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Interval-Valued Pseudo Overlap Functions and Application

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Abstract: A class of interval-valued OWA operators can be constructed from interval-valued overlap functions with interval-valued weights, which plays an important role in solving multi-attribute decision making (MADM) problems considering interval numbers as attribute values. Among them, when the importance of multiple attributes is different, it can only be calculated by changing the interval-valued weights. In fact, we can directly abandon the commutativity and extend the interval-valued overlap functions (IO) to interval-valued pseudo overlap functions (IPO) so that function itself implies the weights of the attributes, thus there is no need to calculate the OWA operator, which is more flexible in applications. In addition, the similar generalization on interval-valued pseudo t-norms obtained from interval-valued t-norms further enhances the feasibility of our study. In this paper, we mainly present the notion of interval-valued pseudo overlap functions and a few their qualities, including migrativity and homogeneity, and give some construction theorems and specific examples. Then, we propose the definitions of residuated implications induced by interval-valued pseudo overlap functions, give their equivalent forms, and prove some properties satisfied by them. Finally, two application examples about IPO to interval-valued multi-attribute decision making (I-MADM) are described. The results show that interval-valued pseudo overlap functions can not only be used to obtain the same rankings, but also be more flexible, simple and widely used.

Keywords: interval-valued pseudo overlap functions; interval-valued residuated implications; migrativity; homogeneity; interval-valued decision making

1. Introduction

Interval-valued fuzzy sets (IFS) were first proposed by Zadeh and they then attracted a large number of other scholars who conducted in-depth research on them [1–4]. They are extensively used in many practical problems, for instance, decision-making [5–7], image processing [8–10] and classification [11–13] etc. After that, in order to better deal with the uncertain information, some scholars extended the overlap functions [14–16] to the IFS, which is introduced in [17,18]. In fact, the commutativity of overlap functions on fuzzy sets limits their application to a certain extent. Some scholars have studied non-commutative overlap functions, namely pseudo overlap functions, and achieved some results [19]. In addition, some scholars introduced pairs of interval negations and interval implications [20,21], which are induced by non-commutative property and studied the non-commutative fuzzy interval logic system and defined interval negation [22]. With regard to the interval-valued t-norms [23] associated with interval-valued overlap functions, some scholars also removed their commutativity and studied the theory of interval-valued pseudo t-norms. For example, in [24], the implication pairs based on interval-valued pseudo t-norms are discussed. All these show that the commutativity of binary operations is sometimes unnecessary. Therefore, we also naturally consider extending the pseudo overlap functions to the IFS, studying the interval-valued pseudo overlap functions, removing the commutativity of the IO, and then expanding their application scope.

On the other hand, an interval-valued OWA operator is a kind of OWA operator on interval-valued fuzzy sets, which plays an important role in solving interval-valued

multi-attribute decision making problems [25–27]. In recent years, it has been studied and discussed by many scholars. In [17], Benjamin Bedregal et al. propose a new construction method, which combines the interval-valued aggregation function and IO with the interval-valued weights, which can be used to solve interval-valued decision making problems. However, what is difficult to ignore in multi-attribute decision making problems is that different attributes may have different degrees of importance, and the IO are difficult to realize the difference because of their commutativity. For the above reasons, in this article, we propose the conception of IPO, research their properties, and apply them to interval-valued multi-attribute decision making to illustrate their flexibility.

The composition of this paper is as below. We introduce some basic concepts about interval-valued functions and their properties, as well as interval-valued fuzzy implication in Section 2. As for Section 3, we put forward the definition of IPO and their induced interval-valued residuated implications, briefly illustrate the filiation between IPO and pseudo t-norms on IFS, afterwards we give a few examples. Then, the representability of IPO is proposed and several equivalent characterizations are given, furthermore, we also analyze the properties of interval-valued residuated implications induced by IPO which are representable. Moreover, we elaborate the related propositions of interval-valued pseudo overlap functions satisfying migrativity or homogeneity. Finally, we expand the interval-valued pseudo overlap functions to n-dimension, list some examples, and show two specific cases concerning how they are applied to I-MADM problems in Section 4. The results indicate that using interval-valued pseudo overlap functions can not only get the same results as other methods, but can also be more simple and flexible. Conclusions and references are given at the end.

2. Preliminaries

We first give some basic concepts about interval and interval-valued functions. Define $IV([0,1]) = \{[a,b] : 0 \leq a \leq b \leqslant 1\}$ as the family of all closed subintervals based on $[0,1]$, the concept of projections are given by: $\underline{[a,b]} = a, \overline{[a,b]} = b$. Some operations on $IV([0,1])$ are defined as follows [28]: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$, $XY = [\underline{XY}, \overline{XY}]$, $X/Y = [\underline{X}/\overline{Y}, \overline{X}/\underline{Y}]$ where $Y \neq 0$, $X^c = [1 - \overline{X}, 1 - \underline{X}]$, $X \wedge Y = [\min\{\underline{X}, \underline{Y}\}, \min\{\overline{X}, \overline{Y}\}]$, $X \vee Y = [\max\{\underline{X}, \underline{Y}\}, \max\{\overline{X}, \overline{Y}\}]$, $X^{[k_1,k_2]} = [\underline{X}^{k_2}, \overline{X}^{k_1}]$ for $0 < k_1 \leqslant k_2$.

Definition 1 ([29]). *Given an interval-valued function (IF) $F: IV([0,1])^n \rightarrow IV([0,1])$, it is defined as inclusion increasing when it is monotonically increasing about the inclusion order, where inclusion order is defined as: $X \subseteq Y$ iff $\underline{Y} \leq \underline{X}, \overline{X} \leq \overline{Y}$.*

In fact, there is another common order on $IV([0,1])$, that is, the product order, which is defined as: $[a,b] \leq [c,d]$ iff $a \leq c$ and $b \leq d$ for arbitrary $[a,b], [c,d] \in IV([0,1])$ [2,17]. If there is no special emphasis, the order mentioned in this paper refers to the product order. The following are some definitions, lemmas and propositions used in this paper.

Definition 2 ([1,30]). *Given the mapping f: $X^n \rightarrow Y$ is a real function, we call IF $\widehat{f}: IV(X)^n \rightarrow IV(Y)$ as the best interval representation composed of f, where $\widehat{f}$ defined as below:*

$$\widehat{f}(A) = [\inf\{f(a) : a \in A\}, \sup\{f(a) : a \in A\}].$$

Definition 3 ([19]). *The binary operator O: $[0,1]^2 \rightarrow [0,1]$ is a pseudo overlap function if for any $x, y \in [0,1]$, it meets these requirements:*

(PO1) $O(x,y) = 0$ *iff* $x = 0$ *or* $y = 0$;
(PO2) $O(x,y) = 1$ *iff* $x = y = 1$;
(PO3) *O is non-decreasing;*
(PO4) *O is continuous about each argument.*

Remark 1. *Similar to the concepts of 0-overlap function and 1-overlap function, we have definitions of 0-pseudo overlap function (0-PO) and 1-pseudo overlap function (1-PO). Based on the above definition, when the (PO1) becomes if $xy = 0$ then $O(x, y) = 0$ (PO1'), and other conditions remain unchanged, we call the mapping O is a 0-PO. In the same way, when the (PO2) becomes if $xy = 1$ then $O(x, y) = 1$ (PO2'), and other conditions remain unchanged, we call the mapping O is a 1-PO.*

Definition 4 ([17,31]). *A mapping M: $IV([0, 1])^n \rightarrow IV([0, 1])$ is called an interval-valued aggregation function if it has the below properties:*

(M1) *M is increasing, i.e., for each $i = 1, \ldots, n$, if $X_i \leq Y_i$, it holds that $M(X_1, \ldots, X_n) \leq M(Y_1, \ldots, Y_n)$;*

(M2) *Boundary condition, is defined as: $M([0, 0], \ldots, [0, 0]) = [0, 0], M([1, 1], \ldots, [1, 1]) = [1, 1]$.*

Definition 5 ([17,30]). *Given the mapping O on $IV([0, 1])$, it is an IO when and only when it meets the below requirements:*

(O1) *$O(X, Y) = O(Y, X)$;*

(O2) *$O(X, Y) = [0, 0]$ when and only when $X = [0, 0]$ or $Y = [0, 0]$;*

(O3) *$O(X, Y) = [1, 1]$ when and only when $X = Y = [1, 1]$;*

(O4) *O is monotonic about every element, i.e., if $Y \leq Z$, then $O(X, Y) \leq O(X, Z)$;*

(O5) *O is Moore continuous, i.e., for any $(X_1, X_2), (Y_1, Y_2) \in IV([0, 1])^2$, and any $\varepsilon > 0$, there is $\delta > 0$ such that $d((X_1, X_2), (Y_1, Y_2)) < \delta \Rightarrow d(O(X_1, X_2), O(Y_1, Y_2)) < \varepsilon$, where $d((X_1, X_2), (Y_1, Y_2)) = \sqrt{(\max\{|\underline{X_1} - \underline{Y_1}|, |\overline{X_1} - \overline{Y_1}|\})^2 + (\max\{|\underline{X_2} - \underline{Y_2}|, |\overline{X_2} - \overline{Y_2}|\})^2}$, $d(O(X_1, X_2), O(Y_1, Y_2)) = \max\{|\underline{O(X_1, X_2)} - \underline{O(Y_1, Y_2)}|, |\overline{O(X_1, X_2)} - \overline{O(Y_1, Y_2)}|\}$ (see [32]).*

Definition 6 ([24]). *The mapping T defined on $IV([0, 1])$ is called an interval-valued pseudo t-norm (IPtm) when T meets the terms below:*

(T1) *T is associative;*

(T2) *T is monotonic about the product order and the inclusion order;*

(T3) *T has [1, 1] as an identity element, i.e., for arbitrary $X \in IV([0, 1])$, $T(X, [1, 1]) = T([1, 1], X) = X$.*

Definition 7 ([17]). *An IPtm T on $IV([0, 1])$ is called positive when it meets $T([a, b], [c, d]) = [0, 0]$ when and only when $[a, b] = [0, 0]$ or $[c, d] = [0, 0]$.*

Before introducing the following lemma, we firstly state a definition of the IF. Given two real functions $i, j: X^n \rightarrow Y$ satisfying $i \leq j$, then the IF $IV([i, j]): IV(X)^n \rightarrow IV(Y)$ defined as $IV([i, j])(A) = [i(\underline{A}), j(\overline{A})]$ for arbitrary $A \in IV(X)^n$.

Lemma 1 ([33]). *Suppose that $i, j: X^n \rightarrow Y$ are monotonous real functions and they satisfy $i \leq j$. Then the following sentences are equal in value:*

(1) *IV([i, j]) is Moore continuous;*

(2) *i and j are continuous.*

Definition 8 ([17]). *Given a monotonic IF $F: IV([0, 1])^n \rightarrow IV([0, 1])$. We call the left and right projections of F the mappings $\underline{F}, \overline{F}: [0, 1]^n \rightarrow [0, 1]$ defined by*

$$\underline{F}(x_1, \ldots, x_n) = \underline{F([x_1, x_1], \ldots, [x_n, x_n])}, \overline{F}(x_1, \ldots, x_n) = \overline{F([x_1, x_1], \ldots, [x_n, x_n])} \quad (1)$$

respectively.

Proposition 1 ([1]). *Given a monotonically increasing (or decreasing) IF $F: IV(X)^n \rightarrow IV(Y)$, then it is monotonic about inclusion order if and only if it holds that $F = IV([\underline{F}, \overline{F}])$.*

Definition 9 ([17]). *Given an IF F: $IV([0,1])^2 \to IV([0,1])$, it is called migrative when for arbitrary $[m,n],[a,b],[c,d] \in IV([0,1])$, it satisfies Equation $F([m,n][a,b],[c,d]) = F([a,b],[m,n][c,d])$.*

Definition 10 ([17]). *Given an IF F: $IV([0,1])^2 \to IV([0,1])$, it is called K-order homogeneous, where $K = [k_1, k_2]$ with $0 < k_1 \le k_2$ when for arbitrary $[m,n],[a,b],[c,d] \in IV([0,1])$, it satisfies Equation $F([m,n][a,b],[m,n][c,d]) = [m,n]^K F([a,b],[c,d])$.*

Definition 11. *Given an IF F: $IV([0,1])^2 \to IV([0,1])$, it is called idempotent if for arbitrary $X \in IV([0,1])$, it holds that $F(X,X) = X$.*

Besides, some related propositions with regard to the properties satisfied by interval-valued functions are given below.

