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Solvability of Conformable Type Frictionless Contact Problem via Hemivariational Inequalities

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Abstract: In this paper, we study a class of conformable frictionless contact problems with the surface traction driven by the conformable impulsive differential equation. The existence of a mild solution for conformable impulsive hemivariational inequality is obtained by the Rothe method, subjectivity of multivalued pseudomonotone operators and the property of the conformable derivative. Notice that we imply some new fractional viscoelastic constitutive laws.

Keywords: conformable impulsive differential equation; conformable hemivariational inequality; conformable frictionless contact problem; Rothe method

MSC: 49J40; 34H05

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1. Introduction

Variational inequalities were proposed by Signorini in the 1950s in the context of applications to the rigid contact problem in elasticity and mainly involved convex energy functionals or nonempty closed convex sets. Variational inequalities represent powerful mathematical tools with application to contact mechanics, see [1–4]. With the development of solid contact mechanics, the researchers realized that not all contact problems are modeled by variational inequalities. Hemivariational inequalities were firstly introduced by Panagiotopoulos in the 1980s, which was mainly used to deal with the mechanical problems of nonsmooth and nonconvex functions, see [5]. During the last 30 years, hemivariational inequalities have been substantially developed in both application and mathematics, see [6–15]. Moreover, numerical analysis of hemivariational inequalities has achieved good results, see [16–18]. In particular, the PDAS approach [17] provides suitable analytical and numerical tools for the solution of hemivariational inequalities. This method is of great significance for solving crack problems.

Systems consisting of variational inequalities and differential equations were first studied by Pang and Stewart [19] in 2008 in a framework of finite-dimensional spaces. They called this complex system a differential variational inequality (DVI, for short) and are mainly applied to electrical circuits with ideal diodes, dynamic contact mechanics, economical dynamics, dynamic traffic networks and so on. Differential hemivariational inequalities (DHVIs, for short) as extensions of differential variational inequalities are mainly organized by hemivariational inequalities and differential equations. Obviously, DHVI mainly involves nonsmooth and nonconvex functions. The mathematical results on DHVIs have found applications to contact problems, see [20–22].

As we all know, the traditional constitutive laws of viscoelastic materials are obtained by linear springs and Newton dashpot in series and parallel. However, the creep and relaxation processes of viscoelastic materials can not be accurately described by this scheme. Abel dashpot modeled by fractional time derivative can effectively characterize the creep and relaxation processes of viscoelastic materials. In recent years, the frictional contact problems for a viscoelastic body with the time-fractional viscoelastic Kelvin–Voigt constitutive law have attracted widespread attention. Zeng and Migórski [23,24] used the Rothe method to study a frictional quasistatic contact problem. Based on the Rothe method and numerical analysis, Weng et al. considered a new class of fractional differential hemivariational inequality with an application in [25]. In real life, the traditional fractional derivative sometimes causes great difficulties in analysis and calculation. To overcome the above difficulties, conformable derivatives as a new fractional derivative are proposed in [26]. Recently, conformable derivatives have attracted attention in mathematics and applications, see [27–32]. It can describe Newtonian mechanics [29], the logical model [30] and cobweb model [31]. It is worth mentioning that Ma et al. [32] established a grey model with a conformal derivative in view of the computational complexity of the existing fractional grey model and the new model was superior to the existing fractional grey model in predicting the natural gas consumption of 11 countries and is more effective than the existing model in nonsmooth time-series prediction.

In this paper, we consider the quasistatic conformable viscoelastic to frictionless contact problem with the surface traction driven by the conformable impulsive differential equation. Next, we briefly present our innovation in this paper.

We give an analysis of constitutive laws by the property of conformable derivative and study the quasistatic conformable viscoelastic to frictionless contact problem. Because a conformable derivative is a form of fractional derivative, our model can accurately describe the creep and relaxation processes of viscoelastic materials. It is necessary to mention that our viscoelastic body is the generalized fractional Kelvin–Voigt constitutive of the conformable type and the contact boundary is modeled with the Clarke subdifferential of a nonconvex and nonsmooth function. In addition, the contact surface is vulnerable to impact, and this phenomenon is described by impulsive differential equations. Therefore, our new contact model leads to a new class of fractional hemivariational inequalities. Based on the literature [23–25], we will use the Rothe method to solve the existence of weak solutions for hemivariational inequality.

This paper is organized as follows. In Section 2, we give the basic notations and some important results. In Section 3, we obtain the new fractional viscoelastic constitutive laws. In Section 4, we give the quasistatic contact problem. Finally, we obtain the existence solvability for the contact problem.

2. Preliminaries

In this section, we recall the basic notations and preliminary results.

Let V and $\mathcal{W}$ be reflexive and separable Banach spaces and $\mathbb{J} = [0, T]$. $\langle\cdot,\cdot\rangle$ denote the duality of V and V^*. We use the notation $C(\mathbb{J}; V)$ to denote the spaces of all continuous functions. $PC(\mathbb{J}; \mathcal{W})$ the space of function $g : \mathbb{J} \to \mathcal{W}$ such that $g : \mathbb{J} / \bigcup_{j=0,1,2,\cdots,m,m+1}\{\tau_j\} \to \mathcal{W}$ is continuous and $g(\tau_j^-)$ and $g(\tau_j^+)$ exist with $g(\tau_j) = g(\tau_j^-)$, where $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$. The norms of the above spaces are defined separately as follows:

$$\|x\|_{C(\mathbb{J};V)} = \sup_{t\in\mathbb{J}} \|x(t)\|_V$$

and

$$\|g\|_{PC(\mathbb{J};\mathcal{W})} = \sup_{t\in\mathbb{J}} \|g(t)\|_{\mathcal{W}}.$$

Next, let us recall some conformable definitions and conclusions.

Definition 1 ([28])**.** *The conformable derivative with lower index a of a function $y : [a, +\infty) \to \mathbb{R}$ is defined as:*

$$\mathfrak{D}_\beta^a y(t) = \lim_{\varepsilon\to 0} \frac{y(t + \varepsilon(t-a)^{1-\beta}) - y(t)}{\varepsilon}, t > a, 0 < \beta < 1,$$

and

$$\mathfrak{D}^a_\beta y(a) = \lim_{t\to a^+} \mathfrak{D}^a_\beta y(t).$$

Definition 2 ([28])**.** *The conformable integral with lower index a of a function $y : [a, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}^a_\beta y(t) = \int_a^t (s-a)^{\beta-1} y(s)\mathrm{d}s.$$

Lemma 1 ([28])**.** *For $y \in C^1([a,\infty), \mathbb{R})$, one has*

$$\mathfrak{I}^a_\beta \mathfrak{D}^a_\beta y(t) = y(t) - y(a), t > a, 0 < \beta < 1.$$

Remark 1 ([26])**.** *If $\mathfrak{D}^a_\beta y(t_0)$ exists and is finite, we say that y is $\beta-$differentiable at t_0. When $y \in C^1((a,\infty], \mathbb{R})$, we have $\mathfrak{D}^a_\beta y(t) = (t-a)^{1-\beta} y'(t)$. Indeed, note that for $t > a$, the conformable derivative $\mathfrak{D}^a_\beta y(t)$ exists if and only if y is differentiable at t and $\mathfrak{D}^a_\beta y(t) = (t-a)^{1-\beta} y'(t)$.*

For more details about conformable derivatives, we can refer to [26,27] and the references therein. Then, we will review the definition of generalized clarke for a locally Lipschitz functional $F : X \to \mathbb{R}$ on a Banach space X. According to [13], we denote by $F^0(x; v)$ the clarke generalized directional derivative of F at x in the direction v, that is

$$F^0(x; v) = \limsup_{y\to x, \lambda\to 0^+} \frac{F(y + \lambda v) - F(y)}{\lambda}.$$

Recall also that the generalized clarke subdifferential of F at x, denoted by ∂F, is a subset X^* given by

$$\partial F(x) = \{x^* \in X^* : F^0(x; v) \geqslant \langle x^*, v\rangle\}, \tag{1}$$

for all $v \in X$.

