, \mathfrak{G}^*] = [\mathfrak{B}_* + \mathfrak{G}_*, \mathfrak{B}^* + \mathfrak{G}^*], \tag{1}$$

$$[\mathfrak{B}_*, \mathfrak{B}^*] \times [\mathfrak{G}_*, \mathfrak{G}^*] = \begin{bmatrix} \min\{\mathfrak{B}_*\mathfrak{G}_*, \mathfrak{B}^*\mathfrak{G}_*, \mathfrak{B}_*\mathfrak{G}^*, \mathfrak{B}^*\mathfrak{G}^*\}, \\ \max\{\mathfrak{B}_*\mathfrak{G}_*, \mathfrak{B}^*\mathfrak{G}_*, \mathfrak{B}_*\mathfrak{G}^*, \mathfrak{B}^*\mathfrak{G}^*\} \end{bmatrix}, \tag{2}$$

$$\rho.[\mathfrak{B}_*, \mathfrak{B}^*] = \begin{cases} [\rho\mathfrak{B}_*, \rho\mathfrak{B}^*] & \text{if } \rho > 0 \\ \{0\} & \text{if } \rho = 0 \\ [\rho\mathfrak{B}^*, \rho\mathfrak{B}_*] & \text{if } \rho < 0. \end{cases} \tag{3}$$

For $[\mathfrak{B}_*, \mathfrak{B}^*]$, $[\mathfrak{G}_*, \mathfrak{G}^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by

$$[\mathfrak{B}_*, \mathfrak{B}^*] \subseteq [\mathfrak{G}_*, \mathfrak{G}^*] \text{ if and only if } \mathfrak{G}_* \leq \mathfrak{B}_*, \mathfrak{B}^* \leq \mathfrak{G}^*. \tag{4}$$

Remark 1. *The relation* " $\leq_p$ " *defined on $\mathcal{K}_C$ by*

$$[\mathfrak{B}_*, \mathfrak{B}^*] \leq_p [\mathfrak{G}_*, \mathfrak{G}^*] \text{ if and only if } \mathfrak{B}_* \leq \mathfrak{G}_*, \mathfrak{B}^* \leq \mathfrak{G}^*, \tag{5}$$

for all $[\mathfrak{B}_*, \mathfrak{B}^*]$, $[\mathfrak{G}_*, \mathfrak{G}^*] \in \mathcal{K}_C$, it is an order relation. This relation is also known as left and right relation, see [27].

Moore [5] initially proposed the concept of Riemann integral for IVF, which is defined as follows:

Theorem 1. ([5]). *If $\mathcal{U} : [\sigma, \varsigma] \subset \mathbb{R} \to \mathcal{K}_C$ is an IVF on such that $\mathcal{U}(\imath) = [\mathcal{U}_*(\imath), \mathcal{U}^*(\imath)]$. Then $\mathcal{U}$ is Riemann integrable (IR) over $[\sigma, \varsigma]$ if and only if, $\mathcal{U}_*$ and $\mathcal{U}^*$ both are Riemann integrable over $[\sigma, \varsigma]$ such that*

$$(IR) \int_\sigma^\varsigma \mathcal{U}(\imath)d\imath = [(R) \int_\sigma^\varsigma \mathcal{U}_*(\imath)d\imath, \ (R) \int_\sigma^\varsigma \mathcal{U}^*(\imath)d\imath]. \tag{6}$$

The collection of all Riemann-integrable interval-valued functions is denoted by $\mathcal{IR}_{([\sigma, \varsigma])}$.

Definition 1. ([29]). *Let K be an invex set and $\lambda : [0, 1] \to \mathbb{R}$ such that $\lambda(\imath) > 0$. Then IVF $\mathcal{U} : K \to \mathcal{K}_C^+$ is said to be left and right λ-preinvex on K with respect to ϖ if*

$$\mathcal{U}(\imath + (1 - \sigma)\varpi(\jmath, \imath)) \leq_p \lambda(\sigma)\mathcal{U}(\imath) + \lambda(1 - \sigma)\mathcal{U}(\jmath), \tag{7}$$

for all $\imath, \jmath \in K, \sigma \in [0, 1]$, where $\mathcal{U}(\imath) \geq_p 0$, $\varpi : K \times K \to \mathbb{R}$. $\mathcal{U}$ is named as left and right λ-preconcave on K with respect to ϖ if inequality (7) is reversed. $\mathcal{U}$ is named as affine left and right λ-preinvex on K with respect to ϖ if

$$\mathcal{U}(\imath + (1 - \sigma)\varpi(\jmath, \imath)) = \lambda(\sigma)\mathcal{U}(\imath) + \lambda(1 - \sigma)\mathcal{U}(\jmath), \tag{8}$$

for all $\imath, \jmath \in K, \sigma \in [0, 1]$, where $\mathcal{U}(\imath) \geq_p 0$, $\varpi : K \times K \to \mathbb{R}$.

Remark 2. *The left and right λ-preinvex IVFs have some very nice properties similar to preinvex IVF:*
If $\mathcal{U}$ is left and right λ-preinvex IVF, then Y is also left and right λ-preinvex for $Y \geq 0$.
If $\mathcal{U}$ and ψ both are left and right λ-preinvex IVFs, then $max(\mathcal{U}(\imath), \psi(\imath))$ is also left and right λ-preinvex IVF.
Now we discuss some new special cases of λ-preinvex IVFs:

(i) *If $\lambda(\sigma) = \sigma^s$, then left and right λ-preinvex IVF becomes left and right s-preinvex IVF, that is*

$$\mathcal{U}(\imath + (1 - \sigma)\varpi(\jmath, \imath)) \leq_p \sigma^s\mathcal{U}(\imath) + (1 - \sigma)^s\mathcal{U}(\jmath), \forall \imath, \jmath \in K, \sigma \in [0, 1]. \tag{9}$$

If $\varpi(\jmath, \imath) = \jmath - \imath$, then $\mathcal{U}$ is named as left and right s-convex IVF.

(ii) *If $\lambda(\sigma) = \sigma$, then left and right λ-preinvex IVF becomes left and right preinvex IVF, that is*

$$\mathcal{U}(\imath + (1 - \sigma)\varpi(\jmath, \imath)) \leq_p \sigma\mathcal{U}(\imath) + (1 - \sigma)\mathcal{U}(\jmath), \forall \imath, \jmath \in K, \sigma \in [0, 1]. \tag{10}$$

If $\varpi(\jmath, \imath) = \jmath - \imath$, then $\mathcal{U}$ is named as left and right convex IVF.

(iii) *If $\lambda(\sigma) \equiv 1$, then left and right λ-preinvex IVF becomes left and right P IVF, that is*

$$\mathcal{U}(\imath + (1 - \sigma)\varpi(\jmath, \imath)) \leq_p \mathcal{U}(\imath) + \mathcal{U}(\jmath), \forall \imath, \jmath \in K, \sigma \in [0, 1]. \tag{11}$$

If $\varpi(\jmath, \imath) = \jmath - \imath$, then $\mathcal{U}$ is named as left and right P IVF.

