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An Avant-Garde Construction for Subclasses of Analytic Bi-Univalent Functions

Feras Yousef [1,*], Ala Amourah [2], Basem Aref Frasin [3] and Teodor Bulboacă [4]

[1] Department of Mathematics, The University of Jordan, Amman 11942, Jordan
[2] Department of Mathematics, Irbid National University, Irbid 21110, Jordan; ala.amourah@siswa.ukm.edu.my
[3] Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan; bafrasin@aabu.edu.jo
[4] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania; bulboaca@math.ubbcluj.ro
* Correspondence: fyousef@ju.edu.jo

Abstract: The zero-truncated Poisson distribution is an important and appropriate model for many real-world applications. Here, we exploit the zero-truncated Poisson distribution probabilities to construct a new subclass of analytic bi-univalent functions involving Gegenbauer polynomials. For functions in the constructed class, we explore estimates of Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, and next, we solve the Fekete–Szegő functional problem. A number of new interesting results are presented to follow upon specializing the parameters involved in our main results.

Keywords: analytic bi-univalent functions; zero-truncated Poisson distribution; Gegenbauer polynomials; Fekete–Szegő functional problem

MSC: 30C45; 33C45; 60E05

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1. Introduction

In discrete probability distributions, the Poisson distribution has found an extensive and varied application in formulating probability models for a wide variety of real-life phenomena dealing with counts of rare events, such as reliability theory, queueing systems, epidemiology, medicine, industry, and many others. In some practical situations, only positive counts would be available and the zero count is ignored or is impossible to be observed at all. For instance: the length of stay in a hospital is recorded as a minimum of at least one day, the number of journal articles published in different disciplines, the number of occupants in passenger cars, etc. An appropriate Poisson distribution that applies to such a case is called a zero-truncated Poisson distribution.

The probability density function of a discrete random variable X that follows a zero-truncated Poisson distribution can be written as

$$P_m(X=s)=\frac{m^s}{(e^m-1)s!},\ s=1,2,3,\ldots,$$

where the parameter mean $m>0$.

Now, we introduce a novel power series whose coefficients are probabilities of the zero-truncated Poisson distribution

$$\mathbb{P}(m,z):=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(e^m-1)(n-1)!}z^n,\ z\in\mathbb{U},$$

where $m>0$ and $\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\}$ is the *open unit disk*. By ratio test, it is clear that the radius of convergence of the above series is infinity.

Orthogonal polynomials have been extensively studied in recent years from various perspectives due to their importance in mathematical statistics, probability theory, mathematical physics, approximation theory, and engineering. From a mathematical point of view, orthogonal polynomials often arise from solutions of ordinary differential equations under certain conditions imposed by certain model. Orthogonal polynomials that appear most commonly in applications are the classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials, and Jacobi polynomials). The general subclass of Jacobi polynomials is the set of Gegenbauer polynomials, this class includes Legendre polynomials and Chebyshev polynomials as subclasses. To study the basic definitions and the most important properties of the classical orthogonal polynomials, we refer the reader to [1–4]. For a recent connection between the classical orthogonal polynomials and geometric function theory, we mention [5–10].

Gegenbauer polynomials $C_n^{\alpha}(x)$ for $n = 2, 3, \dots$, and $\alpha > -\frac{1}{2}$ are defined by the following three-term recurrence formula

$$\begin{aligned} C_0^{\alpha}(x) &= 1; \\ C_1^{\alpha}(x) &= 2\alpha x; \\ C_n^{\alpha}(x) &= \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-2}^{\alpha}(x)\right]. \end{aligned} \tag{1}$$

It is worth mentioning that by setting $\alpha = \frac{1}{2}$ and $\alpha = 1$ in Equation (1), we immediately obtain Legendre polynomials $P_n(x) = C_n^{\frac{1}{2}}(x)$ and Chebyshev polynomials of the second kind $U_n(x) = C_n^{1}(x)$, respectively.