Proposition 2 ([17,34]). *Given an IF F: $IV([0,1])^2 \to IV([0,1])$, then it satisfies migrativity when and only when it satisfies $F([a,b],[c,d]) = F([1,1],[a,b][c,d])$.*

Proposition 3 ([17]). *Given an IF F: $IV([0,1])^2 \to IV([0,1])$, the formulations as below are established:*

(1) *If F is migrative, then it is commutative;*
(2) *If F is K-order homogeneous, then it holds that $F([0,0],[0,0]) = [0,0]$;*
(3) *F satisfies idempotency when F is K-order homogeneous where $K = [1,1]$ and satisfies $F([1,1],[1,1]) = [1,1]$;*
(4) *F is K-order homogeneous where $K = [1,1]$ when F satisfies migrativity and idempotency;*
(5) *For arbitrary $[a,b] \in IV([0,1])$, F is K-order homogeneous where $K = [1,1]$ when F satisfies migrativity and $F([1,1],[a,b]) = F([a,b],[1,1]) = [a,b]$.*

Definition 12 ([19]). *Given a pseudo overlap function PO: $[0,1]^2 \to [0,1]$, the following $R_O^{(1)}, R_O^{(2)} : [0,1]^2 \to [0,1]$ are said to be left (right) residuated operators, for arbitrary $x, y \in [0,1]$:*

$$R_O^{(1)} = \sup\{z \in [0,1] \mid PO(z,x) \le y\}, \tag{2}$$

$$R_O^{(2)} = \sup\{z \in [0,1] \mid PO(x,z) \le y\}. \tag{3}$$

Definition 13 ([35]). *An interval-valued fuzzy implication is defined to be a function I: $IV([0,1])^2 \to IV([0,1])$ that satisfies the requirements as below:*

(I1′) *boundary conditions: $IV([0,0],[0,0]) = IV([0,0],[1,1]) = IV([1,1],[1,1]) = [1,1], IV([1,1],[0,0]) = [0,0]$;*
(I2′) *an extension of the fuzzy implication, i.e., if $IV([x,x],[y,y]) = [a,b]$ then $a = b$;*
(I3′) *decreasing about the first element, i.e., if $[a,b] \le [a',b']$ then $IV([a,b],[m,n]) \ge IV([a',b'],[m,n])$;*
(I4′) *increasing about the second element, i.e., if $[m,n] \le [m',n']$ then $IV([a,b],[m,n]) \le IV([a,b],[m',n'])$.*

3. Interval-Valued Pseudo Overlap Functions and Interval-Valued Residuated Implications

3.1. IPO

In this subchapter, basic notions of IPO as well as the interval-valued residuated implications induced by them are introduced, some concrete examples are given, and then we briefly analyze the relevance between IPO and IPtm.

Definition 14. *An IF O: $IV([0,1])^2 \to IV([0,1])$ is called an IPO when O meets the below statements:*

(O1′) *$O(X,Y) = [0,0]$ when and only when $XY = [0,0]$;*
(O2′) *$O(X,Y) = [1,1]$ when and only when $XY = [1,1]$;*
(O3′) *If $Y \le Z$, then $O(Y,X) \le O(Z,X)$ and $O(X,Y) \le O(X,Z)$;*

(O4$'$)   *O is Moore continuous.*

Obviously, if an IPO is commutative, then it is an IO. Next, some examples of interval-valued pseudo overlap functions are given as follows.

Example 1. (1)   *Any IO is an IPO.*
(2)   *The mapping PO:* $IV([0,1])^2 \to IV([0,1])$ *defined as* $PO([a,b],[c,d]) = [a \wedge c, b \wedge d]$ *is an* IPO, *and also an* IO.
(3)   *The mapping PO:* $IV([0,1])^2 \to IV([0,1])$ *defined as* $PO([a,b],[c,d]) = [a^2c, bd\frac{b+d}{2}]$ *is an* IPO, *but not an* IO.

For any two interval-valued pseudo overlap functions $O_1, O_2 : IV([0,1])^2 \to IV([0,1])$, we mark $O_1 \leq O_2$ when and only when for arbitrary $X, Y \in IV([0,1])$, $O_1(X,Y) \leq O_2(X,Y)$ is established.

Lemma 2. *Given the mapping O is an* IPO. *If* $M \neq N$, *then it holds that* $O([1,1], M) \neq O([1,1], N)$ *and* $O(M, [1,1]) \neq O(N, [1,1])$ *for any* $M, N \in IV([0,1])$.

Proof. Assume that $O([1,1], M) = O([1,1], N)$. Then we have that $d(O([1,1], M), O([1,1], N)) = 0$ and because $M \neq N \Rightarrow d(X,Y) > 0$, we can obtain O is not Moore continuous, which is contradictory. Similarly, we have that $O(M, [1,1]) \neq O(N, [1,1])$.   □

As we all know, overlap functions on fuzzy sets are closely related to t-norms. Therefore, we also discuss the correlation between pseudo overlap functions and pseudo t-norms on IFS.

Proposition 4. *Given the mapping PO is an* IPO. *If PO satisfies associative law, then PO is a positive and Moore continuous interval-valued pseudo t-norm.*

Proof. Assume that $PO([1,1], X) \neq X$ for some $X \in IV([0,1])$, according to (O2$'$) and Lemma 2, since PO is associative, it is clear that $PO([1,1], X) \neq PO([1,1], PO([1,1], X)) = PO(PO([1,1], [1,1]), X) = PO([1,1], X)$, which is contradictory. Similarly, suppose that $PO(X, [1,1]) \neq X$ for some $X \in IV([0,1])$, we also have a contradiction. So it is clear that $PO([1,1], X) = X = PO(X, [1,1])$ for all $X \in IV([0,1])$, i.e., PO takes [1, 1] as the unit element. Therefore, according to the definition of IPO, it is monotonic, positive and continuous, so PO is a positive and Moore continuous interval-valued pseudo t-norm.   □

Note that in the opposite sense, any positive and Moore continuous interval-valued pseudo t-norm is an associative interval-valued pseudo overlap function with [1, 1] as the unit element.

Next, we give a method to make up the IPO.

Theorem 1. *Given two interval-valued pseudo overlap functions* O_1, O_2, *then the* IF *PO:* $IV([0,1])^2 \to IV([0,1])$ *defined as* $PO(X,Y) = O_1(X,Y)O_2(X,Y)$ *satisfies* (O1$'$) $\sim$ (O4$'$), *i.e., it is an IPO.*

Proof. Since $X = [0,0]$ or $Y = [0,0] \Leftrightarrow O_1(X,Y) = O_2(X,Y) = [0,0] \Leftrightarrow PO(X,Y) = [0,0]$, we have that PO satisfies (O1$'$). Similarly, we can get PO satisfies (O2$'$). For arbitrary $X, Y, Z \in IV([0,1])$, when $X \leq Y$, since O_1 and O_2 are non-decreasing, it holds that $PO(Z,X) = O_1(Z,X)O_2(Z,X) \leq O_1(Z,Y)O_2(Z,Y) = PO(Z,Y)$ and $PO(X,Z) \leq PO(Y,Z)$ analogously, so PO is increasing. Finally, since O_1 and O_2 are Moore continuous, by definition, it is obvious that PO is Moore continuous.   □

In addition, another method of obtaining the interval-valued pseudo overlap function using the best interval representation of the pseudo overlap function is as follows.

Theorem 2. *Given a pseudo overlap function O, the function $\widehat{O}$: $IV([0,1])^2 \to IV([0,1])$ defined as $\widehat{O}(X,Y) = [\inf\{O(x,y): x \in X, y \in Y\}, \sup\{O(x,y): x \in X, y \in Y\}]$ satisfies (O1')~(O4').*

Proof. (O1') $\widehat{O}(X,Y) = [0,0] \Leftrightarrow$ for any $x \in X$ and $y \in Y$, $\inf O(x,y) = \sup O(x,y) = 0 \Leftrightarrow$ for any $x \in X$ and $y \in Y$, $O(x,y) = 0 \Leftrightarrow$ for any $x \in X$ and $y \in Y$, $xy = 0 \Leftrightarrow X = [0,0]$ or $Y = [0,0]$. (O2') In a similar way to the above, we can also get $\widehat{O}$ satisfies (O2'). (O3') For arbitrary $X,Y,Z \in IV([0,1])$, when $X \leq Y$, it is clear that $\widehat{O}(Z,X) = [\inf\{O(z,x): z \in Z, x \in X\}, \sup\{O(z,x): z \in Z, x \in X\}] \leq [\inf\{O(z,y): z \in Z, y \in Y\}, \sup\{O(z,y): z \in Z, y \in Y\}] = \widehat{O}(Z,Y)$, similarly, we also have $\widehat{O}(X,Z) \leq \widehat{O}(Y,Z)$. (O4') for arbitrary $X,Y, \in IV([0,1])$, because O satisfies (PO4), we have that $\widehat{O}(X,Y) = [\inf\{O(x,y): x \in X, y \in Y\}, \sup\{O(x,y): x \in X, y \in Y\}]$, i.e., for arbitrary $x \in X, y \in Y$, it equals $[O(\inf x, \inf y), O(\sup x, \sup y)] = [O(\underline{X}, \underline{Y}), O(\overline{X}, \overline{Y})]$, so $\widehat{O} = IV([O,O])$. Then it is clear that $\widehat{O}$ is Moore continuous by Lemma 1. $\square$

When studying interval-valued fuzzy operators, it is inevitable to discuss the interval-valued fuzzy implication induced by them. So we also give the definition of interval-valued residuated implications induced by IPO as follows.

Definition 15. *Given the mapping PO on $IV([0,1])$ is an IPO. We call the interval-valued residuated implications induced by PO to be the interval-valued functions $IR^{(1)}$ and $IR^{(2)}$, where they are defined by*

$$IR^{(1)}(X,Y) = \sup\{Z \in IV([0,1]) \mid PO(Z,X) \leq Y\}, \tag{4}$$

$$IR^{(2)}(X,Y) = \sup\{Z \in IV([0,1]) \mid PO(X,Z) \leq Y\} \tag{5}$$

respectively, for any $X,Y \in IV([0,1])$.

Example 2. *The concrete examples of interval-valued residuated implications corresponding to the examples in Example 1 above are as follows:*

(1)

$$IR^{(1)}([a,b],[c,d]) = IR^{(2)}([a,b],[c,d]) = \begin{cases} [1,1] & a \leq c, b \leq d \\ [c,1] & a > c, b \leq d \\ d & a \leq c, b > d \\ [c,d] & a > c, b > d \end{cases} \tag{6}$$

(2)

$$IR^{(1)}([a,b],[c,d]) = \begin{cases} [1,1] & a \leq c, b \leq d \\ [\sqrt{\frac{c}{a}},1] & a > c, b \leq d \\ \min\{\frac{\sqrt{b^2+8bd}-b^2}{2b},1\} & a \leq c, b > d \\ [\min\{\sqrt{\frac{c}{a}}, \frac{\sqrt{b^2+8bd}-b^2}{2b}\}, \min\{\frac{\sqrt{b^2+8bd}-b^2}{2b},1\}] & a > c, b > d \end{cases} \tag{7}$$

$$IR^{(2)}([a,b],[c,d]) = \begin{cases} [1,1] & a \leq c, b \leq d \\ [\min\{\frac{c}{a^2},1\},1] & a > c, b \leq d \\ \min\{\frac{\sqrt{b^2+8bd}-b^2}{2b},1\} & a \leq c, b > d \\ [\min\{\frac{c}{a^2}, \frac{\sqrt{b^2+8bd}-b^2}{2b},1\}, \min\{\frac{\sqrt{b^2+8bd}-b^2}{2b},1\}] & a > c, b > d \end{cases} \tag{8}$$

3.2. Representable Interval-Valued Pseudo Overlap Functions

Similar to the representability of IO, we can also research the representability of the IPO.

