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On Fractional Inequalities Using Generalized Proportional Hadamard Fractional Integral Operator

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Abstract: The main objective of this paper is to use the generalized proportional Hadamard fractional integral operator to establish some new fractional integral inequalities for extended Chebyshev functionals. In addition, we investigate some fractional integral inequalities for positive continuous functions by employing a generalized proportional Hadamard fractional integral operator. The findings of this study are theoretical but have the potential to help solve additional practical problems in mathematical physics, statistics, and approximation theory.

Keywords: extended Chebyshev functional; generalized proportional Hadamard fractional integral operator

MSC: 26D10; 26A33; 05A30; 26D53



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1. Introduction

Fractional calculus has a new orientation not only with respect to mathematics but also to physics, statistics, engineering, and other applied sciences. Its birth dates back to ancient times, and its development has been rapid in recent years, gaining new momentum, especially with the definition of new fractional integral and derivative operators. New fractional operators lead to useful applications and generalizations in the field, and their kernel structures and properties give them an advantage over classical derivative and integral operators.

Recently, many mathematicians have worked with slightly different fractional integral formulas. For example, see [1–7] for Riemann–Liouville fractional integral operators, [8] for Hadamard fractional integral operators, [9–12] for Saigo fractional integral operators, [13–15] for conformable fractional integral operators, [16–18] for generalized Katugampola fractional operators, and [19–22] for k -generalized (in terms of hypergeometric function) fractional integral operators. In [3,20], the authors investigated fractional integral inequalities for extended Chebyshev functionals by employing Riemann–Liouville and generalized k -fractional integral fractional integrals, respectively. Recently, many mathematicians have examined several kinds of fractional integral and derivative operators with different types of kernels, such as logarithmic kernels, non-singular exponential kernels, etc. During the past few years, numerous analyses of real-world problems, mathematical models, and numerical methods have been resolved by fractional derivatives and integrals [13,15,23–33]. Anber et al. [34] presented some fractional integral inequalities similar to the Minkowski fractional integral inequality, using the Riemann–Liouville fractional integral. In [35], Panchal et al. studied weighted fractional integral inequalities using a generalized Katugampola fractional integral operator. In [36], Andric et al. proposed the reverse fractional Minkowski integral inequality using the extended Mittag-Leffler function with the corresponding fractional integral operator, which was proved together with

several related Minkowski-type inequalities. Rahman et al. [37–39] investigated the Minkowski inequality and some other fractional inequalities for convex functions by employing fractional proportional integral operators. Atangana and Baleanu proposed a new fractional derivative operator with a non-local and non-singular kernel [40]. In [41], Jarad et al. proposed fractional conformable integral and derivative operators. In [42–44], Jarad et al. and Rahman presented the concepts of non-local fractional proportional and generalized Hadamard proportional integrals involving exponential functions in their kernels. In [14,45–47], the authors explored various integral inequalities by employing conformable and generalized conformable fractional integrals. Caputo and Fabrizio [48] introduced new fractional derivatives and integrals without singular kernels. Later, Lasada and Niteto proposed certain properties of fractional derivatives without a singular kernel [49]. Nale et al. and Rahaman et al. [44,50] investigated some Minkowski-type inequalities and other integral inequalities by considering the generalized proportional Hadamard fractional integral operator. Kukushkin [51] examined the final terms of a differential operator with a fractional integro-differential operator composition on a bounded domain of n -dimensional Euclidean space, as well as on the real axis. One of the central points was the relation connecting fractional powers of m -accretive operators and fractional derivatives in the most general sense. By virtue of such an approach, we express fractional derivatives in terms of infinitesimal generators. In this regard, operators such as the Kipriyanov operator, Riesz potential, and difference operator are considered. In addition, in [52], Yosida studied the semigroup that generates the involved fractional integro-differential operators due to the Balakrishnyan formula. We think that it would be interesting to illustrate the relevance of the topic by presenting and comparing the well-known Chebyshev inequality in L_1 . In [53], the Chebyshev functional for two integrable functions u and v on $[a, b]$ is defined as follows:

$$T[u(x), v(x)] = \frac{1}{b-a} \int_a^b u(x)v(x)dx - \frac{1}{b-a} \left(\int_a^b u(x)dx \right) \frac{1}{b-a} \left(\int_a^b v(x)dx \right). \quad (1)$$

