

Article

A Valid Quantization of a Half-Harmonic Oscillator Field Theory

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Abstract: The usual full- and half-harmonic oscillators are turned into field theories, and that behavior is examined using canonical and affine quantization. The result leads to a valid affine quantization of the half harmonic oscillator field theory, which points toward further valid quantizations of more realistic field theory models.

Keywords: half-harmonic oscillators; affine and canonical quantizations; model field theories

1. Introduction

The conventional classical harmonic oscillator, with $-\infty < p \text{ \& } q < \infty$, and a classical Hamiltonian, $H = (p^2 + q^2)/2$, has been well quantized using canonical quantization (CQ), and it has eigenvalues, $E_n = \hbar(n + 1/2)$, where $n = 0, 1, 2, 3, \dots$, leading to equal spacing of the eigenvalues. It is noteworthy that the ground state eigenvalue can be removed by using $(p^2 + q^2)/2 = (p + iq)(p - iq)/2 \rightarrow (P + iQ)(P - iQ)/2 = (P^2 + Q^2 + i[Q, P])/2 = (P^2 + Q^2 - \hbar)/2$. This kind of rearranged variables eliminates the ground state eigenvalue but the spacing between all the eigenvalues remains unchanged. We will exploit a similar formulation in Section 3 to deal with quantum field infinities.

Affine quantization (see Section 1.1) cannot correctly solve the full-harmonic oscillator, nor is it supposed to solve it.

The half-harmonic oscillator, which has the same classical Hamiltonian, $H = (p^2 + q^2)/2$, and $-\infty < p < \infty$, but now $0 < q < \infty$. This model fails a CQ quantization which requires that, for $-\infty < q < 0$, a virtual infinite wall that crushes all wave functions to have zero values for all $q < 0$. However, the half-harmonic oscillator receives a valid affine quantization (AQ), while no virtual infinite wall is required, and the eigenvalues for the half-harmonic oscillator are $E'_n = 2\hbar(n + 1)$, for the same $n = 0, 1, 2, 3, \dots$, again with equal spacing, now a doubled amount, and also, now a quadrupling of the ground state value. These positive features lead to accepting this result as a valid quantization of the half-harmonic oscillator by AQ [1].

Briefly stated, CQ works perfectly for the full harmonic oscillator, while AQ works perfectly for the half-harmonic oscillator.

1.1. A Brief Account of Affine Quantization

Canonical quantization requires classical variables that run the whole real line so that $p \rightarrow P = P^\dagger$ and $q \rightarrow Q = Q^\dagger$, and that means they both are self-adjoint, which is required for a proper quantum Hamiltonian, e.g., in our case, $\mathcal{H} = (P^2 + Q^2)/2$.

Now, we choose that $Q > 0$, which implies that $P^\dagger \neq P$, and that opens the door to many, different, quantum Hamiltonians, beginning with $\mathcal{H}_1 = (P^\dagger P + Q^2)/2$ and $\mathcal{H}_2 = (PP^\dagger + Q^2)/2$. To fix that, we introduce $d = pq \rightarrow D = (P^\dagger Q + QP)/2 = D^\dagger$. Dirac has noted [2] that for CQ it takes special coordinates (i.e., a Cartesian, flat curvature) so that $\mathcal{H}(p, q) = H(p, q)$ will achieve valid quantum results. Now, it has been shown [3] that, for a different set of special coordinates (i.e., a constant negative curvature), then



Citation: Klauder, J.R. A Valid Quantization of a Half-Harmonic Oscillator Field Theory. *Axioms* **2022**, *11*, 360. <https://doi.org/10.3390/axioms11080360>