Theorem 3. *Given two pseudo overlap functions O_1 and O_2 satisfying $O_1 \leq O_2$, then the function $\widetilde{O_1 O_2}$: $IV([0,1])^2 \to IV([0,1])$ defined as $\widetilde{O_1 O_2}(X, Y) = [O_1(\underline{X}, \underline{Y}), O_2(\overline{X}, \overline{Y})]$ is an IPO.*

Proof. We confirm that the function $\widetilde{O_1 O_2}$ satisfies (O1')~(O4') as follows.

(O1') $\widetilde{O_1 O_2}(X, Y) = [0,0] \Leftrightarrow [O_1(\underline{X}, \underline{Y}), O_2(\overline{X}, \overline{Y})] = [0,0] \Leftrightarrow O_1(\underline{X}, \underline{Y}) = 0$ as well as $O_2(\overline{X}, \overline{Y}) = 0 \Leftrightarrow \overline{X} = 0$ or $\overline{Y} = 0$, since $\underline{X} \leq \overline{X}$ and $\underline{Y} \leq \overline{Y}$, when and only when $X = [0,0]$ or $Y = [0,0]$.

(O2') Similar to the above, we have $\widetilde{O_1 O_2}(X, Y) = [1,1] \Leftrightarrow X = [1,1]$ and $Y = [1,1]$.

(O3') For arbitrary $X, Y, Z \in IV([0,1]), X \leq Y \Rightarrow \underline{X} \leq \underline{Y}$ as well as $\overline{X} \leq \overline{Y}$, since O_1, O_2 are increasing, so we have $\widetilde{O_1 O_2}(X, Z) = [O_1(\underline{X}, \underline{Z}), O_2(\overline{X}, \overline{Z})] \leq [O_1(\underline{Y}, \underline{Z}), O_2(\overline{Y}, \overline{Z})] = \widetilde{O_1 O_2}(Y, Z)$, similarly, it holds that $\widetilde{O_1 O_2}(Z, X) \leq \widetilde{O_1 O_2}(Z, Y)$.

(O4') Since O_1, O_2 are non-decreasing and continuous, and $O_1 \leq O_2$, it is clear that $\widetilde{O_1 O_2}(X, Y) = IV([O_1, O_2])(X, Y)$ and it is Moore continuous by Lemma 1. □

According to the above theorem, the definition of representable IPO as below can be generated.

Definition 16. *Given the mapping O is an IPO on $IV([0,1])$, we call it representable when there are two pseudo overlap functions O_1 and O_2 satisfying $O = \widetilde{O_1 O_2}$, where O_1, O_2 are called representatives of O.*

A few examples of the interval-valued pseudo overlap functions obtained by the pseudo overlap function are given below.

Example 3. (1) *The mapping $O_{mM} : IV([0,1])^2 \to IV([0,1])$ defined as*

$$O_{mM}([a,b], [c,d]) = [\min\{a,c\}\max\{a^2, c\}, \min\{b,d\}\max\{b^2, d\}] \tag{9}$$

is an IPO.
(2) *The mapping $O_{pq} : IV([0,1])^2 \to IV([0,1])$ defined as*

$$O_{pq}([a,b], [c,d]) = [a^p \cdot c^q, b^p \cdot d^q], p, q > 0 \tag{10}$$

is an IPO.
(3) *The mapping $O : IV([0,1])^2 \to IV([0,1])$ defined as*

$$O([a,b], [c,d]) = [\frac{2a^p c^q}{1 + a^p c^q}, \frac{2b^p d^q}{1 + b^p d^q}] \tag{11}$$

is an IPO.
(4) *The mapping $O : IV([0,1])^2 \to IV([0,1])$ defined as*

$$O([a,b], [c,d]) = [\min\{a^p, c^q\}, \min\{b^p, d^q\}], p, q > 0 \tag{12}$$

is an IPO.

Remark 2. *There exists some IPO that are not representable. For instance, the mapping $O(X, Y) = [\max\{\underline{X} + \underline{Y} - 1, 0\}, \overline{XY}^2]$ is not representable. Suppose that there are pseudo overlap functions O_1 and O_2 satisfying $O_1 \leq O_2$ and $O = \widetilde{O_1 O_2}$, then for arbitrary $X, Y \in IV([0,1])$, there is $O(X, Y) = \widetilde{O_1 O_2}(X, Y) = [O_1(\underline{X}, \underline{Y}), O_2(\overline{X}, \overline{Y})]$, in particular, when $\underline{X} = 0$ and $\underline{Y} = 0.5, O_1(\underline{X}, \underline{Y}) = 0$, which is in contradiction with (PO1).*

In order to describe the representability of IPO, we give propositions below.

Proposition 5. *Given the mapping PO is an IPO. For arbitrary $z \in (0,1]$, if PO satisfies that $\underline{XY} = 0$ when $PO(X,Y) = [0,z]$ (strongly positive), then $\underline{PO}$ and $\overline{PO}$ are pseudo overlap functions.*

Proof. $\underline{PO}(x_1, x_2) = 0 \Rightarrow \underline{PO}([x_1,x_1],[x_2,x_2]) = 0 \Rightarrow PO([x_1,x_1],[x_2,x_2]) = [0,z]$ is established for any $z \in [0,1]$, when $z = 0$, since PO satisfies (O1'), then $[x_1,x_1] = [0,0]$ or $[x_2,x_2] = [0,0]$, i.e., $x_1 x_2 = 0$, if $z > 0$, from the property satisfied by PO in the proposition, we have that $x_1 = 0$ or $x_2 = 0$. On the other hand, if $x_1 > 0, x_2 > 0$, then $PO([x_1,x_1],[x_2,x_2]) > [0,0] \Rightarrow \underline{PO}(x_1,x_2) \geq 0$. However, when $\underline{PO}(x_1,x_2) = 0$, considering the above evidence, it holds that $x_1 = 0$ or $x_2 = 0$, which is contrary to $x_1 > 0$ and $x_2 > 0$. So the function $\underline{PO}$ satisfies (PO1). $\underline{PO}(x_1,x_2) = \underline{PO}([x_1,x_1],[x_2,x_2]) = 1 \Leftrightarrow PO([x_1,x_1],[x_2,x_2]) = [1,1]$, since PO satisfies (O2'), it is clear that $[x_1,x_1] = [1,1]$ and $[x_2,x_2] = [1,1]$, i.e., $x_1 = 1$ and $x_2 = 1$. So the function $\underline{PO}$ satisfies (PO2). Since PO satisfies (O3'), $\underline{PO}$ clearly satisfies (PO3). Finally, since $\underline{PO}$ is the left projection of PO, which are continuous functions, $\underline{PO}$ is continuous. Similarly, $\overline{PO}$ is also a pseudo overlap function. □

Proposition 6. *If the mapping O is an IPO and it is representable, then $O = \widetilde{\underline{O}\,\overline{O}}$.*

Proof. Assume that O is representable, then according to Theorem 3, there are two pseudo overlap functions O_1 and O_2 satisfying $O_1 \leq O_2$ and $O = \widetilde{O_1 O_2}$. Therefore, $O_1(x,y) = [O_1(x,y), O_2(x,y)] = \widetilde{O_1 O_2}([x,x],[y,y]) = O([x,x],[y,y]) = \underline{O}(x,y)$, similarly, it is clear that $O_2(x,y) = \overline{O}(x,y)$. So $O = \widetilde{\underline{O}\,\overline{O}}$. □

Remark 3. *The inverse of the above proposition is not necessarily true, that is, if there is $O = \widetilde{\underline{O}\,\overline{O}}$ for an IPO, it is not necessarily representable. For example, O is an IPO and defined as $O([x_1,x_2],[y_1,y_2]) = [\max\{x_1 + y_1 - 1, 0\}, \min\{x_2^p, y_2^q\}]$, where $p, q > 0$. However, we have that $\underline{O} = \max\{x_1 + y_1 - 1, 0\}$, which does not satisfy (PO1), i.e., it is not a pseudo overlap function, so O is not representable.*

Point at the above proposition, we can add a condition to make it reversible. The conclusion is as follows.

Theorem 4. *For an interval-valued pseudo overlap function O, the following two statements are equivalent:*

(1)　*O is representable;*

(2)　*O is strongly positive and $O = \widetilde{\underline{O}\,\overline{O}}$.*

Proof. (1) $\Rightarrow$ (2) The latter is obvious from Proposition 6. When O is representable, it holds that $O = \widetilde{O_1 O_2}$ and O_1, O_2 are pseudo overlap functions. Then for some $e \in (0,1]$, it is clear that $O(X,Y) = [0,e] \Rightarrow \widetilde{O_1 O_2}(X,Y) = [0,e] \Rightarrow O_1(\underline{X},\underline{Y}) = 0 \Rightarrow \underline{X} = 0$ or $\underline{Y} = 0$, i.e., O is strongly positive.
(2) $\Rightarrow$ (1) It is obvious from Proposition 5. □

Theorem 5. *Given the mapping O on $IV([0,1])$ is a representable IPO, then it is inclusion monotonic.*

Proof. By Proposition 6, since O is representable, it is clear that $O = \widetilde{\underline{O}\,\overline{O}}$, i.e., $O(X,Y) = [\underline{O}(\underline{X},\underline{Y}), \overline{O}(\overline{X},\overline{Y})] = IV([\underline{O},\overline{O}])(X,Y)$, by Proposition 1, we have that O is monotonic about the inclusion order. □

It is important to note that the inverse of the above theorem is also not necessarily true. We can take the example in Remark 3 above to explain. The mapping O is an IPO and inclusion increasing, but it is not representable where $\underline{O}$ is not a pseudo overlap function.

In addition, a weak equivalent characterization of a representable IPO is given below.

Theorem 6. *Given an inclusion increasing* IF *O on* $IV([0,1])$. *In this way, it is an* IPO *if and only if* $O = IV([O_1, O_2])$, *where* O_1 *is a 0-PO and* O_2 *is a 1-PO with* $O_1 \leq O_2$. *Particularly, we have that* $O_1 = \underline{O}$ *and* $O_2 = \overline{O}$.

Proof. ($\Rightarrow$) Presume that an IPO $O : IV([0,1])^2 \to IV([0,1])$ satisfies monotonic increment about inclusion order. Consider two functions $O_1, O_2 : [0,1]^2 \to [0,1] : O_1(x,y) = \underline{O([x,x],[y,y])}$, $O_2(x,y) = \overline{O([x,x],[y,y])}$, which are clearly definable. Then we certificate that O_1 is a 0-PO. (*i*) if $xy = 0$, then $O_1(x,y) = \underline{O([x,x],[y,y])} = \underline{[0,0]} = 0$; (*ii*) if $xy = 1$, then $O_1(x,y) = \underline{O([x,x],[y,y])} = \underline{[1,1]} = 1$, conversely, if $O_1(x,y) = 1$, then $\underline{O([x,x],[y,y])} = 1 \Rightarrow O([x,x],[y,y]) = [1,1]$, since O satisfies (O2'), $xy = 1$; (*iii*) for arbitrary $a,b,c \in [0,1]$, suppose that $b \leq c$, since O is monotonic, it holds that $O_1(b,a) = \underline{O([b,b],[a,a])} \leq \underline{O([c,c],[a,a])} = O_1(c,a)$, similarly we have that $O_1(a,b) \leq O_1(a,c)$; (*iv*) According to Proposition 1, $O = IV([\underline{O},\overline{O}])$, since O satisfies (O4'), $O_1 = \underline{O}$ and it satisfies (PO4) by Lemma 1. Similarly, we have that O_2 is a 1-pseudo overlap function and $O_1 \leq O_2$, $O = IV([O_1, O_2])$.