Many applications and several inequalities related to Chebyshev functionals can be found in [6,54–56]. Let us now consider the following extended Chebyshev functional [3]:

$$\begin{aligned} T[u(x), v(x), p(x), q(x)] &= \int_a^b q(x)dx \int_a^b p(x)u(x)v(x)dx + \int_a^b p(x)dx \int_a^b q(x)u(x)v(x)dx \\ &- \left(\int_a^b p(x)u(x)dx \right) \left(\int_a^b q(x)v(x)dx \right) \\ &- \left(\int_a^b q(x)u(x)dx \right) \left(\int_a^b p(x)v(x)dx \right), \end{aligned} \quad (2)$$

where u and v are two integrable functions on $[a, b]$, and p and q are positive integrable functions on $[a, b]$. In order to present a famous inequality for this function, let us now introduce the concept of synchronous (asynchronous) functions.

Definition 1. Two functions u and v are called synchronous (asynchronous) functions on $[a, b]$ if

$$\left(u(\tau) - u(\sigma) \right) \left(v(\tau) - v(\sigma) \right) \geq (\leq) 0, \quad \tau, \sigma \in [a, b]. \quad (3)$$

Hence, if u and v are synchronous on $[a, b]$, then $T[u(x), v(x), p(x), q(x)] \geq 0$.

Motivated by [3,19,34,38,39,43,44], our purpose in this paper is to obtain fractional integral inequalities for the extended Chebyshev functional and other fractional inequalities, using the generalized Hadamard proportional integral. The assumption of synchronous (asynchronous) functions will sometimes be made.

The paper has been organized as follows. In Section 2, we recall basic definitions, remarks, and lemmas related to generalized Hadamard proportional integrals. In Section 3, we obtain fractional integral inequalities for extended Chebyshev functionals using generalized Hadamard proportional integrals. In Section 4, we present some other fractional integral inequalities using generalized Hadamard proportional integrals. In Section 5, we give the concluding remarks.

2. Preliminary

Here, we present some important definitions, remarks, and lemmas of the generalized proportional Hadamard fractional integral operator, which will be used throughout this paper. Recently, Rahman et al. [43] presented the left and right-sided generalized proportional integral operators as follows:

Definition 2. The left- and right-sided generalized proportional fractional integrals are, respectively, defined by

$${}_a\mathfrak{J}^{\alpha,\beta}[z(x)](x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_a^x e^{\left[\frac{\beta-1}{\beta}(x-t)\right]} (x-t)^{\alpha-1} z(t) dt, \quad a < x \tag{4}$$

(here, the first x between the brackets refers to the variable of the function $z(x)$, and the second x between the brackets refers to the integral upper bound; other notations are possible), and

$$\mathfrak{J}_b^{\alpha,\beta}[z(x)](x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_x^b e^{\left[\frac{\beta-1}{\beta}(t-x)\right]} (t-x)^{\alpha-1} z(t) dt, \quad x < b, \tag{5}$$

where the proportionality index is $\beta \in (0, 1]$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and $\Gamma(\alpha)$ is the classical well-known gamma function.

Remark 1. If we consider $\beta = 1$ in Equations (4) and (5), then we obtain the well-known left- and right-sided Riemann–Liouville integrals, which are, respectively, defined by

$${}_a\mathfrak{J}^\alpha[z(x)](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} z(t) dt, \quad a < x \tag{6}$$

and

$$\mathfrak{J}_b^\alpha[z(x)](x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} z(t) dt, \quad x < b, \tag{7}$$

where $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.

On the other hand, recently, Rahman et al. [44] proposed the following generalized Hadamard proportional fractional integrals.

Definition 3. The left-sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by

$${}_a\mathcal{H}^{\alpha,\beta}[z(x)](x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_a^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln t)\right]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, \quad a < x. \tag{8}$$

Definition 4. The right-sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by

$$\mathcal{H}_b^{\alpha,\beta}[z(x)](x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_x^b e^{\left[\frac{\beta-1}{\beta}(\ln t - \ln x)\right]} (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, \quad x < b. \tag{9}$$

Remark 2. If we consider $a = 1$ in Equation (8), then we obtain

$${}_1\mathcal{H}^{\alpha,\beta}[z(x)](x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln t)\right]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, \quad x > 1. \quad (10)$$

Hereafter, to lighten the notation, we set $\mathcal{H}_{1,x}^{\alpha,\beta}[z(x)] = {}_1\mathcal{H}^{\alpha,\beta}[z(x)](x)$.