Academic Editors: Francesco dell'Isola, Giorgio Nardo and Hovik Matevosian

Received: 17 June 2022

Accepted: 20 July 2022

Published: 24 July 2022

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$\mathcal{H}'(pq, q) = H(pq, q)$ can also achieve valid quantum results using AQ. In particular, a promotion of the half-harmonic oscillator classical Hamiltonian, then becomes

$$H = (d^2/q^2 + q^2)/2 \rightarrow \mathcal{H} = (D(Q^{-2})D + Q^2)/2 \\ = (P^2 + (3/4)\hbar^2/Q^2 + Q^2)/2, \quad (1)$$

in which, although $P^\dagger \neq P$, it follows that P^\dagger and $(P^\dagger)^2$ act like P and P^2 thanks to the additional \hbar -term. This special \hbar -term provides a ‘real quantum wall’ that forces all wave functions to have a continuous zero value, at $x = 0$, and which also has continuous first and second derivatives.

The equation for eigenfunctions is given by

$$[-\hbar^2(d^2/dx^2) + (3/4)\hbar^2/x^2 + x^2]/2 \zeta_l(x) = E_l \zeta_l(x), \quad (2)$$

with two solutions, $\zeta_0(x) = x^{3/2} e^{-x^2/2\hbar}$ and $\zeta_1(x) = x^{3/2} (1 - x/2) e^{-x^2/2\hbar}$.

1.2. Multiple Independent Harmonic Oscillators

We introduce a large number of independent, identical, harmonic oscillators with the classical Hamiltonian

$$H_N = \sum_{k=1}^N (p_k^2 + q_k^2)/2 a, \quad (3)$$

provided that, say, $Na = 100$, or some other positive number, which applies to both the full and half set of coordinates, such as $|q_k| < \infty$ for the full oscillator and $0 < q_k < \infty$ for the half oscillator. Ignoring normalization, two eigenfunctions for this quantum Hamiltonian are a ground state $\psi_f = \sum_{n=1}^N f_n e^{-x_n^2/2\hbar}$, and the next state is given by $\psi_g = \sum_{n=1}^N g_n x_n e^{-x_n^2/2\hbar}$, assuming that $\sum_{n=1}^N [|f_n|^2 + |g_n|^2] < \infty$.

These two cases are fully independent, and the sum, $H_{N \rightarrow \infty} = H_c$, still with $Na = 100$, must lead to a finite result. Evidently, the result is represented by the integral

$$H_c = \int_0^{100} [\pi(x)^2 + \varphi(x)^2]/2 dx, \quad (4)$$

which must remain finite. Of course, there are situations where $\int_0^{100} |x - 1|^{-2/3} dx < \infty$. Mathematics may accept that, but perhaps, physics should instead require that $|\pi(x)| + |\varphi(x)| < \infty$.

A Common Example Where Infinities Really Appear

For a moment, we examine a different model,

$$H_p = \int_0^{100} \{[\pi(x)^2 + \varphi(x)^2]/2 + g |\varphi(x)|^p\} dx, \quad (5)$$

where $p > 2$. The domain of functions that admit finite integrals for arbitrary fields when $g = 0$ is reduced if $g > 0$, even when $p = 2.0001$, etc. However, when $g \rightarrow 0$, the reduced domain would not recover what was lost from the larger original domain. We need some process to ensure that $|\pi(x)| + |\varphi(x)| < \infty$ so that the original domain, when $g = 0$, would retain its initial domain when $g > 0$, and to do so, then every p , with $2 < p < \infty$, maintains the initial domain as well.

1.3. A Plan to Ensure That $|\pi(x)| + |\varphi(x)| < \infty$

Let us first accept that a toy example, such as $AB = C$, makes good mathematics provided two of the terms are known, and that should provide the third term. However, if $B = 0$, then $C = 0$, and A is unknown. If $B = \infty$ then $C = \infty$, and again A is unknown. Unless a zero is permitted by physics, it is reasonable to accept that rigorous mathematics requires that $0 < |A|, |B|, |C| < \infty$.