Theorem 2. ([29]). *Let K be an invex set and $\lambda : [0, 1] \subseteq K \to \mathbb{R}^+$ such that $\lambda > 0$, and let $\mathcal{U} : K \to \mathcal{K}_C^+$ be a IVF with $\mathcal{U}(\imath) \geq_p 0$ such that*

$$\mathcal{U}(\imath) = [\mathcal{U}_*(\imath), \mathcal{U}^*(\imath)], \forall \imath \in K. \tag{12}$$

for all $\imath \in K$. Then $\mathcal{U}$ is left and right λ-preinvex IVF on K if and only if $\mathcal{U}_*(\imath)$ and $\mathcal{U}^*(\imath)$ both are λ-preinvex functions.

Example 1. *We consider $\lambda(\sigma) = \sigma$, for $\sigma \in [0, 1]$ and the IVF $\mathcal{U} : \mathbb{R}^+ \to \mathcal{K}_C^+$ defined by $\mathcal{U}(\imath) = [(e^\imath, 2e^\imath]$. Since $\mathcal{U}_*(\imath), \mathcal{U}^*(\imath)$ are λ-preinvex functions $\varpi(\jmath, \imath) = \jmath - \imath$. Hence, $\mathcal{U}(\imath)$ is left and right λ-preinvex IVF.*

3. Main Results

We use the crucial assumption about bifunction $\varpi : K \times K \to \mathbb{R}$ that has been provided to demonstrate the main conclusions of this study.

Condition C 1 (see [14]). *Let K be an invex set with respect to ϖ. For any $\varrho, \sigma \in K$ and $\sigma \in [0, 1]$,*

$$\begin{cases} \varpi(\varsigma, \varrho + \sigma\varpi(\varsigma, \varrho)) = (1 - \sigma)\varpi(\varsigma, \varrho) \\ \varpi(\varrho, \varrho + \sigma\varpi(\varsigma, \varrho)) = -\sigma\varpi(\varsigma, \varrho) \end{cases}, \tag{13}$$

Clearly, for $\sigma = 0$, we have $\varpi(\varsigma, \sigma) = 0$ if and only if $\varsigma = \sigma$, for all $\sigma, \varsigma \in K$. For the applications of Condition C, see [14–16].

Theorem 3. *Let $\mathcal{U} : [\sigma, \sigma + \varpi(\varsigma, \sigma)] \to \mathcal{K}_C^+$ be a left and right λ-preinvex IVF with $\lambda : [0, 1] \to \mathbb{R}^+$ and $\lambda\left(\dfrac{1}{2}\right) \not\equiv 0$ such that $\mathcal{U}(i) = [\mathcal{U}_*(i), \mathcal{U}^*(i)]$ for all $i \in [\sigma, \sigma + \varpi(\varsigma, \sigma)]$. If $\mathcal{U} \in \mathcal{IR}_{([\sigma, \sigma + \varpi(\varsigma, \sigma)])}$, then*

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq_p \frac{1}{\varpi(\varsigma, \sigma)} \, (IR) \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}(i) di \leq_p [\mathcal{U}(\sigma) + \mathcal{U}(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma. \tag{14}$$

If $\mathcal{U}$ is left and right λ-preconcave IVF, then (14) is reversed such that

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \geq_p \frac{1}{\varpi(\varsigma, \sigma)} \, (IR) \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}(i) di \geq_p [\mathcal{U}(\sigma) + \mathcal{U}(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma. \tag{15}$$

Proof. Let $\mathcal{U} : [\sigma, \sigma + \varpi(\varsigma, \sigma)] \to \mathcal{K}_C^+$ be a left and right λ-preinvex IVF. Then, by hypothesis, we have

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathcal{U}\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq_p \mathcal{U}(\sigma + (1 - \sigma)\varpi(\varsigma, \sigma)) + \mathcal{U}(\sigma + \sigma\varpi(\varsigma, \sigma)).$$

Therefore, we have

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathcal{U}_*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq \mathcal{U}_*(\sigma + (1 - \sigma)\varpi(\varsigma, \sigma)) + \mathcal{U}_*(\sigma + \sigma\varpi(\varsigma, \sigma)),$$

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathcal{U}^*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq \mathcal{U}^*(\sigma + (1 - \sigma)\varpi(\varsigma, \sigma)) + \mathcal{U}^*(\sigma + \sigma\varpi(\varsigma, \sigma)).$$

Then

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \int_0^1 \mathcal{U}_*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) d\sigma \leq \int_0^1 \mathcal{U}_*(\sigma + (1 - \sigma)\varpi(\varsigma, \sigma)) d\sigma + \int_0^1 \mathcal{U}_*(\sigma + \sigma\varpi(\varsigma, \sigma)) d\sigma,$$

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \int_0^1 \mathcal{U}^*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) d\sigma \leq \int_0^1 \mathcal{U}^*(\sigma + (1 - \sigma)\varpi(\varsigma, \sigma)) d\sigma + \int_0^1 \mathcal{U}^*(\sigma + \sigma\varpi(\varsigma, \sigma)) d\sigma.$$

It follows that

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathcal{U}_*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq \frac{2}{\varpi(\varsigma, \sigma)} \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}_*(i) di,$$

$$\frac{1}{\lambda\left(\frac{1}{2}\right)} \mathcal{U}^*\left(\frac{2\sigma + \varpi(\varsigma, \sigma)}{2}\right) \leq \frac{2}{\varpi(\varsigma, \sigma)} \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}^*(i) di.$$

That is,

$$\frac{1}{\lambda\left(\frac{1}{2}\right)}\left[\mathcal{U}_*\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right),\mathcal{U}^*\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right)\right] \leq_p \frac{2}{\omega(\varsigma,\sigma)}\left[\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}_*(\imath)d\imath, \int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}^*(\imath)d\imath\right].$$

Thus,

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath. \tag{16}$$

In a similar way as above, we have

$$\frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq_p [\mathcal{U}(\sigma)+\mathcal{U}(\varsigma)]\int_0^1 \lambda(\sigma)d\sigma. \tag{17}$$

Combining (16) and (17), we have

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq_p [\mathcal{U}(\sigma)+\mathcal{U}(\varsigma)]\int_0^1 \lambda(\sigma)d\sigma,$$

which complete the proof. □

Note that, inequality (14) is known as fuzzy-interval *H-H* inequality for left and right λ-preinvex IVF.