Remark 3. If we consider $\beta = 1$, then Equations (8)–(10) will lead to the following well known Hadamard fractional integrals, indicated as

$${}_a\mathcal{H}^\alpha[z(x)](x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, \quad a < x, \quad (11)$$

$$\mathcal{H}_b^\alpha[z(x)](x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, \quad x < b, \quad (12)$$

and

$$\mathcal{H}_{1,x}^{\alpha,\beta}[z(x)] = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, \quad x > 1. \quad (13)$$

One can easily prove the following results.

Lemma 1. With the special function: $z(x) = e^{\left[\frac{\beta-1}{\beta}(\ln x)\right]} (\ln x)^{\lambda-1}$, we have

$$\mathcal{H}_{1,x}^{\alpha,\beta} \left[e^{\left[\frac{\beta-1}{\beta}(\ln x)\right]} (\ln x)^{\lambda-1} \right] = \frac{\Gamma(\lambda)}{\beta^\alpha\Gamma(\alpha + \lambda)} e^{\left[\frac{\beta-1}{\beta}(\ln x)\right]} (\ln x)^{\alpha+\lambda-1}, \quad (14)$$

and the following semigroup property holds:

$$\mathcal{H}_{1,x}^{\alpha,\beta} \left[\mathcal{H}_{1,x}^{\lambda,\beta}[z(x)] \right] = \mathcal{H}_{1,x}^{\alpha+\lambda,\beta}[z(x)]. \quad (15)$$

Remark 4. If $\beta = 1$, then Equation (14) reduces to the result of [57] as defined by

$$\mathcal{H}_{1,x}^\alpha \left[(\ln x)^{\lambda-1} \right] = \frac{\Gamma(\lambda)}{\Gamma(\alpha + \lambda)} (\ln x)^{\alpha+\lambda-1}. \quad (16)$$

3. Fractional Integral Inequalities for Extended Chebyshev Functional

In this section, we establish a fractional integral inequality involving generalized proportional Hadamard fractional integral operators. We now prove the following lemma.

Lemma 2. Let f and g be two integrable and synchronous functions on $[1, \infty)$, and $u, v : [1, \infty) \rightarrow [0, \infty)$. Then, for all $x > 1$, $\alpha > 0$ and $\beta \in (0, 1]$, we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[vf(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[v(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[vg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[vf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]. \end{aligned} \quad (17)$$

It is understood that, for instance, $vf(x) = v(x)f(x)g(x)$.

Proof. Since f and g are synchronous functions on $[1, \infty)$, for all $\tau \geq 0$ and $\sigma \geq 0$, the following inequality holds:

$$\left(f(\tau) - f(\sigma) \right) \left(g(\tau) - g(\sigma) \right) \geq 0. \quad (18)$$

Then, Equation (18) becomes

$$f(\tau)g(\tau) + f(\sigma)g(\sigma) \geq f(\tau)g(\sigma) + f(\sigma)g(\tau). \tag{19}$$

Let us now consider

$$\psi(x, \tau) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1}. \tag{20}$$

We can clearly state that the function $\psi(x, \tau)u(\tau)$ remains positive, because for all $\tau \in (1, x)$, $(x > 1)$, $\alpha, \beta > 0$. Multiplying both sides of Equation (19) by $\psi(x, \tau)$, then integrating the resulting identity with respect to τ from 1 to x , we obtain

$$\begin{aligned} & \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} u(\tau) f(\tau) g(\tau) \frac{d\tau}{\tau} \\ & + \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} u(\tau) f(\sigma) g(\sigma) \frac{d\tau}{\tau} \\ & \geq \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} u(\tau) f(\tau) g(\sigma) \frac{d\tau}{\tau} \\ & + \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} u(\tau) f(\sigma) g(\tau) \frac{d\tau}{\tau}. \end{aligned} \tag{21}$$

Consequently,

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] + f(\sigma)g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)] \\ & \geq g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)] + f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]. \end{aligned} \tag{22}$$

Taking both sides of Equation (22) and multiplying them by $\psi(x, \sigma)v(\sigma)$, which remains positive because for all $\sigma \in (1, x)$, $(x > 1)$, $\alpha, \beta > 0$, then integrating the resulting identity with respect to σ from 1 to x , we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\alpha-1} v(\sigma) \frac{d\sigma}{\sigma} \\ & + \mathcal{H}_{1,x}^{\alpha,\beta}[u(x)] \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\alpha-1} v(\sigma) f(\sigma) g(\sigma) \frac{d\sigma}{\sigma} \\ & \geq \mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)] \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\alpha-1} v(\sigma) g(\sigma) \frac{d\sigma}{\sigma} \\ & + \mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)] \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\alpha-1} v(\sigma) f(\sigma) \frac{d\sigma}{\sigma}. \end{aligned} \tag{23}$$

This completes the proof of Inequality (17). \square

We present below the major result of the paper.

