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Dynamic Behaviors of an Obligate Commensal Symbiosis Model with Crowley–Martin Functional Responses

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Abstract: A two species obligate commensal symbiosis model with Crowley–Martin functional response was proposed and studied in this paper. For an autonomous case, local and global dynamic behaviors of the system were investigated, respectively. The conditions that ensure the existence of the positive equilibrium is coincident to the conditions of global stability of a positive equilibrium. For nonautonomous cases, persistent and extinction properties of the system are investigated.

Keywords: commensal symbiosis model; Crowley–Martin functional response; stability

JEL Classification: 92D25; 34D20



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1. Introduction

During the last decades, many scholars investigated the dynamic behaviors of the commensalism model; see [1–37] and the references cited therein. Topics such as the stability of the system [1–10,37], the existence of periodic solution or almost periodic solution [11,15,25,36], the influence of harvesting [1–3,12–14,26–28], the influence of stage structure ([16]), the influence of Allee effect [9,17–20,24,32,33], the bifurcation phenomenon of the system [9,29,32,33], the influence of feedback control [8,23], the persistent property of the system [34,35,37], the influence of commensalism to the ecological network [6,7], and the influence of stochastic disturbance [5] were extensively investigated by many scholars. However, the commensalism model is not well studied in the sense that, to this day, still a few works on commensal symbiosis model with one party cannot survive independently [34–37].

Yang et al. [34] proposed the following non-autonomous obligate commensalism model:

$$\begin{aligned} \dot{x} &= x(-a_1(t) - b_1(t)x + c_1(t)y), \\ \dot{y} &= y(a_2(t) - b_2(t)y). \end{aligned} \quad (1)$$

where $a_1(t)$, $a_2(t)$, $b_1(t)$, $c_1(t)$, and $b_2(t)$ are all continuous functions bounded above and below by positive constants. They paid attention to the persistent, extinction, and stability of the system. Chen et al. [35,36] proposed and studied a discrete commensal symbiosis model.

Recently, stimulated by the concept of functional response of the predator prey system, Wu et al. [37] proposed the following obligate commensalism model with ratio-dependent functional responses.

$$\begin{aligned} \frac{dx}{dt} &= x\left(-a_1 - b_1x + \frac{c_1y}{x+y}\right), \\ \frac{dy}{dt} &= y(a_2 - b_2y). \end{aligned} \quad (2)$$

They provided a thoroughly investigation about the dynamic behaviors of the system (2).

As for as functional response is considered, there are many type functional responses

on predator prey system. In his pioneering work, Holling [38] argued that, in the predator prey system, functional response should take into consideration; from then on, numerous works (see, for example, [39,40]) were been performed on Holling type II and III functional response. Noting that the predators hunt for food resources and as a consequence they have to share food or involve intra-specific competition, to describe this phenomenon, a ratio-dependent functional response [41] was introduced, and the functional response is dependent on both predator and prey species instead of only prey dependent functional response. However, scholars argued that the predator prey model with a ratio-dependent functional response has curious dynamic behaviors; to overcome the drawback of the system, the Beddington–DeAngelis functional response, which can be seen as the generalization of the ratio-dependent and Holling II functional response functional response [42], was introduced. In 1989, Crowley and Martin [43] proposed a functional response, which is similar to the Beddington–DeAngelis response function, but it includes one more term explaining mutual interferences of predators at the high density of its prey. Many scholars performed works on predator prey system with Crowley–Martin functional responses; see [44] and the references cited therein. The Crowley–Martin functional response can be seen as the generalization of the Holling II functional response, ratio-dependent functional response, and Beddington–DeAngelis functional response. Noting that to this day, still no scholars propose and study the commensalism model with Crowley–Martin functional response. This motivated us to propose the following model.

$$\begin{aligned} \frac{dx}{dt} &= x\left(-a_1 - b_1x + \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy}\right), \\ \frac{dy}{dt} &= y(a_2 - b_2y). \end{aligned} \tag{3}$$

Throughout this paper, we assume that (H_1) or (H_2) hold; here, (H_1) $a_i, b_i, i = 1, 2$ and c_1, d_1, e_1, f_1, g_1 are all positive constants. (H_2) $a_i(t), b_i(t), i = 1, 2$ and $c_1(t), d_1(t), e_1(t), f_1(t), g_1(t)$ are all continuous functions bounded above and below by some positive constants.

