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Some Cardinal and Geometric Properties of the Space of Permutation Degree

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Abstract: This paper is devoted to the investigation of cardinal invariants such as the hereditary density, hereditary weak density, and hereditary Lindelöf number. The relation between the spread and the extent of the space $\mathsf{SP}^2(\mathbb{R}, \tau(A))$ of permutation degree of the Hattori space is discussed. In particular, it is shown that the space $\mathsf{SP}^2(\mathbb{R}, \tau_S)$ contains a closed discrete subset of cardinality $\mathfrak{c}$. Moreover, it is shown that the functor SP^n_G preserves the homotopy and the retraction of topological spaces. In addition, we prove that if the spaces X and Y are homotopically equivalent, then the spaces $\mathsf{SP}^n_G X$ and $\mathsf{SP}^n_G Y$ are also homotopically equivalent. As a result, it has been proved that the functor SP^n_G is a covariant homotopy functor.

Keywords: extent; Lindelöf number; hereditary cardinal invariant; homotopically equivalent; covariant homotopy functor; retract

MSC: 18F60; 18A05; 54A25; 55P99

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1. Introduction

The cardinal invariants are considered as topological invariants with values in the class of all cardinal numbers, and are used to describe various topological properties of spaces. For example, the weight, π-weight, network weight, density, character, Lindelöf number, tightness, and cellularity of a topological space X are some classical cardinal invariants. Many researches have been devoted to the investigation of cardinal invariants and hereditary cardinal invariants (see, for example, [1–6]) and their important role in topology. Recall that a function $\varphi : \mathrm{Top} \to \mathrm{Card}$ from the class Top of topological spaces to the set Card of infinite cardinals such that $\varphi(X) = \varphi(Y)$ whenever X and Y are homeomorphic, is called a cardinal function (or cardinal invariant). The hereditary version of a cardinal function φ, denoted $h\varphi$, is defined as $h\varphi(X) = \sup\{\varphi(Y) : Y \subset X\}$ [1,4–6].

In recent research, the interest in the theory of cardinal invariants and their behavior under the influence of various covariant functors is increasing (see, for example, [7–9]). In [10], the authors investigated several cardinal invariants under the influence of some seminormal and normal functors. In the investigations in [11,12], the concept of symmetric product of a topological space is introduced. In particular, in [13] the functor SP^n_G is studied, and some cardinal and topological properties of this functor were investigated. In [14], some propositions about homotopy properties of the topological spaces were proved. For instance, it was proved that, contractibility, connectedness, and pathwise connectedness are homotopy properties of the spaces. In our work, we prove that if the mappings $f, g : X \to Y$ are homotopic, then the mappings $\mathsf{SP}^n_G f, \mathsf{SP}^n_G g : \mathsf{SP}^n_G X \to \mathsf{SP}^n_G Y$ are also homotopic.

The current paper is devoted to the investigation of hereditary cardinal invariants (such as the hereditary density, the hereditary weak density, and the hereditary Lindelöf number) in the space of permutation degree. Additionally, the relation between the spread and the extent of the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ of the permutation degree of Hattori space $(\mathbb{R}, \tau(A))$ is studied. Moreover, it is shown that the functor $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}$ preserves the homotopy and the retraction of topological spaces. As a consequence, it has been proved that the functor $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}$ is a covariant homotopy functor. Our research complements and extends existing results in the fields of cardinal invariants and the theory of covariant functors.

The paper is organized as follows. In Section 2, we recall basic notions and notation that will be used in the rest of the study. In Section 3, we study hereditary cardinal invariants and obtain some results for the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ of permutation degree of the Hattori space $(\mathbb{R}, \tau(A))$. Finally, in Section 4, we study some geometric properties of the space $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$ of permutation degree of a space X.