Remark 3. *If one takes* $\lambda(\sigma) = \sigma^s$, *then from (14), one can obtain the result for left and right s-preinvex IVF:*

$$2^{s-1}\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq_p \frac{1}{s+1}[\mathcal{U}(\sigma)+\mathcal{U}(\varsigma)]. \tag{18}$$

If one takes $\lambda(\sigma) = \sigma$, *then from (14), one can obtain the result for left and right preinvex IVF:*

$$\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq_p \frac{\mathcal{U}(\sigma)+\mathcal{U}(\varsigma)}{2}. \tag{19}$$

If one takes $\lambda(\sigma) \equiv 1$, *then from (14), one can obtain the result for left and right P IVF:*

$$\frac{1}{2}\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma,\sigma)}\,(IR)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq_p \mathcal{U}(\sigma)+\mathcal{U}(\varsigma). \tag{20}$$

If one takes $\mathcal{U}_*(\imath) = \mathcal{U}^*(\imath)$, *then from (14), one can acquIRe the result for λ-preinvex function, see* [16]:

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)}\mathcal{U}\left(\frac{2\sigma+\omega(\varsigma,\sigma)}{2}\right) \leq \frac{1}{\omega(\varsigma,\sigma)}\,(R)\int_\sigma^{\sigma+\omega(\varsigma,\sigma)}\mathcal{U}(\imath)d\imath \leq [\mathcal{U}(\sigma)+\mathcal{U}(\varsigma)]\int_0^1 \lambda(\sigma)d\sigma. \tag{21}$$

Note that, if $\omega(\varsigma,\sigma) = \varsigma - \sigma$, *then integral Inequalities (18)–(21) reduce to classical ones.*

Example 2. *We consider* $\lambda(\sigma) = \sigma$ *for* $\sigma \in [0, 1]$ *and the IVF* $\mathcal{U} : [\sigma, \sigma+\omega(\varsigma,\sigma)] = [0, \omega(2, 0)] \to \mathcal{K}_C^+$, *defined by* $\mathcal{U}(\imath) = [2\imath^2, 4\imath^2]$. *Since* $\mathcal{U}_*(\imath) = 2\imath^2$, $\mathcal{U}^*(\imath) = 4\imath^2$ *are λ-preinvex functions with respect to* $\omega(\varsigma,\sigma) = \varsigma - \sigma$. *Hence,* $\mathcal{U}(\imath)$ *is left and right λ-preinvex*

IVF with respect to $\omega(\varsigma, \sigma) = \varsigma - \sigma$. *Since* $\mathcal{U}_*(\iota) = 2\iota^2$ *and* $\mathcal{U}^*(\iota) = 4\iota^2$ *then, we compute the following*

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}_*\left(\frac{2\sigma + \omega(\varsigma, \sigma)}{2}\right) \leq \frac{1}{\omega(\varsigma, \sigma)} \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}_*(\iota) d\iota \leq [\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma.$$

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}_*\left(\frac{2\sigma + \omega(\varsigma, \sigma)}{2}\right) = \mathcal{U}_*(1) = 2,$$

$$\frac{1}{\omega(\varsigma, \sigma)} \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}_*(\iota) d\iota = \frac{1}{2} \int_0^2 2\iota^2 d\iota = \frac{8}{3},$$

$$[\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma = 4,$$

That means

$$2 \leq \frac{8}{3} \leq 4.$$

Similarly, it can be easily shown that

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}^*\left(\frac{2\sigma + \omega(\varsigma, \sigma)}{2}\right) \leq \frac{1}{\omega(\varsigma, \sigma)} \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}^*(\iota) d\iota \leq [\mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma.$$

such that

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}^*\left(\frac{2\sigma + \omega(\varsigma, \sigma)}{2}\right) = \mathcal{U}_*(1) = 4,$$

$$\frac{1}{\omega(\varsigma, \sigma)} \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}^*(\iota) d\iota = \frac{1}{2} \int_0^2 4\iota^2 d\iota = \frac{16}{3},$$

$$[\mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma = 8.$$

From which it follows that

$$4 \leq \frac{16}{3} \leq 8,$$

That is,

$$[2,\ 4] \leq_p \left[\frac{8}{3},\ \frac{16}{3}\right] \leq_p [4,\ 8].$$

Hence,

$$\frac{1}{2\lambda\left(\frac{1}{2}\right)} \mathcal{U}\left(\frac{2\sigma + \omega(\varsigma, \sigma)}{2}\right) \leq_p \frac{1}{\omega(\varsigma, \sigma)} (IR) \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}(\iota) d\iota \leq_p [\mathcal{U}(\sigma) + \mathcal{U}(\varsigma)] \int_0^1 \lambda(\sigma) d\sigma,$$

and Theorem 3 is verified.

Theorem 4. *Let* $\mathcal{U}, \psi : [\sigma,\ \sigma + \omega(\varsigma, \sigma)] \to \mathcal{K}_C^+$ *be two left and right* λ_1- *and left and right* λ_2-*preinvex IVFs with* $\lambda_1, \lambda_2 : [0,\ 1] \to \mathbb{R}^+$ *such that* $\mathcal{U}(\iota) = [\mathcal{U}_*(\iota), \mathcal{U}^*(\iota)]$ *and* $\psi(\iota) = [\psi_*(\iota), \psi^*(\iota)]$ *for all* $\sigma \in [\sigma,\ \sigma + \omega(\varsigma, \sigma)]$. *If* $\mathcal{U} \times \psi \in \mathcal{IR}_{([\sigma,\ \sigma + \omega(\varsigma, \sigma)])}$, *then*

$$\frac{1}{\omega(\varsigma, \sigma)} (IR) \int_\sigma^{\sigma + \omega(\varsigma, \sigma)} \mathcal{U}(\iota) \times \psi(\iota) d\iota \leq_p \mathfrak{S}(\sigma, \varsigma) \int_0^1 \lambda_1(\sigma) \lambda_2(\sigma) d\sigma + \mathfrak{Q}(\sigma, \varsigma) \int_0^1 \lambda_1(\sigma) \lambda_2(1 - \sigma) d\sigma,$$

where $\mathfrak{S}(\sigma, \varsigma) = \mathcal{U}(\sigma) \times \psi(\sigma) + \mathcal{U}(\varsigma) \times \psi(\varsigma)$, $\mathfrak{Q}(\sigma, \varsigma) = \mathcal{U}(\sigma) \times \psi(\varsigma) + \mathcal{U}(\varsigma) \times \psi(\sigma)$ *with* $\mathfrak{S}(\sigma, \varsigma) = [\mathfrak{S}_*((\sigma, \varsigma)),\ \mathfrak{S}^*((\sigma, \varsigma))]$ *and* $\mathfrak{Q}(\sigma, \varsigma) = [\mathfrak{Q}_*((\sigma, \varsigma)),\ \mathfrak{Q}^*((\sigma, \varsigma))]$.