3. Conformable Type Viscoelastic Constitutive Laws

Before implying the conformable constitutive laws, let us review the constitutive model. According to rheology, the idealized elastic and viscous properties of the substance can be ideally modeled with the linear spring and Newton dashpot. The linear spring obeies the Hooke's law $\sigma_e = E\varepsilon_e$ and the Newton dashpot obeies Newton's law $\sigma_v = \eta\dot{\varepsilon_v}$, where the coefficient $E > 0$ denotes the Young modulus of elasticity and $\eta > 0$ is the Newtonian Viscosity. $\dot{\varepsilon_v}$ is the time derivative of the strain ε_v, and σ_v denotes the stress of the dashpot element. Then, people produce the Maxwell model, see Figure 1, and the Kelvin–Voigt model, Figure 2, by combining the linear spring and Newton dashpot. In order to improve the quality of models, more elements are used, for example, the generalized Maxwell and Voigt model and the Burgers model. However, the constitutive laws are influenced by a larger number of parameters, from which many difficulties in theoretical and numerical analyses arise. In order to overcome the above difficulties, fractional constitutive models have been used. Therefore, the fractional model is a natural extension of the integral order case. For the fractional constitutive models, we can refer to [33,34]. In particular, Han et al. [35] implied the creep and relaxation behavior for THE Caputo fractional-order Maxwell model and the fractional Kelvin–Voigt model.

Motivated by the above works, we imply the conformable constitutive laws. The basic conformable element is called the conformable Scott-Blair dashpot, and it satisfies the law of the form

$$\sigma_v(t) = \eta(\mathfrak{D}^0_\beta(\varepsilon_v(t))), \;\; 0 < \beta < 1, t \geqslant 0,$$

where σ_v and ε_v are the stress and strain of the Scott-Blair dashpot element, and $\mathfrak{D}^0_\beta$ is the conformable fractional integral with a lower index of 0.

If $\sigma_v(t) = \sigma_0$ for $t \geqslant 0$, according to the definition of conformable integral, we have

$$\varepsilon_v(t) = \varepsilon_v(0) + \frac{\sigma_0 t^\beta}{\eta\beta}.$$

Thus, we imply the creep compliance of the Scott-Blair model

$$J(t) = \frac{t^\beta}{\eta\beta}, \ \ 0 < \beta < 1, t \geqslant 0.$$

We can choose a different number $\beta \in (0, 1)$. Therefore, we get a series of strain creep compliance for the conformable Scott-Blair model.

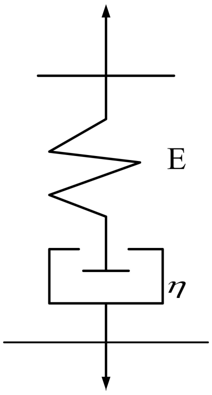

Figure 1. The Maxwell model.

Now, we can deduce that the conformable Maxwell model, Figure 3, and the conformable Kelvin–Voigt model, Figure 4, by combining the linear spring with the conformable Scott-Blair dashpot. As is known to all, if a system of elements is connected in parallel, their stresses coincide, and the total strain equals the sum of stresses in separate elements. It is obvious that the fractional Kelvin–Voigt constitutive law in the one-dimensional case is described as follows

$$\begin{aligned} \sigma(t) &= E\varepsilon(t) + \eta(\mathfrak{D}^0_\beta(\varepsilon(t))) \\ &= E\varepsilon(t) + \eta\varepsilon'(t)t^{1-\beta}, \end{aligned} \tag{2}$$

for $0 < \beta < 1$ and $t > 0$. On the other hand, according to Figure 3, we know that the elements are connected in serial. Thus, the fractional Maxwell constitutive law in the one-dimensional case can be formulated by

$$\begin{aligned} \varepsilon &= \varepsilon_e + \varepsilon_v \\ &= \frac{\sigma}{E} + \frac{1}{\eta}\int_0^t \frac{\sigma(s)}{s^{1-\beta}} \mathrm{d}s + \varepsilon_0. \end{aligned}$$

Similarly, in order to improve the quality of models, we can use more elements, which implies the generalized conformable Maxwell and conformable Voigt models, the conformable Burgers model, etc. Next, we push to creep the conformable Kelvin–Voigt model.

Using creep compliance of the conformable Kelvin–Voigt model: we obtain from $\sigma_s(t) = \sigma_0$ for $t \geqslant 0$ that

$$J_{fk} = \frac{t^\beta \sigma_0}{\eta\beta} + \frac{\sigma_0}{E} + \varepsilon_0.$$

Similarly, we can also choose a different number $\beta \in (0, 1)$. Meanwhile, we get a series of strain creep compliance for the conformable Kelvin–Voigt model.

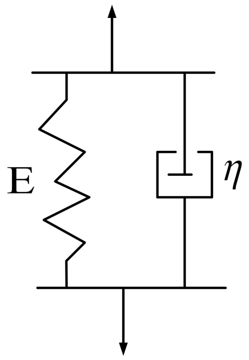

Figure 2. The Kelvin–Voigt model.

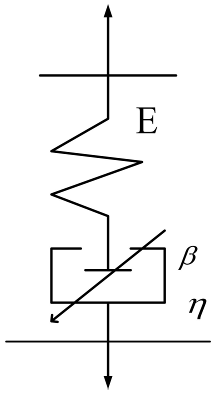

Figure 3. The Conformable Maxwell model.

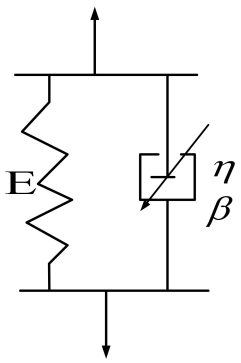

Figure 4. The Conformable Kelvin–Voigt model.

4. Quasistatic Conformable Viscoelastic Frictionless Contact Problem

In real life, the surface traction Γ_N will suddenly change due to external interference, see Figures 5 and 6. We regard this phenomenon as an impulsive effect. On the other hand, the surface traction Γ_N is influenced by a larger number of parameters, for example, the pointwise fractional density of active bonds and the roughness of the surface traction Γ_N. Therefore, the force $\boldsymbol{f}_N$ on the surface Γ_N of the viscoelastic body can not be changed uniformly. Namely, the change rate of the force $\boldsymbol{f}_N$ on each place of the contact surface Γ_N is also different. Based on the above fact and discussion in the third part, we study the quasistatic conformable viscoelastic frictionless contact problem for a viscoelastic body with Kelvin–Voigt constitutive law. Meanwhile, we use a conformable impulsive differential equation to describe the influence on the surface traction Γ_N, see Figure 6.