Throughout the paper, all spaces are assumed to be completely regular; τ denotes an infinite cardinal number; and by ω and $\mathfrak{c}$ we denote the countable cardinal number and the cardinality of continuum, respectively. The real line with the Sorgenfrey topology [4] is denoted by $(\mathbb{R}, \tau_S)$; and for $A \subset \mathbb{R}$ by $(\mathbb{R}, \tau(A))$ we denote the Hattori space over A. Recall that in [15], the following generalization of the Sorgengfey line was defined: if $A \subset \mathbb{R}$, then $\tau(A)$ denotes the topology on $\mathbb{R}$, in which each point $a \in A$ has the usual Euclidean neighborhoods, and basic neighborhoods of a point $x \in \mathbb{R} \setminus A$ are of the form $[x, \varepsilon)$, $\varepsilon > 0$. Notice that for $A \subset \mathbb{R}$, the topology $\tau(A)$ is finer than the usual Euclidean topology on $\mathbb{R}$ and weaker than the Sorgenfrey topology τ_S [16].

2. Preliminaries

For convenience of the reader, we give some notation, concepts, and statements that are widely used in this article. For a space X, the group of all permutations of X is denoted by $S(X)$ and called the permutation group of X. If $X = \{1, 2 \ldots, n\}$, then we write S_n instead of $S(X)$.

Let X be a space. The permutation group S_n acts on the n-th power X^n of X as the permutation of coordinates: the points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$ are equivalent if there exists a permutation $\sigma \in S_n$ for which $y_i = x_{\sigma(i)}$. This equivalence relation is called the *symmetric equivalence relation* [17], and the set of all orbits of the action of S_n on X^n with the quotient topology is denoted by $\mathrm{SP}^{\mathrm{n}}X$ and called the *space of n-permutation degree* of X [17].

The following generalization of the permutation degree will be also used in what follows. If G be a subgroup of the group S_n, then G also acts on X^n as the group of permutations of coordinates, and generates an equivalence relation called the *G-symmetric equivalence relation* [17]. The quotient space of X^n under this relation is called *G-permutation degree* of X and is denoted by $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$, and the quotient mapping from X^n to $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$ is denoted by $\mathrm{ß}^{\mathrm{s}}_{\mathrm{n,G}}$. Observe that $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}$ is a covariant functor in the category of compact spaces and is called the *functor of G-permutation degree* [17]. Clearly, if $G = S_n$, then $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}} = \mathrm{SP}^{\mathrm{n}}$, and if G contains only the identity element, $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X = X^n$[17].

In [12], it is proved that the quotient mapping $\mathrm{ß}^{\mathrm{s}}_{\mathrm{n,G}} : X^n \to \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X$ is continuous, open, and closed surjection.

For every mapping $f : X \to Y$, the mapping $\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}f : \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}X \to \mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}Y$ is defined [17] by the formula

$$\mathrm{SP}^{\mathrm{n}}_{\mathrm{G}}f[(x_1, x_2, \ldots, x_n)]_G = [(f(x_1), f(x_2), \ldots, f(x_n))]_G.$$

A set $A \subset X$ is dense in X if $\overline{A} = X$. The *density* of X, denoted by $d(X)$, is defined as $d(X) = \min\{|A| : A \text{ is dense in } X\}$ [4]. A collection $\mathcal{B}$ of nonempty open sets in X is said to be a *π-base* of X if for every nonempty open set $G \subset X$ there is a $B \in \mathcal{B}$ with $B \subset G$. The *π-weight* of a space X is defined as $\pi w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base of } X\}$ [1,4,6]. The *weak density* of a space X, denoted by $wd(X)$, is the smallest cardinal number $\tau \geq \omega$ such that there is a π-base $\mathcal{B} = \cup\{\mathcal{B}_\alpha : \alpha < \tau\}$ in X, and for each $\alpha < \tau$, $\mathcal{B}_\alpha$ is a centered system of open sets in X [7,9,10].

For definitions of the following cardinal functions, see [1,4–6].

The *extent* of a space X, denoted by $e(X)$, is defined as $e(X) = \sup\{|Y| : Y$ is a closed discrete subspace in $X\}$. The *Souslin number* or *cellularity* of the space X, denoted by $c(X)$, is the smallest cardinal number $\tau \geq \omega$ such that every family or pairwise, disjoint, non-empty open subset of X has cardinality $\leq \tau$. The *Lindelöf number* $l(X)$ of X is the smallest cardinal number τ such that each open cover of X has a subcover of cardinality $\leq \tau$.