Example 3. *We consider* $\lambda_1(\sigma) = \sigma$, $\lambda_2(\sigma) \equiv 1$, *for* $\sigma \in [0, 1]$, *and the IVFs* $\mathcal{U}$, ψ : $[\varrho, \varrho + \varpi(\varsigma, \varrho)] = [0, \varpi(1, 0)] \to \mathcal{K}_C^+$ *defined by* $\mathcal{U}(\imath) = [2\imath^2, 4\imath^2]$ *and* $\psi(\imath) = [\imath, 2\imath]$. *Since* $\mathcal{U}_*(\imath) = 2\imath^2$ *and* $\mathcal{U}^*(\imath) = 4\imath^2$ *are both* λ_1*-preinvex functions and* $\psi_*(\imath) = \imath$, *and* $\psi^*(\imath) = 2\imath$ *are also both* λ_2*-preinvex functions with respect to the same* $\varpi(\varsigma, \varrho) = \varsigma - \varrho$, $\mathcal{U}$ *and* ψ *both are left and right* λ_1*- and left and right* λ_2*-preinvex IVFs, respectively. Since* $\mathcal{U}_*(\imath) = 2\imath^2$ *and* $\mathcal{U}^*(\imath) = 4\imath^2$, *and* $\psi_*(\imath) = \imath$, *and* $\psi^*(\imath) = 2\imath$, *then*

$$\frac{1}{\varpi(\varsigma, \varrho)} \int_\varrho^{\varrho+\varpi(\varsigma,\varrho)} \mathcal{U}_*(\imath) \times \psi_*(\imath) d\imath = \int_0^1 (2\imath^2)(\imath) d\imath = \frac{1}{2},$$

$$\frac{1}{\varpi(\varsigma, \varrho)} \int_\varrho^{\varrho+\varpi(\varsigma,\varrho)} \mathcal{U}^*(\imath) \times \psi^*(\imath) d\imath = \int_0^1 (4\psi^2)(2\psi) d\imath = 2,$$

$$\mathfrak{S}_*((\varrho,\varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma = 1,$$

$$\mathfrak{S}^*((\varrho,\varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma = 4,$$

$$\mathfrak{Q}_*((\varrho,\varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma = 0$$

$$\mathfrak{Q}^*((\varrho,\varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma = 0,$$

That means

$$\frac{1}{2} \leq 1 + 0 = 1,$$

$$2 \leq 4 + 0 = 4,$$

Hence, Theorem 4 is verified.

Theorem 5. *Let* $\mathcal{U}, \psi : [\varrho, \varrho + \varpi(\varsigma, \varrho)] \to \mathcal{K}_C^+$ *be two left and right* λ_1*- and left and right* λ_2*-preinvex IVFs with* $\lambda_1, \lambda_2 : [0, 1] \to \mathbb{R}^+$ *given by* $\mathcal{U}(\imath) = [\mathcal{U}_*(\imath), \mathcal{U}^*(\imath)]$ *and* $\psi(\imath) = [\psi_*(\imath), \psi^*(\imath)]$ *for all* $\imath \in [\varrho, \varrho + \varpi(\varsigma, \varrho)]$. *If* $\mathcal{U} \times \psi \in \mathcal{IR}_{([\varrho, \varrho+\varpi(\varsigma, \varrho)])}$ *and condition C hold for* ϖ , *then*

$$\frac{1}{2\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)} \mathcal{U}\left(\frac{2\varrho + \varpi(\varsigma, \varrho)}{2}\right) \times \psi\left(\frac{2\varrho + \varpi(\varsigma, \varrho)}{2}\right) \leq_p \frac{1}{\varpi(\varsigma, \varrho)} (IR) \int_\varrho^{\varrho+\varpi(\varsigma,\varrho)} \mathcal{U}(\imath) \times \psi(\imath) d\imath$$

$$+ \mathcal{M}(\varrho, \varsigma) \int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma + \mathfrak{Q}(\varrho, \varsigma) \int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma,$$

where $\mathfrak{S}(\varrho, \varsigma) = \mathcal{U}(\varrho) \times \psi(\varrho) + \mathcal{U}(\varsigma) \times \psi(\varsigma)$, $\mathfrak{Q}(\varrho, \varsigma) = \mathcal{U}(\varrho) \times \psi(\varsigma) + \mathcal{U}(\varsigma) \times \psi(\varrho)$, *and* $\mathfrak{S}(\varrho, \varsigma) = [\mathfrak{S}_*((\varrho,\varsigma)), \mathfrak{S}^*((\varrho,\varsigma))]$ *and* $\mathfrak{Q}(\varrho, \varsigma) = [\mathfrak{Q}_*((\varrho,\varsigma)), \mathfrak{Q}^*((\varrho,\varsigma))]$.

Proof. Using condition C, we can write

$$\varrho + \frac{1}{2}\varpi(\varsigma, \varrho) = \varrho + \sigma\varpi(\varsigma, \varrho) + \frac{1}{2}\varpi(\varrho + (1-\sigma)\varpi(\varsigma, \varrho), \ \varrho + \sigma\varpi(\varsigma, \varrho)).$$

By hypothesis, we have

$$
\mathcal{U}_*\left(\frac{2\sigma + \wp(\varsigma, \sigma)}{2}\right) \times \psi_*\left(\frac{2\sigma + \wp(\varsigma, \sigma)}{2}\right)
$$

$$
\mathcal{U}^*\left(\frac{2\sigma + \wp(\varsigma, \sigma)}{2}\right) \times \psi^*\left(\frac{2\sigma + \wp(\varsigma, \sigma)}{2}\right)
$$

$$
= \mathcal{U}_*\left(\sigma + \sigma\wp(\varsigma, \sigma) + \frac{1}{2}\wp(\sigma + (1-\sigma)\wp(\varsigma, \sigma),\ \sigma + \sigma\wp(\varsigma, \sigma))\right)
$$

$$
\times \psi_*\left(\sigma + \sigma\wp(\varsigma, \sigma) + \frac{1}{2}\wp(\sigma + (1-\sigma)\wp(\varsigma, \sigma),\ \sigma + \sigma\wp(\varsigma, \sigma))\right),
$$

$$
= \mathcal{U}^*\left(\sigma + \sigma\wp(\varsigma,\ \sigma) + \frac{1}{2}\wp(\sigma + (1-\sigma)\wp(\varsigma,\ \sigma),\ \sigma + \sigma\wp(\varsigma,\ \sigma))\right)
$$

$$
\times \psi^*\left(\sigma + \sigma\wp(\varsigma,\ \sigma) + \frac{1}{2}\wp(\sigma + (1-\sigma)\wp(\varsigma,\ \sigma),\ \sigma + \sigma\wp(\varsigma,\ \sigma))\right),
$$

$$
\leq \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi_*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}_*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ \mathcal{U}^*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi^*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix},
$$

$$
\leq \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi^*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}^*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ \mathcal{U}_*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi_*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix},
$$

