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On r-Ideals and m-k-Ideals in BN-Algebras

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Abstract: A BN-algebra is a non-empty set X with a binary operation "$*$" and a constant 0 that satisfies the following axioms: $(B1)$ $x * x = 0$, $(B2)$ $x * 0 = x$, and (BN) $(x * y) * z = (0 * z) * (y * x)$ for all $x,\ y,\ z\ \in X$. A non-empty subset I of X is called an ideal in BN-algebra X if it satisfies $0 \in X$ and if $y \in I$ and $x * y \in I$, then $x \in I$ for all $x, y \in X$. In this paper, we define several new ideal types in BN-algebras, namely, r-ideal, k-ideal, and m-k-ideal. Furthermore, some of their properties are constructed. Then, the relationships between ideals in BN-algebra with r-ideal, k-ideal, and m-k-ideal properties are investigated. Finally, the concept of r-ideal homomorphisms is discussed in BN-algebra.

Keywords: ideal; r-ideal; k-ideal; m-k-ideal; BN-algebra; homomorphism

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1. Introduction

J. Neggers and H.S. Kim introduced the B-algebra, which is a non-empty set X with a binary operation $*$ and a constant 0, denoted by $(X;\ *,\ 0)$, that fulfills the axioms $(B1)$ $x * x = 0$, $(B2)$ $x * 0 = x$, and $(B3)$ $(x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$ (see [1]). H.S. Kim and H.G. Park discuss a special form of B-algebra, called 0-commutative B-algebra, which also satisfies a further axiom, namely, $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$ (see [2]). Furthermore, C. B. Kim constructed the related BN-algebra, which is an algebra $(X;\ *,\ 0)$ that satisfies axioms $(B1)$ and $(B2)$, as well as (BN) $(x * y) * z = (0 * z) * (y * x)$ for all $x, y, z \in X$ (see [3]). For example, let $X = \{0,\ 1,\ 2\}$ be a set with a binary operation "$*$" on X as shown in Table 1.

Table 1. Cayley's table for $(X;\ *, 0)$.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then, $(X;\ *, 0)$ is a BN-algebra.

A BN-algebra $(X;\ *, 0)$ that satisfies $(x * y) * z = x * (z * y)$ for all $x,\ y, z\ \in X$ is said to be a BN-algebra with condition D. A. Walendziak introduced another special form of BN-algebra, namely, a BN_1-algebra, which is a BN-algebra $(X;\ *, 0)$ that satisfies $x = (x * y) * y$ for all $x,\ y\ \in X$ (see [4]). Furthermore, the new QM-BZ-algebras were proposed by Y. Du and X. Zhang (see [5]). The relationship between B-algebra and BN-algebra is that every 0-commutative B-algebra is a BN-algebra, and a BN-algebra with condition D is a B-algebra. The relationship between a BN-algebra and other algebras can be seen in Figure 1.

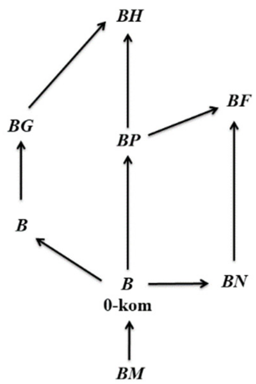

Figure 1. The relationship of *BN*-algebra with other algebras.

In 2017, E. Fitria et al. discussed the concept of prime ideals in *B*-algebras, which produces a definition and various prime ideals and their properties in *B*-algebras, including that a non-empty subset I is said to be ideal in a *B*-algebra X if it satisfies $0 \in X$ and if $y \in I$, $x * y \in I$ applies to $x \in I$ for all $x, y \in X$ (see [6]). Moreover, I is called a prime ideal of X if it satisfies $A \cap B \subseteq I$; then, $A \subseteq I$ or $B \subseteq I$ for all A and B are two ideals in X. The concept of the ideal was also discussed in *BN*-algebras by G. Dymek and A. Walendziak, and the resulting definition of an ideal in *BN*-algebras is the same as in *B*-algebras, but their properties differ (see [7]).