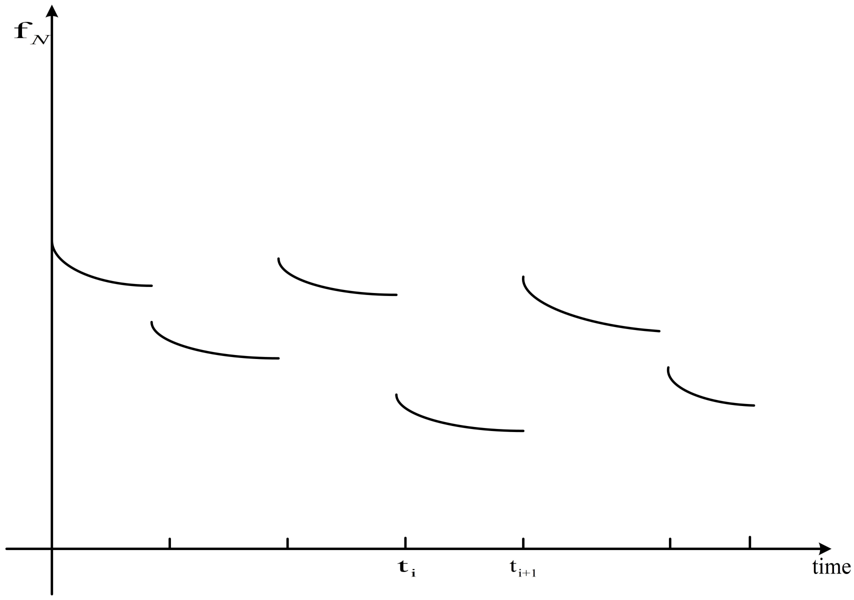

Figure 5. The force f_N on surface traction Γ_N.

Next, we review the physical setting of the contact problem and introduce the basic notations. We study a deformable viscoelastic body, which occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with the boundary $\Gamma = \partial\Omega$. The boundary Γ consists of three disjoint measurable parts Γ_D, Γ_N and Γ_C with means $\Gamma_D > 0$, see Figure 6.

Let v be a unit outward normal vector, and $\mathbb{S}^d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$. The inner products and corresponding norms in $\mathbb{S}^d$ and $\mathbb{R}^d$ are denoted by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \ \ \|\mathbf{v}\|_{\mathbb{R}^d} = (\mathbf{v} \cdot \mathbf{v})^{1/2} \ \ for\ all\ \mathbf{u} = (u_i)\mathbf{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \ \ \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \ \ for\ all\ \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

where $\mathbf{u} = (u_i), \boldsymbol{\sigma} = (\sigma_{ij})$, and $\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(u))$ denote the displacement tensors, the stress tensor, and the linearized strain tensor, the indices i, j, k, l run from 1 to d and the summation convention over repeated indices is used. On the other hand, we denote by $\boldsymbol{u}_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - \boldsymbol{u}_\nu$ the normal and tangential components of the displacement $\boldsymbol{u}$. The normal and tangential components of the displacement $\boldsymbol{u}$ on Γ is denoted by $\boldsymbol{\sigma}_\nu = (\sigma\nu) \cdot v$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \sigma_\nu\nu$.

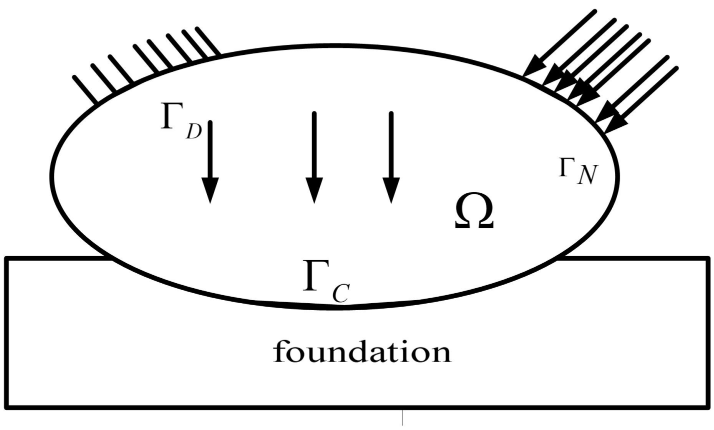

Figure 6. A deformable body in contact with a foundation.

This paper considers the following new conformable contact problem with the surface traction by a conformable impulsive differential equation.

Problem 1. *Find a displacement filed* $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ *and a surface traction density* $\boldsymbol{f}_N : \Gamma_N \times [0, T] \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\mathfrak{D}^0_\beta(\boldsymbol{u}(t))) + \mathcal{B}(\varepsilon(\boldsymbol{u}(t))) \qquad in\ \Omega \times [0, T], \tag{3}$$

$$Div\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \qquad in\ \Omega \times [0, T], \tag{4}$$

$$\boldsymbol{u}(t) = 0 \qquad on\ \Gamma_D \times [0, T], \tag{5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{v} = \boldsymbol{f}_N(t) \qquad on\ \Gamma_N \times [0, T], \tag{6}$$

$$\mathfrak{D}^0_\alpha(\boldsymbol{f}_N(t)) = g(t, \boldsymbol{f}_N(t), \boldsymbol{u}(t)), \quad t \neq \tau_j, j = 1, 2, \cdots, m, \qquad on\ \Gamma_N \times [0, T], \tag{7}$$

$$\wedge \boldsymbol{f}_N(\tau_j) = \Theta_j(\boldsymbol{f}_N(\tau_j^-)), \quad j = 1, 2, \cdots, m, \qquad on\ \Gamma_N \times [0, T], \tag{8}$$

$$\boldsymbol{f}_N(0) = \boldsymbol{f}^N_0 \qquad on\ \Gamma_N \times [0, T], \tag{9}$$

$$-\boldsymbol{\sigma}_\nu(t) \in \partial j_\nu(u_\nu(t)) \qquad on\ \Gamma_C \times [0, T], \tag{10}$$

$$\boldsymbol{\sigma}_\tau = 0 \qquad on\ \Gamma_C \times [0, T], \tag{11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad in\ \Omega. \tag{12}$$

We give a brief description. Equation (3) is the conformable Kelvin–Voigt viscoelastic constitutive law. It is made of the linear spring and the conformable Scott-Blair dashpot in parallel. We extend (2) to the general d-dimensional case to obtain (3). The conformable Kelvin–Voigt viscoelastic constitutive law presents elastic and viscous features. Elastic properties are described by springs, and viscous properties are modeled by dashpots. $\mathcal{A}$ and $\mathcal{B}$ stand for the viscosity and elasticity operators. On the other hand,

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\ \ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{i,j}),\ i, j = 1, \cdots, d.$$

denotes the linearized or the small strain tensor. Equation (4) denotes the equilibrium equation, and Equation (5) denotes that the body is clamped on Γ_D. Equations (6)–(9) show that the traction is acted on Γ_N and the density of the surface traction is governed by a conformable impulsive differential equation, Θ_j is an impulsive function with $j = 1, 2, \ldots, m$, and $\wedge \boldsymbol{f}_N(\tau_j) = \boldsymbol{f}_N(\tau_j^+) - \boldsymbol{f}_N(\tau_j^-)$ and $\boldsymbol{f}_N(\tau_j^-) = \boldsymbol{f}_N(\tau_j)$ with $\boldsymbol{f}_N(\tau_j^+)$ and $\boldsymbol{f}_N(\tau_j^-)$ being the left and the right limit of $\boldsymbol{f}_N$ at $t = \tau_j$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$. Equation (10) denotes contact condition, where j_ν is locally Lipschitz functionals. We need to explain the contact conditions. Equation (11) represents the frictionless contact problem.