A continuous mapping $f : [0,1] \to X$ is called a *path* in X. $f(0)$ is called the initial point, and $f(1)$ the final point of this path. If $x \in X$, then the constant path $e_x : I \to X$ is defined by $e_x(t) = x$ for all $t \in I$. A space X is *path connected* if for any two points $x_0, x_1 \in X$ there is a path from x_0 to x_1 [18].

Continuous mappings $f, g : X \to Y$ are *homotopic*, denoted by $f \simeq g$, if there is a continuous mapping $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. F is called a homotopy between f and g [18].

Example 1. *Consider the mappings* $f(x) = (\cos(\pi x), \sin(\pi x))$ *and* $g(x) = (\cos(\pi x), -\sin(\pi x))$. *These mappings are homotopic. We can define the homotopy* $F : I \times I \to R^2$ *between* f *and* g *as follows:* $F(x,t) = (\cos(\pi x), (1-2t)\sin(\pi x))$. *Indeed,* F *is continuous and* $F(x,0) = (\cos(\pi x), \sin(\pi x)) = f(x)$, $F(x,1) = (\cos(\pi x), -\sin(\pi x)) = g(x)$ *(see [18]).*

A continuous mapping $f : X \to Y$ is said to be a *homotopy equivalence* [18] if there exists a continuous mapping $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity mappings on X and Y, respectively. Two topological spaces X and Y are said to be *homotopically equivalent* (notation $X \simeq Y$) if there exists a homotopy equivalence $f : X \to Y$ [18].

By a *covariant homotopy functor* [17], we mean an operator ϕ which assigns to each topological space X a space $\phi(X)$, and to each continuous mapping $f : X \to Y$, a mapping $\phi(f) : \phi(X) \to \phi(Y)$ satisfying the following three conditions:

(i) ϕ preserves the identity mapping; that is, if f is the identity mapping of X, then $\phi(f)$ is the identity mapping of $\phi(X)$.

(ii) ϕ preserves compositions; that is, if $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then

$$\phi(g \circ f) = \phi(g) \circ \phi(f).$$

(iii) ϕ preserves homotopy; that is, if a mapping $F(x,t)$ is a homotopy between the continuous mappings $f, g : X \to Y$, then $\phi(F(x,t))$ is a homotopy between the mappings $\phi(f), \phi(g) : \phi(X) \to \phi(Y)$.

A space X which is homotopy equivalent to a point is called *contractible*. A subset A of a space X is a *retract* of X if there exists a continuous mapping $r : X \to A$, called a *retraction*, such that $r|A = 1_A$ [18].

A property P of topological spaces is called a *homotopy property* if it is preserved by all homotopy equivalences. More precisely, P is a homotopy property if and only if for an arbitrary homotopy equivalence $f : X \to Y$, if X has P, then Y also has P [18].

3. Some Cardinal Properties of the Space of Permutation Degree

In this section, we study some (hereditary) cardinal invariants (the spread, extent, density, weak density, π-weight) of the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ of permutation degree of the Hattori space $(\mathbb{R}, \tau(A))$. Let us observe that the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ has a Sorgenfrey-type topology.

We begin with the following two lemmas.

Lemma 1. *The space* $\mathrm{SP}^2(\mathbb{R}, \tau_S)$ *contains a closed discrete subset of cardinality* $\mathfrak{c}$.

Proof. Note that the subset $Y = \{(x,y) \in (\mathbb{R}, \tau_S)^2 : x \geq y\}$ of $(\mathbb{R}, \tau_S)^2$ is homeomorphic to the space $\mathrm{SP}^2(\mathbb{R}, \tau_S)$, and the set $Z = \{(x,y) \in Y : y = -x, x > 0\}$ is closed and discrete in Y and has cardinality $\mathfrak{c}$. $\square$

Lemma 2. *Let Y be a subset of a Hausdorff topological space X and $Z = \{F \in \mathrm{SP}^2X : F \subset Y\} \subset \mathrm{SP}^2X$. Then:*

(i) The space SP^2Y is homeomorphic to the subspace Z of the space SP^2X;
(ii) The set Z is open in SP^2X whenever Y is open in X;
(iii) The set Z is closed in SP^2X whenever Y is closed in X;
(iv) The set Z is clopen in SP^2X whenever Y is clopen in X.