Guided by the last paragraph, we introduce the dilation field, $\kappa(x) = \pi(x) \varphi(x)$. In this case, we assume that $\varphi(x) \neq 0$, because it could represent a substance of nature, (We used ‘nature’ to emphasize that something like a heavy rain and a snow storm, for example, are different until both of them vanish, which signals that they can even be ignored.) while $\pi(x)$ takes the role of time derivative of $\varphi(x)$, and $\kappa(x)$ plays the role of ‘momentum’. We now replace $\pi(x)$ by $\kappa(x)/\varphi(x)$.

The classical Hamiltonian is now given in these affine variables by

$$H_a = \int_0^{100} [\kappa(x)^2 / \varphi(x)^2 + \varphi(x)^2] / 2 \, dx, \quad (6)$$

an equation in which $0 < |\varphi(x)| < \infty$ and $|\kappa(x)| < \infty$ to properly represent $\pi(x)$. Even if the term $0 < g |\varphi(x)|^p < \infty$ were added, there would be no infinities, and the initial domain would remain valid because the Hamiltonian density, i.e., $H(x) < \infty$, for all x .

2. Quantization of the Full-Harmonic Oscillator Field Theory

This example is common and well known. We choose CQ, and promote classical variables to quantum operators, $\hat{\pi}(x)$ and $\hat{\varphi}(x)$, with the commutator $[\hat{\varphi}(x), \hat{\pi}(y)] = i\hbar\delta(x - y)\mathbb{I}$. Our quantum Hamiltonian then is

$$\mathcal{H} = \int_0^{100} [\hat{\pi}(x)^2 + \hat{\varphi}(x)^2] / 2 \, dx. \quad (7)$$

Accepting Schrödinger’s representation, the quantum Hamiltonian becomes

$$H = \int_0^{100} [-\hbar^2 \delta^2 / \delta \varphi(x)^2 + \varphi(x)^2] / 2 \, dx. \quad (8)$$

An unnormalized ground state is $\Psi_f(\varphi) = \int_0^{100} f(y) e^{-\varphi(y)^2/2\hbar} \, dy$, providing $\int_0^{100} f(y)^2 \, dy < \infty$, which leads its first derivative to be

$$-i\hbar \delta \Psi(\varphi) / \delta \varphi(x) = i\varphi(x) \Psi(\varphi). \quad (9)$$

However, the second derivative leads to

$$-\hbar^2 \delta^2 \Psi(\varphi) / \delta \varphi(x)^2 + \varphi(x)^2 \Psi(\varphi) = \hbar \delta(0) \Psi(\varphi), \quad (10)$$

for which Dirac’s delta function, $\delta(0) = \infty$. This situation points toward changing the quantum Hamiltonian so as to become

$$\mathcal{H} = \int_0^{100} [\hat{\pi}(x)^2 + \hat{\varphi}(x)^2 - \hbar \delta(0)] / 2 \, dx \quad (11)$$

in order to preserve a proper ground state.

How can we get rid of this divergence? Instead of promoting $\pi(x)^2 + \varphi(x)^2 \rightarrow \hat{\pi}(x)^2 + \hat{\varphi}(x)^2$, let us promote

$$\begin{aligned} (\pi(x)^2 + \varphi(x)^2) &= (\pi(x) + i\varphi(x))(\pi(x) - i\varphi(x)) \\ &\rightarrow (\hat{\pi}(x) + i\hat{\varphi}(x))(\hat{\pi}(x) - i\hat{\varphi}(x)) \quad (\text{YES!}) \\ &= (\hat{\pi}(x)^2 + \hat{\varphi}(x)^2 + i[\hat{\varphi}(x), \hat{\pi}(x)]) \\ &= [\hat{\pi}(x)^2 + \hat{\varphi}(x)^2 - \hbar \delta(0)], \end{aligned} \quad (12)$$

and the (YES!) line is the way to correctly quantize, and avoid infinities for this example using CQ.