In this paper, we consider the following spaces V, H and $\mathcal{H}$.

$$\mathcal{V} = \{v \in H^1(\Omega; \mathbb{R}^d) | v = 0 \text{ on } \Gamma_D\},\ H = L^2(\Omega; \mathbb{R}^d) \text{ and } \mathcal{H} = L^2(\Omega; \mathbb{S}^d).$$

The space $\mathcal{H}$ is endowed with the Hilbert structure by the inner product by

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\mathrm{d}x \text{ for } \sigma, \tau \in \mathcal{H},$$

and the associated norm $\|\cdot\|_{\mathcal{H}}$. For space $\mathcal{V}$, we consider the inner product by

$$(u, v)_{\mathcal{V}} = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \text{ for } u, v \in \mathcal{V}.$$

and the associated norm $\|\cdot\|_{\mathcal{V}}$, and it is well known that $\mathcal{V}$ is a real Hilbert space with the inner product. From the Sobolev trace theorem [23,24,35], there exists the smallest constant r_0, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant r_0\|\boldsymbol{v}\|_{\mathcal{V}} \text{ for } v \in \mathcal{V}.$$

In order to study the solution to Problem 1, we give some hypotheses on the relevant dates.

$H(\mathcal{A})$: the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:
(a) $\mathcal{A}(\boldsymbol{x}, \varepsilon) = a(\boldsymbol{x})\varepsilon$ for $a.e. x \in \Omega$ and $\varepsilon \in \mathbb{S}^d$;
(b) $a(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x}))$ with $a_{ijkl} \in L^\infty(\Omega)$;
(c) $a_{ijkl}(\boldsymbol{x})\varepsilon_{ij}\varepsilon_{kl} \geqslant m_a \|\varepsilon\|^2_{\mathbb{S}^d}$ for a.e. $x \in \Omega$ and all $\varepsilon = (\varepsilon_{ij})$ with $m_a > 0$.
$H(\mathcal{B})$: the elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:
(a) $\mathcal{B}(\boldsymbol{x}, \varepsilon) = b(\boldsymbol{x})\varepsilon$ for a.e $x \in \Omega$ and $\varepsilon \in \mathbb{S}^d$;
(b) $b(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x}))$ with $b_{ijkl} \in L^\infty(\Omega)$.
$H(j_\nu)$: the function $j_\nu : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ is such that
(a) $j_\nu(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$, and there exists $e_\nu \in L^1(\Gamma_C)$ such that $j_\nu(\cdot, e_v) \in L^1(\Gamma_C)$;
(b) $j_\nu(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e $x \in \Gamma_C$;
(c) there exists $c_{j_\nu} > 0$ such that $|\partial j_\nu(x, \zeta)| \leqslant c_{j_\nu}(1 + |\zeta|)$ for all $\xi \in \mathbb{R}$;
(d) $j^0_\nu(x, r_1; r_2 - r_1) + j^0_\nu(x, r_2; r_1 - r_2) \leqslant \alpha_{j_\nu}|r_1 - r_2|^2$ for $a.e. x \in \Gamma_C$, all $r_1, r_2 \in \mathbb{R}$ with $\alpha_{j_\nu} > 0$.
(e) $s \to j^0_\nu(x, s; \eta)$ is upper semicontinuous for all $\eta \in \mathbb{R}$ and a.e.$x \in \Gamma_C$.
$H(g)$: the function $g : \Gamma_N \times (0, T) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(a) $g(\cdot, \cdot, \xi, r)$ is measurable on $\Gamma_N \times (0, T)$ for all $(\xi, r) \in \mathbb{R}^d \times \mathbb{R}$;
(b) $|g(x, t, \xi_1, r_1) - g(x, t, \xi_2, r_2)| \leqslant L_g(\|\xi_1 - \xi_2\| + |r_1 - r_2|)$ for a.e. $(x, t) \in \Gamma_N \times (0, T)$ and all $(\xi_i, r_i) \in \mathbb{R}^d \times \mathbb{R}, i = 1, 2$ with $L_g > 0$;
(c) there exists $\pi \in L^p[0, T]$ satisying

$$\|g(t, x, z, y)\| \leqslant \pi(t), for\ all\ (t, z, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}, a.e. x \in \Gamma_N,$$

where $p > \frac{1}{\alpha}$.
$H(I)$: $\Theta_j : L^2(\Gamma_N; \mathbb{R}^d) \to L^2(\Gamma_N; \mathbb{R}^d)$ is bounded, and there exists $d_j > 0$ satisfying

$$\|\Theta_j(z_1) - \Theta_j(z_2)\| \leqslant d_j \|z_1 - z_2\|_{L^2(\Gamma_N;\mathbb{R}^d)}, \quad \forall z_1, z_2 \in L^2(\Gamma_N; \mathbb{R}^d)$$

and

$$\sum_{i=1}^{m} d_i \leqslant \frac{1}{2}.$$

$H(f_0)$: the densities of body force $\boldsymbol{f}_0$ satisfies the following condition:

$$\boldsymbol{f}_0 \in C([0, T]; L^2(\Omega, \mathbb{R}^d)).$$

$H(I_1)$ $m_a > c_0\|B\| + c_{j_\nu} r_0^2 \frac{T^\beta}{\beta}$.
It follows from the Green formula that

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}} = \langle \boldsymbol{f}_0(t), \boldsymbol{v} \rangle_H + \int_\Gamma \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma.$$

On the other hand, we have

$$\int_\Gamma \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma = \int_{\Gamma_D} \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma.$$

By (10) and the definition of subgradient (1), we have

$$-\boldsymbol{\sigma}_\nu(t) \boldsymbol{v}_\nu \leqslant j^0_\nu(\boldsymbol{u}_\nu(t); \boldsymbol{v}_\nu).$$

Combined with the Green formula, we infer that

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}} = \langle \boldsymbol{f}_0(t), \boldsymbol{v} \rangle_H + \int_{\Gamma_C} \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t) v \cdot \boldsymbol{v} \, d\Gamma.$$

Then, we have the following variational inequality

$$\langle \mathcal{A}(\varepsilon(\mathfrak{D}^0_\beta(\boldsymbol{u}(t)))) + (\mathcal{B}\varepsilon(\boldsymbol{u}(t))), \varepsilon(\boldsymbol{v})\rangle_{\mathcal{H}} + \int_{\Gamma_C} j^0_\nu(\boldsymbol{u}_\nu(t); \boldsymbol{v}_\nu)\mathrm{d}\Gamma$$
$$\geqslant \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_H + \langle \boldsymbol{f}_N(t), \boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}.$$

Therefore, we get the following variational formulation of Problem 1.