Proof. (i) It is known [17] and easy to check that the space $\exp_2 Y$ is homeomorphic to the subspace Z of the space $\exp_2 X$. In [17], it is shown that the space SP^2Y is homeomorphic to the space $\exp_2 Y$. Hence, we have that the space SP^2Y is homeomorphic to $Z \subset \mathrm{SP}^2X$.

(ii), (iii), and (iv) follow from the fact that the set Z is open (closed, clopen) in $\exp_2 X$ whenever Y is open (closed, clopen) in X [17] and the mentioned result that SP^2X is homeomorphic to $\exp_2 X$. □

From Lemmas 1 and 2 we have:

Proposition 1. *Let A be a subset of $\mathbb{R}$ and $B \subseteq \mathbb{R} \setminus A$. If B is a (closed) subset of $(\mathbb{R}, \tau(A))$ which is homeomorphic to the space $(\mathbb{R}, \tau_S)$, then the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ contains a (closed) discrete subset of cardinality $\mathfrak{c}$.*

Proof. In [16], it was shown that if $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R} \setminus A$, then $\tau(A)|_B = \tau_S|_B$. Recall that the set $(\mathbb{R}, \tau(A))$ is a closed subset of $\mathrm{SP}^2(\mathbb{R}, \tau(A))$. Hence, each (closed) discrete subset M of $(\mathbb{R}, \tau(A))$ with cardinality $\mathfrak{c}$ (which exists by Lemma 1) is a (closed) discrete subset of $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ with cardinality $\mathfrak{c}$. □

Proposition 2. *Let A be a subset of $\mathbb{R}$, and Y be a subspace of $(\mathbb{R}, \tau(A))$. Then, $|\tau(A)_Y| \leq \mathfrak{c}$. In addition, $|\mathrm{SP}^2Y| \leq \mathfrak{c}$.*

Proof. It is enough to show that $|\tau(A)| \leq \mathfrak{c}$. Let $\mathcal{B}$ be a base for $(\mathbb{R}, \tau(A))$ of cardinality $\leq \mathfrak{c}$. Since the space $(\mathbb{R}, \tau(A))$ is hereditary Lindelöf, each open subset of $(\mathbb{R}, \tau(A))$ is an union of countably many elements of $\mathcal{B}$. Hence, $|\tau(A)| \leq \mathfrak{c}^\omega = (2^\omega)^\omega = 2^\omega = \mathfrak{c}$. □

Corollary 1. *Let A be a subset of $\mathbb{R}$ and $B \subseteq \mathbb{R} \setminus A$. If B is a closed subset of $(\mathbb{R}, \tau(A))$ which is homeomorphic to the space $(\mathbb{R}, \tau_S)$, then $s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = e(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = \mathfrak{c}$.*

Proof. By Proposition 1 we have $e(\mathrm{SP}^2(\mathbb{R}, \tau(A))) \geq \mathfrak{c}$. Note that

$$e(\mathrm{SP}^2(\mathbb{R}, \tau(A))) \leq s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) \leq |(\mathrm{SP}^2(\mathbb{R}, \tau(A)))|.$$

It follows from Proposition 2 that

$$|(\mathrm{SP}^2(\mathbb{R}, \tau(A)))| \leq |exp(\mathbb{R}, \tau(A))| \leq \mathfrak{c}.$$

Thus, we get

$$s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = e(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = \mathfrak{c}.$$

□

Corollary 2. *Let A be a subset of $\mathbb{R}$ and $B \subseteq \mathbb{R} \setminus A$. If B is a subset of $(\mathbb{R}, \tau(A))$ which is homeomorphic to the space $(\mathbb{R}, \tau_S)$, then $s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = \mathfrak{c}$.*

Proof. By Proposition 1 we have $s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) \geq \mathfrak{c}$. Note that $s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) \leq |(\mathrm{SP}^2(\mathbb{R}, \tau(A)))|$. It follows from Proposition 2 that $|(\mathrm{SP}^2(\mathbb{R}, \tau(A)))| \leq \mathfrak{c}$. Thus, we get $s(\mathrm{SP}^2(\mathbb{R}, \tau(A))) = \mathfrak{c}$. □