In [3], the definition of a homomorphism in *BN*-algebras was given: for two *BN*-algebras $(X;\, *,\, 0)$ and $(Y;\, *,\, 0)$, a mapping $\varphi\, : X\ \to Y$ is called a homomorphism of X to Y if it satisfies $\varphi(x * y) = \varphi(x) * \varphi(y)$ for all $x, y \in X$. In [7], G. Dymek and A. Walendziak stated that the kernel of φ is an ideal of X. In addition, G. Dymek and A. Walendziak also investigated the kernel by letting X and Y be a *BN*-algebra and a *BM*-algebra, respectively, such that the kernel φ is a normal ideal. The concepts of ideals are also discussed in [8].

In 2020, S. Gemawati et al. discussed the concept of a complete ideal (briefly, *c*-ideal) of *BN*-algebra and introduced the concept of an *n*-ideal in *BN*-algebra (see [9]). From this research, several interesting properties were obtained that showed the relationship between an ideal, *c*-ideal, and *n*-ideal, as well as the relationship between a subalgebra and a normal with a *c*-ideal and *n*-ideal in *BN*-algebras. The research also discussed the concepts of a *c*-ideal and *n*-ideal in a homomorphism of *BN*-algebra and *BM*-algebra. In 2016, M. A. Erbay et al. defined the concept of an *r*-ideal in commutative semigroups (see [10]). Furthermore, M. M. K. Rao defined the concept of an *r*-ideal and *m*-*k*-ideal in an incline (see [11]). An incline is a non-empty set M with two binary operations, addition (+) and multiplication $(\cdot)$, satisfying certain axioms. For example, let $M = [0, 1]$ be subject to a binary operation "+" defined by $a + b = \max\{a, b\}$ for all $a, b \in M$, and multiplication defined by $xy = \min\{x, y\}$ for all $x, y \in M$. Then, M is an incline. However, interesting properties were obtained from the concepts of an *r*-ideal and *m*-*k*-ideal in an incline, such as a relationship between an ideal, *r*-ideal, and *m*-*k*-ideal in an incline, as well as properties of these ideals in a homomorphism of incline.

Based on this description, the concepts of an *r*-ideal, a *k*-ideal, and a *m*-*k*-ideal in *BN*-algebras are discussed and their properties determined, followed by the properties of homomorphism in *BN*-algebras.

2. Preliminaries

In this section, some definitions that are needed to construct the main results of the study are given. We start with some definitions and theories about *B*-algebra and *BN*-algebra. Then, we give the concepts of an *r*-ideal in a semigroup, and a *k*-ideal and *m*-*k*-ideal in an incline, as discussed in [1–4,6,10,11].

Definition 1 ([1])**.** *A B-algebra is a non-empty set X with a constant0 and a binary operation "$*$" that satisfies the following axioms for all x, y, z* $\in$ *X:*

$(B1)$ $x * x = 0$;
$(B2)$ $x * 0 = x$;
$(B3)$ $(x * y) * z = x * (z * (0 * y))$.

Definition 2 ([3])**.** *A BN-algebra is a non-empty set X with a constant0 and a binary operation "$*$" that satisfies axioms* $(B1)$ and $(B2)$*, as well as* (BN) $(x * y) * z = (0 * z) * (y * x)$*, for all x, y, $z \in X$.*

Theorem 1 ([3])**.** *Let $(X;\ *,\ 0)$ be a BN-algebra, then for all $x, y, z \in X$:*

(i) $0 * (0 * x) = x$;
(ii) $y * x = (0 * x) * (0 * y)$
(iii) $(0 * x) * y = (0 * y) * x$;
(iv) *If* $x * y = 0$, *then* $y * x = 0$;
(v) *If* $0 * x = 0 * y$, *then* $x = y$;
(vi) $(x * z) * (y * z) = (z * y) * (z * x)$.