$$
\leq \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}_*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi_*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}(\lambda_1(\sigma)\mathcal{U}_*(\sigma) + \lambda_1(1-\sigma)\mathcal{U}_*(\varsigma)) \\ \times (\lambda_2(1-\sigma)\psi_*(\sigma) + \lambda_2(\sigma)\psi_*(\varsigma)) \\ +(\lambda_1(1-\sigma)\mathcal{U}_*(\sigma) + \lambda_1(\sigma)\mathcal{U}_*(\varsigma)) \\ \times (\lambda_2(\sigma)\psi_*(\sigma) + \lambda_2(1-\sigma)\psi_*(\varsigma))\end{bmatrix},
$$

$$
\leq \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}^*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi^*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}(\lambda_1(\sigma)\mathcal{U}^*(\sigma) + \lambda_1(1-\sigma)\mathcal{U}^*(\varsigma)) \\ \times (\lambda_2(1-\sigma)\psi^*(\sigma) + \lambda_2(\sigma)\psi^*(\varsigma)) \\ +(\lambda_1(1-\sigma)\mathcal{U}^*(\sigma) + \lambda_1(\sigma)\mathcal{U}^*(\varsigma)) \\ \times (\lambda_2(\sigma)\psi^*(\sigma) + \lambda_2(1-\sigma)\psi_*(\varsigma))\end{bmatrix},
$$

$$
= \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi_*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}_*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi_*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ 2\lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\{\lambda_1(\sigma)\lambda_2(\sigma) + \lambda_1(1-\sigma)\lambda_2(1-\sigma)\}\mathfrak{Q}_*((\sigma, \varsigma)) \\ +\{\lambda_1(\sigma)\lambda_2(1-\sigma) + \lambda_1(1-\sigma)\lambda_2(\sigma)\}\mathfrak{S}_*((\sigma, \varsigma))\end{bmatrix},
$$

$$
= \lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\mathcal{U}^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \times \psi^*(\sigma + (1-\sigma)\wp(\varsigma, \sigma)) \\ +\mathcal{U}^*(\sigma + \sigma\wp(\varsigma, \sigma)) \times \psi^*(\sigma + \sigma\wp(\varsigma, \sigma))\end{bmatrix}
$$

$$
+ 2\lambda_1(\tfrac{1}{2})\lambda_2(\tfrac{1}{2})\begin{bmatrix}\{\lambda_1(\sigma)\lambda_2(\sigma) + \lambda_1(1-\sigma)\lambda_2(1-\sigma)\}\mathfrak{Q}^*((\sigma, \varsigma)) \\ +\{\lambda_1(\sigma)\lambda_2(1-\sigma) + \lambda_1(1-\sigma)\lambda_2(\sigma)\}\mathfrak{S}^*((\sigma, \varsigma))\end{bmatrix},
$$

Integrating over $[0, 1]$, we have

$$\frac{1}{2\lambda_1(\frac{1}{2})\lambda_2(\frac{1}{2})}\mathcal{U}_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \times \psi_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right)$$

$$\leq \frac{1}{\varpi(\varsigma,\ \mathscr{o})}\int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}_*(\mathscr{o}) \times \psi_*(\iota)d\iota$$

$$+\mathfrak{S}_*((\mathscr{o}\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma + \mathfrak{Q}_*((\mathscr{o},\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(\lambda)d\sigma,$$

$$\frac{1}{2\lambda_1(\frac{1}{2})\lambda_2(\frac{1}{2})}\mathcal{U}^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \times \psi^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right)$$

$$\leq \frac{1}{\varpi(\varsigma,\ \mathscr{o})}\int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}^*(\iota) \times \psi^*(\iota)d\iota$$

$$+\mathfrak{S}^*((\mathscr{o},\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma + \mathfrak{Q}^*((\mathscr{o},\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma,$$

from which we have

$$\frac{1}{2\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}\left[\mathcal{U}_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\mathscr{o})}{2}\right) \times \psi_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\mathscr{o})}{2}\right), \mathcal{U}^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\mathscr{o})}{2}\right) \times \psi^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\mathscr{o})}{2}\right)\right]$$

$$\leq_p \frac{1}{\varpi(\varsigma,\mathscr{o})}\left[\int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}_*(\iota) \times \psi_*(\iota)d\iota\ ,\ \int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}^*(\iota) \times \psi^*(\iota)d\iota\right]$$

$$+ \int_0^1 \lambda_1(\sigma)\lambda_2(1-\text{ val}-\text{valued functions}\sigma)d\sigma[\mathfrak{S}_*((\mathscr{o},\varsigma)),\ \mathfrak{S}^*((\mathscr{o},\varsigma))]$$

$$+[\mathfrak{Q}_*((\mathscr{o},\varsigma)),\ \mathfrak{Q}^*((\mathscr{o},\varsigma))]\int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma,$$

that is,

$$\frac{1}{2\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}\mathcal{U}\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \times \psi\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \leq_p \frac{1}{\varpi(\varsigma,\ \mathscr{o})}\ (IR)\int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}(\iota) \times \psi(\iota)d\iota$$

$$+\mathfrak{S}(\mathscr{o},\varsigma)\int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma + \mathfrak{Q}(\mathscr{o},\varsigma)\int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma,$$

This completes the result. $\square$

Example 4. *We consider $\lambda_1(\sigma) = \sigma$, $\lambda_2(\sigma) \equiv 1-\sigma$, for $\sigma \in [0, 1]$ and the IVFs $\mathcal{U}$, ψ : $[\mathscr{o},\ \mathscr{o}+\varpi(\varsigma,\ \mathscr{o})] = [0,\ \varpi(1, 0)] \to \mathcal{K}_C^+$ defined by $\mathcal{U}(\iota) = [2\iota^2, 4\iota^2]$ and $\psi(\iota) = [\iota, 2\iota]$, as in Example 3, and $\mathcal{U}(\iota)$, $\psi(\iota)$ both are left and right λ_1- and left and right λ_2-preinvex IVFs with respect to $\varpi(\varsigma,\ \mathscr{o}) = \varsigma - \mathscr{o}$, respectively. Since $\mathcal{U}_*(\iota) = 2\iota^2, \mathcal{U}^*(\iota) = 4\iota^2$ and $\psi_*(\iota) = \iota$, $\psi^*(\iota) = 2\iota$, we have*

$$\frac{1}{2\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}\ \mathcal{U}_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \times \psi_*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) = \frac{1}{2},$$

$$\frac{1}{2\lambda_1\left(\frac{1}{2}\right)\lambda_2\left(\frac{1}{2}\right)}\ \mathcal{U}^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) \times \psi^*\left(\frac{2\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}{2}\right) = 2,$$

$$\frac{1}{\varpi(\varsigma,\ \mathscr{o})}\ \int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}_*(\iota) \times \psi_*(\iota)d\iota = \frac{1}{2}$$

$$\frac{1}{\varpi(\varsigma,\ \mathscr{o})}\ \int_{\mathscr{o}}^{\mathscr{o}+\varpi(\varsigma,\ \mathscr{o})}\mathcal{U}^*(\iota) \times \psi^*(\iota)d\iota = 2,$$

$$\mathfrak{S}_*((\mathscr{o},\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma = \frac{1}{3},$$

$$\mathfrak{S}^*((\mathscr{o},\varsigma))\int_0^1 \lambda_1(\sigma)\lambda_2(1-\sigma)d\sigma = \frac{4}{3},$$

$$\mathfrak{Q}_*((\sigma, \varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma = 0,$$
$$\mathfrak{Q}^*((\sigma, \varsigma)) \int_0^1 \lambda_1(\sigma)\lambda_2(\sigma)d\sigma = 0,$$