Problem 2. *Find a displacement vector $\boldsymbol{u} \in W^{1,2}(0,T;\mathcal{V})$ and a surface traction density $\boldsymbol{f}_N : \Gamma_N \times [0,T] \to \mathbb{R}^d$ such that*

$$\begin{cases} \langle \mathcal{A}(\varepsilon(\mathfrak{D}^0_\beta(\boldsymbol{u}(t)))) + (\mathcal{B}\varepsilon(\boldsymbol{u}(t))), \varepsilon(\boldsymbol{v})\rangle + \int_{\Gamma_C} j^0_\nu(\boldsymbol{u}_\nu(t); \boldsymbol{v}_\nu)\mathrm{d}\Gamma \\ \geqslant (\boldsymbol{f}_0(t), v)_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{L^2(\Gamma_N;R^d)}, \\ \mathfrak{D}^0_\alpha(\boldsymbol{f}_N(t)) = g(t, \boldsymbol{f}_N(t), \boldsymbol{u}(t)) \ \ on\ \Gamma_N \times [0,T], \\ \wedge \boldsymbol{f}_N(\tau_j) = \Theta_j(\boldsymbol{f}_N(\tau_j^-)), j = 1,2,\cdots,m,\ on\ \Gamma_N \times [0,T], \\ \boldsymbol{f}_N(0) = \boldsymbol{f}^N_0 \ \ on\ \Gamma_N \times [0,T]. \end{cases}$$

5. The Conformable Impulsive Hemivariational Inequality

To prove Problem 2, we need to consider the following conformable impulsive differential problem

$$\begin{cases} \mathfrak{D}^0_\alpha(\boldsymbol{f}_N(t)) = g(t, \boldsymbol{f}_N(t), \boldsymbol{u}(t)),\ t \in (0,T],\ t \neq \tau_j,\ j = 1,2,\cdots,m, \\ \boldsymbol{f}_N(0) = \boldsymbol{f}^0_N, \wedge \boldsymbol{f}_N(\tau_j) = \Theta_j(\boldsymbol{f}_N(\tau_j^-)),\ j = 1,2,\cdots,m \end{cases}$$

According to [36], Lemma 1 and Defination 2, the above problem is equivalent to the following integral equation

$$\boldsymbol{f}_N(t) = \boldsymbol{f}^0_N + \sum_{i=1}^{j} \Theta_i(\boldsymbol{f}_N(\tau_i^-)) + \int_0^t g(t, \boldsymbol{f}_N(s), \boldsymbol{u}(s)) s^{\alpha-1}\mathrm{d}s,\ \ \forall t \in (t_j, t_{j+1}]. \tag{13}$$

Then, the assumption $H(g)$ implies that $\boldsymbol{f}_N(t)$ is well defined. Meanwhile, we assert that $\boldsymbol{f}_N \in C([\tau_j, \tau_{j+1}); L^2(\Gamma_N;\mathbb{R}^d))$. Indeed, for $\kappa > 0$ and $t + \kappa \in [\tau_0, \tau_1)$, it follows from Equation (13) and condition $H(g)$ that

$$\begin{aligned} & \|\boldsymbol{f}_N(t+\kappa) - \boldsymbol{f}_N(t)\|_{C([\tau_0,\tau_1);L^2(\Gamma_N;\mathbb{R}^d))} \\ \leqslant & \left\| \int_t^{t+\kappa} g(s, \boldsymbol{f}_N(s), \boldsymbol{u}(s)) s^{\alpha-1}\mathrm{d}s \right\|_{C([\tau_0,\tau_1);L^2(\Gamma_N;\mathbb{R}^d))} \\ \leqslant & \int_t^{t+\kappa} \pi(s) s^{\alpha-1}\mathrm{d}s \\ \leqslant & \left(\int_t^{t+\kappa} (\pi(s))^p\right)^{1/p} \left(\int_t^{t+\kappa} (s^{\alpha-1})^{\frac{p}{p-1}}\mathrm{d}s\right)^{1-1/p} \\ \leqslant & \left(\int_t^{t+\kappa} (\pi(s))^p\right)^{1/p} \left(\int_t^{t+\kappa} s^{\frac{p\alpha-p}{p-1}}\mathrm{d}s\right)^{1-1/p} \\ = & \left(\int_t^{t+\kappa} (\pi(s))^p\right)^{1/p} \left(\frac{p-1}{p\alpha-1}((t+\kappa)^{(p\alpha-1)/(p-1)} - t^{((p\alpha-1)/(p-1))})\right)^{1-1/p} \\ \longrightarrow & \ 0,\ \ as\ \kappa \to 0. \end{aligned}$$

On the other hand, since $\boldsymbol{f}_N(\tau_1^-) = \boldsymbol{f}_N(\tau_1)$. Then, we have $\boldsymbol{f}_N \in C([\tau_0, \tau_1]; L^2(\Gamma_N;\mathbb{R}^d))$. When $t \in (\tau_1, \tau_2)$ and $t + \kappa \in (\tau_1, \tau_2)$, we have

$$\begin{aligned} & \|\boldsymbol{f}_N(t+\kappa) - \boldsymbol{f}_N(t)\|_{C((\tau_1,\tau_2);L^2(\Gamma_N;\mathbb{R}^d))} \\ \leqslant & \left(\int_t^{t+\kappa} (\pi(s))^p\right)^{1/p} \left(\frac{p-1}{p\alpha-1}((t+\kappa)^{(p\alpha-1)/(p-1)} - t^{(p\alpha-1)/(p-1)})\right)^{1-1/p} \to 0. \end{aligned}$$

Since $\boldsymbol{f}_N(\tau_2^-) = \boldsymbol{f}_N(\tau_2)$. Thus, $\boldsymbol{f}_N \in C((\tau_1, \tau_2]; L^2(\Gamma_N; \mathbb{R}^d))$.

When $t \in (\tau_j, \tau_{j+1})$ and $t + \kappa \in (\tau_j, \tau_{j+1}), j = 1, 2, \cdots m$, we can show that

$$\begin{aligned}
&\|\boldsymbol{f}_N(t+\kappa) - \boldsymbol{f}_N(t)\|_{C((\tau_j,\tau_{j+1});L^2(\Gamma_N;\mathbb{R}^d))} \\
\leqslant \quad &\left(\int_t^{t+\kappa} (\pi(s))^p\right)^{1/p} \left(\frac{p-1}{p\alpha - 1}((t+\kappa)^{(p\alpha-1)/(p-1)} - t^{(p\alpha-1)/(p-1)})\right)^{1-1/p} \to 0.
\end{aligned}$$

Since $\boldsymbol{f}_N(\tau_{j+1}^-) = \boldsymbol{f}_N(\tau_{j+1})$. Thus $\boldsymbol{f}_N \in C((\tau_j, \tau_{j+1}]; L^2(\Gamma_N; \mathbb{R}^d))$.

To study Problem 2, we first consider the following conformable differential hemivariational inequality.

Lemma 2. *For any given $\boldsymbol{f}_N \in C((\tau_j, \tau_{j+1}]; L^2(\Gamma_N; \mathbb{R}^d))$, find $\boldsymbol{u} \in W^{1,2}((\tau_j, \tau_{j+1}]; \mathcal{V})$ such that*

$$\begin{cases} \mathcal{A}(\varepsilon(\mathfrak{D}_\beta^0(\boldsymbol{u}(t)))) + (\mathcal{B}\varepsilon(\boldsymbol{u}(t))) + r_0^*\partial J(r_0\boldsymbol{u}(t)) \ni \boldsymbol{f}_0(t) + \boldsymbol{f}_N(t), \quad t \in (\tau_j, \tau_{j+1}], \\ \boldsymbol{u}(0) = u_0. \end{cases} \tag{14}$$

Here $J(\boldsymbol{u}) = \int_{\Gamma_C} j_\nu(\boldsymbol{x}, u_\nu(t)) \mathrm{d}\Gamma$.