Corollary 3. *Let A be a subset of $\mathbb{R}$ and $\varphi \in \{d, e, c\}$ (resp. $\varphi \in \{wd, l, \pi w\}$). If B is a subset of $(\mathbb{R}, \tau(A))$ which is homeomorphic to the space $(\mathbb{R}, \tau_S)$, then $h\varphi(\mathsf{SP}^2(\mathbb{R}, \tau(A))) = \mathfrak{c}$ (viz., $h\varphi(\mathsf{SP}^2(\mathbb{R}, \tau(A))) \geq \mathfrak{c}$).*

Proof. In fact, note that $he = s$ and $hc = s$. Since $hc(\mathsf{SP}^2(\mathbb{R}, \tau(A))) \geq \mathfrak{c}$ (by Proposition 1), we have the equalities. The inequalities also trivially follow from Proposition 1. □

Remark 1. *Let $A \subseteq \mathbb{R}$. Note that the family $\mathcal{B} = \{(r_1, r_2) : r_1, r_2 \in \mathbb{Q}, r_1 < r_2\}$ is a π-base for the space $(\mathbb{R}, \tau(A))$ and $\mathcal{B}^n = \{\prod_{i=1}^n B_i : B_i \in \mathcal{B}\}$ is also a π-base for the space $(\mathbb{R}, \tau(A))^n$. Hence, $\pi w(\mathbb{R}, \tau(A))^n = c(\mathbb{R}, \tau(A))^n = \omega$. Similarly, the family $\mathsf{SP}^n \mathcal{B}^n = \{\mathsf{SP}^n B_i = \mathfrak{B}_n^s(B_i) : B_i \in \mathcal{B}^n\}$ is a π-base for the space $\mathsf{SP}^n(\mathbb{R}, \tau(A))$. This shows that $\pi w(\mathsf{SP}^n(\mathbb{R}, \tau(A))) = c(\mathsf{SP}^n(\mathbb{R}, \tau(A))) = \omega$.*

4. Some Geometric Properties of the Space of Permutation Degree

Now we study some geometric properties of the space $\mathsf{SP}^n_G X$ of permutation degree. In particular, we show the functor SP^n_G preserves the homotopy and the retraction of topological spaces. In fact, we prove that if spaces X and Y are homotopically equivalent, then the spaces $\mathsf{SP}^n_G X$ and $\mathsf{SP}^n_G Y$ are also homotopically equivalent, and conclude that the functor SP^n_G is a covariant homotopy functor.

Theorem 1. *If the mappings $f, g : X \to Y$ are homotopic, then the mappings $\mathsf{SP}^n_G f, \mathsf{SP}^n_G g : \mathsf{SP}^n_G X \to \mathsf{SP}^n_G Y$ are also homotopic.*

Proof. Assume that the mappings $f, g : X \to Y$ are homotopic. Then there exists a continuous mapping $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. On the other hand, we have

$$\mathsf{SP}^n_G f[(x_1, x_2, \ldots, x_n)]_G = [(f(x_1), f(x_2), \ldots, f(x_n))]_G,$$
$$\mathsf{SP}^n_G g[(x_1, x_2, \ldots, x_n)]_G = [(g(x_1), g(x_2), \ldots, g(x_n))]_G.$$

Now we define the mapping

$$\mathsf{SP}^n_G F([(x_1, x_2, \ldots, x_n)]_G, t) = [(F(x_1, t), F(x_2, t), \ldots, F(x_n, t))]_G.$$

It is clear that since the mapping F is continuous, the mapping $\mathsf{SP}^n_G F$ is also continuous. Now we will show that the mapping $\mathsf{SP}^n_G F$ is a homotopy between the mappings $\mathsf{SP}^n_G f$ and $\mathsf{SP}^n_G g$. Indeed,

$$\begin{aligned}\mathsf{SP}^n_G F([(x_1, x_2, \ldots, x_n)]_G, 0) &= [(F(x_1, 0), F(x_2, 0), \ldots, F(x_n, 0))]_G \\ &= [(f(x_1), f(x_2), \ldots, f(x_n))]_G \\ &= \mathsf{SP}^n_G f[(x_1, x_2, \ldots, x_n)]_G;\end{aligned}$$

and

$$\begin{aligned}\mathsf{SP}^n_G F([(x_1, x_2, \ldots, x_n)]_G, 1) &= [(F(x_1, 1), F(x_2, 1), \ldots, F(x_n, 1))]_G \\ &= [(g(x_1), g(x_2), \ldots, g(x_n))]_G \\ &= \mathsf{SP}^n_G g[(x_1, x_2, \ldots, x_n)]_G.\end{aligned}$$