That means

$$\frac{1}{2} \leq \frac{1}{2} + 0 + \frac{1}{3} = \frac{5}{6},$$
$$2 \leq 2 + 0 + \frac{4}{3} = \frac{10}{3},$$

Hence, Theorem 5 is demonstrated.

Using Condition C, we will present weighted extensions of Theorems 3 for left and right λ-preinvex IVF in the following findings.

Theorem 6. *Let $\mathcal{U} : [\sigma, \sigma + \omega(\varsigma, \sigma)] \to \mathcal{K}_C^+$ be a left and right λ -preinvex IVF with $\sigma < \sigma + \omega(\varsigma, \sigma)$ and $\lambda : [0, 1] \to \mathbb{R}^+$ given by $\mathcal{U}(\iota) = [\mathcal{U}_*(\iota), \mathcal{U}^*(\iota)]$ for all $\iota \in [\sigma, \sigma + \omega(\varsigma, \sigma)]$. If $\mathcal{U} \in \mathcal{IR}_{([\sigma, \sigma+\omega(\varsigma, \sigma)])}$ and $\mathfrak{X} : [\sigma, \sigma + \omega(\varsigma, \sigma)] \to \mathbb{R}$, $\mathfrak{X}(\iota) \geq 0$, symmetric with respect to $\sigma + \frac{1}{2}\omega(\varsigma, \sigma)$, then*

$$\frac{1}{\omega(\varsigma, \sigma)} (IR) \int_\sigma^{\sigma+\omega(\varsigma,\sigma)} \mathcal{U}(\iota)\mathfrak{X}(\iota)d\iota \leq_p [\mathcal{U}(\sigma) + \mathcal{U}(\varsigma)] \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))d\sigma. \tag{22}$$

Proof. Let $\mathcal{U}$ be a left and right λ-preinvex IVF. Then, we have

$$\mathcal{U}_*(\sigma + (1 - \sigma) \quad \omega(\varsigma,\sigma))\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))$$
$$\leq (\lambda(\sigma)\mathcal{U}_*(\sigma) + \lambda(1 - \sigma)\mathcal{U}_*(\sigma))\mathfrak{X}(+(1 - \sigma\omega(\varsigma,\sigma)),$$
$$\mathcal{U}^*(\sigma + (1 - \sigma) \quad \omega(\varsigma,\sigma))\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))$$
$$\leq (\lambda(\sigma)\mathcal{U}^*(\sigma) + \lambda(1 - \sigma)\mathcal{U}^*(\sigma))\mathfrak{X}(+(1 - \sigma\omega(\varsigma,\sigma)), \tag{23}$$

And

$$\mathcal{U}_*(\sigma + \sigma\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma)) \leq (\lambda(1 - \sigma)\mathcal{U}_*(\sigma) + \lambda(\sigma)\mathcal{U}_*(\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma)),$$
$$\mathcal{U}^*(\sigma + \sigma\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma)) \leq (\lambda(1 - \sigma)\mathcal{U}^*(\sigma) + \lambda(\sigma)\mathcal{U}^*(\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma)). \tag{24}$$

After adding (23) and (24), and integrating over [0, 1], we get

$$\int_0^1 \mathcal{U}_*(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))d\sigma$$
$$+ \int_0^1 \mathcal{U}_*(\sigma + \sigma\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))d\sigma$$
$$\leq \int_0^1 \begin{bmatrix} \mathcal{U}_*(\sigma)\{\lambda(\sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma)) + \lambda(1 - \sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))\} \\ +\mathcal{U}_*(\varsigma)\{\lambda(1 - \sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma)) + \lambda(1 - \sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))\} \end{bmatrix} d\sigma,$$
$$\int_0^1 \mathcal{U}^*(\sigma + \sigma\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))d\sigma$$
$$+ \int_0^1 \mathcal{U}^*(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma))d\sigma$$
$$\leq \int_0^1 \begin{bmatrix} \mathcal{U}^*(\sigma)\{\lambda(\sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma)) + \lambda(1 - \sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))\} \\ +\mathcal{U}^*(\varsigma)\{\lambda(1 - \sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma,\sigma)) + \lambda(1 - \sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma,\sigma))\} \end{bmatrix} d\sigma,$$
$$= 2\mathcal{U}_*(\sigma) \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma, \sigma))d\sigma + 2\mathcal{U}_*(\varsigma) \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma, \sigma))d\sigma,$$
$$= 2\mathcal{U}^*(\sigma) \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + (1 - \sigma)\omega(\varsigma, \sigma))d\sigma + 2\mathcal{U}^*(\varsigma) \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\omega(\varsigma, \sigma))d\sigma.$$

Since $\mathfrak{X}$ is symmetric, then

$$= 2[\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma)] \int_0^1 \lambda(\sigma) X(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma, \tag{25}$$

$$= 2[\mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma) X(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma.$$

Since

$$\int_0^1 \mathcal{U}_*(\sigma + (1-\sigma)\varpi(\varsigma,\sigma))\mathfrak{X}(\sigma + (1-\sigma)\varpi(\varsigma,\sigma))d\sigma$$

$$= \int_0^1 \mathcal{U}_*(\sigma + \sigma\varpi(\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\sigma))d\sigma$$

$$= \frac{1}{\varpi(\varsigma,\sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\sigma)} \mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath$$

$$\int_0^1 \mathcal{U}^*(\sigma + \sigma\varpi)\varsigma,\sigma))\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\sigma))d\sigma \tag{26}$$

$$= \int_0^1 \mathcal{U}^*(\sigma + (1-\sigma)\varpi(\varsigma,\sigma))\mathfrak{X}(\sigma + (1-\sigma)\varpi(\varsigma,\sigma))d\sigma$$

$$= \frac{1}{\varpi(\varsigma,\sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\sigma)} \mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath$$

From (25) and (26), we have

$$\frac{1}{\varpi(\varsigma,\ \sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath \leq [\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma)] \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma,$$

$$\frac{1}{\varpi(\varsigma,\ \sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath \leq [\mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma,$$