Ignoring normalization, two solutions are $\Psi_f(\varphi) = \int_0^{100} f(y) e^{-\varphi(y)^2/2\hbar} \, dy$ and $\Psi_g(\varphi) = \int_0^{100} g(y) \varphi(y) e^{-\varphi(y)^2/2\hbar} \, dy$ for the full-harmonic oscillator field theory, provided that $\int_0^{100} [f(y)^2 + g(y)^2] \, dy < \infty$.

3. Quantization of the Half-Harmonic Oscillator Field Theory

The topic in this section uses AQ and focusses on examples for which $\varphi(x) > 0 \rightarrow \hat{\varphi}(x) > 0$. Now we are faced with $\hat{\pi}(x)^\dagger \neq \hat{\pi}(x)$. We accept that, and introduce $\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger \hat{\varphi}(x) + \hat{\varphi}(x) \hat{\pi}(x)]/2 (= \hat{\kappa}(x)^\dagger)$. While the CQ operators obey $[\hat{\varphi}(x), \hat{\pi}(y)] = i\hbar\delta(x-y)\mathbb{I}$, it follows that the AQ operators obey $[\hat{\varphi}(x), \hat{\kappa}(y)] = i\hbar\delta(x-y)\hat{\varphi}(x)$. While $\kappa(x)^2/\varphi(x)^2 = \pi(x)^2$, it follows that $\hat{\kappa}(x)(\hat{\varphi}(x)^{-2})\hat{\kappa}(x) = \hat{\pi}(x)^2 + (3/4)\hbar^2\delta(0)^2/\hat{\varphi}(x)^2$, where here $\hat{\pi}(x)^\dagger$ acts like $\hat{\pi}(x)$ thanks to the \hbar -term. Again we are faced with $\delta(0)$ -type divergences, and let us find a way to eliminate them.

For a moment, let us accept that $\delta(0) \rightarrow A^2$ which is GIGANTIC, but not yet infinity. That allows us to smoothly rescale the present quantum Hamiltonian. For scaling, we introduce $\hat{\pi}(x) \rightarrow A\hat{\pi}(x)$, $\hat{\varphi}(x) \rightarrow A\hat{\varphi}(x)$, and $\hat{\kappa}(x) \rightarrow A^2\hat{\kappa}(x)$. Now we have

$$\begin{aligned}\mathcal{H}_A &= \int_0^{100} \{A^2 \hat{\kappa}(x)(\hat{\varphi}(x)^{-2})\hat{\kappa}(x) + A^2 \hat{\varphi}(x)^2\}/2 dx \\ &= \int_0^{100} \{A^2 \hat{\pi}(x)^2 + (3/4)\hbar^2 A^4/A^2 \hat{\varphi}(x)^2 + A^2 \hat{\varphi}(x)^2\}/2 dx,\end{aligned}\quad (13)$$

which leads us to

$$\mathcal{H}_{almost} \equiv \lim_{A \rightarrow \infty} A^{-2} \mathcal{H}_A = \int_0^{100} \{\hat{\pi}(x)^2 + (3/4)\hbar^2/\hat{\varphi}(x)^2 + \hat{\varphi}(x)^2\}/2 dx,\quad (14)$$

and finally we choose

$$\mathcal{H}_{DONE} = \int_0^{100} \{(\hat{\pi}(x) + i\hat{\varphi}(x))(\hat{\pi}(x) - i\hat{\varphi}(x)) + (3/4)\hbar^2/\hat{\varphi}(x)^2\}/2 dx,\quad (15)$$

in which each and every spacial point should obey the quantization of the single particle point of the half-harmonic oscillator that has been correctly quantized by affine procedures [1], just as how Equation (5) points to how the full-harmonic oscillator field theory has already been solved.