The generating function of Gegenbauer polynomials is given as

$$H_{\alpha}(x,z) = \frac{1}{(1-2xz+z^2)^{\alpha}},$$

where $x \in [-1,1]$ and $z \in \mathbb{U}$. For fixed x, the function H_{α} is analytic in $\mathbb{U}$, so it can be expanded in a Taylor–Maclaurin series, as follows:

$$H_{\alpha}(x,z) = \sum_{n=0}^{\infty} C_n^{\alpha}(x) z^n, \; z \in \mathbb{U}. \tag{2}$$

2. Preliminaries and Definitions

Let $\mathcal{A}$ denote the class of all normalized analytic functions f written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in \mathbb{U}. \tag{3}$$

Differential subordination of analytic functions provides excellent tools for study in geometric function theory. The earliest problem in differential subordination was introduced by Miller and Mocanu [11], see also [12]. The book of Miller and Mocanu [13] sums up most of the advancement in the field and the references to the date of its publication.

Definition 1. *Let f and g be two analytic functions in $\mathbb{U}$. The function f is said to be subordinate to g, written as $f(z) \prec g(z)$, if there is an analytic function ω in $\mathbb{U}$ with the properties*

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \; z \in \mathbb{U},$$

such that

$$f(z) = g(\omega(z)), \; z \in \mathbb{U}.$$

Definition 2. *A single-valued one-to-one function f defined in a simply connected domain is said to be a univalent function.*

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$, given by (3), that are univalent in $\mathbb{U}$. Hence, every function $f \in \mathcal{S}$ has an inverse given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \dots. \tag{4}$$

Definition 3. *A univalent function f is said to be bi-univalent in $\mathbb{U}$ if its inverse function $f^{-1}(w)$ has an analytic univalent extension in $\mathbb{U}$.*

Let Σ denote the class of all functions $f \in \mathcal{A}$ that are bi-univalent in $\mathbb{U}$ given by (3). For interesting subclasses of functions in the class Σ, see [14–24].

The coefficient functional

$$\Delta_\eta(f) = a_3 - \eta a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\eta}{2}\left(f''(0)\right)^2\right) \tag{5}$$

of the analytic function f given by (3) is very important in the theory of analytic and univalent functions. Thus, it is quite natural to ask about inequalities for $\Delta_\eta(f)$ corresponding to subclasses of bi-univalent functions in the open unit disk $\mathbb{U}$. The problem of maximizing the absolute value of the functional $\Delta_\eta(f)$ is called the Fekete–Szegö problem [25]. There are now several results of this type in the literature, each of them dealing with $|a_3 - \eta a_2^2|$ for various classes of functions defined in terms of subordination (see, e.g., [26–31]).

Now, let us define the linear operator

$$\chi : \mathcal{A} \to \mathcal{A}$$

by

$$\chi_m f(z) := \mathbb{P}(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m - 1)(n-1)!} a_n z^n, \ z \in \mathbb{U},$$

where the symbol "$*$" denotes the Hadamard product of the two series.

To obtain our results we need the following lemma:

Lemma 1 ([32], p. 172)**.** *Assume that $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$, $z \in \mathbb{U}$, is an analytic function in $\mathbb{U}$ such that $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Then, $|\omega_1| \le 1, \quad |\omega_n| \le 1 - |\omega_1|^2, \ n = 2, 3, \dots.$*

Motivated essentially by the earlier work of Amourah et al. [33], we construct, in the next section, a new subclass of bi-univalent functions governed by the zero-truncated Poisson distribution series and Gegenbauer polynomials. Then, we investigate the optimal bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and solve the Fekete–Szegő functional problem for functions in our new subclass.

3. The Class $\zeta_\Sigma(x, \alpha, \delta, \mu)$

Consider the function $f \in \Sigma$ given by (3), the function $g = f^{-1}$ given by (4), and H_α is the generating function of Gegenbauer polynomials given by (2). Now, we are ready to define our new subclass of bi-univalent functions $\zeta_\Sigma(x, \alpha, \delta, \mu)$ as follows.

Definition 4. *A function f is said to be in the class $\zeta_\Sigma(x, \alpha, \delta, \mu)$, if the following subordinations are fulfilled:*

$$(1-\mu)\frac{\chi_m f(z)}{z} + \mu(\chi_m f(z))' + \delta z(\chi_m f(z))'' \prec H_\alpha(x, z),$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w} + \mu(\chi_m g(w))' + \delta w(\chi_m g(w))'' \prec H_\alpha(x, w),$$

where $\alpha > 0$, $\mu, \delta \ge 0$, and $x \in \left(\frac{1}{2}, 1\right]$.

Upon allocating the parameters μ and δ, one can obtain several new subclasses of Σ, as illustrated in the following two examples.