That is,

$$\left[\frac{1}{\varpi(\varsigma,\ \sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath,\ \frac{1}{\varpi(\varsigma,\ \sigma)} \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath \right]$$

$$\leq_p [\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma),\ \mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma,$$

Hence,

$$\frac{1}{\varpi(\varsigma,\ \sigma)} (IR) \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}(\imath)\mathfrak{X}(\imath)d\imath \leq_p [\mathcal{U}(\sigma) + \mathcal{U}(\varsigma)] \int_0^1 \lambda(\sigma)\mathfrak{X}(\sigma + \sigma\varpi(\varsigma,\ \sigma))d\sigma.$$

This completes the proof. $\square$

Theorem 7. *Let* $\mathcal{U} : [\sigma,\ \sigma+\varpi(\varsigma,\ \sigma)] \to \mathcal{K}_C^+$ *be a left and right* λ-*preinvex IVF with* $\sigma < \sigma + \varpi(\varsigma,\ \sigma)$ *and* $\lambda : [0,\ 1] \to \mathbb{R}^+$, *such that* $\mathcal{U}(\imath) = [\mathcal{U}_*(\imath),\ \mathcal{U}^*(\imath)]$ *for all* $\imath \in [\sigma,\ \sigma+\varpi(\varsigma,\ \sigma)]$. *If* $\mathcal{U} \in \mathcal{IR}_{([\sigma,\ \sigma+\varpi(\varsigma,\ \sigma)])}$ *and* $\mathfrak{X} : [\sigma,\ \sigma+\varpi(\varsigma,\ \sigma)] \to \mathbb{R}$, $\mathfrak{X}(\imath) \geq 0$, *symmetric with respect to* $\sigma + \frac{1}{2}\varpi(\varsigma,\ \sigma)$, *and* $\int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathfrak{X}(\imath)d\imath > 0$, *and Condition C for* ϖ, *then*

$$\mathcal{U}\left(\sigma + \frac{1}{2}\varpi(\varsigma,\sigma)\right) \leq_p \frac{2\lambda\left(\dfrac{1}{2}\right)}{\int_\sigma^{\sigma+\varpi(\varsigma,\sigma)} \mathfrak{X}(\imath)d\imath} (IR) \int_\sigma^{\sigma+\varpi(\varsigma,\ \sigma)} \mathcal{U}(\imath)\mathfrak{X}(\imath)d\imath. \tag{27}$$

Proof. Using Condition C, we can write

$$\sigma + \frac{1}{2}\varpi(\varsigma,\sigma) = \sigma + \sigma\varpi(\varsigma,\sigma) + \frac{1}{2}\varpi(\sigma + (1-\sigma)\varpi(\varsigma,\sigma),\ \sigma + \sigma\varpi(\varsigma,\sigma)).$$

Since $\mathcal{U}$ is a left and right λ-preinvex, we have

$$
\begin{aligned}
\mathcal{U}_*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\mathit{o})\right) &= \mathcal{U}_*\left(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o})+\frac{1}{2}\varpi(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o}),\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\right), \\
&\leq \lambda\left(\frac{1}{2}\right)\left(\mathcal{U}_*(\mathit{o}+(1-\sigma)\varpi(\varsigma,\ \mathit{o}))+\mathcal{U}_*(\mathit{o}+\sigma\varpi(\varsigma,\ \mathit{o}))\right), \\
\mathcal{U}^*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\mathit{o})\right) &= \mathcal{U}^*\left(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o})+\frac{1}{2}\varpi(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o}),\ \mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\right), \\
&\leq \lambda\left(\frac{1}{2}\right)\left(\mathcal{U}^*(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o}))+\mathcal{U}^*(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\right),
\end{aligned}
\tag{28}
$$

By multiplying (28) by $\mathfrak{X}(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})) = \mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))$ and integrate it by σ over $[0,\ 1]$, we obtain

$$
\begin{aligned}
\mathcal{U}_*&\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\mathit{o})\right)\int_0^1\mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))d\sigma \\
&\leq \lambda(\frac{1}{2})\left(\begin{array}{c}\int_0^1\mathcal{U}_*\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})\mathfrak{X}\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})d\sigma \\ +\int_0^1\mathcal{U}_*(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))d\sigma\end{array}\right), \\
\mathcal{U}^*&\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\mathit{o})\right)\int_0^1\mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))d\sigma \\
&\leq \lambda(\frac{1}{2})\left(\begin{array}{c}\int_0^1\mathcal{U}^*\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})\mathfrak{X}\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})d\sigma \\ +\int_0^1\mathcal{U}^*(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))d\sigma\end{array}\right).
\end{aligned}
\tag{29}
$$

Since

$$
\begin{aligned}
\int_0^1 &\mathcal{U}_*\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})\mathfrak{X}\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o})d\sigma \\
&= \int_0^1\mathcal{U}_*(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))\mathfrak{X}(\mathit{o}+\sigma\varpi(\varsigma,\mathit{o}))d\sigma, \\
&= \frac{1}{\varpi(\varsigma,\mathit{o})}\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\mathit{o})}\mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath \\
\int_0^1 &\mathcal{U}^*\mathit{o}+\sigma\varpi(\varsigma,\mathit{o})\mathfrak{X}\mathit{o}+\sigma\varpi(\varsigma,\mathit{o})d\sigma \\
&= \int_0^1\mathcal{U}^*(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o}))\mathfrak{X}(\mathit{o}+(1-\sigma)\varpi(\varsigma,\mathit{o}))d\sigma, \\
&= \frac{1}{\varpi(\varsigma,\mathit{o})}\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\mathit{o})}\mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath,
\end{aligned}
\tag{30}
$$

From (29) and (30), we have

$$
\begin{aligned}
\mathcal{U}_*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\ \mathit{o})\right) &\leq \frac{2\lambda\left(\frac{1}{2}\right)}{\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathfrak{X}(\imath)d\imath}\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath, \\
\mathcal{U}^*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\ \mathit{o})\right) &\leq \frac{2\lambda\left(\frac{1}{2}\right)}{\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathfrak{X}(\imath)d\imath}\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath.
\end{aligned}
$$

From which, we have

$$
\begin{aligned}
&\left[\mathcal{U}_*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\ \mathit{o})\right),\mathcal{U}^*\left(\mathit{o}+\frac{1}{2}\varpi(\varsigma,\ \mathit{o})\right)\right] \\
&\leq_p \frac{2\lambda\left(\frac{1}{2}\right)}{\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathfrak{X}(\imath)d\imath}[\int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathcal{U}_*(\imath)\mathfrak{X}(\imath)d\imath,\ \int_\mathit{o}^{\mathit{o}+\varpi(\varsigma,\ \mathit{o})}\mathcal{U}^*(\imath)\mathfrak{X}(\imath)d\imath]
\end{aligned}
$$

That is,

$$\mathcal{U}\left(\sigma + \frac{1}{2}\varpi(\varsigma, \sigma)\right) \leq_p \frac{2\lambda\left(\frac{1}{2}\right)}{\int_\sigma^{\sigma+\varpi(\varsigma, \sigma)} \mathfrak{X}(\iota)d\iota} \ (IR) \int_\sigma^{\sigma+\varpi(\varsigma, \sigma)} \mathcal{U}(\iota)\mathfrak{X}(\iota)d\iota,$$

Then we complete the proof. □

Remark 4. *If one takes $\lambda(\sigma) = \sigma$, then (22) and (27) reduce to the result for left and right preinvex IVFs.*

If one takes $\mathcal{U}_*(\iota) = \mathcal{U}^*(\iota)$, then (22) and (27) reduce to the classical first and second H-H–Fejér inequality for λ-preinvex function, see [28].