Let $(X;\ *,\ 0)$ be an algebra. A non-empty set S is called a subalgebra or BN-subalgebra of X if it satisfies $x * y\ \in S$ for all $x, y \in S$, and a non-empty set N of X is called normal in X if it satisfies $(x * a) * (y * b) \in N$ for all $x * y,\ a * b \in N$. Let $(X;\ *,\ 0)$ and $(Y;\ *, 0)$ be BN-algebras. A map $\varphi : X\ \to Y$ is called a homomorphism of X to Y if it satisfies $\varphi(x * y) = \varphi(x) * \varphi(y)$ for all $x, y \in X$. A homomorphism of X to itself is called an endomorphism.

Definition 3 ([7])**.** A non-empty subset I of BN-algebra X is called an ideal of X if satisfies

(i) $0 \in I$;
(ii) $x * y \in I$ *and* $y \in I$ *implies* $x \in I$, *for all* $x, y \in X$.

An ideal I of a BN-algebra X is called a closed ideal if $a * b \in I$ for all $a, b \in I$. In the following, some properties of ideals in BN-algebra are as given in [7].

Proposition 1. *If I is a normal ideal in BN-algebra A, then I is a subalgebra of A.*

Proposition 2. *Let A be a BN-algebra and $S \subseteq A$. S is a normal subalgebra of A if and only if S is a normal ideal.*

Definition 4 ([3])**.** *An algebra $(X;\ *, 0)$ is called* 0*-commutative if, for all $x,\ y \in X$,*

$$x * (0 * y) = y * (0 * x).$$

A semigroup is a non-empty set G, together with an associative binary operation, we can write $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$. An ideal of semigroup G is a subset A of G such that $A \cdot G$ and $G \cdot A$ is contained in G. Any element x of G is a zero divisor if $ann(x) = \{g \in G : g \cdot x = 0\} \neq 0$.

Definition 5 ([10])**.** *Let G be a semigroup. A proper ideal A of G is said to be an r-ideal of G if when $x \cdot y \in A$ with $ann(x) = 0$, then $y \in A$ for all$x$, $y \in G$.*

Definition 6 ([11])**.** *An incline is a non-empty set M with two binary operations, namely, addition (+) and multiplication (·), satisfying the following axioms for all x, y, $z \in X$:*

(i) $x + y = y + x$;
(ii) $x + x = x$;
(iii) $x + xy = x$;
(iv) $y + xy = y$;
(v) $x + (y + z) = (x + y) + z$;

(*vi*) $x(yz) = (xy)z$;
(*vii*) $x(y+z) = xy + xz$;
(*viii*) $(x+y)z = xz + yz$;
(*ix*) $x1 = 1x = x$;
(*x*) $x + 0 = 0 + x = x$.

A subincline of an incline M is a non-empty subset I of M that is closed under addition and multiplication. Note that $x \leq y$ iff $x + y = y$ for all $x, y \in M$.

Definition 7 ([11]). *Let M be an incline and I a subincline of M. I is called an ideal of M if when $x \in I$, $y \in M$, and $y \leq x$, then $y \in I$.*

Definition 8 ([11]). *Let M be an incline and I a subincline of M. I is said to be a left r-ideal of M if $MI \subseteq I$ and I is said to be a right r-ideal of M if $IM \subseteq I$. If I is a left and right r-ideal of M, then I is called an r-ideal of M.*

Definition 9 ([11]). *Let M be an incline and I be a subincline of M. I is said to be a k-ideal of M if when $x + y \in I$ and $y \in I$, then $x \in I$.*

Definition 10 ([11]). *Let M be an incline and I be an ideal of M. I is said to be an m-k-ideal of M if $xy \in I$, $x \in I$, and $1 \neq y \in M$, then $y \in I$.*

3. *r*-Ideal in *BN*-Algebra

In this section, the main results of the study are given. Starting from the definition of an *r*-ideal in *BN*-algebras, which was constructed based on the concept of *r*-ideal in a semigroup. Then, some properties of *r*-ideals in *BN*-algebras are investigated.

Definition 11. *Let $(X; *, 0)$ be a BN-algebra and I be a proper ideal of X. I is called an r-ideal of X if when $x * y \in I$ and $0 * x = 0$, then $y \in I$ for all $x, y \in X$.*

Example 1. *Let $A = \{0, 1, 2, 3\}$ be a set. Define a binary operation "$*$" with the* Table 2.