($\Leftarrow$) we prove that O satisfies (O1') $\sim$ (O4'). (O1') $O(X,Y) = [0,0] \Leftrightarrow [O_1(\underline{X},\underline{Y}), O_2(\overline{X},\overline{Y})] = [0,0] \Leftrightarrow O_1(\underline{X},\underline{Y}) = 0$ and $O_2(\overline{X},\overline{Y}) = 0 \Leftrightarrow \overline{X}\overline{Y} = 0 \Leftrightarrow X = [0,0]$ or $Y = [0,0]$. (O2') Similar to the above, we have O satisfies (O2'). (O3') for arbitrary $X, Y, Z \in IV([0,1])$, when $Z \geq Y$, $O(X,Z) = [O_1(\underline{X},\underline{Z}), O_2(\overline{X},\overline{Z})] \geq [O_1(\underline{X},\underline{Y}), O_2(\overline{X},\overline{Y})] = O(X,Y)$ and $O(Y,X) = [O_1(\underline{Y},\underline{X}), O_2(\overline{Y},\overline{X})] \leq [O_1(\underline{Z},\underline{X}), O_2(\overline{Z},\overline{X})] = O(Z,X)$, i.e., O is monotonic. (O4') According to Lemma 1, it is obvious that O satisfies (O4'). $\square$

In fact, the above theorem is equivalent to a method of using 0-pseudo overlap function and 1-pseudo overlap function to construct an interval-valued pseudo overlap function. Here are some examples.

Example 4. (1) *A mapping* $O : IV([0,1])^2 \to IV([0,1])$ *defined as*

$$O([a,b],[c,d]) = [\max\{a + c - 1, 0\}, \frac{2b^p d^q}{1 + b^p d^q}], p, q > 0 \tag{13}$$

is an IPO.
(2) *A mapping* $O : IV([0,1])^2 \to IV([0,1])$ *defined as*

$$O([a,b],[c,d]) = [\frac{2a^p c^q}{1 + a^p c^q}\max\{a + c - 1, 0\}, \frac{2b^p d^q}{1 + b^p d^q}], p, q > 0 \tag{14}$$

is an IPO.
(3) *A mapping* $O : IV([0,1])^2 \to IV([0,1])$ *defined as*

$$O([a,b],[c,d]) = [\max\{a + c - 1, 0\}, b^2 d + b^2 d(1 - b)(1 - d)] \tag{15}$$

is an IPO.
(4) *A mapping* $O : IV([0,1])^2 \to IV([0,1])$ *defined as*

$$O([a,b],[c,d]) = [\frac{2a^p c^q}{1 + a^p c^q}, \min\{2bd, 1\}], p, q > 0 \tag{16}$$

is an IPO.

Next, we consider some features of interval-valued residuated implications induced by representable IPO.

As O is representable, it holds that $O = \widetilde{O_1 O_2}$, where O_1 and O_2 are two pseudo overlap functions, then $O(Z,X) = [O_1(\underline{Z},\underline{X}), O_2(\overline{Z},\overline{X})]$, $O(X,Z) = [O_1(\underline{X},\underline{Z}), O_2(\overline{X},\overline{Z})]$. Therefore, we can use the residuated implications induced by pseudo overlap functions O_1 and O_2 to represent the interval-valued residuated implication induced by O. The theorem is as follows.

Theorem 7. *Given a representable IPO O on $IV([0,1])$ and O_1, O_2 are two representatives of O such that $O_1 \leq O_2$. Then the interval-valued residuated implications $IR^{(1)}$, $IR^{(2)}$ induced by O have the following forms,*

$$IR^{(1)}(X,Y) = [R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})] \tag{17}$$

$$IR^{(2)}(X,Y) = [R^{(2)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(2)}_{O_2}(\overline{X},\overline{Y}), R^{(2)}_{O_2}(\overline{X},\overline{Y})] \tag{18}$$

where $R^{(1)}_{O_1}$, $R^{(2)}_{O_1}$ are residuated implications induced by O_1, $R^{(1)}_{O_2}$, $R^{(2)}_{O_2}$ are residuated implications induced by O_2.

Proof. First, according to definition, $IR^{(1)}(X,Y) = \sup\{Z \in IV([0,1]) \mid O(Z,X) \leq Y\} = \sup\{Z \in IV([0,1]) \mid O_1(\underline{Z},\underline{X}) \leq \underline{Y} \text{ and } O_2(\overline{Z},\overline{X}) \leq \overline{Y}\}$. We record the set $\{Z \in IV([0,1]) \mid O_1(\underline{Z},\underline{X}) \leq \underline{Y} \text{ and } O_2(\overline{Z},\overline{X}) \leq \overline{Y}\}$ as S, then we prove that $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})]$ is the minimum upper bound of S.

(*i*) Since $R^{(1)}_{O_1}(\underline{X},\underline{Y}) = \sup\{z \mid O_1(z,\underline{X}) \leq \underline{Y}\}$, $R^{(1)}_{O_2}(\overline{X},\overline{Y}) = \sup\{z \mid O_2(z,\overline{X}) \leq \overline{Y}\}$, it holds that for any $Z \in S$, $O_1(\underline{Z},\underline{X}) \leq \underline{Y}$ and $O_2(\overline{Z},\overline{X}) \leq \overline{Y} \Rightarrow R^{(1)}_{O_1}(\underline{X},\underline{Y}) \geq \underline{Z}$ and $R^{(1)}_{O_2}(\overline{X},\overline{Y}) \geq \overline{Z} \geq \underline{Z}$. So $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})] \geq [\underline{Z},\overline{Z}] = Z$, i.e., $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})]$ is the upper bound of S.

(*ii*) Suppose that there is $[a,b]$ another upper bound of S, i.e., for any $Z \in S$, $[a,b] \geq Z$, and such that $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})] \not\leq [a,b]$. At this point, we consider the following two cases. (1) $a < R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}) \Rightarrow a < R^{(1)}_{O_1}(\underline{X},\underline{Y})$ and $a < R^{(1)}_{O_2}(\overline{X},\overline{Y}) \Rightarrow a < \sup\{z \mid O_1(z,\underline{X}) \leq \underline{Y}\}$, $a < \sup\{z \mid O_2(z,\overline{X}) \leq \overline{Y}\} \Rightarrow \exists z_1, z_2 > a$, s.t. $O_1(z_1,\underline{X}) \leq \underline{Y}$ and $O_2(z_2,\overline{X}) \leq \overline{Y}$, we take $m = z_1 \wedge z_2$, then $m > a$ and $O_1(m,\underline{X}) \leq \underline{Y}$, $O_2(m,\overline{X}) \leq \overline{Y}$, i.e., $[m,m] \in S$ and $[m,m] \not\leq [a,b]$, which is a contradiction. (2) $a \geq R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y})$ and $b < R^{(1)}_{O_2}(\overline{X},\overline{Y})$, since $a \leq b$, $a < R^{(1)}_{O_2}(\overline{X},\overline{Y})$, then it holds that $a \geq R^{(1)}_{O_1}(\underline{X},\underline{Y})$. So we have that $a \geq \sup\{z \mid O_1(z,\underline{X}) \leq \underline{Y}\}$, we take n such that $O_1(n,\underline{X}) \leq \underline{Y}$, then $n \leq a$. On the other hand, $b < R^{(1)}_{O_2}(\overline{X},\overline{Y}) \Rightarrow b < \sup\{z \mid O_2(z,\overline{X}) \leq \overline{Y}\} \Rightarrow \exists z' > b \geq a$, $O_2(z',\overline{X}) \leq \overline{Y}$. So we have that $n \leq a \leq b < z'$ and $O_1(n,\underline{X}) \leq \underline{Y}$, $O_2(z',\overline{X}) \leq \overline{Y}$, i.e., $[n,z'] \in S$ and $[n,z'] \not\leq [a,b]$, which is also a contradiction. So $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})]$ is the minimum upper bound of S, i.e., $[R^{(1)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(1)}_{O_2}(\overline{X},\overline{Y}), R^{(1)}_{O_2}(\overline{X},\overline{Y})] = \sup\{Z \in IV([0,1]) \mid O(Z,X) \leq Y\}$. Similarly, $[R^{(2)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(2)}_{O_2}(\overline{X},\overline{Y}), R^{(2)}_{O_2}(\overline{X},\overline{Y})]$ is the minimum upper bound of the set $\{Z \in L([0,1]) \mid O_1(\underline{X},\underline{Z}) \leq \underline{Y} \text{ and } O_2(\overline{X},\overline{Z}) \leq \overline{Y}\}$, i.e., $[R^{(2)}_{O_1}(\underline{X},\underline{Y}) \wedge R^{(2)}_{O_2}(\overline{X},\overline{Y}), R^{(2)}_{O_2}(\overline{X},\overline{Y})] = \sup\{Z \in IV([0,1]) \mid O(X,Z) \leq Y\}$. □

Below we give some concrete examples of interval-valued residuated implications.

Example 5. *(1) Given an IPO defined as $O_{pq}(X,Y) = [\underline{X}^p \cdot \underline{Y}^q, \overline{X}^p \cdot \overline{Y}^q]$, where $p,q > 0$, the function $IR^{(1)}$, $IR^{(2)}$ defined as*

$$IR^{(1)}(X,Y) = \begin{cases} [1,1] & X^{[q,q]} \leq Y \\ [\sqrt[p]{\dfrac{\overline{Y}}{\overline{X}^q}}, \sqrt[p]{\dfrac{\overline{Y}}{\overline{X}^q}}] & \underline{X}^q \leq \underline{Y}, \overline{X}^q > \overline{Y} \\ [\sqrt[p]{\dfrac{\underline{Y}}{\underline{X}^q}}, 1] & \underline{X}^q > \underline{Y}, \overline{X}^q \leq \overline{Y} \\ [\sqrt[p]{\dfrac{\overline{Y}}{\overline{X}^q}} \wedge \sqrt[p]{\dfrac{\underline{Y}}{\underline{X}^q}}, \sqrt[p]{\dfrac{\overline{Y}}{\overline{X}^q}}] & X^{[q,q]} > Y \end{cases} \tag{19}$$

$$IR^{(2)}(X,Y) = \begin{cases} [1,1] & X^{[p,p]} \leq Y \\ [\sqrt[q]{\frac{\overline{Y}}{\overline{X}^p}}, \sqrt[q]{\frac{\overline{Y}}{\underline{X}^p}}] & \underline{X}^p \leq \underline{Y}, \overline{X}^p > \overline{Y} \\ [\sqrt[q]{\frac{\underline{Y}}{\underline{X}^p}}, 1] & \underline{X}^p > \underline{Y}, \overline{X}^p \leq \overline{Y} \\ [\sqrt[q]{\frac{\overline{Y}}{\overline{X}^p}} \wedge \sqrt[q]{\frac{\underline{Y}}{\underline{X}^p}}, \sqrt[q]{\frac{\overline{Y}}{\underline{X}^p}}] & X^{[p,p]} > Y \end{cases} \tag{20}$$

are interval-valued residuated implications induced by O_{pq}.

(2) *Given an IPO defined by $O(X,Y) = [\min\{\underline{X}^p, \underline{Y}^q\}, \min\{\overline{X}^p, \overline{Y}^q\}]$, where $p, q > 0$, the function $IR^{(1)}, IR^{(2)}$ defined by*

$$IR^{(1)}(X,Y) = \begin{cases} [1,1] & X^{[q,q]} \leq Y \\ [\sqrt[p]{\overline{Y}}, \sqrt[p]{\overline{Y}}] & \underline{X}^q \leq \underline{Y}, \overline{X}^q > \overline{Y} \\ [\sqrt[p]{\underline{Y}}, 1] & \underline{X}^q > \underline{Y}, \overline{X}^q \leq \overline{Y} \\ [\sqrt[p]{\underline{Y}}, \sqrt[p]{\overline{Y}}] & X^{[q,q]} > Y \end{cases} \tag{21}$$

$$IR^{(2)}(X,Y) = \begin{cases} [1,1] & X^{[p,p]} \leq Y \\ [\sqrt[q]{\overline{Y}}, \sqrt[q]{\overline{Y}}] & \underline{X}^p \leq \underline{Y}, \overline{X}^p > \overline{Y} \\ [\sqrt[q]{\underline{Y}}, 1] & \underline{X}^p > \underline{Y}, \overline{X}^p \leq \overline{Y} \\ [\sqrt[q]{\underline{Y}}, \sqrt[q]{\overline{Y}}] & X^{[p,p]} > Y \end{cases} \tag{22}$$

are interval-valued residuated implications induced by O.