Next, we define the following operators.

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{A}(\varepsilon(\boldsymbol{u})), \varepsilon(\boldsymbol{v})\rangle_H.$$

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{V}^* \times \mathcal{V}} = \langle \mathcal{B}(\varepsilon(\boldsymbol{u})), \varepsilon(\boldsymbol{v})\rangle_H.$$

$$\langle h(t, \boldsymbol{f}_N(t)), \boldsymbol{v}\rangle_{\mathcal{V}^* \times \mathcal{V}} = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_N(t), \boldsymbol{v})_{L^2(\Gamma_N;\mathbb{R}^d)}.$$

According to conditions $H(\mathcal{A})$ and $H(\mathcal{B})$, we imply that the operators A and B satisfy the following conditions.

$H(A) : A \in L(\mathcal{V}, \mathcal{V}^*)$. There exists a constant $m_a > 0$ such that

$$\langle Av, v\rangle \geqslant m_a\|v\|^2 \text{ for all } v \in \mathcal{V}.$$

$H(B) : B \in L(\mathcal{V}, \mathcal{V}^*)$.

Thus, system (14) is equivalent to the following system

$$\begin{cases} A(\mathfrak{D}_\beta^0(\boldsymbol{u}(t))) + B(\boldsymbol{u}(t)) + r_0^*\partial J(r_0\boldsymbol{u}(t)) \ni h(t, \boldsymbol{f}_N(t)), \quad t \in (\tau_j, \tau_{j+1}], \\ \boldsymbol{u}(0) = u_0. \end{cases} \tag{15}$$

According to Remark 1, we have

$$\mathfrak{D}_\beta^{\tau_j}(\boldsymbol{u}(t)) = (t - \tau_j)^{1-\beta}\boldsymbol{u}'(t), t \in (\tau_j, \tau_{j+1}].$$

Let $\kappa(t) = t^{1-\beta}\boldsymbol{u}'(t)$. Then, we infer that

$$\boldsymbol{u}(t) = \boldsymbol{u}(0) + \int_0^t t^{\beta-1}\kappa(s)\mathrm{d}s. \tag{16}$$

Then, system (15) is equivalent to the following problem.

Problem 3. *Find $\kappa \in L^1(0, \tau_1; \mathcal{V})$ such that*

$$\begin{aligned}
&A(\kappa(t))) + B(\boldsymbol{u}(0) + \int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s) + r_0^*\partial J(r_0(\boldsymbol{u}(0) \\
&+ \int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s)) \ni h(t, \boldsymbol{f}_N(t)), t \in (\tau_j, \tau_{j+1}].
\end{aligned}$$

Let $h_\iota^k = \frac{1}{\iota}\int_{t_{\iota-1}^k}^{t_\iota^k} h(s)\mathrm{d}s$ and $t_k = k\iota$. Next, we discuss the following Rothe problem.

Problem 4. *Find* $\{\kappa_\iota^k\}_{k=1}^N \subset \mathcal{V}$ *such that* $\boldsymbol{u}_0 = \boldsymbol{u}(0)$ *and*

$$A(\kappa_\iota^k) + B(\beta_\iota^k) + r_0^*\eta_\iota^k = h_\iota^k, \tag{17}$$

with $\eta_\iota^k \in \partial J(r_0\beta_\iota^k)$. *Here,*

$$\beta_\iota^k = \kappa_0 + \frac{\iota^\beta}{\beta}\sum_{j=1}^{k} \kappa_\iota^j[(k-j+1)^\beta - (k-j)^\beta]. \tag{18}$$

Lemma 3. *If the conditions* $H(A), H(B), H(j_\nu), H(h)$ *hold. Then there exists* $\kappa_0 > 0$ *such that Problem 4 has at least one solution for all* $\iota \in (0, \kappa_0)$.

Proof. Given $\kappa_\iota^0, \kappa_\iota^1, ..., \kappa_\iota^{n-1}$, we will show that there exist $\kappa_\iota^n \in \mathcal{X}_1, \eta_\iota^n \in \mathcal{V}^*$ such that Equations (17) and (18) hold. We claim that operator

$$\mathcal{V} \ni v \to Av + B(v_0 + c_0 v) + r_0^*\partial J(r_0(v_0 + c_0 v)) \subset \mathcal{V}^*$$

is pseudomonotone. Where

$$v_0 = \boldsymbol{u}_0 + \frac{\iota^\beta}{\beta}\sum_{j=1}^{n-1} \kappa_\iota^j[(n-j+1)^{\alpha_0} - (n-j)^\beta] \text{ and } c_0 = \frac{\iota^\beta}{\beta}.$$

It follows from conditions $H(A)$ and $H(B)$ that

$$\langle A(v) + B(\boldsymbol{u}_0 + c_0 v) - Az - B(\boldsymbol{u}_0 + c_0 z), v - z\rangle_{\mathcal{V}^*\times\mathcal{V}} \geqslant (m_a - c_0\|B\|)\|v - z\|_{\mathcal{V}}^2,$$

where $v, z \in \mathcal{V}_1$. By $H(A)$ and $H(B)$, operator

$$\mathcal{V} \ni v = Av + B(\boldsymbol{u}_0 + c_0 v) \in \mathcal{V}^*$$

is bounded and continuous. Thus, we infer that the above operator is pseudomonotone. On the other hand, It follows from condition $H(j)$ that $v \to r_0^*\partial J(r_0 v)$ is pseudomonotone. Thus, by (Corollary 7, [24]), we have $v \to Av + B(u_0 + c_0 v) + \iota^\beta r_0^*\partial J(r_0(u_0 + c_0 v))$ as a pseudomonotone. Next, we will show that the operator $L : v \to Av + B(u_0 + c_0 v) + \iota^\beta r_0^*\partial J(r_0(\boldsymbol{u}_0 + c_0 v))$ is coercive. Indeed, by conditions $H(A), H(B)$ and $H(j)$, we have

$$\begin{aligned}&\langle Av + B(\boldsymbol{u}_0 + c_0 v) + r_0^*\partial J(r_0(\boldsymbol{u}_0 + c_0 v)), v\rangle \geqslant (m_a - c_0\|B\| - c_0 c_{j_\nu} r_0^2)\|v\|^2 \\ &-\|\boldsymbol{u}_0\|\|B\|\|v\| - c_{j_\nu}(1 + \|\boldsymbol{u}_0\|)\|v\|.\end{aligned}$$

By condition $H(I_1)$, we have $m_a > c_0\|B\| + c_0 c_{j_\nu} r_0^2$. Taking $\iota_0 = \left(\frac{\beta\|B\| + c_{j_\nu}\beta r_0^2}{m_a}\right)^{1/\beta}$, we have $m_a > c_0\|B\| + c_0 c_{j_\nu} r_0^2$ for all $\iota \in (0, \iota_0)$. Meanwhile, the operator L is coercive. By (Theorem 1.3.70, [37]), system (17) has at least one solution for all $\iota \in (0, \iota_0)$. The proof is complete. □

Lemma 4. *Assume that conditions* $H(A), H(B), H(h)$ *hold. Then, we have*

$$\max_{j=1,2,...,N} \|\kappa_\iota^j\|_{\mathcal{V}} \leqslant c;$$
$$\max_{j=1,2,...,N} \|\beta_\iota^j\|_{\mathcal{V}} \leqslant c;$$
$$\max_{j=1,2,...,N} \|\eta_\iota^j\|_{\mathcal{V}^*} \leqslant c$$

where $\eta_\iota^j \in \partial J(r_0\beta_\iota^j)$.