Example 1. *A function f is said to be in the class $\zeta_\Sigma(x,\alpha,\mu) := \zeta_\Sigma(x,\alpha,0,\mu)$, if the following subordinations are fulfilled:*

$$(1-\mu)\frac{\chi_m f(z)}{z} + \mu(\chi_m f(z))' \prec H_\alpha(x,z),$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w} + \mu(\chi_m g(w))' \prec H_\alpha(x,w),$$

where $\alpha > 0$, $\mu \geq 0$, and $x \in \left(\frac{1}{2},1\right]$.

Example 2. *A function f is said to be in the class $\zeta_\Sigma(x,\alpha) := \zeta_\Sigma(x,\alpha,0,1)$, if the following subordinations are fulfilled:*

$$(\chi_m f(z))' \prec H_\alpha(x,z),$$

and

$$(\chi_m g(w))' \prec H_\alpha(x,w),$$

where $\alpha > 0$ and $x \in \left(\frac{1}{2},1\right]$.

4. Main Results

Theorem 1. *If the function f belongs to the class $\zeta_\Sigma(x,\alpha,\delta,\mu)$, then*

$$|a_2| \leq \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(1+2\mu+6\delta)(e^m-1)-2(1+\alpha)(1+\mu+2\delta)^2\right]x^2+(1+\mu+2\delta)^2\right|}}, \tag{6}$$

and

$$|a_3| \leq \frac{4\alpha^2x^2(e^m-1)^2}{m^2(1+\mu+2\delta)^2} + \frac{4\alpha x(e^m-1)}{m^2(1+2\mu+6\delta)}.$$

Proof. If $f \in \zeta_\Sigma(x,\alpha,\delta,\mu)$, from the Definition 4 there exist two analytic functions in $\mathbb{U}$ that are w and v, such that $w(0) = v(0) = 0$ and $|\omega(z)| < 1$, $|v(w)| < 1$ for all $z, w \in \mathbb{U}$, and

$$(1-\mu)\frac{\chi_m f(z)}{z} + \mu(\chi_m f(z))' + \delta z(\chi_m f(z))'' = H_\alpha(x,\omega(z)),\ z \in \mathbb{U}, \tag{7}$$

and

$$(1-\mu)\frac{\chi_m g(w)}{w} + \mu(\chi_m g(w))' + \delta w(\chi_m g(w))'' = H_\alpha(x,v(w)),\ w \in \mathbb{U}, \tag{8}$$

From the equalities (7) and (8), we obtain

$$\begin{aligned}&(1-\mu)\frac{\chi_m f(z)}{z} + \mu(\chi_m f(z))' + \delta z(\chi_m f(z))'' \\ &= 1 + C_1^\alpha(x)c_1 z + \left[C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2\right]z^2 + \ldots,\ z \in \mathbb{U},\end{aligned} \tag{9}$$

and

$$\begin{aligned}&(1-\mu)\frac{\chi_m g(w)}{w} + \mu(\chi_m g(w))' + \delta w(\chi_m g(w))'' \\ &= 1 + C_1^\alpha(x)d_1 w + \left[C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2\right]w^2 + \ldots,\ w \in \mathbb{U},\end{aligned} \tag{10}$$

where

$$\omega(z) = \sum_{j=1}^{\infty} c_j z^j, \ z \in \mathbb{U}, \quad \text{and} \quad v(w) = \sum_{j=1}^{\infty} d_j w^j, \ w \in \mathbb{U}. \tag{11}$$

According to Lemma 1, if the above function ω and v has the form (11), then

$$|c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \tag{12}$$

Thus, upon comparing and equating the corresponding coefficients in (9) and (10), we have

$$\frac{(1+\mu+2\delta)m}{e^m - 1} a_2 = C_1^{\alpha}(x) c_1, \tag{13}$$

$$\frac{(1+2\mu+6\delta)m^2}{2(e^m - 1)} a_3 = C_1^{\alpha}(x) c_2 + C_2^{\alpha}(x) c_1^2, \tag{14}$$

$$-\frac{(1+\mu+2\delta)m}{e^m - 1} a_2 = C_1^{\alpha}(x) d_1, \tag{15}$$

and

$$\frac{(1+2\mu+6\delta)m^2}{2(e^m - 1)} \left[2a_2^2 - a_3\right] = C_1^{\alpha}(x) d_2 + C_2^{\alpha}(x) d_1^2. \tag{16}$$