Table 2. Cayley's table for $(A; *, 0)$.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then, $(A; *, 0)$ is a *BN*-algebra. We obtain that $I_1 = \{0, 2\}$, $I_2 = \{0, 3\}$, and $I_3 = \{0, 2, 3\}$ are *r*-ideals in A.

In the following, the properties of an *r*-ideal in *BN*-algebras are given.

Theorem 2. *Let $(X; *, 0)$ be a BN-algebra. If I is a closed ideal of X, then I is an r-ideal of X.*

Proof. Since I is an ideal of X, then $0 \in I$; furthermore, if $y \in I$ and $x * y \in I$, then $x \in I$ for all $x, y \in X$. Let $x * y \in I$ and $0 * x = 0$ for all $x, y \in X$. Since I is closed, if we can prove that $x \in I$, then it shows that $y \in I$. By Theorem 1 (ii) and Axiom B2, we obtain

$$x * y = (0 * y) * (0 * x) = (0 * y) * 0 = 0 * y \tag{1}$$

Furthermore, by (1), Theorem 1 (i), and by all axioms of *BN*-algebra, we obtain

$$y * x = (y * x) * 0 = (0 * 0) * (x * y) = 0 * (0 * y) = y \tag{2}$$

By (1) and (2), we obtain $x = 0 \in I$. Thus, we obtain $y \in I$. Therefore, I is an r-ideal of X. □

The converse of Theorem 2 does not hold in general. In Example 1, I_1 and I_2 are two closed ideals in A, and thus, I_1 and I_2 are clearly r-ideals. Meanwhile, $I_3 = \{0,\ 2,\ 3\}$ is an ideal in A, but it is not a closed ideal. However, I_3 is an r-ideal in A. It should be noted that not all ideals are r-ideals. To be clear, consider the following example.

Example 2. *Let $X = (\mathbb{Z}; -,\ 0)$ be a set of integers $\mathbb{Z}$ with a subtraction operation. Then, X is a BN-algebra. Let subset $\mathbb{Z}^+$ of X be positive integers, then $I = \mathbb{Z}^+ \cup \{0\}$ is an ideal of X, but I is not a closed ideal and it is not an r-ideal of X.*

Theorem 3. *Let $(X;\ *,\ 0)$ be a BN-algebra. If I is a normal ideal of X, then I is a normal r-ideal of X.*

Proof. Since I is a normal ideal of X, then, by Proposition 1, we have that I is a *BN*-subalgebra of X, which for all $x,\ y \in I$, $x * y \in I$ implies that I is closed. Furthermore, by Theorem 2, we obtain that I is an r-ideal of X. Since I is normal, then I is a normal r-ideal of X. □

Theorem 4. *Let $(X;\ *,\ 0)$ be a BN-algebra and f be an endomorphism of X. If I is an r-ideal of X, then $f(I)$ is an r-ideal of X.*

Proof. Let I be an r-ideal of X, *then* clearly $I \subset X$ and I is a proper ideal of X such that $0 \in I$ and $f(I) \subset X$. Since f is an endomorphism of X and by Axiom *B1*, for all $x \in I$, we obtain

$$f(0) = f(x * x) = f(x) * f(x) = 0 \in I.$$

Let $f(y) \in f(I)$ and $f(x * y) \in f(I)$. Since I is an ideal of X, then $x \in I$; consequently, $f(x) \in f(I)$. Thus, $f(I)$ is an ideal of X. Let $f(x * y) \in f(I)$ and $0 * f(x) = 0$. Since I is an r-ideal of X, then $y \in I$ implies $f(y) \in f(I)$. Therefore, $f(I)$ is an r-ideal of X. □

The converse of Theorem 4 does hold in general.