Again, ignoring normalization, two eigenfunctions, using $\varphi(x) > 0$, for the half-harmonic oscillator field theory, are $\Psi_f(\varphi) = \int_0^{100} f(y) \varphi(y)^{3/2} e^{-\varphi(y)^2/2\hbar} dy$ and $\Psi_g(\varphi) = \int_0^{100} g(y) \varphi(y)^{3/2} (1 - \varphi(y)/2) e^{-\varphi(y)^2/2\hbar} dy$, provided $\int_0^{100} [f(y)^2 + g(y)^2] dy < \infty$.

4. The Relevance of This Work for Covariant Scalar Fields

Conventional quantization of scalar field theories, using both CQ or AQ for the formulations, have used several Monte Carlo calculations to seek results. This work has carefully studied standard field models, such as φ_4^4 and φ_3^{12} . Here, φ_n^p uses p as the power of the interaction term, and $n = s + 1$ in which s is the number of spatial coordinates, and 1 stands for the time. The results for using CQ have led to “free results”, as if the interaction term was absent when in fact it was present. On the other hand, the results using AQ for the same models, have shown strong presence of the interaction term when it is present, as can be seen in [4–10]. Thus, CQ has clearly found only invalid results, while AQ, quite possibly, has found valid results.

The simple plan to choose a scalar field that has $\varphi(x) \neq 0$, where now $x = (x_1, x_2, \dots, x_s)$ still leads to $\hat{\pi}(x)^\dagger \neq \hat{\pi}(x)$, which again points toward $\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger \varphi(x) + \varphi(x) \hat{\pi}(x)]/2$, just as in the earlier sections. Instead of insisting that $\varphi(x) > 0$, we keep both sides, i.e., $\varphi(x) < 0$ and $\varphi(x) > 0$, which maintains continuity thanks to the gradient term. Moreover, with $s > 1$, there are paths that can avoid the removed coordinate points, which leads to continuity almost everywhere.

The principal reason to study the half-harmonic oscillator field has been to find a valid quantization of the field theory version, which then may shed light on the validity of affine quantizations of covariant classical field theories.

As a possible proposal, we suggest a quantization of the standard classical Hamiltonian given by

$$H = \int \left\{ \frac{1}{2} [\pi(x)^2 + (\vec{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] + g \varphi(x)^p \right\} d^s x. \quad (16)$$

Adopting an affine quantization of this example, where $\varphi(x) \neq 0 \rightarrow \hat{\varphi}(x) \neq 0$, we are led to

$$\mathcal{H} = \int \left\{ \frac{1}{2} [(\hat{\pi}(x) + i m \hat{\varphi}(x))(\hat{\pi}(x) - i m \hat{\varphi}(x)) + X \hbar^2 / \hat{\varphi}(x)^2 + (\vec{\nabla} \hat{\varphi}(x))^2] + g \hat{\varphi}(x)^p \right\} d^s x, \quad (17)$$

where the numerical constant X here is somewhat under investigation for its value, a topic that is touched on in the next section. This suggestion is not so different from models that are already under examination.

5. Conclusions

A string of valid quantizations, from particles to fields, including half fields, like $\varphi(x) > 0$, and effectively full fields, like $\varphi(x) \neq 0$, may profit from a somewhat different \hbar -term. For the half oscillator fields, the \hbar -term is $(3/4)\hbar^2/\hat{\varphi}(x)^2$, while for the effectively full oscillator fields, we suggest consideration of a new \hbar -term, which is $2\hbar^2/\hat{\varphi}(x)^2$. This alternative \hbar -term can be seen in Equations (7), (10) and (11) in [11], followed by letting the constant $\Phi^2 \rightarrow 0$. The purpose of this procedure is to *guarantee* that the field $\varphi(x)$ is present when $\varphi(x) < 0$ as well as when $\varphi(x) > 0$, guided by continuity thanks to the gradient term. This suggestion may be considered as a worthy \hbar -term proposal alongside the standard ‘(3/4)-term’.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The author thanks R. Fantoni for his helpful comments. Thanks are also extended to L. Gouba for her useful suggestions.

Conflicts of Interest: The author states that there is no conflict of interest.

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