(3) *Given an IPO defined as $O_{mM}(X,Y) = [\min\{\underline{X}, \underline{Y}\}\max\{\underline{X}^2, \underline{Y}\},$ $\min\{\overline{X}, \overline{Y}\}\max\{\overline{X}^2, \overline{Y}\}]$, the function $IR^{(1)}, IR^{(2)}$ defined by*

$$IR^{(1)}(X,Y) = \begin{cases} [1,1] & X \leq Y \\ \min\{\overline{X}, \frac{\overline{Y}}{\overline{X}}\} & \underline{X} \leq \underline{Y}, \overline{Y} < \overline{X} \\ [\underline{X}, 1] & \underline{X}^2 \leq \underline{Y} < \underline{X}, \overline{X} \leq \overline{Y} \\ X & X^{[2,2]} \leq Y < X \\ [\underline{X} \wedge \frac{\overline{Y}}{\overline{X}}, \frac{\overline{Y}}{\overline{X}}] & \underline{X}^2 \leq \underline{Y} < \underline{X}, \overline{Y} < \overline{X}^2 \\ [\frac{\underline{Y}}{\underline{X}}, 1] & \underline{Y} < \underline{X}^2, \overline{X} \leq \overline{Y} \\ [\frac{\underline{Y}}{\underline{X}} \wedge \overline{X}, \overline{X}] & \underline{Y} < \underline{X}^2, \overline{X}^2 \leq \overline{Y} < \overline{X} \\ [\frac{\underline{Y}}{\underline{X}} \wedge \frac{\overline{Y}}{\overline{X}}, \frac{\overline{Y}}{\overline{X}}] & Y < X^{[2,2]} \end{cases} \tag{23}$$

$$IR^{(2)}(X,Y) = \begin{cases} \min\{1, \frac{\overline{Y}}{\overline{X}}\} & \underline{X} \leq \underline{Y}, \overline{X}^2 \leq \overline{Y} \\ \min\{\frac{\overline{Y}}{\overline{X}^2}, \sqrt{\overline{Y}}\} & \underline{X} \leq \underline{Y}, \overline{X}^2 > \overline{Y} \\ [\frac{\underline{Y}}{\underline{X}}, 1] & \underline{X}^2 \leq \underline{Y} < \underline{X}, \overline{X} \leq \overline{Y} \\ [\frac{\underline{Y}}{\underline{X}} \wedge \frac{\overline{Y}}{\overline{X}}, \frac{\overline{Y}}{\overline{X}}] & X^{[2,2]} \leq Y < X \\ [\min\{\frac{\underline{Y}}{\underline{X}}, \frac{\overline{Y}}{\overline{X}^2}, \sqrt{\overline{Y}}\}, \min\{\frac{\overline{Y}}{\overline{X}^2}, \sqrt{\overline{Y}}\}] & \underline{X}^2 \leq \underline{Y} < \underline{X}, \overline{Y} < \overline{X}^2 \\ [\min\{\frac{\underline{Y}}{\underline{X}^2}, \sqrt{\underline{Y}}\}, 1] & \underline{Y} < \underline{X}^2, \overline{X} \leq \overline{Y} \\ [\min\{\frac{\underline{Y}}{\underline{X}^2}, \sqrt{\underline{Y}}\}, \frac{\overline{Y}}{\overline{X}}], \frac{\overline{Y}}{\overline{X}}] & \underline{Y} < \underline{X}^2, \overline{X}^2 \leq \overline{Y} < \overline{X} \\ [\min\{\frac{\underline{Y}}{\underline{X}^2}, \sqrt{\underline{Y}}\}, \frac{\overline{Y}}{\overline{X}^2}\}, \min\{\frac{\overline{Y}}{\overline{X}^2}, \sqrt{\overline{Y}}\}] & Y < X^{[2,2]} \end{cases} \tag{24}$$

are interval-valued residuated implications induced by O_{mM}.

(4) *Given an IPO defined as*

$$O(X,Y) = \begin{cases} [0, \overline{X}\overline{Y}^2] & \alpha\underline{X} + (2-\alpha)\underline{Y} = 0 \\ [\frac{2\underline{X}\underline{Y}}{\alpha\underline{X} + (2-\alpha)\underline{Y}}, \overline{X}\overline{Y}^2] & otherwise \end{cases} \tag{25}$$

the function $IR^{(1)}$, $IR^{(2)}$ defined by

$$IR^{(1)}(X,Y) = \begin{cases} [1,1] & (2-\alpha)\underline{XY} \geq 2\underline{X} - \alpha\underline{Y}, \overline{X}^2 \leq \overline{Y} \\ \frac{\overline{Y}}{\overline{X}^2} & (2-\alpha)\underline{XY} \geq 2\underline{X} - \alpha\underline{Y}, \overline{X}^2 > \overline{Y} \\ [\frac{(2-\alpha)\underline{XY}}{2\underline{X}-\alpha\underline{Y}}, 1] & (2-\alpha)\underline{XY} < 2\underline{X} - \alpha\underline{Y}, \overline{X}^2 \leq \overline{Y} \\ [\frac{(2-\alpha)\underline{XY}}{2\underline{X}-\alpha\underline{Y}} \wedge \frac{\overline{Y}}{\overline{X}^2}, \frac{\overline{Y}}{\overline{X}^2}] & otherwise \end{cases} \tag{26}$$

$$IR^{(2)}(X,Y) = \begin{cases} [1,1] & \alpha\underline{XY} \geq 2\underline{X} - (2-\alpha)\underline{Y}, \overline{X} \leq \overline{Y} \\ \sqrt{\frac{\overline{Y}}{\overline{X}}} & \alpha\underline{XY} \geq 2\underline{X} - (2-\alpha)\underline{Y}, \overline{X} > \overline{Y} \\ [\frac{\alpha\underline{XY}}{2\underline{X}-(2-\alpha)\underline{Y}}, 1] & \alpha\underline{XY} < 2\underline{X} - (2-\alpha)\underline{Y}, \overline{X} \leq \overline{Y} \\ [\frac{\alpha\underline{XY}}{2\underline{X}-(2-\alpha)\underline{Y}} \wedge \sqrt{\frac{\overline{Y}}{\overline{X}}}, \sqrt{\frac{\overline{Y}}{\overline{X}}}] & otherwise \end{cases} \tag{27}$$

are interval-valued residuated implications induced by O.

Below a concrete proof that the interval-valued residuated implications defined by us are interval-valued fuzzy implications is given.

Proposition 7. *Given a representable* IPO *O on* $IV([0,1])$, $IR^{(1)}$ *and* $IR^{(2)}$ *are interval-valued residuated implications induced by O, then* $IR^{(1)}$ *and* $IR^{(2)}$ *satisfy conditions* (I1') $\sim$ (I4'), *i.e., they are interval-valued fuzzy implications.*

Proof. (*i*)   Since O is interval-valued pseudo overlap function, it is obvious that $IR^{(1)}([0,0], [0,0]) = \sup\{Z \in IV([0,1]) \mid O(Z, [0,0]) \leq [0,0]\} = \sup\{Z \in IV([0,1]) \mid [0,0] \leq [0,0]\} = [1,1]$, $IR^{(1)}([0,0], [1,1]) = \sup\{Z \in IV([0,1]) \mid O(Z, [0,0]) \leq [1,1]\} = [1,1]$, $IR^{(1)}([1,1], [1,1]) = \sup\{Z \in IV([0,1]) \mid O(Z, [1,1]) \leq [1,1]\} = [1,1]$, $IR^{(1)}([1,1], [0,0]) = \sup\{Z \in IV([0,1]) \mid O(Z, [1,1]) \leq [0,0]\} = [0,0]$, so $IR^{(1)}$ satisfies (I1').

(*ii*)   If $IR^{(1)}([x,x], [y,y]) = [a,b]$, by Theorem 7, it holds that $IR^{(1)}([x,x], [y,y]) = [R_{O_1}^{(1)}(x,y) \wedge R_{O_2}^{(1)}(x,y), R_{O_2}^{(1)}(x,y)] = [a,b]$, then $a = R_{O_1}^{(1)}(x,y) \wedge R_{O_2}^{(1)}(x,y), b = R_{O_2}^{(1)}(x,y)$. Since $R_{O_1}^{(1)}(x,y) = \sup\{z \mid O_1(z,x) \leq y\}, R_{O_2}^{(1)}(x,y) = \sup\{z \mid O_2(z,x) \leq y\}, O_1 \leq O_2 \Rightarrow O_1(z,x) \leq O_2(z,x)$, i.e., $O_2(z,x) \leq y \Rightarrow O_1(z,x) \leq y$, so $\sup\{z \mid O_2(z,x) \leq y\} \leq \sup\{z \mid O_1(z,x) \leq y\}$, that is, $R_{O_2}^{(1)}(x,y) \leq R_{O_1}^{(1)}(x,y)$, then $a = R_{O_1}^{(1)}(x,y) \wedge R_{O_2}^{(1)}(x,y) = R_{O_2}^{(1)}(x,y) = b$, so $IR^{(1)}$ satisfies (I2').

(*iii*) According to the definition we have that $IR^{(1)}([x,y], [a,b]) = \sup\{Z \in IV([0,1]) \mid O(Z, [x,y]) \leq [a,b]\}$, as well as $IR^{(1)}([x',y'], [a,b]) = \sup\{Z \in IV([0,1]) \mid O(Z, [x',y']) \leq [a,b]\}$. If $[x,y] \leq [x',y']$, then $O(Z, [x,y]) \leq O(Z, [x',y'])$ for any $Z \in IV([0,1])$, it is clear that $O(Z, [x',y']) \leq [a,b] \Rightarrow O(Z, [x,y]) \leq [a,b]$, so $\sup\{Z \in IV([0,1]) \mid O(Z, [x',y']) \leq [a,b]\} \leq \sup\{Z \in IV([0,1]) \mid O(Z, [x,y]) \leq [a,b]\} \Rightarrow IR^{(1)}([x',y'], [a,b]) \leq IR^{(1)}([x,y], [a,b])$, thus $IR^{(1)}$ satisfies (I3').

(*iv*) When $[x,y] \leq [x',y']$, it is clear that $O(Z, [a,b]) \leq [x,y] \Rightarrow O(Z, [a,b]) \leq [x',y']$, that is, $IR^{(1)}([a,b], [x,y]) = \sup\{Z \in IV([0,1]) \mid O(Z, [a,b]) \leq [x,y]\} \leq \sup\{Z \in IV([0,1]) \mid O(Z, [a,b]) \leq [x',y']\} = IR^{(1)}([a,b], [x',y'])$, so $IR^{(1)}$ satisfies (I4').

Similarly, we have that $IR^{(2)}$ also satisfies (I1')$\sim$(I4').   □

The following proposition expounds that the interval-valued residuated implications induced by the representable interval-valued pseudo overlap function satisfy the residuation property.

Proposition 8. *Given a representable* IPO *O on* $IV([0,1])$, *and* $IR^{(1)}$, $IR^{(2)}$ *are interval-valued residuated implications induced by O. Then the pair* $(O, IR^{(1)})$ *satisfies the residuation property*

$O(X, Y) \leq Z \Leftrightarrow IR^{(1)}(Y, Z) \geq X$ (RP1), *as well as the pair (O, IR$^{(2)}$) satisfies the residuation property* $O(X, Y) \leq Z \Leftrightarrow IR^{(2)}(X, Z) \geq Y$ (RP2).

Proof. (*i*) $O(X, Y) \leq Z \Rightarrow X \in \{Z' \in IV([0, 1]) \mid O(Z', Y) \leq Z\} \Rightarrow X \leq \sup\{Z' \in IV([0, 1]) \mid O(Z', Y) \leq Z\} = IR^{(1)}(Y, Z)$, i.e., $IR^{(1)}(Y, Z) \geq X$.