It follows from (13) and (15) that

$$c_1 = -d_1, \tag{17}$$

and

$$\frac{2(1+\mu+2\delta)^2 m^2}{(e^m - 1)^2} a_2^2 = [C_1^{\alpha}(x)]^2 \left(c_1^2 + d_1^2\right). \tag{18}$$

If we add (14) and (16), we get

$$\frac{(1+2\mu+6\delta)m^2}{(e^m - 1)} a_2^2 = C_1^{\alpha}(x)(c_2 + d_2) + C_2^{\alpha}(x)\left(c_1^2 + d_1^2\right). \tag{19}$$

Substituting the value of $\left(c_1^2 + d_1^2\right)$ from (18) in the right hand side of (19), we deduce that

$$\left[(1+2\mu+6\delta) - \frac{2(1+\mu+2\delta)^2}{(e^m - 1)} \frac{C_2^{\alpha}(x)}{\left[C_1^{\alpha}(x)\right]^2}\right] \frac{m^2}{(e^m - 1)} a_2^2 = C_1^{\alpha}(x)(c_2 + d_2). \tag{20}$$

Now, using (1), (12) and (20), we find that (6) holds.
Moreover, if we subtract (16) from (14), we obtain

$$\frac{(1+2\mu+6\delta)m^2}{(e^m - 1)} \left(a_3 - a_2^2\right) = C_1^{\alpha}(x)(c_2 - d_2) + C_2^{\alpha}(x)\left(c_1^2 - d_1^2\right). \tag{21}$$

Then, in view of (17) and (18), Equation (21) becomes

$$a_3 = \frac{(e^m - 1)^2 \left[C_1^{\alpha}(x)\right]^2}{2m^2(1+\mu+2\delta)^2} \left(c_1^2 + d_1^2\right) + \frac{(e^m - 1) C_1^{\alpha}(x)}{m^2(1+2\mu+6\delta)} (c_2 - d_2).$$

Thus, applying (1), we conclude that

$$|a_3| \leq \frac{4\alpha^2 x^2 (e^m - 1)^2}{m^2(1+\mu+2\delta)^2} + \frac{4\alpha x (e^m - 1)}{m^2(1+2\mu+6\delta)},$$

and the proof of the theorem is complete. $\square$

The following result addresses the Fekete–Szegő functional problem for functions in the class $\zeta_{\Sigma}(x,\alpha,\delta,\mu)$.

Theorem 2. *If the function f belongs to the class $\zeta_{\Sigma}(x,\alpha,\delta,\mu)$, then*

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{4\alpha x(e^m-1)}{m^2(1+2\mu+6\delta)}, & \text{if} \quad |\eta-1| \le M, \\ \frac{8\alpha^2 x^3 (e^m-1)^2|1-\eta|}{\left|m^2\left\{\left[2\alpha(1+2\mu+6\delta)(e^m-1)-2(1+\alpha)(1+\mu+2\delta)^2\right]x^2+(1+\mu+2\delta)^2\right\}\right|}, & \text{if} \quad |\eta-1| \ge M, \end{cases}$$

where

$$M := \left|1-\frac{(1+\mu+2\delta)^2\left[2(1+\alpha)x^2-1\right]}{2\alpha x^2(e^m-1)(1+2\mu+6\delta)}\right|.$$

Proof. If $f \in \zeta_{\Sigma}(x,\alpha,\delta,\mu)$, from (20) and (21) we get

$$\begin{aligned} a_3-\eta a_2^2 = (1-\eta)&\frac{(e^m-1)^2\left[C_1^{\alpha}(x)\right]^3(c_2+d_2)}{m^2\left[(e^m-1)(1+2\mu+6\delta)\left[C_1^{\alpha}(x)\right]^2-2(1+\mu+2\delta)^2C_2^{\alpha}(x)\right]} \\ &+\frac{(e^m-1)C_1^{\alpha}(x)}{m^2(1+2\mu+6\delta)}(c_2-d_2) \\ = C_1^{\alpha}(x)&\left[h(\eta)+\frac{(e^m-1)}{m^2(1+2\mu+6\delta)}\right]c_2+\left[h(\eta)-\frac{(e^m-1)}{m^2(1+2\mu+6\delta)}\right]d_2, \end{aligned}$$

where

$$h(\eta)=\frac{(e^m-1)^2\left[C_1^{\alpha}(x)\right]^2(1-\eta)}{m^2\left[(e^m-1)(1+2\mu+6\delta)\left[C_1^{\alpha}(x)\right]^2-2(1+\mu+2\delta)^2C_2^{\alpha}(x)\right]}.$$