Proof. According to Lemma 3, Equation (17) and taking $k = n$, we have

$$\langle A(\kappa_\iota^n), \kappa_\iota^n\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle B(\beta_\iota^n), \kappa_\iota^n\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle r_0^*\eta_\iota^n, \kappa_\iota^n\rangle_{\mathcal{V}^*\times\mathcal{V}} = \langle h_\iota^n, \kappa_\iota^n\rangle_{\mathcal{V}^*\times\mathcal{V}}.$$

Combined with conditions $H(A), H(B), H(j)$ and Equation (18), we have

$$\begin{aligned}&\|h_\iota^n\|_{\mathcal{V}^*} + \frac{\iota^\beta(\|B\| + c_{j_\nu}\|r_0\|^2)}{\beta}\sum_{j=1}^{n-1}\kappa_\iota^j[(n-j+1)^\beta - (n-j)^\beta]\\ &+\|B(u_0)\|_{\mathcal{V}} + c_{j_\nu}\|r_0\| + c_{j_\nu}\|r_0\|^2\|u_0\| \geqslant \left(m_a - \frac{\iota^\beta(\|B\| + c_{j_\nu}\|r_0\|^2)}{\beta}\right)\|\kappa_\iota^n\|_{\mathcal{V}}\end{aligned}$$

Taking $\iota_0 = \left(\frac{\beta\|B\| + c_{j_\nu}\beta r_0^2}{2m_a}\right)^{1/\beta}$, we have

$$m_a - \frac{\iota^\beta(\|B\| + c_{j_\nu}\|r_0\|^2)}{\beta} \geqslant \frac{m_a}{2}.$$

By condition $H(h)$, there exists constant $\mathcal{M} > 0$, for all $\iota \in (0, \iota_0)$ such that

$$\|h_\iota^n\|_{\mathcal{V}*} \leqslant \mathcal{M}.$$

Then, we have

$$\begin{aligned}&\|\kappa_\iota^n\|_{\mathcal{V}} \leqslant \frac{2(\mathcal{M} + \|B(\boldsymbol{u}_0)\|_{\mathcal{V}^*} + c_{j_\nu}\|r_0\| + c_{j_\nu}\|r_0\|^2\|\boldsymbol{u}_0\|)}{m_a}\\ &exp\left(\frac{2\iota^\beta(\|B\| + c_{j_\nu}\|r_0\|^2)}{m_a\beta}\sum_{j=1}^{n-1}[(n-j+1)^{\alpha_0} - (n-j)^{\alpha_0}]\right)\\ &\leqslant c_{\kappa_1} := \frac{2(\mathcal{M} + \|B(\boldsymbol{u}_0)\|_{\mathcal{V}^*} + c_{j_\nu}\|r_0\| + c_{j_\nu}\|r_0\|^2\|\boldsymbol{u}_0\|)}{m_a}exp\left(\frac{2T^\beta(\|B\| + c_{j_\nu}\|r_0\|^2)}{m_a\beta}]\right),\end{aligned}$$

By Equation (18) and condition $H(j)$, we can easily obtain

$$\|\beta_\iota^n\|_{\mathcal{V}} \leqslant \mathcal{M} \text{ and } \|\eta_\iota^n\|_{\mathcal{V}^*} \leqslant \mathcal{M}.$$

Thus, we complete the proof of the lemma. □

Theorem 1. *If hypotheses* $H(A), H(B), H(j), H(h)$ *hold. Let* $\{\iota_k\}$ *be a sequence with* $\iota_k \to 0$ *as* $k \to \infty$. *Then we have*

$$\kappa_{\iota_k} \to \kappa \text{ weakly in } L^{1/\mu}(0, \tau_1; \mathcal{V}); \tag{19}$$
$$\eta_{\iota_k} \to \eta \text{ weakly in } \mathcal{V}^*, \tag{20}$$

where $0 < \mu < \beta$ *and* $(\kappa, \eta) \in L^{1/\mu}(0, \tau_1; \mathcal{V}) \times \mathcal{V}^*)$ *is a solution to Problem 3.*

Proof. By Lemma 4, there exists a constant $\mathcal{M} > 0$ such that $\|\kappa_{\iota_n}\|_{L^{1/\mu}(0,\tau_1;\mathcal{V})} \leqslant \mathcal{M}$. We claim that there exists $\kappa \in L^{1/\mu}(0, \tau_1; \mathcal{V})$ such that

$$\int_0^t s^{\beta-1}\kappa_{\iota_k}(s)ds \to \int_0^t s^{\beta-1}\kappa(s)ds \text{ weakily in } \mathcal{V}.$$

Then we have

$$\begin{aligned} & \int_0^t \langle s^{\beta-1}(\kappa_{\iota_k}(s)-\kappa(s)), v\rangle_{\mathcal{V}\times\mathcal{V}^*}\mathrm{d}s \\ = & \int_0^t \langle (\kappa_{\iota_k}(s)-\kappa(s)), s^{\beta-1}v\rangle_{\mathcal{V}\times\mathcal{V}^*}\mathrm{d}s \\ = & \int_0^t \langle (\kappa_{\iota_k}(s)-\kappa(s)), s^{\beta-1}\chi_{[0,t]}v\rangle_{\mathcal{V}\times\mathcal{V}^*}\mathrm{d}s, \end{aligned}$$

It follows from $\frac{(\beta-1)}{\mu^1} > -1, \mu+\mu^1 = 1$ and $\mu < \beta$ that $s^{\beta-1}\chi_{[0,t]}v \in L^{1/\mu^1}(0,T;\mathcal{V}^*)$ and $\int_0^t (s^{\beta-1}\chi_{[0,t]}v)^{1/\mu^1}\mathrm{d}s < \infty$. Thus, we complete the proof of the assertion. By (Theorem 16, [24]), we infer that

$$\beta_{\iota_k} \to \int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + u_0 \text{ weakly in } \mathcal{V}, \text{as } \iota_k \to 0. \tag{21}$$

By Lemma 4, the sequence $\{\eta_{\iota_k}\}$ is bounded in $\mathcal{V}^*$. There is subsequence again $\{\eta_{\iota_k}\}$ such that $\eta_{\iota_k} \to \eta$ weakly in $\mathcal{V}^*$. On the other hand, we have

$$r_0\beta_{\iota_k} \to r_0(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0) \text{ strongly in } \mathcal{V}, \text{as } \iota_k \to 0.$$