The aim of this paper is to provide a thorough investigation about the dynamic behaviors of the system (3). For the autonomous case, we will investigate the local stability property of the equilibria in the next section. The global stability property is then investigated in Section 3. For the nonautonomous case, we investigate persistent and extinction properties in Section 4. Some numeric simulations are carried out in Section 5 to show the feasibility of our results. We end this paper by a brief discussion.

2. The Existence and Local Stability of the Equilibria

Now let us consider the autonomous case; i.e., assume that (H_1) holds.

Concerned with the existence of the equilibria, we have the following result.

Theorem 1. *System (3) always admits the boundary equilibrium $A_0(0, 0)$ and $A_1(0, \frac{a_2}{b_2})$. Assume further that the inequality of the follow:*

$$c_1 > a_1f_1 + \frac{a_1d_1b_2}{a_2} \tag{4}$$

holds; then, system (3) admits a unique positive equilibrium $A_2(x^, y^*)$, where the following is the case.*

$$x^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \quad y^* = \frac{a_2}{b_2}, \tag{5}$$

Here, we have the following.

$$\begin{aligned} A_1 &= a_2b_1g_1 + b_1b_2e_1 > 0, \\ A_2 &= a_1a_2g_1 + a_1b_2e_1 + a_2b_1f_1 + b_1b_2d_1, \\ A_3 &= a_1f_1a_2 + a_1d_1b_2 - c_1a_2 < 0. \end{aligned} \tag{6}$$

Proof. The equilibria of system (3) is determined by the following system.

$$\begin{aligned} x \left(-a_1 - b_1x + \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy} \right) &= 0, \\ y(a_2 - b_2y) &= 0. \end{aligned} \tag{7}$$

System (3) always admits the boundary equilibrium $A_0(0,0)$ and $A_1(0, \frac{a_2}{b_2})$. Now, let us consider the positive equilibrium. From the second equation of (4), the following immediately follows.

$$y = \frac{a_2}{b_2}. \tag{8}$$

Substituting (8) into the first equation of (4) and simplify, we finally obtain the following:

$$A_1x^2 + A_2x + A_3 = 0, \tag{9}$$

where $A_i, i = 1, 2, 3$ are defined by (6). Now let us consider the following function.

$$F(x) = A_1x^2 + A_2x + A_3, \tag{10}$$

Noting that $A_1 > 0$ implies that $F(-\infty) = F(+\infty) = +\infty$, $A_3 < 0$ leads to $F(0) = A_3 < 0$, form the continuity of function F , and fact F has at most two real solutions where F has a unique solution on $(0, +\infty)$. Hence, Equation (9) has a unique positive solution.

$$x^* = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}. \tag{11}$$

Consequently, system (3) has a unique positive solution $A_2(x^*, y^*)$. This ends the proof of Theorem 1. \square

Obviously, $A_2(x^*, y^*)$ satisfies the following equation.

$$\begin{aligned} -a_1 - b_1x^* + \frac{c_1y^*}{d_1 + e_1x^* + f_1y^* + g_1x^*y^*} &= 0, \\ a_2 - b_2y^* &= 0. \end{aligned} \tag{12}$$

Concerned with the local stability property of the above three equilibria, we have the following.

Theorem 2. $A_0(0,0)$ is unstable; $A_1(0, \frac{a_2}{b_2})$ is unstable if $c_1 > a_1f_1 + \frac{a_1d_1b_2}{a_2}$ holds and locally stable if $c_1 < a_1f_1 + \frac{a_1d_1b_2}{a_2}$ holds; if $A_2(x^*, y^*)$ exists, it is locally stable.

Proof. The Jacobian matrix of system (3) is calculated as follows:

$$J(x, y) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & -2b_2y + a_2 \end{pmatrix}, \tag{13}$$

where the following is the case.

$$\begin{aligned}
 A_{11} &= -a_1 - b_1x + \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy} + x \left(-b_1 - \frac{c_1y(g_1y + e_1)}{(d_1 + e_1x + f_1y + g_1xy)^2} \right), \\
 A_{12} &= \frac{xc_1(e_1x + d_1)}{(d_1 + e_1x + f_1y + g_1xy)^2}.
 \end{aligned}
 \tag{14}$$

Then, the Jacobian matrix of the system (3) about equilibrium $A_0(0,0)$ is given by the following.

$$\begin{pmatrix} -a_1 & 0 \\ 0 & a_2 \end{pmatrix}.
 \tag{15}$$

The corresponding eigenvalues are $\lambda_1 = -a_1 < 0$, $\lambda_2 = a_2 > 0$. Hence, $A_0(0,0)$ is unstable.