This means that $\mathsf{SP}^n_G f \simeq \mathsf{SP}^n_G g$. □

Corollary 4. *If the spaces X and Y are homotopically equivalent, then the spaces $\mathsf{SP}^n_G X$ and $\mathsf{SP}^n_G Y$ are also homotopically equivalent.*

Proof. Suppose that the spaces X and Y are homotopically equivalent. Then there exist two continuous mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. This means that there are two homotopy $F(y,t)$ and $H(x,t)$ such that

$$F(y,0) = (f \circ g)(y), F(y,1) = y \text{ and } (x,0) = (g \circ f)(x), H(x,1) = x.$$

Consider the compositions $\mathrm{SP}^n_G f \circ \mathrm{SP}^n_G g : \mathrm{SP}^n_G Y \to \mathrm{SP}^n_G Y$ and $\mathrm{SP}^n_G g \circ \mathrm{SP}^n_G f : \mathrm{SP}^n_G X \to \mathrm{SP}^n_G X$ of the mappings $\mathrm{SP}^n_G f : \mathrm{SP}^n_G X \to \mathrm{SP}^n_G Y$ and $\mathrm{SP}^n_G g : \mathrm{SP}^n_G Y \to \mathrm{SP}^n_G X$ defined by

$$(SP^n_G f \circ SP^n_G g)[(y_1, y_2, \dots, y_n)]_G = [((f \circ g)(y_1), (f \circ g)(y_2), \dots, (f \circ g)(y_n))]_G$$

and

$$(SP^n_G g \circ SP^n_G f)[(x_1, x_2, \dots, x_n)]_G = [((f \circ g)(x_1), (f \circ g)(x_2), \dots, (f \circ g)(x_n))]_G.$$

One can easily check that the mapping

$$SP^n_G F([(y_1, y_2, \dots, y_n)]_G, t) = [(F(y_1,t), F(y_2,t), \dots, F(y_n,t))]_G$$

is a homotopy between $SP^n_G f \circ SP^n_G g$ and $id_{SP^n_G Y}$.

Similarly,

$$SP^n_G H([(x_1, x_2, \dots, x_n)]_G, t) = [(H(x_1,t), H(x_2,t), \dots, H(x_n,t))]_G$$

is a homotopy between $SP^n_G g \circ SP^n_G f$ and $id_{SP^n_G X}$.

By Theorem 1, $SP^n_G X$ and $SP^n_G Y$ are homotopically equivalent. □

Proposition 3. *If a set A is a retract of the topological space X, then the set $\mathrm{SP}^n_G A$ is a retract of the topological space $\mathrm{SP}^n_G X$.*

Proof. Suppose that a set A is a retract of X. Then there exists a continuous mapping $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Now we consider the mapping $\mathrm{SP}^n_G r : \mathrm{SP}^n_G X \to \mathrm{SP}^n_G A$. For every $[(a_1, a_2, \dots, a_n)]_G \in \mathrm{SP}^n_G A$ we have

$$\mathrm{SP}^n_G r([(a_1, a_2, \dots, a_n)]_G) = [(r(a_1), r(a_2), \dots, r(a_n))]_G = [(a_1, a_2, \dots, a_n)]_G.$$

This means that the mapping $\mathrm{SP}^n_G r : \mathrm{SP}^n_G X \to \mathrm{SP}^n_G A$ is a retraction. Hence, the set $\mathrm{SP}^n_G A$ is a retract of the space $\mathrm{SP}^n_G X$. □

Theorem 2. *The functor SP^n_G is a covariant homotopy functor.*

Proof. Now we will show that the functor SP^n_G satisfies the above three conditions.

(i) Let id_X be identity mapping in the topological space X. Then we have that $\mathrm{SP}^n_G id_X[(x_1, x_2, \dots, x_n)]_G = [(id_X(x_1), id_X(x_2), \dots, id_X(x_n))]_G = [(x_1, x_2, \dots, x_n)]$. This means that the mapping $\mathrm{SP}^n_G id_X$ is the identity mapping in the topological space $\mathrm{SP}^n_G X$.