(*ii*) Conversely, by Theorem 7, $IR^{(1)}(Y, Z) \geq X \Rightarrow [R_{O_1}^{(1)}(\underline{Y}, \underline{Z}) \wedge R_{O_2}^{(1)}(\overline{Y}, \overline{Z}), R_{O_2}^{(1)}(\overline{Y}, \overline{Z})] \geq X \Rightarrow [\sup\{x' \in [0, 1] \mid O_1(x', \underline{Y}) \leq \underline{Z}\} \wedge \sup\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}, \sup\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}] \geq X$, because $R_{O_1}^{(1)}$ and $R_{O_2}^{(1)}$ are residuated implications satisfying residuation property, it equals $[\max\{x' \in [0, 1] \mid O_1(x', \underline{Y}) \leq \underline{Z}\} \wedge \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}, \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}] \geq X \Rightarrow \underline{X} \leq \max\{x' \in [0, 1] \mid O_1(x', \underline{Y}) \leq \underline{Z}\} \wedge \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}$ and $\overline{X} \leq \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}$, since $\underline{X} \leq \overline{X}$, i.e., $\overline{X} \leq \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\} \Rightarrow \underline{X} \leq \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}$, so we have that $\underline{X} \leq \max\{x' \in [0, 1] \mid O_1(x', \underline{Y}) \leq \underline{Z}\}$ and $\overline{X} \leq \max\{x' \in [0, 1] \mid O_2(x', \overline{Y}) \leq \overline{Z}\}$, then by residuation property of residuated implication induced by pseudo overlap function, it holds that $O_1(\underline{X}, \underline{Y}) \leq \underline{Z}$ and $O_2(\overline{X}, \overline{Y}) \leq \overline{Z}$, i.e., $O(X, Y) = [O_1(\underline{X}, \underline{Y}), O_2(\overline{X}, \overline{Y})] \leq Z$.

Similarly, it is clear that $(O, IR^{(2)})$ satisfies the residuation property (RP2). □

Remark 4. *Given an* IPO *O on IV$([0, 1])$, when it is not representable, the pair (O, IR) composed of it and its induced interval-valued residuated implications may not satisfy the residuation property. For example, $O(X, Y) = [\underline{XY}\overline{X}, \overline{XY}]$ is an IPO, and it is not representable. Suppose that $X = [\sqrt{0.4}, 0.8], Y = [0.2, 1], Z = [0.1, 0.8]$, then $IR^{(1)}(Y, Z) = [\sqrt{0.5}, 0.8] \geq X$, but $O(X, Y) = [\sqrt{0.01024}, 0.8] \nleq Z$.*

At the end of this subsection, we spread some properties contented by interval-valued residuated implications.

Proposition 9. *Given a representable interval-valued pseudo overlap function O, some properties satisfied by interval-valued residuated implications $IR^{(1)}, IR^{(2)}$ are as follows:*

(1) $IR^{(1)}([0, 0], [a, b]) = IR^{(2)}([0, 0], [a, b]) = [1, 1]$ *for any* $[a, b] \in IV([0, 1])$;

(2) $IR^{(1)}([a, b], [1, 1]) = IR^{(2)}([a, b], [1, 1]) = [1, 1]$ *for any* $[a, b] \in IV([0, 1])$;

(3) $IR^{(1)}([1, 1], [x, y]) = IR^{(2)}([1, 1], [x, y]) = [x, y]$ *if O take [1, 1] as unit element, for any* $[x, y] \in IV([0, 1])$;

(4) $IR^{(2)}(X, IR^{(2)}(Y, Z)) = IR^{(2)}(Y, IR^{(2)}(X, Z))$ *if O is associative;*

(5) *When* $X \neq [0, 0]$, *it is established that* $IR^{(1)}(X, [0, 0]) = IR^{(2)}(X, [0, 0]) = [0, 0]$;

(6) $IR^{(1)}([x, y], [x, y]) = [1, 1]$ *if and only if* $O([1, 1], [x, y]) \leq [x, y]$, $IR^{(2)}([x, y], [x, y]) = [1, 1]$ *if and only if* $O([x, y], [1, 1]) \leq [x, y]$, *for any* $[x, y] \in IV([0, 1])$;

(7) $IR^{(1)}([a, b], [c, d]) = [1, 1] \Leftrightarrow [a, b] \leq [c, d]$ *when and only when* $O([1, 1], [a, b]) = [a, b]$, $IR^{(2)}([a, b], [c, d]) = [1, 1] \Leftrightarrow [a, b] \leq [c, d]$ *if and only if* $O([a, b], [1, 1]) = [a, b]$, *for any* $[a, b], [c, d] \in IV([0, 1])$;

(8) $IR^{(1)}([a, b], [c, d]) \geq [c, d]$, $IR^{(2)}([a, b], [c, d]) \geq [c, d]$ *when O takes [1, 1] as a unit element, for any* $[a, b], [c, d] \in IV([0, 1])$.

Proof. (1) $IR^{(1)}([0, 0], [a, b]) = \sup\{Z \in IV([0, 1]) \mid O(Z, [0, 0]) \leq [a, b]\} = \sup\{Z \in IV([0, 1]) \mid [0, 0] \leq [a, b]\} = [1, 1], IR^{(2)}([0, 0], [a, b]) = \sup\{Z \in IV([0, 1]) \mid O([0, 0], Z) \leq [a, b]\} = \sup\{Z \in IV([0, 1]) \mid [0, 0] \leq [a, b]\} = [1, 1]$.

(2) Since O is an interval-valued pseudo overlap function, it is clear that $IR^{(1)}([a, b], [1, 1]) = \sup\{Z \in IV([0, 1]) \mid O(Z, [a, b]) \leq [1, 1]\} = [1, 1], IR^{(2)}([a, b], [1, 1]) = \sup\{Z \in IV([0, 1]) \mid O([a, b], Z) \leq [1, 1]\} = [1, 1]$.

(3) Since O takes [1, 1] as a unit element, $IR^{(1)}([1, 1], [x, y]) = \sup\{Z \in IV([0, 1]) \mid O(Z, [1, 1]) \leq [x, y]\} = \sup\{Z \in IV([0, 1]) \mid Z \leq [x, y]\} = [x, y], IR^{(2)}([1, 1], [x, y]) = \sup\{Z \in IV([0, 1]) \mid O([1, 1], Z) \leq [x, y]\} = \sup\{Z \in IV([0, 1]) \mid Z \leq [x, y]\} = [x, y]$.

(4) Suppose that $O(X, O(Y, Z)) = O(Y, O(X, Z))$. By the residuation property, $IR^{(2)}(X, IR^{(2)}(Y, Z)) = \sup\{Z' \in IV([0,1]) \mid O(X, Z') \leq IR^{(2)}(Y, Z)\} = \sup\{Z' \in IV([0,1]) \mid O(Y, O(X, Z')) \leq Z\}$ (by RP2) $= \sup\{Z' \in IV([0,1]) \mid O(X, O(Y, Z')) \leq Z\} = \sup\{Z' \in IV([0,1]) \mid O(Y, Z') \leq IR^{(2)}(X, Z)\}$ (by RP2) $= IR^{(2)}(Y, IR^{(2)}(X, Z))$.

(5) $IR^{(1)}(X, [0,0]) = \sup\{Z \in IV([0,1]) \mid O(Z, X) \leq [0,0]\}$, it is clear that if $X \neq [0,0]$ then $Z = [0,0]$, so $IR^{(1)}(X, [0,0]) = [0,0]$. Similarly, $IR^{(2)}(X, [0,0]) = \sup\{Z \in IV([0,1]) \mid O(X, Z) \leq [0,0]\} = [0,0]$.

(6) $IR^{(1)}([x,y], [x,y]) = [1,1] \Leftrightarrow \sup\{Z \in IV([0,1]) \mid O(Z, [x,y]) \leq [x,y]\} = [1,1] \Leftrightarrow O([1,1], [x,y]) \leq [x,y], IR^{(2)}([x,y], [x,y]) = [1,1] \Leftrightarrow \sup\{Z \in IV([0,1]) \mid O([x,y], Z) \leq [x,y]\} = [1,1] \Leftrightarrow O([x,y], [1,1]) \leq [x,y]$.

(7) If $O([1,1], [a,b]) = [a,b]$, then $IR^{(1)}([a,b], [c,d]) = [1,1] \Leftrightarrow \sup\{Z \in IV([0,1]) \mid O(Z, [a,b]) \leq [c,d]\} = [1,1] \Leftrightarrow O([1,1], [a,b]) \leq [c,d] \Leftrightarrow [a,b] \leq [c,d]$, conversely, since $IR^{(1)}([a,b], [c,d]) = [1,1] \Leftrightarrow [a,b] \leq [c,d]$, we have $[a,b] \leq [a,b] \Rightarrow IR^{(1)}([a,b], [a,b]) = [1,1] \Rightarrow \sup\{Z \in IV([0,1]) \mid O(Z, [a,b]) \leq [a,b]\} = [1,1] \Rightarrow O([1,1], [a,b]) \leq [a,b]$, and since $IR^{(1)}([a,b], O([1,1], [a,b])) = \sup\{Z \in IV([0,1]) \mid O(Z, [a,b]) \leq O([1,1], [a,b])\} = [1,1] \Rightarrow [a,b] \leq O([1,1], [a,b])$, so $O([1,1], [a,b]) = [a,b]$, similarly, $IR^{(2)}([a,b], [c,d]) = [1,1] \Leftrightarrow [a,b] \leq [c,d]$ if and only if $O([a,b], [1,1]) = [a,b]$.

(8) Suppose that O takes $[1, 1]$ as a unit element, since $IR^{(1)}$ is non-increasing about the first element, $[a,b] \leq [1,1] \Rightarrow IR^{(1)}([a,b], [c,d]) \geq IR^{(1)}([1,1], [c,d]) = \sup\{Z \in IV([0,1]) \mid O(Z, [1,1]) \leq [c,d]\} = \sup\{Z \in IV([0,1]) \mid Z \leq [c,d]\} = [c,d]$, similarly, $IR^{(2)}$ is also non-increasing about the first element, so $IR^{(2)}([a,b], [c,d]) \geq [c,d]$.
□

3.3. Migrativity and Homogeneity of Interval-Valued Pseudo Overlap Functions

There have been some studies on the migrativity of mappings ([17] for IFS and [36] for fuzzy sets). In the following, some properties of IPO are discussed, mainly migrativity and homogeneity.

Proposition 10. *Given a migrative IPO O on $IV([0,1])$. Then when $p + q = s + t$ for any $p, q, s, t \in \{0, 1, 2, \ldots\}$, it is established that $O(\alpha^p M, \alpha^q N) = O(\alpha^s M, \alpha^t N)$.*

Proof. Since O is migrative, we have that $O(\alpha^p M, \alpha^q N) = O(M, \alpha^p \alpha^q N) = O(M, \alpha^{p+q} N) = O(M, \alpha^{s+t} N) = O(M, \alpha^s \alpha^t N) = O(\alpha^s M, \alpha^t N)$. □

Proposition 11. *Given a migrative IPO O on $IV([0,1])$ satisfing $O(X, [e,e]) = O([e,e], X) = X$. Then O is associative if and only if $e = 1$, that is, $O(X, Y) = XY$, for any $X, Y \in IV([0,1])$.*

Proof. $(\Rightarrow)$ Suppose that $0 < e < 1$, and we take $x = e^2$ and $y = 1$, then $[x,x], [y,y] \in IV([0,1])$, it is clear that $O(O([x,x], [1,1]), [y,y]) = O([e,e], [y,y]) = [y,y] = [1,1], O([x,x], O([1,1], [y,y])) = O([x,x], [1,1]) = [e,e] \Rightarrow O(O([x,x], [1,1]), [y,y]) \neq O([x,x], O([1,1], [y,y]))$, which is a contradiction, so $e = 1$.
$(\Leftarrow)$ Since $O(O(X, Y), Z) = O(XY, Z) = XYZ$ and $O(X, O(Y, Z)) = O(X, YZ) = XYZ$, O is associative. □

Proposition 12. *Given an IPO O on $IV([0,1])$, when it satisfies migrativity, it is an IO.*

Proof. Directly from Proposition 3. □

Theorem 8. *Given an IPO O on $IV([0,1])$, it is representable if it satisfies migrativity.*

Proof. Since O is migrative, it holds that O is an IO. Then by ([17], Theorem 3.6), it is obvious. □

Proposition 13. *Given a representable* IPO O *on* $IV([0,1])$, *then O satisfies migrativity if and only if $\underline{O}, \overline{O}$ are migrative pseudo overlap functions.*

Proof. ($\Rightarrow$) Since O is representable, $O = \widetilde{\underline{O}\overline{O}}$, so $\underline{O}, \overline{O}$ are pseudo overlap functions. For some $\alpha \in [0,1]$, $\underline{O}(\alpha x, y) = O([\alpha x, \alpha x], [y,y]) = O([\alpha, \alpha][x,x], [y,y]) = O([x,x], [\alpha, \alpha][y,y]) = O([x,x], [\alpha y, \alpha y])$, i.e., $\underline{O}$ is migrative. Similarly, $\overline{O}$ is also migrative.