Theorem 1. Let f and g be two integrable and synchronous functions on $[1, \infty)$, and $r, p, q : [1, \infty) \rightarrow [0, \infty)$ (so are positive). Then, for all $x > 1$, $\alpha > 0$ and $\beta \in (0, 1]$, we have

$$\begin{aligned} & 2\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \right] + \\ & 2\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] \right] + \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right] + \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right]. \end{aligned} \tag{24}$$

Proof. To prove this theorem, we put $u = p$ and $v = q$ into Lemma 2, and we obtain

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]. \end{aligned} \tag{25}$$

Now, multiplying both sides of Equation (25) by $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right]. \end{aligned} \tag{26}$$

Again, putting $u = r$ and $v = q$, into Lemma 2, we obtain

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)], \end{aligned} \tag{27}$$

Multiplying both sides of Equation (27) by $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right]. \end{aligned} \tag{28}$$

With the same arguments as in Equations (26) and (28), we can write

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right]. \end{aligned} \tag{29}$$

Adding Inequalities (26), (28) and (29), we obtain Inequality (24). \square

Lemma 3. Let f and g be two integrable and synchronous functions on $[1, \infty)$, and $u, v : [1, \infty) \rightarrow [0, \infty)$. Then, for all $x > 1$, $\beta, \varphi \in (0, 1]$, and $\alpha, \phi > 0$, we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[v(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[vf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]. \end{aligned} \tag{30}$$

Proof. Multiplying both sides of Equation (22) by $\frac{1}{\varphi^\phi\Gamma(\phi)}e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]}(\ln x - \ln \sigma)^{\phi-1}$, $\sigma \in (1, x)$, $x > 1$, $\phi, \varphi > 0$, which remains positive (in view of the argument mentioned above in the proof of Lemma 2). Then, integrating the resulting identity with respect to σ from 1 to x , we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\frac{1}{\varphi^\phi\Gamma(\phi)}\int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]}(\ln x - \ln \sigma)^{\phi-1}v(\sigma)\frac{d\sigma}{\sigma} \\ &+ \mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\frac{1}{\varphi^\phi\Gamma(\phi)}\int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]}(\ln x - \ln \sigma)^{\phi-1}v(\sigma)f(\sigma)g(\sigma)\frac{d\sigma}{\sigma} \\ &\geq \mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\frac{1}{\varphi^\phi\Gamma(\phi)}\int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]}(\ln x - \ln \sigma)^{\phi-1}v(\sigma)g(\sigma)\frac{d\sigma}{\sigma} \\ &+ \mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]\frac{1}{\varphi^\phi\Gamma(\phi)}\int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]}(\ln x - \ln \sigma)^{\phi-1}v(\sigma)f(\sigma)\frac{d\sigma}{\sigma}. \end{aligned} \tag{31}$$

This completes the proof of Inequality (30). \square

Theorem 2. Let f and g be two integrable and synchronous functions on $[1, \infty)$, and $r, p, q : [1, \infty) \rightarrow [0, \infty)$. Then, for all $x > 1$, $\beta, \varphi \in (0, 1]$, and $\alpha, \phi > 0$, we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[pfg(x)] + 2 \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] \right. \\ & \left. + \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \right] \\ & + \left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[p(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \right] \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \tag{32} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] \right] + \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right] + \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right]. \end{aligned}$$

Proof. To prove this theorem, we put $u = p$ and $v = q$ into Lemma 3, and we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \geq \tag{33} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]. \end{aligned}$$

Now, multiplying both sides of Equation (33) by $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \right] \geq \tag{34} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] \right], \end{aligned}$$

Now, putting $u = r$ and $v = q$ into Lemma 3, we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \tag{35} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]. \end{aligned}$$

Multiplying both sides of Equation (35) by $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \right] \geq \tag{36} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right]. \end{aligned}$$

Arguing as for Equations (34) and (36), we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[pfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[p(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \right] \geq \tag{37} \\ & \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)] \left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)] \mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)] \mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)] \right]. \end{aligned}$$

Adding Inequalities (34), (36) and (37), we obtain Inequality (32). \square

Remark 5. We assume f, g, r, p and q satisfy the following conditions:

1. the functions f and g are asynchronous on $[1, \infty)$;
2. the functions r, p, q are negative on $[1, \infty)$;
3. two of the functions r, p, q are positive and the third is negative on $[1, \infty)$.