If one takes $\mathcal{U}_*(\iota) = \mathcal{U}^*(\iota) \ \varpi(\varsigma, \sigma) = \varsigma - \sigma$, then (22) and (27) reduce to the classical second H-H–Fejér inequality for λ-convex function, see [28].

Example 5. *We consider $\lambda(\sigma) = \sigma$, for $\sigma \in [0, 1]$ and the IVF $\mathcal{U} : [1, 1 + \partial(4, 1)] \to \mathcal{K}_C^+$ defined by $\mathcal{U}(\iota) = [1, 4]e^\iota$. Since $\mathcal{U}_*(\iota)$ and $\mathcal{U}^*(\iota)$ are λ-preinvex functions $\varpi(\jmath, \iota) = \jmath - \iota$, then $\mathcal{U}(\iota)$ is left and right λ-preinvex IVF. If*

$$\mathfrak{X}(\iota) = \begin{cases} \iota - 1, \ \mathfrak{B} \in \left[1, \frac{5}{2}\right] \\ 4 - \iota, \ \mathfrak{B} \in \left(\frac{5}{2}, 4\right], \end{cases}$$

Then, we have

$$\frac{1}{\varpi(4,1)} \int_1^{1+\varpi(4,1)} \mathcal{U}_*(\iota)\mathfrak{X}(\iota)d(\iota) = \frac{1}{3}\int_1^4 \mathcal{U}_*(\iota)\mathfrak{X}(\iota)d(\iota) = \frac{1}{3}\int_1^{\frac{2}{5}} \mathcal{U}_*(\iota)\mathfrak{X}(\iota)d(\iota)$$
$$+ \frac{1}{3}\int_{\frac{5}{2}}^4 \mathcal{U}_*(\iota)\mathfrak{X}(\iota)d(\iota),$$

$$\frac{1}{\varpi(4,1)} \int_1^{1+\varpi(4,1)} \mathcal{U}^*(\iota)\mathfrak{X}(\iota)d(\iota) = \frac{1}{3}\int_1^4 \mathcal{U}^*(\iota)\mathfrak{X}(\iota)d(\iota) = \frac{1}{3}\int_1^{\frac{2}{5}} \mathcal{U}^*(\iota)\mathfrak{X}(\iota)d(\iota)$$

$$+ \frac{1}{3}\int_{\frac{5}{2}}^4 \mathcal{U}^*(\iota)\mathfrak{X}(\iota)d(\iota), \tag{31}$$

$$= \frac{1}{3}\int_1^{\frac{5}{2}} e^\iota(\iota - 1)d\iota + \frac{1}{3}\int_{\frac{5}{2}}^4 e^\iota(4 - \iota)d\iota \approx 11,$$

$$= \frac{4}{3}\int_1^{\frac{5}{2}} e^\iota(\iota - 1)d\iota + \frac{2}{3}\int_{\frac{5}{2}}^4 e^\iota(4 - \iota)d\iota \approx 42,$$

and

$$[\mathcal{U}_*(\sigma) + \mathcal{U}_*(\varsigma)] \int_0^1 \lambda(\sigma) X(\sigma + \sigma\varpi(\varsigma, \sigma))d\sigma$$

$$[\mathcal{U}^*(\sigma) + \mathcal{U}^*(\varsigma)] \int_0^1 \lambda(\sigma) X(\sigma + \sigma\varpi(\varsigma, \sigma))d\sigma$$

$$= \left[e + e^4 \right] \left[\int_0^{\frac{1}{2}} 3\sigma^2 d\imath + \int_{\frac{1}{2}}^1 \sigma(3 - 3\sigma) d\sigma \right] \approx 21.5$$

$$(32)$$

$$= 4 \left[e + e^4 \right] \left[\int_0^{\frac{1}{2}} 3\sigma^2 d\imath + \int_{\frac{1}{2}}^1 \sigma(3 - 3\sigma) d\sigma \right] \approx 96.$$

From (31) and (32), we have

$$[11,\ 42] \leq_p [21.5,\ 96].$$

Hence, Theorem 6 is verified.
   For Theorem 7, we have

$$\mathcal{U}_* \left(\sigma + \frac{1}{2} \varpi(\varsigma, \sigma) \right) \approx 12.8,$$

$$\mathcal{U}^* \left(\sigma + \frac{1}{2} \varpi(\varsigma, \sigma) \right) \approx 49,$$

$$(33)$$

$$\int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathfrak{X}(\imath) d\imath = \int_1^{\frac{5}{2}} (\imath - 1) d\imath + \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} (4 - \imath) d\imath \approx \frac{9}{4},$$

$$\frac{2\lambda\left(\frac{1}{2}\right)}{\int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathfrak{X}(\imath) d\imath} \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}_*(\imath) \mathfrak{X}(\imath) d\imath \approx 14.6$$

$$\frac{2\lambda\left(\frac{1}{2}\right)}{\int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathfrak{X}(\imath) d\imath} \int_\sigma^{\sigma + \varpi(\varsigma, \sigma)} \mathcal{U}^*(\imath) \mathfrak{X}(\imath) d\imath \approx 58.5$$

$$(34)$$

From (33) and (34), we have

$$[12.8,\ 49] \leq_p [14.6,\ 58.5].$$

Hence, Theorem 7 is verified.

4. Conclusions

Over the past three decades, there has been an increase in interest in the field of convex mathematical inequalities. Novel findings are being added to the theory of inequalities as a result of the researchers' search for new generalizations of convex functions. A number of conclusions that hold for convex functions have been generalized in the current study using left and right λ-preinvex IVF. In this work, we developed several new mappings in order to obtain the innovative results. In addition to obtaining further modifications of the Hermite–Hadamard- and Fejér-type inequalities previously established for left and right λ-preinvex IVF, we have highlighted several intriguing characteristics of these mappings. The findings of this study, in our opinion, may serve as a source of motivation for mathematicians working in this area and for future researchers considering a career in this exciting area of mathematics.

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