(ii) Let $f : X \to Y$, $g : Y \to Z$ be continuous mappings. Then it follows that $\mathrm{SP}^n_G(g \circ f)[(x_1, x_2, \dots, x_n)]_G = [((g \circ f)(x_1), (g \circ f)(x_2), \dots, (g \circ f)(x_n))]_G = \mathrm{SP}^n_G g[(f(x_1), f(x_2), \dots, f(x_n))]_G = \mathrm{SP}^n_G g \circ \mathrm{SP}^n_G f$.

(iii) It follows easily from Theorem 1. □

In [14], some propositions about homotopy properties of the topological spaces were given. For instance, it was proved that contractibility, connectedness, and pathwise connectedness are homotopy property of the spaces.

Corollary 5. *If a topological space X is contractible, then the space $\mathrm{SP}^n_G X$ is also contractible.*

Corollary 6. *If a topological space X is connected (viz., pathwise connected), then the space $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ is also connected (viz., pathwise connected).*

If f and g are two paths in X with $f(1) = g(0)$, then by the product of f and g we mean the path $f * g$, which is defined by

$$(f * g)(t) = \begin{cases} f(2t), & \text{if } 0 \le t \le 1/2, \\ g(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Let $f \simeq g$ and $g \simeq h$, where F is a homotopy from f to g and G is a homotopy from g to h. Then, $f \simeq h$. Define a homotopy $H : X \times I \to Y$ between f and h as follows:

$$H(x,t) = \begin{cases} F(x,2t), & \text{if } 0 \le t \le 1/2, \\ G(x,2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Corollary 7. *If the mappings $f_i : I \to X$ are paths in X from the points x_0^i to the points x_1^i, $i = 1,2,\ldots,n$, respectively, then the mapping $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f^n : I \to \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ defined by $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f^n(t) = [(f_1(t), f_2(t), \ldots, f_n(t))]_G$ is also a path from the point $[(x_0^1, x_0^2, \ldots, x_0^n)]_G$ to the point $[(x_1^1, x_1^2, \ldots, x_1^n)]_G$ in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$.*

Corollary 8. *Let $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f^n$ and $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}g^n$ be paths from $[(x_0^1, x_0^2, \ldots, x_0^n)]_G$ to $[(x_1^1, x_1^2, \ldots, x_1^n)]_G$ and from $[(x_2^1, x_2^2, \ldots, x_2^n)]_G$ to $[(x_3^1, x_3^2, \ldots, x_3^n)]_G$, respectively, with $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f^n(1) = \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}g^n(0)$. Then we define the multiplication of the paths in $\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}X$ as follows:*

$$(\mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}f^n * \mathsf{SP}^{\mathsf{n}}_{\mathsf{G}}g^n)(t) = [((f_1 * g_1)(t), (f_2 * g_2)(t), \ldots, (f_n * g_n)(t))]_G.$$

This path sharing the points $[(x_0^1, x_0^2, \ldots, x_0^n)]_G$ and $[(x_3^1, x_3^2, \ldots, x_3^n)]_G$.

In [18], it is shown that the multiplication of equivalence classes of paths is associative; in other words, $([f][g])[h] = [f]([g][h])$.

Let f and g be paths from the initial point x_0 to the final point x_1. If there is a homotopy F from f to g such that for each $t \in I$, $F(0,t) = x_0$ and $F(1,t) = x_1$, then f and g are said to be *path-homotopic* [18]. For a path f, $[f]$ denotes the equivalence class of all paths path-homotopic to f.

The operation $*$ defined above can be applied to homotopy classes as well. Let $f : I \to X$ be a path from x_0 to x_1 and $g : I \to X$ a path from x_1 to x_2. Then, one defines $[f] * [g] = [f * g]$.

Let X be a space and $x_0 in X$. A path in X beginning and ending at x_0 is called a *loop* [18] based at x_0. Denote by $\pi_1(X, x_0)$ the set of all equivalence classes $[f]$ of loops in X based at x_0. $\pi_1(X, x_0)$ with the operation $*$ is a group, where the identity element of the group is $[e_x]$ and the inverse element of $[f]$ is $[\overline{f}(t)] = [f(1-t)]$. We call $\pi_1(X, x_0)$ the *fundamental group* [18].