Corollary 1. *Let $(X;\ *,\ 0)$ be a BN-algebra and$f$ be an endomorphism of X. If I is a closed r-ideal of X, then $f(I)$ is a closed r-ideal of X.*

Proof. Follows directly from Theorem 4. □

Example 3. *Let $A = \{0,\ 1,\ 2,\ 3\}$ be aBN-algebra in Example 1. Define a map $f : A \to A$ by*

$$f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x = 1 \\ 3 \text{ if } x = 2 \\ 2 \text{ if } x = 3 \end{cases}$$

Then, f is an endomorphism. By Example 1, we obtain that $I_1 = \{0, 2\}$, $I_2 = \{0, 3\}$, and $I_3 = \{0,\ 2,\ 3\}$ are r-ideals in A. It easy to check that $f(I_1) = \{0, 3\}$ and $f(I_2) = \{0, 2\}$ are two closed r-ideals of A. However, $f(I_3) = \{0, 2, 3\}$ is an r-ideal of A, but it is not closed.

4. *m*-*k*-Ideals in *BN*-Algebras

This section gives the main results of the study. We start by defining the concepts of k-ideal and m-k-ideal in a *BN*-algebra, which is constructed based on the concept of a

k-ideal and m-k-ideal in an incline. The properties of k-ideals and m-k-ideals in a BN-algebra are given.

Definition 12. *Let* $(X;\, *,\, 0)$*be a BN-algebra and* I *be a BN-subalgebra of* X. I *is called a k-ideal in* X *if when* $y \in I,\ \ x \in X$, *and* $x * y \in I$, *then* $x \in I$.

Example 4. *Let* $B = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ *be a set. Define a binary operation "* $*$*" with the* Table 3.

Table 3. Cayley's table for $(B;\, *, 0)$.

*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Then $(B;\, *, 0)$ is a BN-algebra. It is easy to check that $I_1 = \{0, 1\}$, $I_2 = \{0, 2\}$, $I_3 = \{0, 3\}$, $I_4 = \{0, 4\}$, $I_5 = \{0, 5\}$, $I_6 = \{0, 6\}$, $I_7 = \{0, 7\}$, and $I_8 = \{0,\ 1,\ 2,\ 3\}$ are closed ideals in B and also BN-subalgebras in B. Thus, we can prove that they are k-ideals in B.

Some properties of a k-ideal in BN-algebras are given.

Theorem 5. *Let* $(X;\, *,\, 0)$ *be a BN-algebra. If* I *is a closed ideal of* X, *then* I *is a k-ideal of* X.

Proof. Let $(X;\, *,\, 0)$ be a BN-algebra. Let I be a closed ideal of X. Then, I is a BN-subalgebra of X, and if $y \in I$, $x \in X$, and $x * y \in I$, then $x \in I$. Therefore, I is a k-ideal of X. □

Theorem 6. *Let* $(X;\, *,\, 0)$ *be a BN-algebra. If* I *is a k-ideal of* X, *then* I *is a closed ideal of* X.

Proof. Let $(X;\, *,\, 0)$ be a BN-algebra. Since I is a k-ideal of X, then I is a BN-subalgebra of X. Consequently, I is closed and for all $x \in I$, $x * x = 0 \in I$. Moreover, since I is a k-ideal of X that is obtained when $y \in I,\ \ x \in X$, and $x * y \in I$, then $x \in I$. Thus, I is a closed ideal of X. □

Corollary 2. *Let* $(X;\, *,\, 0)$ *be a BN-algebra.* I *is a closed ideal of* X *if and only if* I *is a k-ideal of* X.

Proof. Follows directly from Theorems 5 and 6. □

Theorem 7. *Let* $(X;\, *,\, 0)$ *be a BN-algebra. If* N *is a normal BN-subalgebra of* X, *then* N *is a normal k-ideal of* X.

Proof. Since N is a normal BN-subalgebra of X, then, by Proposition 2, it is obtained that N is a normal ideal of X. We know that N is a BN-subalgebra such that it is a closed ideal of X. Consequently, by Theorem 5, it is obtained that N is a k-ideal of X. Since N is normal, then N is a normal k-ideal of X. □

Definition 13. *Let* $(X;\, *,\, 0)$ *be a BN-algebra and* I *be an ideal of* X. I *is called an m-k-ideal of* X *if when* $x \in I$, $0 \neq y \in X$, *and* $x * y \in I$, *then* $y \in I$.