Then, the Jacobian matrix of the system (3) about equilibrium $A_1(0, \frac{a_2}{b_2})$ is given by the following.

$$\begin{pmatrix} -\frac{a_1a_2f_1 + a_1b_2d_1 - a_2c_1}{a_2f_1 + b_2d_1} & 0 \\ 0 & -a_2 \end{pmatrix}.
 \tag{16}$$

The corresponding eigenvalues are $\lambda_1 = -\frac{a_1a_2f_1 + a_1b_2d_1 - a_2c_1}{a_2f_1 + b_2d_1}$, $\lambda_2 = -a_2 < 0$.

Obviously, if $c_1 < a_1f_1 + \frac{a_1d_1b_2}{a_2}$, then $\lambda_1 < 0$, in this case, $A_1(0, \frac{a_2}{b_2})$ is locally stable;

$A_1(0, \frac{a_2}{b_2})$ is unstable if $c_1 > a_1f_1 + \frac{a_1d_1b_2}{a_2}$.

By using (12), the Jacobian matrix about the positive equilibrium A_2 is given by the following.

$$\begin{pmatrix} -x^* \left(b_1 + \frac{c_1y^*(g_1y^* + e_1)}{(d_1 + e_1x^* + f_1y^* + g_1x^*y^*)^2} \right) & \frac{x^*c_1(e_1x^* + d_1)}{(d_1 + e_1x^* + f_1y^* + g_1x^*y^*)^2} \\ 0 & -b_2y^* \end{pmatrix}.
 \tag{17}$$

The eigenvalues of the above matrix are $\lambda_1 = -x^* \left(b_1 + \frac{c_1y^*(g_1y^* + e_1)}{(d_1 + e_1x^* + f_1y^* + g_1x^*y^*)^2} \right) < 0$, $\lambda_2 = -b_2y^* < 0$. Hence, $A_2(x^*, y^*)$ is locally stable.

This ends the proof of Theorem 2. \square

3. Global Stability of the Equilibria

We also assume that (H_1) holds in this section.

We will investigate the global stability property of the equilibria in this section.

Theorem 3. Assume that $c_1 < a_1f_1 + \frac{a_1d_1b_2}{a_2}$ holds; then, $A_1(0, \frac{a_2}{b_2})$ is globally attractive.

Proof. Inequality $c_1 < a_1f_1 + \frac{a_1d_1b_2}{a_2}$ is equivalent to the following.

$$c_1 \frac{a_2}{b_2} < a_1f_1 \frac{a_2}{b_2} + a_1d_1.
 \tag{18}$$

The above inequality also equivalent to the following.

$$a_1 > \frac{c_1 \frac{a_2}{b_2}}{f_1 \frac{a_2}{b_2} + d_1}.
 \tag{19}$$

From (19), for small enough $\varepsilon > 0$, the following inequality:

$$a_1 > \frac{c_1\left(\frac{a_2}{b_2} + \varepsilon\right)}{f_1\left(\frac{a_2}{b_2} + \varepsilon\right) + d_1} \tag{20}$$

holds.

Noting that the second equation of (3) takes the following form.

$$\frac{dy}{dt} = y(a_2 - b_2y). \tag{21}$$

System (18) has a unique globally attractive positive equilibrium $y^* = \frac{a_2}{b_2}$.

$$\lim_{t \rightarrow +\infty} y(t) = y^*. \tag{22}$$

For $\varepsilon > 0$ that is small enough, which satisfies (20), it follows from (22) that there exists a large enough $T_1 > 0$ such that the following is the case.

$$y(t) < \frac{a_2}{b_2} + \varepsilon. \tag{23}$$

Now let us consider the following function.

$$F(y) = \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy} \tag{24}$$

Note the following.

$$\frac{dF(y)}{dy} = \frac{c_1(e_1x + d_1)}{(d_1 + e_1x + f_1y + g_1xy)^2} > 0. \tag{25}$$

Hence, $F(y)$ is the strictly increasing function of y ; hence, from the first equation of (3) and (23), for $t > T_1$, we have the following.

$$\begin{aligned} \frac{dx}{dt} &= x\left(-a_1 - b_1x + \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy}\right) \\ &\leq x\left(-a_1 + \frac{c_1y}{d_1 + f_1y}\right) \\ &\leq x\left(-a_1 + \frac{c_1\left(\frac{a_2}{b_2} + \varepsilon\right)}{d_1 + f_1\left(\frac{a_2}{b_2} + \varepsilon\right)}\right), \end{aligned} \tag{26}$$

Hence, the following is the case.

$$x(t) \leq x(T_1) \exp\left\{\left(-a_1 + \frac{c_1\left(\frac{a_2}{b_2} + \varepsilon\right)}{d_1 + f_1\left(\frac{a_2}{b_2} + \varepsilon\right)}\right)(t - T_1)\right\} \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{27}$$

(22) and (27) show that $A_1(0, \frac{a_2}{b_2})$ is globally attractive. This ends the proof of Theorem 3. \square

Theorem 4. Assume that $c_1 > a_1f_1 + \frac{a_1d_1b_2}{a_2}$ holds; then, $A_2(x^*, y^*)$ is globally stable.