Then, Inequalities (24) and (32) are reversed.

4. Some Other Fractional Integral Inequalities

Now, we give some other fractional integral inequalities using generalized proportional Hadamard fractional integral operators.

Theorem 3. Suppose that $f, g,$ and h are positive and continuous functions on $[1, \infty)$, such that

$$(g(\tau) - g(\sigma)) \left(\frac{f(\sigma)}{h(\sigma)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0, \quad \tau, \sigma \in (1, x) \quad x > 1. \tag{38}$$

Then, for all $x > 1, \alpha > 0$ and $\beta \in (0, 1]$, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)]}. \tag{39}$$

Proof. Since $f, g,$ and h are three positive and continuous functions on $[1, \infty)$, by Equation (38) we obtain

$$g(\tau) \frac{f(\sigma)}{h(\sigma)} + g(\sigma) \frac{f(\tau)}{h(\tau)} - g(\sigma) \frac{f(\sigma)}{h(\sigma)} - g(\tau) \frac{f(\tau)}{h(\tau)} \geq 0, \quad \tau, \sigma \in (0, x) \quad x > 0. \tag{40}$$

Multiplying both sides of Equation (40) by $h(\sigma)h(\tau)$, we have

$$g(\tau)f(\sigma)h(\tau) - g(\tau)f(\tau)h(\sigma) - g(\sigma)f(\sigma)h(\tau) + g(\sigma)f(\tau)h(\sigma) \geq 0. \tag{41}$$

Now, multiplying Equation (41) by $\psi(x, \tau)$ defined by Equation (20), then integrating the resulting identity with respect to τ from 1 to x , we obtain

$$\begin{aligned} & \frac{f(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} g(\tau)h(\tau) \frac{d\tau}{\tau} \\ & - \frac{h(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} g(\tau)f(\tau) \frac{d\tau}{\tau} \\ & - \frac{f(\sigma)g(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \\ & + \frac{h(\sigma)g(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \geq 0. \end{aligned} \tag{42}$$

It follows from Equation (42) that

$$\begin{aligned} & f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)] + g(\sigma)h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \\ & - g(\sigma)f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] - h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)] \geq 0. \end{aligned} \tag{43}$$

Again, let us multiply Equation (43) by $\psi(x, \sigma)$ as defined by Equation (20), which remains positive because for all $\sigma \in (1, x), (x > 1), \alpha, \beta > 0$. Then, integrating the resulting identity with respect to σ from 1 to x , we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)] - \mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)] \\ & - \mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \geq 0, \end{aligned} \tag{44}$$

which implies that

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)] \geq \mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)]. \tag{45}$$

This completes the proof of the theorem. \square

Theorem 4. Suppose that $f, g,$ and h are positive and continuous functions on $[1, \infty)$, such that

$$(g(\tau) - g(\sigma)) \left(\frac{f(\sigma)}{h(\sigma)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0, \quad \tau, \sigma \in (1, x) \quad x > 1, \tag{46}$$

Then, for all $x > 1, \beta, \phi \in (0, 1],$ and $\alpha, \phi > 0,$ we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\phi}[gf(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\phi}[gf(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)]} \geq 1. \tag{47}$$

Proof. Multiplying Equation (43) by $\frac{1}{\phi^\phi \Gamma(\phi) \sigma} e^{\left[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1}, \sigma \in (1, x),$ $x > 1, \phi, \phi > 0,$ which is always positive, then integrating the resulting identity with respect to σ from 1 to $x,$ we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[hg(x)] \frac{1}{\phi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} f(\sigma) \frac{d\sigma}{\sigma} \\ & - \mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)] \frac{1}{\phi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} h(\sigma) \frac{d\sigma}{\sigma} \\ & - \mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] \frac{1}{\phi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} gf(\sigma) \frac{d\sigma}{\sigma} \\ & + \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \frac{1}{\phi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\phi-1}{\phi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} fh(\sigma) \frac{d\sigma}{\sigma} \geq 0. \end{aligned} \tag{48}$$

This gives us the following relation:

$$\begin{aligned} & \mathcal{H}_{1,x}^{\phi,\phi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)] - \mathcal{H}_{1,x}^{\phi,\phi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)] \\ & - \mathcal{H}_{1,x}^{\phi,\phi}[gf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[gh(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \geq 0. \end{aligned} \tag{49}$$

From Equation (49), we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\phi,\phi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gh(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[gh(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)] \\ & \geq \mathcal{H}_{1,x}^{\phi,\phi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[gf(x)] + \mathcal{H}_{1,x}^{\phi,\phi}[gf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]. \end{aligned} \tag{50}$$

This yields Inequality (47). This completes the proof of the theorem. \square

Remark 6. If we take $\alpha = \phi$ and $\beta = \phi$ in Theorem 4, then we obtain Theorem 3.