Theorem 8. *Let $(X;\ *,\ 0)$ be a BN-algebra. If I is a k-ideal of X, then I is an m-k-ideal.*

Proof. Let $(X;\ *,\ 0)$ be a BN-algebra. Since I is a k-ideal of X, then by Theorem 6, I is a closed ideal of X such that if $y \in I$, $x \in X$, and $x * y \in I$, then $x \in I$. Furthermore, since I is closed, it must be the case that if $x \in I$, $\ 0 \neq y \in X$, and $x * y \in I$, then $y \in I$. Hence, we prove that I is an m-k-ideal of X. □

The converse of Theorem 8 does not hold in general. Let $A = \{0,\ 1,\ 2,\ 3\}$ be a BN-algebra in Example 1. It is easy to check that $I_1 = \{0, 2\}$ and $I_2 = \{0, 3\}$ are k-ideals and m-k-ideals of A. Meanwhile, $I_3 = \{0,\ 2,\ 3\}$ is an m-k-ideal in A, but I_3 is not k-ideal because it is not a BN-subalgebra of A.

Theorem 9. *Let $(X;\ *,\ 0)$ be a BN-algebra. If I is a closed ideal of X, then I is an m-k-ideal.*

Proof. Follows directly from Theorems 5 and 8. □

Theorem 10. *Let $(X;\ *,\ 0)$ be a BN-algebra. If I is a k-ideal of X, then I is an r-ideal.*

Proof. Since I is a k-ideal of X, by Theorem 6, we obtain that I is a closed ideal of X such that by Theorem 2, we obtain that I is an r-ideal of X. □

The converse of Theorem 10 does not hold in general since, in Example 1, we have I_3 as an r-ideal in A, but it is not a k-ideal.

Theorem 11. *Let $(X;\ *,\ 0)$ be a BN-algebra. If I is a closed r-ideal of X, then I is a k-ideal.*

Proof. Since I is an r-ideal of X, clearly I is a proper ideal of X. Since I is closed, then by Theorem 5, we obtain that I is a k-ideal of X. □

By Theorem 10, we know that the converse of Theorem 11 does hold in general. In Example 1, I_1 and I_2 are two closed r-ideals in A and also k-ideals.

Proposition 3. *Let $(X;\ *,\ 0)$ be a BN-algebra and f be an endomorphism of X. If I is a k-ideal of X, then $f(I)$ is an r-ideal of X.*

Proof. Follows directly from Theorems 4 and 10. □

The converse of Proposition 3 does not hold in general.

Proposition 4. *Let $(X;\ *,\ 0)$ be a BN-algebra and f be an endomorphism of X. If $f(I)$ is a closed r-ideal of X, then I is a k-ideal of X.*

Proof. Follows directly from Corollary 1 and Theorem 11. □

5. Conclusions and Future Work

In this paper, we defined the concepts of an r-ideal, k-ideal, and m-k-ideal in BN-algebras and investigated several properties. We obtained the relationships between a closed ideal, r-ideal, k-ideal, and m-k-ideal in a BN-algebra. Some of its properties are every closed ideal in BN-algebras is an r-ideal, a k-ideal, and an m-k-ideal. Every k-ideal is an r-ideal and an m-k-ideal of BN-algebras. Moreover, if I is an r-ideal or k-ideal of a BN-algebra, then $f(I)$ is an r-ideal, where f is an endomorphism of the BN-algebra.

We did this research to build complete concepts of an r-ideal, k-ideal, and m-k-ideal in BN-algebras. These results can be used by researchers in the field of abstract algebra to discuss more deeply about types of ideals in BN-algebras.

In future work, we will consider the concept of an r-ideal and m-k-ideal in QM-BZ-algebra and quasi-hyper BZ-algebra, investigating several properties and the relationship between an r-ideal and m-k-ideal in a QM-BZ-algebra and quasi-hyper BZ-algebra.

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