Proof. In the proof of Theorem 3, we showed that $\lim_{t \rightarrow +\infty} y(t) = \frac{a_2}{b_2}$. That is, for any $\varepsilon > 0$ that is small enough, there exists $T > 0$ such that for all $t > T_1$, the following is the case.

$$y^* - \varepsilon < y(t) < y^* + \varepsilon \text{ for all } t > T_1. \tag{28}$$

From the first equation of system (3), we have the following.

$$\frac{dx}{dt} \leq x \left(-a_1 - b_1x + \frac{c_1}{f_1} \right), \tag{29}$$

Thus, the following is the case.

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{\frac{c_1}{f_1} - a_1}{b_1}. \tag{30}$$

That is, there exists a $T_2 > T_1$ such that the following is the case.

$$x(t) < \frac{\frac{c_1}{f_1} - a_1}{b_1} + \varepsilon \text{ for all } t > T_2. \tag{31}$$

Let $D = \{(x, y) \mid \in R_+^2 : x < \frac{\frac{c_1}{f_1} - a_1}{b_1} + \varepsilon, y < \frac{a_2}{b_2} + \varepsilon\}$. Then, every solution of system (3) starting in R_+^2 is uniformly bounded on D . Moreover, from Theorem 2, $A_0(0, 0)$ and $A_1(0, \frac{a_2}{b_2})$ are all unstable, and the unique positive equilibrium $A_2(x^*, y^*)$ is locally stable. To ensure $A_2(x^*, y^*)$ is globally stable in the above area, we consider Dulac function $u(x, y) = x^{-1}y^{-1}$; then, we have the following:

$$\frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} = -\frac{b_1}{y} - \frac{b_2}{x} - \frac{c_1(g_1y + e_1)}{(d_1 + e_1x + f_1y + g_1xy)^2} < 0, \tag{32}$$

where $P(x, y) = x \left(-a_1 - b_1x + \frac{c_1y}{d_1 + e_1x + f_1y + g_1xy} \right)$, $Q(x, y) = y(a_2 - b_2y)$. By Dulac Theorem [31], there is no closed orbit in area D . Thus, $A_2(x^*, y^*)$ is globally asymptotically stable.

This completes the proof of Theorem 4. \square

4. Nonautonomous Case

Now let us consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= x \left(-a_1(t) - b_1(t)x + \frac{c_1(t)y}{d_1(t) + e_1(t)x + f_1(t)y + g_1(t)xy} \right), \\ \frac{dy}{dt} &= y(a_2(t) - b_2(t)y), \end{aligned} \tag{33}$$

where $a_i(t), b_i(t), c_1(t), i = 1, 2$ are all continuous functions bounded above and below by positive constants. For the rest of the paper, for a bounded continuous function g defined on R , let g^L and g^M be defined as follows.

$$g^L = \inf_{t \in R} g(t), \quad g^M = \sup_{t \in R} g(t). \tag{34}$$

As for as system (33) is concerned, the most important thing is to find out the conditions that ensure the permanence of the system, which means that the species could be coexist in the long run. Moreover, in today’s society, more and more species are rapidly reduced or even extinct; hence, it is also important to investigate the extinction property of the system.

The aim of this section is to investigate the extinction and persistent property of the system. Concerned with the extinction of the first species, we have the following result.

Theorem 5. Assume the following:

$$a_1^L > \frac{c_1^M M_2}{d_1^L + f_1^L m_2} \tag{35}$$

holds, where M_2, m_2 are defined in (39) and (41), respectively; then, we have the following:

$$\lim_{t \rightarrow +\infty} x(t) = 0. \tag{36}$$

i.e., the first species will be driven to extinction.