Theorem 5. Suppose that f and h are two positive continuous functions such that $f \leq h$ on $[1, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1, \infty)$, then, for any $p \geq 1, x > 1, \alpha > 0$ and $\beta \in (0, 1],$ we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}. \tag{51}$$

Proof. Now, by taking $g = f^{p-1}$ in Theorem 3, we obtain

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[ff^{p-1}(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)]}. \tag{52}$$

Since $f \leq h$ on $[1, \infty),$ we have

$$hf^{p-1} \leq h^p. \tag{53}$$

Multiplying Equation (53) by $\psi(x, \tau)$ defined by Equation (20), then integrating the resulting identity with respect to τ from 1 to x , we obtain

$$\frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} f^{p-1} h(\tau) \frac{d\tau}{\tau} \tag{54}$$

$$\leq \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} h^p(\tau) \frac{d\tau}{\tau}, \tag{55}$$

which implies that

$$\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]. \tag{56}$$

Thus, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[ff^{p-1}(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)]} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}. \tag{57}$$

From Equations (52) and (57), we obtain Equation (51). \square

Theorem 6. Suppose that f and h are two positive continuous functions such that $f \leq h$ on $[1, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[1, \infty)$, then, for any $p \geq 1, x > 1, \beta, \varphi \in (0, 1], \alpha, \phi > 0$, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[f^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]} \geq 1. \tag{58}$$

Proof. Taking $g = f^{p-1}$ in Theorem 4, we obtain

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[hf^{p-1}(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)]}{\mathcal{H}_{1,x}^{\alpha,\beta}[h(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[f^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[h(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[f^p(x)]} \geq 1, \tag{59}$$

then, by hypothesis, $f \leq h$ on $[1, \infty)$, which implies that

$$hf^{p-1} \leq h^p. \tag{60}$$

Now, multiplying both sides of Equation (60) by $\frac{1}{\varphi^\phi \Gamma(\phi)\sigma} e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1}$, $\sigma \in (1, x), x > 1, \phi, \varphi > 0$, which remains positive. Then, integrating the resulting identity with respect to σ from 1 to x , we have

$$\begin{aligned} & \frac{1}{\varphi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} hf^{p-1}(\sigma) \frac{d\sigma}{\sigma} \\ & \leq \frac{1}{\varphi^\phi \Gamma(\phi)} \int_1^x e^{\left[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)\right]} (\ln x - \ln \sigma)^{\phi-1} h^p(\sigma) \frac{d\sigma}{\sigma}. \end{aligned} \tag{61}$$

Integrating both sides of Equation (61) with respect to σ over 1 to x , we have

$$\mathcal{H}_{1,x}^{\phi,\varphi}[hf^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)]. \tag{62}$$

Multiplying both sides of Equation (62) by $\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]$, we obtain

$$\mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[hf^{p-1}(x)] \leq \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)]. \tag{63}$$

Hence, from Equations (56) and (63), we obtain

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[hf^{p-1}(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[hf^{p-1}(x)] \\ & \leq \mathcal{H}_{1,x}^{\alpha,\beta}[f(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[h^p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[f(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[h^p(x)]. \end{aligned} \tag{64}$$

From Equations (59) and (64), we obtain Equation (58). This ends the proof of the theorem. \square

5. Concluding Remarks

In [42], the authors proposed the concept of generalized proportional fractional integral operators with exponential kernels. Following this, Rahman et al. [44] worked on these operators and established some fractional inequalities for convex functions by considering Hadamard proportional fractional integrals. In this study, we obtained some fractional integral inequalities for the extended Chebyshev function by considering the generalized proportional Hadamard fractional integral operator. The inequalities investigated in this paper represent novel contributions in the fields of fractional calculus and generalized proportional Hadamard fractional integral operators. They are also expected to lead to some applications for determining the uniqueness of fractional differential equation solutions. We also believe that the findings of this study will help to solve additional practical problems in mathematical physics, statistics, and approximation theory.

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