($\Leftarrow$) For some $M, X, Y \in IV([0,1])$, it holds that $O(MX, Y) = \widetilde{\underline{O}\overline{O}}(MX, Y) = [\underline{O}(\underline{MX}, \underline{Y}), \overline{O}(\overline{MX}, \overline{Y})] = [\underline{O}(\underline{X}, \underline{MY}), \overline{O}(\overline{X}, \overline{MY})] = \widetilde{\underline{O}\overline{O}}(X, MY) = O(X, MY)$, so O is migrative. $\square$

According to the above proposition, we can easily get the following inference.

Corollary 1. *Given a representable* IPO O *on* $IV([0,1])$, *if pseudo overlap functions $\underline{O}, \overline{O}$ are migrative, then O is an* IO.

Proposition 14. *Given mappings O_1, O_2 on $IV([0,1])$ are M-order homogeneous and N-order homogeneous interval-valued pseudo overlap functions, respectively, then the mapping O defined as $O(X, Y) = O_1(X, Y)O_2(X, Y)$ is an (M + N)-order homogeneous* IPO.

Proof. It is clear that $O(\lambda X, \lambda Y) = O_1(\lambda X, \lambda Y)O_2(\lambda X, \lambda Y) = \lambda^M O_1(X, Y)\lambda^N O_2(X, Y) = \lambda^{M+N} O_1(X, Y)O_2(X, Y) = \lambda^{M+N}O(X, Y)$, so $O(X, Y)$ is $(M + N)$-order homogeneous. $\square$

Proposition 15. *Given a K-order homogeneous* IPO O *on* $IV([0,1])$, *then it is idempotent if and only if $K = [1,1]$.*

Proof. ($\Rightarrow$) Since $O(\alpha X, \alpha X) = \alpha^K O(X, X)$ and $O(\alpha X, \alpha X) = \alpha X$, $\alpha^K O(X, X) = \alpha^K X = \alpha X \Rightarrow \alpha^K = \alpha$, so $K = [1,1]$.

($\Leftarrow$) It holds that $O([1,1], [1,1]) = [1,1]$, by (3) of Proposition 3, we have that O is idempotent. $\square$

Proposition 16. *Given a Moore continuous and P-order homogeneous interval-valued aggregation function M on $IV([0,1])$ satisfying $M(X_1, X_2) = [0,0]$ only if $X_1 X_2 = [0,0]$ and $M(X_1, X_2) = [1,1]$ only if $X_1 X_2 = [1,1]$. Both O_1 and O_2 are Q-order homogeneous interval-valued pseudo overlap functions. Then $M(O_1, O_2)$ defined by $M(O_1, O_2)(X, Y) = M(O_1(X, Y), O_2(X, Y))$ is a PQ-order homogeneous* IPO.

Proof. We first prove that the function $M(O_1, O_2)$ is an interval-valued pseudo overlap function. $XY = [0,0] \Rightarrow O_1(X, Y) = [0,0], O_2(X, Y) = [0,0] \Rightarrow M(O_1(X, Y), O_2(X, Y)) = M([0,0], [0,0]) = [0,0]$, $M(O_1(X, Y), O_2(X, Y)) = [0,0] \Rightarrow O_1(X, Y)O_2(X, Y) = [0,0] \Rightarrow XY = [0,0]$, so $M(O_1, O_2)$ satisfies (O1′). Similarly, we can get $M(O_1, O_2)$ satisfies (O2′). It is obvious that $M(O_1, O_2)$ is increasing and Moore continuous, i.e., $M(O_1, O_2)$ satisfies (O3′) and (O4′). Then $M(O_1, O_2)(\alpha X, \alpha Y) = M(O_1(\alpha X, \alpha Y), O_2(\alpha X, \alpha Y)) = M(\alpha^Q O_1(X, Y), \alpha^Q O_2(X, Y)) = (\alpha^Q)^P M(O_1(X, Y), O_2(X, Y)) = \alpha^Q)^P M(O_1, O_2)(X, Y)$, so $M(O_1, O_2)$ is PQ-order homogeneous. $\square$

Proposition 17. *Given a representable* IPO O *on* $IV([0,1])$, *then O is K-order homogeneous where $K = [k_1, k_2]$ with $0 < k_1 \leq k_2$ if and only if $\underline{O}$ is a k_2-order homogeneous pseudo overlap function and $\overline{O}$ is a k_1-order homogeneous pseudo overlap function.*

Proof. ($\Rightarrow$) $\underline{O}$ and $\overline{O}$ are pseudo overlap functions according to the representable definition. For some $\alpha \in [0,1]$, since $\underline{O}(\alpha x, \alpha y) = O([\alpha x, \alpha x], [\alpha y, \alpha y]) = O([\alpha, \alpha][x,x], [\alpha, \alpha][y,y]) = [\alpha, \alpha]^K O([x,x], [y,y]) = \alpha^{k_2} O([x,x], [y,y]) = \alpha^{k_2} \underline{O}(x, y)$, $\underline{O}$ is k_2-order homogeneous, similarly, $\overline{O}$ is k_1-order homogeneous.

($\Leftarrow$) For arbitrary $X, Y \in IV([0,1])$ and some $\alpha \in IV([0,1])$, since $O(\alpha X, \alpha Y) = \widetilde{\underline{O}\overline{O}}(\alpha X, \alpha Y) =$

$[\underline{O}(\alpha\underline{X},\alpha\underline{Y}),\overline{O}(\overline{\alpha X},\overline{\alpha Y})] = [\underline{\alpha}^{k_2}\underline{O}(\underline{X},\underline{Y}),\overline{\alpha}^{k_1}\overline{O}(\overline{X},\overline{Y})] = \alpha^K[\underline{O}(\underline{X},\underline{Y}),\overline{O}(\overline{X},\overline{Y})] = \alpha^K\widetilde{OO}(X,Y) = \alpha^K O(X,Y)$, O is K-order homogeneous where $K = [k_1,k_2]$. $\quad\square$

Proposition 18. *Given a pseudo overlap function O:* $[0,1]^2 \to [0,1]$*, then O is idempotent if and only if the interval-valued pseudo overlap function IV*$([O,O])$ *is idempotent.*

Proof. ($\Rightarrow$) It is clear that $IV([O,O])(X,X) = [O(\underline{X},\underline{X}),O(\overline{X},\overline{X})]$, since O is idempotent, $O(x,x) = x$, then $IV([O,O])(X,X) = [\underline{X},\overline{X}] = X$. So $IV([O,O])$ is idempotent.
($\Leftarrow$) Since $IV([O,O])$ is idempotent, $IV([O,O])(X,X) = X$. For arbitrary $x \in [0,1]$, we have $IV([O,O])([x,x],[x,x]) = [O(x,x),O(x,x)] = [x,x]$, so $O(x,x) = x$, i.e., O is idempotent. $\quad\square$

4. Applications to Multi-Attribute Decision Making

In this section, we take into account interval-valued fuzzy multi-attribute decision making problems, we first introduce the n-dimensional IPO, illustrate it through specific examples, and then show its application in interval-valued fuzzy multi-attribute decision-making.

Definition 17. *The mapping* $O : IV([0,1])^n \to IV([0,1])$ *is called an n-dimensional* IPO *if it meets requirements below:*

(O^n1) $O(X_1,\dots,X_n) = [0,0]$ *if and only if* $\prod_{i=1}^{n} X_i = [0,0]$;

(O^n2) $O(X_1,\dots,X_n) = [1,1]$ *if and only if* $\prod_{i=1}^{n} X_i = [1,1]$;

(O^n3) *O is increasing, i.e., if for some* $i = 1,\dots,n$, $X_i \le Y_i$, *then* $O(X_1,\dots,X_{i-1},X_i,X_{i+1},\dots,X_n) \le O(X_1,\dots,X_{i-1},Y_i,X_{i+1},\dots,X_n)$;

(O^n4) *O is Moore continuous.*

A few examples of n-dimensional IPO are given below.

Example 6. (1) *The mapping O:* $IV([0,1])^n \to IV([0,1])$ *defined as*

$$O(X_1,\dots,X_n) = [\underline{X_1} \cdot \underline{X_2} \cdot \dots \cdot \underline{X_{n-1}} \cdot \underline{X_n}^2, \overline{X_1} \cdot \overline{X_2} \cdot \dots \cdot \overline{X_{n-1}} \cdot \overline{X_n}^2] \tag{28}$$

is an n-dimensional IPO.
(2) *The mapping O:* $IV([0,1])^n \to IV([0,1])$ *defined as*

$$O(X_1,\dots,X_n) = [\min\{\underline{X_1},\underline{X_2},\dots,\underline{X_{n-1}},\sqrt{\underline{X_n}}\}, \min\{\overline{X_1},\overline{X_2},\dots,\overline{X_{n-1}},\sqrt{\overline{X_n}}\}] \tag{29}$$

is an n-dimensional IPO.

We first introduce the interval-valued multi-attribute decision making (I-MADM) problem. The I-MADM problem is to discuss the normal multi-attribute decision-making problem on the interval set, in which the information is uncertain and fuzzy, we use interval numbers to represent their attribute values and weights. There are the following basic representations in the I-MADM problem: $X = \{x_1,\dots,x_m\}$ is a set of m feasible alternatives, $A = \{a_1,\dots,a_n\}$ is a set of n attributes and attributes are additively independent, $W = \{w_1,\dots,w_n\}^T$ is a set of weights of attributes, $M = (A_{ij})_{m\times n}$ is a decision matrix of the decision maker, where A_{ij} is the value of the association degree between the alternative x_i and the attribute a_j, and $A_{ij} = [\underline{A_{ij}},\overline{A_{ij}}]$. The purpose of decision makers is to use this information to get the relatively best alternative. In the existing literature [37–41] etc., many scholars have studied the I-MADM problem and given many methods, such as IER, I-TODIM and so on. For the I-MADM problem, we propose the following processing method.

Step 1. Standardized decision matrix

Because attribute types are often divided into benefit type and cost type, and different physical dimensions also have a certain impact on the decision results, we first need to

use the following formula to convert the normal decision matrix into a standard decision matrix $R = (R_{ij})_{m \times n}$:

$$R_{ij} = \frac{A_{ij}}{\sqrt{\sum\limits_{i=1}^{n} A_{ij}^2}}, \text{ if } a_j \text{ is a benefit attribute; } R_{ij} = \frac{\frac{1}{A_{ij}}}{\sqrt{\sum\limits_{i=1}^{n} (\frac{1}{A_{ij}})^2}}, \text{ if } a_j \text{ is a cost attribute}$$

In terms of interval numbers, $R_{ij} = [\underline{R_{ij}}, \overline{R_{ij}}]$, where

$$\underline{R_{ij}} = \frac{\underline{A_{ij}}}{\sqrt{\sum\limits_{i=1}^{n} (\overline{A_{ij}})^2}}, \overline{R_{ij}} = \frac{\overline{A_{ij}}}{\sqrt{\sum\limits_{i=1}^{n} (\underline{A_{ij}})^2}}, \text{ if } a_j \text{ is a benefit attribute;}$$

$$\underline{R_{ij}} = \frac{\frac{1}{\overline{A_{ij}}}}{\sqrt{\sum\limits_{i=1}^{n} (\frac{1}{\overline{A_{ij}}})^2}}, \overline{R_{ij}} = \frac{\frac{1}{\underline{A_{ij}}}}{\sqrt{\sum\limits_{i=1}^{n} (\frac{1}{\underline{A_{ij}}})^2}}, \text{ if } a_j \text{ is a cost attribute.}$$

(If the given matrix is already a standard decision matrix, this step may not be carried out)

Step 2. Get the interval value vector of the comprehensive attribute

We aggregate the interval values of each attribute belonging to the same alternative using the interval-valued pseudo overlap function O to obtain the interval value vector of the comprehensive attribute $Z = (Z_1, \ldots, Z_m)$, where $Z_i = O(R_{i1}, \ldots, R_{in})$, $i = 1, \ldots, m$.