Proof. It follows from (35) that for small enough $\varepsilon > 0$, without a loss of generality, assume that $\varepsilon < \frac{1}{2}m_2$, and the following inequality holds.

$$a_1^L > \frac{c_1^M (M_2 + \varepsilon)}{d_1^L + f_1^L (m_2 - \varepsilon)} \tag{37}$$

Let $(x(t), y(t))$ be any solution of system (33) with initial conditions $x(0) > 0, y(0) > 0$. From the second equation of system (33), we have the following.

$$\dot{y}(t) \leq y(a_2^M - b_2^L y), \tag{38}$$

Thus, the following is the case.

$$\limsup_{t \rightarrow +\infty} y(t) \leq \frac{a_2^M}{b_2^L} \stackrel{\text{def}}{=} M_2. \tag{39}$$

From the second equation of system (33), we have the following.

$$\dot{y}(t) \geq y(a_2^L - b_2^M y), \tag{40}$$

Thus, the following is the case.

$$\liminf_{t \rightarrow +\infty} y(t) \geq \frac{a_2^L}{b_2^M} \stackrel{\text{def}}{=} m_2. \tag{41}$$

For any $\varepsilon > 0$ that is small enough such that inequality (37) holds, it follows from (39) and (41) that there exists a $T > 0$ such that the following is the case.

$$m_2 - \varepsilon < y(t) < M_2 + \varepsilon \text{ for all } t > T. \tag{42}$$

For $t > T$, from the first equation of system (33), the following is the case.

$$\begin{aligned} \dot{x}(t) &= x \left(-a_1(t) - b_1(t)x + \frac{c_1(t)y}{d_1(t) + e_1(t)x + f_1(t)y + g_1(t)xy} \right) \\ &\leq x \left(-a_1(t) - b_1(t)x + \frac{c_1(t)y}{d_1(t) + f_1(t)y} \right) \\ &\leq x \left(-a_1^L - b_1^L x + \frac{c_1^M (M_2 + \varepsilon)}{d_1^L + f_1^L (m_2 - \varepsilon)} \right). \end{aligned} \tag{43}$$

Thus, the following is the case.

$$x(t) \leq x(T) \exp \left\{ \left(-a_1^L - b_1^L x + \frac{c_1^M (M_2 + \varepsilon)}{d_1^L + f_1^L (m_2 - \varepsilon)} \right) (t - T) \right\} \rightarrow 0 \text{ as } t \rightarrow +\infty. \quad (44)$$

This ends the proof of Theorem 5. \square

Lemma 1. Assume that $\frac{c}{d} > a$; then, the following system:

$$\frac{dy}{dt} = y \left(-a - by + \frac{c}{ey + d} \right) \quad (45)$$

admits a unique positive equilibrium x^* , which is globally attractive, where a, b, c, d , and e are all positive constants.

Proof. With some minor revision, the proof of Lemma 1 is similar to the proof of Lemma 3.1 in [37], and we omit the details here. \square

Concerned with the persistent property of the system, we have the following result.

Theorem 6. Assume the following:

$$a_1^M < \frac{c_1^L m_2}{d_1^M + f_1^M M_2} \quad (46)$$

holds, where M_2, m_2 are defined in (39) and (41), respectively; then, system (33) is permanent.

Proof. It follows from (46) that for small enough $\varepsilon > 0$, without loss of generality, assume that $\varepsilon < \frac{1}{2}m_2$; the following inequality holds.

$$a_1^M < \frac{c_1^L (m_2 - \varepsilon)}{d_1^M + f_1^M (M_2 + \varepsilon)}. \quad (47)$$

Let $(x(t), y(t))$ be any solution of system (33) with initial conditions $x(0) > 0, y(0) > 0$. similarly to the analysis of (37)–(40), for any $\varepsilon > 0$ small enough such that inequality (47) holds; there exists a $T > 0$ such that the following is the case.

$$m_2 - \varepsilon < y(t) < M_2 + \varepsilon \text{ for all } t > T. \quad (48)$$

For $t > T$, from the first equation of system (33), the following is the case.

$$\begin{aligned} \dot{x}(t) &= x \left(-a_1(t) - b_1(t)x + \frac{c_1(t)y}{d_1(t) + e_1(t)x + f_1(t)y + g_1(t)xy} \right) \\ &\leq x \left(-a_1^L - b_1^L x + \frac{c_1^M}{f_1^L} \right). \end{aligned} \quad (49)$$

Hence, the following holds

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{\frac{c_1^M}{f_1^L} - a_1^L}{b_1^L}. \quad (50)$$

From the first equation of system (33), the following is the case.