Suppose $(G, *)$ and $(G_1, *_1)$ are groups. A homomorphism is a mapping such that $f(x * y) = f(x) *_1 f(y)$ for all $x, y \in G$. A homomorphism f is called an *isomorphism* if it is bijective.

The fundamental groups of a space and its quotient space are not always isomorphic.

Example 2. *Let $X = [0,1]$ and let S^1 be the unit circle. Clearly, S^1 is the quotient space of the space X, where the quotient mapping is defined as $q(x) = (cos(2\pi x); sin(2\pi x))$. We know that $X = [0,1]$ has the trivial fundamental group (the group consisting of the identity), and the fundamental group of S^1 is isomorphic to the group $(\mathbb{Z}, +)$.*

Corollary 9. *The fundamental groups $\pi_1(X, x_0)$ and $\pi_1(SP^n_G X, [x_0]_G)$ of the topological spaces X and $SP^n_G X$ are not always isomorphic for every $x_0 \in X$, where $[x_0]_G = [(x_0, x_0, \ldots, x_0)]_G = (x_0, x_0, \ldots, x_0)$.*

Let $h : X \to Y$ be a continuous mapping sending the point $x_0 \in X$ to the point $y_0 \in Y$; We denote this fact by $h : (X, x_0) \to (Y, y_0)$. If f is a loop in X based at x_0, then the composition $h \circ f : I \to Y$ is a loop in Y based at y_0. In this way the correspondence $f \to h \circ f$ gives rise to a mapping from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$. In [19] it is proved that h_* is a homomorphism.

Corollary 10. *The fundamental groups $\pi_1(X, x_0)$ and $\pi_1(\mathrm{SP}^n_G X, [x_0])$ of the topological spaces X and $\mathrm{SP}^n_G X$ are homomorphic for every $x_0 \in X$, and the homomorphism is defined by $(\pi^s_{n,G})_*[f] = [\pi^s_{n,G} \circ f]$.*

Proof. We know that $\pi^s_{n,G} : X^n \to SP^n_G X$ is a continuous mapping that carries the point $(x_0, x_0, \ldots, x_0)$ of X^n to the point $[(x_0, x_0, \ldots, x_0)]$ of $SP^n_G X$. If f is a loop in X based at x_0, then f^n is a loop in X^n based at $(x_0, x_0, \ldots, x_0)$ and the composition $\pi^s_{n,G} \circ f^n : I \to SP^n_G X$ is a loop in $SP^n_G X$ based at $[(x_0, x_0, \ldots, x_0)]$. The correspondence $f \to \pi^s_{n,G} \circ f^n$ thus gives rise to a mapping carrying $\pi_1(X, x_0)$ into $\pi_1(SP^n_G X, [(x_0, x_0, \ldots, x_0)])$. Define $\pi^s_{n,G_*} : \pi_1(X, x_0) \to \pi_1(SP^n_G X, [(x_0, x_0, \ldots, x_0)])$ by the equation $\pi^s_{n,G_*}([f]) = [\pi^s_{n,G} \circ f^n]$. The mapping π^s_{n,G_*} is a homomorphism (as we said before). $\square$

5. Conclusions

In this article we continue the study of the functor of permutation degree—one of important functors in topology. Our results extend and complement the existing results in this field. We obtained several relations among cardinal invariants in the space $\mathrm{SP}^2(\mathbb{R}, \tau(A))$ of the permutation degree of the Hattori space $(\mathbb{R}, \tau(A))$. These cardinal invariants include the hereditary density, hereditary weak density, spread, extent, π-weight, and (hereditary) Lindelöf number. Additionally, we proved that if the spaces X and Y are homotopically equivalent, then the spaces $\mathrm{SP}^n_G X$ and $\mathrm{SP}^n_G Y$ are homotopically equivalent too. As a consequence, one obtains that the functor SP^n_G is a covariant homotopy functor. It preserves a few topological properties, including retracts.

We believe that our results can be applied to similar investigation of other topological properties and other functors.

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