It follows from condition $H(j_\mathcal{V})(e)$ that

$$\langle \eta, z\rangle_{\mathcal{V}^*\times\mathcal{V}} = \limsup_{\iota_k\to\infty}\langle \eta_{\iota_k}, z\rangle_{\mathcal{V}^*\times\mathcal{V}} \leqslant \limsup_{\iota_k\to\infty} J^0(r_0\beta_{\iota_k};z) \leqslant J^0(r_0(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0);z).$$

Thus, we infer that

$$\eta \in \partial J\left(r_0(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0)\right).$$

As $\iota_k \to 0$, we have

$$\begin{aligned} \langle A(\kappa_{\iota_k}), \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}} &\to \langle A(\kappa), \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}}; \\ \langle B(\kappa_{\iota_k}), \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}} &\to \langle B(\int_0^t t^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0), \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}}; \\ \langle \eta_{\iota_k}, \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}} &\to \langle \eta, \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}}; \\ \langle h_{\iota_k}, \varpi\rangle_{\mathcal{V}^*\times\mathcal{V}} &\to \langle h, \varpi\rangle_{\mathcal{V}\times\mathcal{V}}; \end{aligned}$$

where $\eta_{\iota_k} \in \partial J((r_0\beta_{\iota_k}))$ and $\eta \in \partial J(r_0(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0))$. Then, we imply that

$$\begin{aligned} 0 \quad \leqslant \quad & \limsup_{\iota_k\to 0}\langle A(\kappa_{\iota_k}), v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \limsup_{\iota_k\to 0}\langle B(\beta_{\iota_k}), v\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & + \limsup_{\iota_k\to 0}\langle r_0^*\eta_{\tau_k}, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \liminf_{\iota_k\to 0}\langle h_{\iota_k}, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ = \quad & \langle A(\kappa), v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle B(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0), v\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ & + \langle r_0^*\eta, v\rangle_{\mathcal{V}^*\times\mathcal{V}} - \langle h, v\rangle_{\mathcal{V}^*\times\mathcal{V}}, \end{aligned}$$

where $\eta \in r_0^*\partial J(r_0(\int_0^t s^{\beta-1}\kappa(s)\mathrm{d}s + \boldsymbol{u}_0))$. Thus, $(\kappa, \eta) \in (L^{1/\mu}(0,\tau_1;\mathcal{V}) \times \mathcal{V}^*)$ is a solution of Problem 3. $\square$

When the contact surface Γ_N is impacted at τ_j, the corresponding displacement $\boldsymbol{u}(\tau_j)$ is given. Similar to the proof of Theorem 1, for any given $\boldsymbol{f}_N \in C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))$, there exists $\boldsymbol{u} \in W^{1,2}((\tau_j,\tau_{j+1}];\mathcal{V})$ such that (14) hold.

Theorem 2. *If* $(\boldsymbol{u}, \boldsymbol{f}_N) \in PC(\mathbb{J}; \mathcal{V}) \times PC(\mathbb{J}; L^2(\Gamma_N; \mathbb{R}^d))$ *satisfies the following system*

$$(M(\boldsymbol{f}_N))(t) = \boldsymbol{f}_N^0 + \sum_{i=1}^{j} \Theta_i(\boldsymbol{f}_N(\tau_i^-)) + \int_0^t g(t, \boldsymbol{f}_N(s), \boldsymbol{u}(s)) s^{\alpha-1} \mathrm{d}s, \ \ \forall t \in (t_j, t_{j+1}], \tag{22}$$

and

$$\begin{cases} A(\mathfrak{D}_\beta^0(\boldsymbol{u}(t))) + B(\boldsymbol{u}(t)) + r_0^* \partial J(r_0 \boldsymbol{u}(t)) \ni h(t, \boldsymbol{f}_N(t)), & t \in (\tau_j, \tau_{j+1}], \\ \boldsymbol{u}(0) = u_0. \end{cases} \tag{23}$$

where $\alpha, \beta \in (0,1)$. *Then* $(\boldsymbol{u}, \boldsymbol{f}_N) \in PC(\mathbb{J}; \mathcal{V}) \times PC(\mathbb{J}; L^2(\Gamma_N; \mathbb{R}^d))$ *is a solution to Problem 2.*

Proof. By Theorem 1, for a given $\boldsymbol{f}_N \in C((\tau_j, \tau_{j+1}]; L^2(\Gamma_N; \mathbb{R}^d))$, we infer that system (23) is satisfied. On the other hand, for any given $\boldsymbol{u} \in W^{1,2}((\tau_j, \tau_{j+1}]; \mathcal{V})$, according to the condition $H(g)$ and Equation (22), we have

$$\begin{aligned} & \|(M\boldsymbol{f}_{N1}(t))(t) - (M\boldsymbol{f}_{N2}(t))(t)\|^2_{C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))} \\ = & \left\| \int_0^t (g(s, \boldsymbol{f}_{N1}(s), \boldsymbol{u}_1(s)) - g(s, \boldsymbol{f}_{N2}(s), \boldsymbol{u}_2(s)) s^{\alpha-1} \mathrm{d}s \right. \\ & \left. + \sum_{i=1}^{j} \Theta_i(\boldsymbol{f}_{N1}(\tau_i^-)) - \sum_{i=1}^{j} \Theta_i(\boldsymbol{f}_{N2}(\tau_i^-)) \right\|^2_{C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))} \\ \leqslant & L_g \frac{4T^\beta}{\beta} \int_0^t (\|\boldsymbol{f}_{N1}(s) - \boldsymbol{f}_{N2}(s)\|^2_{C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))} \\ & + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2_{C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))}) \mathrm{d}s \\ & + 2 \sum_{i=1}^{j} d_i \|\boldsymbol{f}_{N1}(\tau_i^-) - \boldsymbol{f}_{N2}(\tau_i^-)\|^2_{C((\tau_j,\tau_{j+1}];L^2(\Gamma_N;\mathbb{R}^d))}. \end{aligned} \tag{24}$$

Obviously, by inequality (24) and Gronwall's inequality, we infer that

$$\begin{aligned} & (1 - 2\sum_{i=1}^{m} d_i) \|M\boldsymbol{f}_{N1} - M\boldsymbol{f}_{N2}\|^2_{PC(\mathbb{J};L^2(\Gamma_N;\mathbb{R}^d))} \\ \leqslant \ & L_g \frac{4T^\beta}{\beta} \int_0^t (\|\boldsymbol{f}_{N1}(s) - \boldsymbol{f}_{N2}(s)\|^2 + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2) \mathrm{d}s. \end{aligned}$$

and

$$\|\boldsymbol{f}_{N1} - \boldsymbol{f}_{N2}\| \leqslant \left(L_g \frac{4T^\beta}{\beta} \right)^2 \frac{1}{(1 - 2\sum_{i=1}^{m} d_i)} T \int_0^t (\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s))\|^2) \mathrm{d}s. \tag{25}$$

By inequality (25) and [36], $H(I)$, we know the uniqueness of the solution to system (22) and the dependence of this solution on $\boldsymbol{u}_i$. Thus, by Theorem 1 and (16), we complete the proof of Theorem 2. □

6. Conclusions

In this paper, we study a class of conformable frictionless contact problems with the surface traction driven by the conformable impulsive differential equation. Under quite general assumptions on the data and employing a subjectivity theorem for pseudomonotone operators and the Rothe method, we prove that the system has at least one solution. In the future, we plan to apply the theoretical results established in the current paper to conformable frictional contact problems.

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