$$\begin{aligned} \dot{x}(t) &\geq x\left(-a_1^M - b_1^M x + \frac{c_1^L(m_2 - \varepsilon)}{d_1^M + e_1^M x + f_1^M(M_2 + \varepsilon) + g_1^M x(M_2 + \varepsilon)}\right) \\ &= x\left(-a_1^M - b_1^M x + \frac{c_1^L(m_2 - \varepsilon)}{d_1^M + f_1^M(M_2 + \varepsilon) + (e_1^M + g_1^M(M_2 + \varepsilon))x}\right). \end{aligned} \tag{51}$$

Now, let us consider the following equation.

$$\dot{w}(t) = w\left(-a_1^M - b_1^M w + \frac{c_1^L(m_2 - \varepsilon)}{d_1^M + f_1^M(M_2 + \varepsilon) + (e_1^M + g_1^M(M_2 + \varepsilon))w}\right). \tag{52}$$

Since the following is the case:

$$\frac{c_1^L(m_2 - \varepsilon)}{d_1^M + f_1^M(M_2 + \varepsilon)} > a_1^M, \tag{53}$$

it follows from Lemma 1 that (52) admits a unique positive equilibrium w_ε^* , which is globally stable. Thus, by the comparison theorem of the differential equation, one has the following:

$$\liminf_{t \rightarrow +\infty} x(t) \geq w_\varepsilon^* - \varepsilon. \tag{54}$$

and it immediately follows from (48), (50) and (54) that system (3) is permanent. This ends the proof of Theorem 6. \square

5. Numeric Simulations

Now let us consider the following three examples.

Example 1. Consider the following system.

$$\begin{aligned} \frac{dx}{dt} &= x\left(-a_1 - x + \frac{c_1 y}{1 + x + y + xy}\right), \\ \frac{dy}{dt} &= y(1 - y). \end{aligned} \tag{55}$$

In this system, corresponding to system (3), we take $b_1 = d_1 = e_1 = f_1 = g_1 = a_2 = b_2 = 1$.

- (1) Now take $a_1 = 2, c_1 = 1$, then $c_1 = 1 < 4 = a_1 f_1 + \frac{a_1 d_1 b_2}{a_2}$; it follows from Theorem 3 that $(0, 1)$ is globally stable. Numeric simulation (Figure 1) supports this assertion.
- (2) Now take $a_1 = \frac{1}{2}, c_1 = 2$; then, $c_1 = 2 > 1 = a_1 f_1 + \frac{a_1 d_1 b_2}{a_2}$, and it follow from Theorem 4 that the unique positive equilibrium $(0.28, 1)$ is globally stable. Numeric simulation (Figure 2) supports this assertion.

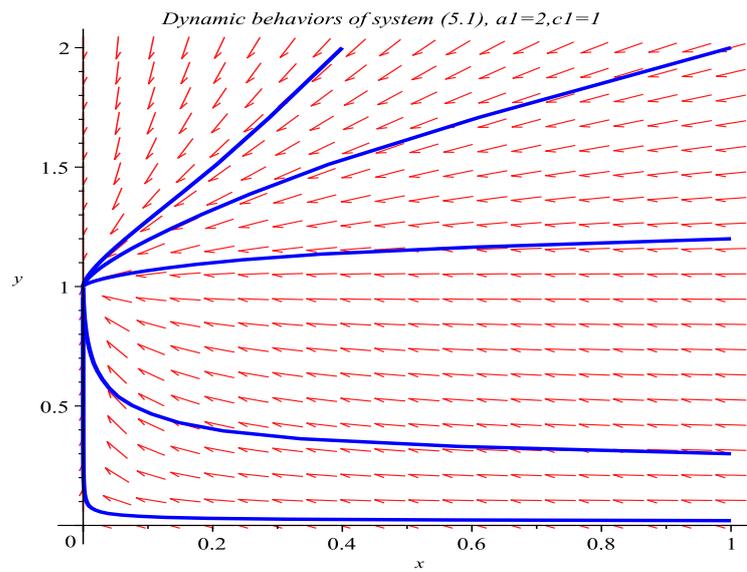


Figure 1. Stability of boundary equilibrium (0,1); here, we take $a_1 = 2, c_1 = 1$, which includes initial conditions $(x(0), y(0)) = (1, 0.3), (0.4, 2), (1, 0.02), (1, 2),$ and $(1, 1.2)$.