Then, in view of (1), we conclude that

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \frac{4\alpha x(e^m-1)}{m^2(1+2\mu+6\delta)}, & \text{if} \quad 0 \le |h(\eta)| \le \frac{(e^m-1)}{m^2(1+2\mu+6\delta)}, \\ 4\alpha x|h(\eta)|, & \text{if} \quad |h(\eta)| \ge \frac{(e^m-1)}{m^2(1+2\mu+6\delta)}, \end{cases}$$

which completes the proof of Theorem 2. □

5. Corollaries and Consequences

Corresponding essentially to the Example 1 (setting $\delta=0$) and Example 2 (setting $\delta=0$ and $\mu=1$), from Theorems 1 and 2 we get the following consequences, respectively.

Corollary 1. *If the function f belongs to the class $\zeta_{\Sigma}(x,\alpha,\mu)$, then*

$$|a_2| \le \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(1+2\mu)(e^m-1)-2(1+\alpha)(1+\mu)^2\right]x^2+(1+\mu)^2\right|}},$$

$$|a_3| \le \frac{4\alpha^2x^2(e^m-1)^2}{m^2(1+\mu)^2}+\frac{4\alpha x(e^m-1)}{m^2(1+2\mu)},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x(e^m-1)}{m^2(1+2\mu)}, & \text{if} \quad |\eta-1| \leq N, \\ \frac{8\alpha^2 x^3 (e^m-1)^2 |1-\eta|}{\left|m^2\left\{\left[2\alpha(1+2\mu)(e^m-1)-2(1+\alpha)(1+\mu)^2\right]x^2+(1+\mu)^2\right\}\right|}, & \text{if} \quad |\eta-1| \geq N, \end{cases}$$

where

$$N := \left|1-\frac{(1+\mu)^2\left[2(1+\alpha)x^2-1\right]}{2\alpha x^2(e^m-1)(1+2\mu)}\right|.$$

Corollary 2. *If the function f belongs to the class $\zeta_\Sigma(x,\alpha)$, then*

$$|a_2| \leq \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[6\alpha(e^m-1)-8(1+\alpha)\right]x^2+4\right|}},$$

$$|a_3| \leq \frac{\alpha^2 x^2 (e^m-1)^2}{m^2}+\frac{4\alpha x(e^m-1)}{3m^2},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x(e^m-1)}{3m^2}, & \text{if} \quad |\eta-1| \leq L, \\ \frac{8\alpha^2 x^3 (e^m-1)^2 |1-\eta|}{\left|m^2\left\{\left[6\alpha(e^m-1)-8(1+\alpha)\right]x^2+4\right\}\right|}, & \text{if} \quad |\eta-1| \geq L, \end{cases}$$

where

$$L := \left|1-\frac{2\left[2(1+\alpha)x^2-1\right]}{3\alpha x^2(e^m-1)}\right|.$$

6. Concluding Remarks

In the present work we have constructed a new subclass $\zeta_\Sigma(x,\alpha,\delta,\mu)$ of normalized analytic and bi-univalent functions governed with the zero-truncated Poisson distribution series and Gegenbauer polynomials. For functions belonging to this class, we have made estimates of Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, and solved the Fekete–Szegő functional problem. Furthermore, by suitably specializing the parameters δ and μ, one can deduce the results for the subclasses $\zeta_\Sigma(x,\alpha,\mu)$ and $\zeta_\Sigma(x,\alpha)$ which are defined, respectively, in Examples 1 and 2.

The results offered in this paper would lead to other different new results for the classes $\zeta_\Sigma(x,1/2,\delta,\mu)$ for Legendre polynomials and $\zeta_\Sigma(x,1,\delta,\mu)$ for Chebyshev polynomials.

It remains an open problem to derive estimates on the bounds of $|a_n|$ for $n \geq 4$, $n \in \mathbb{N}$, for the subclasses that have been introduced here.

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