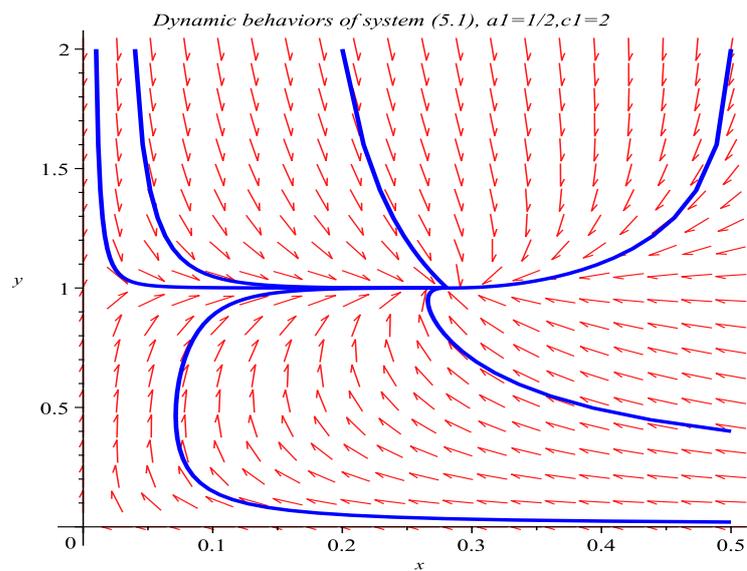


Figure 2. Stability of positive equilibrium (0.28,1); here we take $a_1 = \frac{1}{2}, c_1 = 2$, which includes the initial conditions $(x(0), y(0)) = (0.04, 2), (0.5, 0.2), (0.5, 0.02), (0.5, 0.4), (0.2, 2),$ and $(0.01, 2)$.

Example 2. Consider the following system.

$$\begin{aligned} \frac{dx}{dt} &= x \left(- (3 + \sin(t)) - x + \frac{(0.75 + 0.25 \cos(t))y}{1 + x + y + xy} \right), \\ \frac{dy}{dt} &= y(1 - y). \end{aligned} \tag{56}$$

In this system, corresponding to system (3), we take $b_1 = d_1 = e_1 = f_1 = g_1 = a_2 = b_2 = 1, a_1 = 3 + \sin(t), c_1 = 0.75 + 0.25 \cos(t)$. By simple computation, we have $a_1^L = 2, c_1^M = 1, M_2 = m_2 = 1$. One could easily verify the following.

$$a_1^L = 2 > \frac{1}{2} = \frac{c_1^M M_2}{d_1^L + f_1^L m_2}. \tag{57}$$

Hence, it follows from Theorem 5 that the first species will be driven to extinction. Figure 3 supports this assertion.

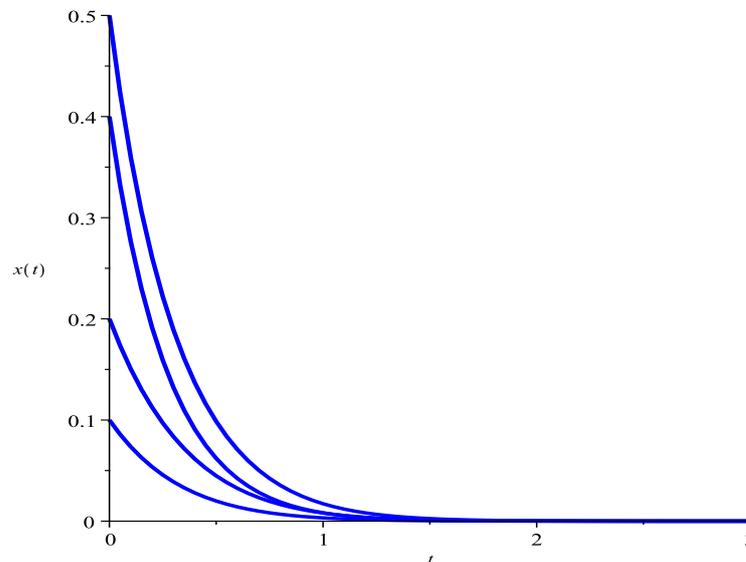


Figure 3. Extinction of the first species; here, we take $a_1 = 3 + \sin(t)$, $c_1 = 0.75 + 0.25 \cos(t)$, which includes the initial conditions $(x(0), y(0)) = (0.5, 2)$, $(0.4, 0.02)$, $(0.1, 0.4)$, and $(0.2, 2)$.

Example 3. Consider the following system.

$$\begin{aligned} \frac{dx}{dt} &= x \left(- \left(1 + \frac{\sin(t)}{2} \right) - x + \frac{(7 + \frac{\cos(t)}{4})y}{1 + x + y + xy} \right), \\ \frac{dy}{dt} &= y \left(1 + 0.2 \sin(t) - (1 - 0.2 \cos(t))y \right). \end{aligned} \tag{58}$$

In this system, corresponding to system (3), we take $b_1 = d_1 = e_1 = f_1 = g_1 = a_2 = b_2 = 1$. $a_1 = 1 + \frac{\sin(t)}{2}$, $c_1 = 7 + \frac{\cos(t)}{4}$, $a_2 = 1 + 0.2 \sin(t)$, and $b_2 = 1 - 0.2 \cos(t)$. By simple computation, we have $a_1^M = \frac{3}{2}$, $c_1^L = \frac{27}{4}$, $m_2 = \frac{2}{3}$, $M_2 = \frac{3}{2}$, $d_1^M = f_1^M = 1$. One could easily verify that the following is the case.

$$a_1^M = \frac{3}{2} < \frac{9}{5} = \frac{c_1^L m_2}{d_1^M + f_1^M M_2}. \tag{59}$$

Hence, it follows from Theorem 6 that the system is permanent. Figure 4 supports this assertion.

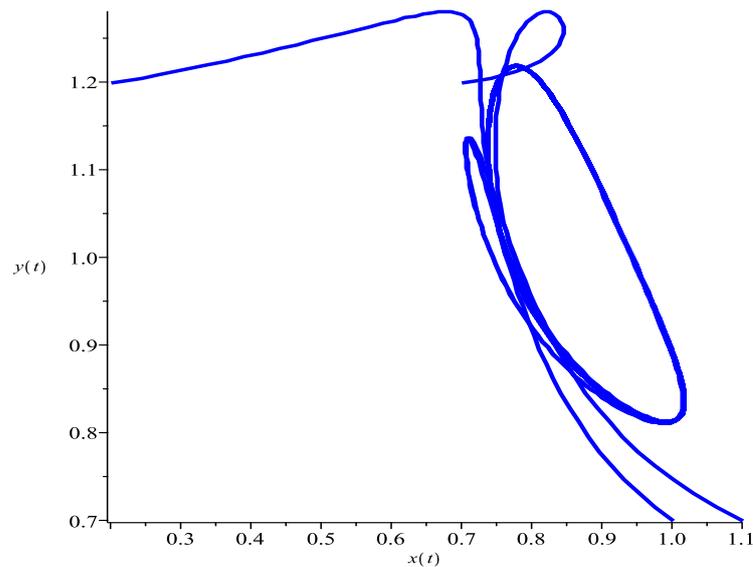


Figure 4. Attractivity of solutions; here, we take $a_1 = 1 + \frac{\sin(t)}{2}$, $c_1 = 7 + \frac{\cos(t)}{4}$, $a_2 = 1 + 0.2 \sin(t)$, and $b_2 = 1 - 0.2 \cos(t)$. The initial conditions are $(x(0), y(0)) = (1.0, 0.7)$, $(1.1, 0.7)$, $(0.7, 1.2)$, and $(0.2, 1.2)$.

6. Conclusions

Stimulated by recent work of Wu and Li [37], we proposed a two-species obligate commensal symbiosis model with Crowley–Martin functional responses. Such forms of functional response take many famous functional responses as its special case: Holling II functional response, ratio-dependent functional response, and Beddington–DeAngelis functional response, etc.

For autonomous case, we showed that the conditions that ensure the existence of the positive equilibrium is coincident to the conditions of the global stability of positive equilibrium, which means that if the positive equilibrium exists, it is a globally stable one; consequently, two species could be coexistent in the long run. In this case, the system has no positive equilibrium, and we showed that the first species involves extinction, while the second species is globally stable. Our results showed that, for the obligate system, the commensal effect may be one of the most important factors to avoid the extinction of the species.

For the nonautonomous case, by using the differential inequality theory, we also could establish sufficient conditions to ensure the persistence or extinction of the system.

We mention here that, in our main results Theorems 3 and 4, coefficients e_1 and g_1 have no influence on the persistent property of the system; that is, the mutual interferences of the first species have no influence on the persistence or extinction property of the system. The strength of the commensalism plays an essential role on the persistence property of the system.

One of the anonymous reviewers thought it is better for us to add a numerical example to show the persistence property of the system; we add Example 3, from numerical simulation (Figure 4) and we found that, indeed, the system admits a unique T periodic solution, which is globally attractive; however, we could not prove this assertion at present, and we leave this for future investigation.

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