Step 3. Set up the possibility degree matrix

We calculate the possibility degree p_{ij} of alternative x_i to alternative x_j and set up the possibility degree matrix $P = (p_{ij})_{m \times m}$ using the following formula:

$$p_{ij} = p(Z_i \geq Z_j) = \frac{min\{\overline{Z_i} + \overline{Z_j} - \underline{Z_i} - \underline{Z_j}, max\{\overline{Z_i} - \underline{Z_j}, 0\}\}}{\overline{Z_i} + \overline{Z_j} - \underline{Z_i} - \underline{Z_j}}$$

Step 4. Calculate the ranking vector of the possibility degree matrix

Finally, we rank all alternatives through calculating the ranking vector V of the possibility matrix, where $V = (v_1, \ldots, v_m)$ and the larger the component, the better the corresponding alternative. The specific formula is as follows:

$$v_i = \frac{\sum\limits_{j=1}^{m} p_{ij} + \frac{m}{2} - 1}{m(m-1)}, i = 1, \ldots, m.$$

4.1. Illustrative Example I

Then we consider an illustrative example in [40] as the first example to show our method. In order to make an assessment of residential properties, the decision maker considered the following eight attributes: localization (a_1), construction area (a_2), quality of construction (a_3), state of conservation (a_4), number of garage spaces (a_5), number of rooms (a_6), attractions (a_7) and security (a_8). The weight vector of the attributes provided by the decision maker is $W = (0.25, 0.15, 0.1, 0.2, 0.05, 0.1, 0.05, 0.1)^T$. There are five candidate residential properties (alternatives x_1, x_2, x_3, x_4, x_5) available for evaluation. The specific data are revealed in the following Table 1.

Table 1. Decision matrix.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
x_1	[7, 8]	[280, 280]	[5, 7]	[6, 7]	[2, 2]	[5, 5]	[6, 7]	[6, 7]
x_2	[5, 6]	[124, 124]	[4, 6]	[5, 9]	[2, 2]	[3, 3]	[4, 5]	[3, 6]
x_3	[6, 8]	[360, 360]	[4, 8]	[7, 8]	[4, 4]	[5, 5]	[6, 8]	[5, 9]
x_4	[4, 7]	[121, 121]	[5, 7]	[4, 5]	[0, 0]	[5, 5]	[2, 4]	[4, 5]
x_5	[5, 6]	[124, 124]	[3, 5]	[5, 9]	[2, 2]	[4, 4]	[4, 5]	[3, 6]

The above eight attributes are benefit attributes. We first standardize the decision matrix and obtain the following matrix $R = [R_1 \ R_2]$ (rounded to three decimal places):

$$R_1 = \begin{pmatrix} [0.444, 0.651] & [0.556, 0.556] & [0.335, 0.734] & [0.346, 0.570] \\ [0.317, 0.488] & [0.246, 0.246] & [0.268, 0.629] & [0.289, 0.732] \\ [0.380, 0.651] & [0.715, 0.715] & [0.268, 0.839] & [0.404, 0.651] \\ [0.253, 0.570] & [0.240, 0.240] & [0.335, 0.734] & [0.231, 0.407] \\ [0.317, 0.488] & [0.246, 0.246] & [0.201, 0.524] & [0.289, 0.732] \end{pmatrix}$$

$$R_2 = \begin{pmatrix} [0.378, 0.378] & [0.500, 0.500] & [0.448, 0.674] & [0.398, 0.718] \\ [0.378, 0.378] & [0.300, 0.300] & [0.299, 0.481] & [0.199, 0.616] \\ [0.756, 0.756] & [0.500, 0.500] & [0.448, 0.770] & [0.332, 0.923] \\ [0.000, 0.000] & [0.500, 0.500] & [0.149, 0.385] & [0.265, 0.513] \\ [0.378, 0.378] & [0.400, 0.400] & [0.299, 0.481] & [0.199, 0.616] \end{pmatrix}$$

Then we use the eight-dimensional interval-valued pseudo overlap function $O_1(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = [\underline{O_1}, \overline{O_1}]$

$$\underline{O_1} = \sqrt{\frac{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}{0.25\underline{X_1} + 0.15\underline{X_2} + 0.1\underline{X_3} + 0.2\underline{X_4} + 0.05\underline{X_5} + 0.1\underline{X_6} + 0.05\underline{X_7} + 0.1\underline{X_8}}},$$

$$\overline{O_1} = \sqrt{\frac{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}{0.25\overline{X_1} + 0.15\overline{X_2} + 0.1\overline{X_3} + 0.2\overline{X_4} + 0.05\overline{X_5} + 0.1\overline{X_6} + 0.05\overline{X_7} + 0.1\overline{X_8}}}$$

to get the interval value vector Z of the comprehensive attribute, as shown in the following Table 2 (rounded to four decimal places).

Table 2. Interval value vector of the comprehensive attribute.

	x_1	x_2	x_3	x_4	x_5
Z	[0.0475, 0.1509]	[0.0120, 0.0608]	[0.0604, 0.3117]	[0.0000, 0.0000]	[0.0119, 0.0641]

After calculating the possibility degree, we establish the possibility degree matrix P as follows (rounded to four decimal places):

$$P = \begin{pmatrix} 0.5000 & 0.9124 & 0.2550 & 0.1000 & 0.4167 \\ 0.0876 & 0.5000 & 0.0013 & 0.1000 & 0.4839 \\ 0.7450 & 0.9987 & 0.5000 & 0.1000 & 0.9878 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 0.5833 & 0.5161 & 0.0122 & 1.0000 & 0.5000 \end{pmatrix}$$

Finally, we use the formula mentioned above to get the ranking vector V, as shown in the following Table 3 (round to four decimal places).

Table 3. Ranking vector of the possibility degree matrix.

	x_1	x_2	x_3	x_4	x_5
V	0.2292	0.1786	0.2866	0.1000	0.2056

From the above table, we can obtain the ranking of five alternatives as follows:

$$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$$

Of course, the above eight-dimensional interval-valued pseudo overlap function O_1 used to get the interval value vector of the comprehensive attribute is not fixed, but can

also be replaced by the following functions:

$$O_2(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = [\underline{O_2}, \overline{O_2}]$$

$$\underline{O_2} = \sqrt{\underline{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}(0.25\underline{X_1} + 0.15\underline{X_2} + 0.1\underline{X_3} + 0.2\underline{X_4} + 0.05\underline{X_5} + 0.1\underline{X_6} + 0.05\underline{X_7} + 0.1\underline{X_8})},$$

$$\overline{O_2} = \sqrt{\overline{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}(0.25\overline{X_1} + 0.15\overline{X_2} + 0.1\overline{X_3} + 0.2\overline{X_4} + 0.05\overline{X_5} + 0.1\overline{X_6} + 0.05\overline{X_7} + 0.1\overline{X_8})}$$

$$O_3(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = [\underline{O_3}, \overline{O_3}]$$

$$\underline{O_3} = \frac{1}{\frac{0.25}{\underline{X_1}} + \frac{0.15}{\underline{X_2}} + \frac{0.1}{\underline{X_3}} + \frac{0.2}{\underline{X_4}} + \frac{0.05}{\underline{X_5}} + \frac{0.1}{\underline{X_6}} + \frac{0.05}{\underline{X_7}} + \frac{0.1}{\underline{X_8}}}, \quad \overline{O_3} = \frac{1}{\frac{0.25}{\overline{X_1}} + \frac{0.15}{\overline{X_2}} + \frac{0.1}{\overline{X_3}} + \frac{0.2}{\overline{X_4}} + \frac{0.05}{\overline{X_5}} + \frac{0.1}{\overline{X_6}} + \frac{0.05}{\overline{X_7}} + \frac{0.1}{\overline{X_8}}}$$

$$O_4(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = [\underline{O_4}, \overline{O_4}]$$

$$\underline{O_4} = \sqrt{\frac{\underline{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}(1 + (1 - 0.25\underline{X_1})(1 - 0.15\underline{X_2})(1 - 0.1\underline{X_3})(1 - 0.2\underline{X_4})(1 - 0.05\underline{X_5})(1 - 0.1\underline{X_6})(1 - 0.05\underline{X_7})(1 - 0.1\underline{X_8}))}{1 + (1 - 0.25)(1 - 0.15)(1 - 0.1)(1 - 0.2)(1 - 0.05)(1 - 0.1)(1 - 0.05)(1 - 0.1)}},$$

$$\overline{O_4} = \sqrt{\frac{\overline{X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8}(1 + (1 - 0.25\overline{X_1})(1 - 0.15\overline{X_2})(1 - 0.1\overline{X_3})(1 - 0.2\overline{X_4})(1 - 0.05\overline{X_5})(1 - 0.1\overline{X_6})(1 - 0.05\overline{X_7})(1 - 0.1\overline{X_8}))}{1 + (1 - 0.25)(1 - 0.15)(1 - 0.1)(1 - 0.2)(1 - 0.05)(1 - 0.1)(1 - 0.05)(1 - 0.1)}}$$

Table 4 contains four rankings obtained by the functions given above and the results in Table 8 of [40] for the same problem.

Table 4. Summary of the rankings obtained from the proposed method and the other methods.

Method	Ranking
I-TODIM	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$
Extended TODIM	$x_3 \succ x_1 \succ x_5 \sim x_2 \succ x_4$
method based on loss aversion	$x_3 \succ x_1 \succ x_5 \sim x_2 \succ x_4$
O_1	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$
O_2	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$
O_3	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$
O_4	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4$

Through observation and analysis, seven methods in Table 4 return two different results, but all of which believe that the best alternative is x_3 and the worst alternative is x_4. Compared with the second and third methods, our proposed method can effectively rank x_2 and x_5 clearly, and compared with the I-TODIM method in [40], our method is simpler and can get the same ranking result.

4.2. Illustrative Example II

The second example is the case in [42]. When purchasing artillery weapons, an army needs to consider the following five indicators (attributes): firepower assault capability index (a_1), reaction capability index (a_2), mobility index (a_3), survivability index (a_4) and cost (a_5). There are four series of guns (alternatives x_1, x_2, x_3, x_4) available for purchase. The specific data are revealed in the Table 5 below.

Table 5. Decision matrix.

	a_1	a_2	a_3	a_4	a_5
x_1	[26,000, 27,000]	[2, 4]	[18,000, 19,000]	[0.7, 0.8]	[15,000, 16,000]
x_2	[60,000, 70,000]	[3, 4]	[16,000, 17,000]	[0.3, 0.4]	[27,000, 28,000]
x_3	[50,000, 60,000]	[2, 3]	[15,000, 16,000]	[0.7, 0.8]	[24,000, 26,000]
x_4	[40,000, 50,000]	[1, 2]	[28,000, 29,000]	[0.4, 0.5]	[15,000, 17,000]

Among them, all attributes except a_5 are benefit attributes.

We first standardize the decision matrix and obtain the following matrix R:

$$R = \begin{pmatrix} [0.240, 0.295] & [0.298, 0.943] & [0.431, 0.477] & [0.538, 0.721] & [0.571, 0.663] \\ [0.554, 0.765] & [0.447, 0.943] & [0.383, 0.426] & [0.231, 0.361] & [0.326, 0.368] \\ [0.462, 0.656] & [0.298, 0.707] & [0.359, 0.401] & [0.538, 0.721] & [0.351, 0.414] \\ [0.369, 0.546] & [0.149, 0.471] & [0.670, 0.728] & [0.308, 0.451] & [0.537, 0.663] \end{pmatrix}$$

Then we use five-dimensional interval-valued pseudo overlap function
$O_5(X_1, X_2, X_3, X_4, X_5) = [\underline{O_5}, \overline{O_5}]$

$$\underline{O_5} = \frac{X_1 X_2 X_3 X_4 X_5}{0.2189\underline{X_1} + 0.2182\underline{X_2} + 0.1725\underline{X_3} + 0.2143\underline{X_4} + 0.1761\underline{X_5}},$$

$$\overline{O_5} = \frac{\overline{X_1 X_2 X_3 X_4 X_5}}{0.2189\overline{X_1} + 0.2182\overline{X_2} + 0.1725\overline{X_3} + 0.2143\overline{X_4} + 0.1761\overline{X_5}}$$

to get the interval value vector of the comprehensive attribute, as shown in the following Table 6 (rounded to four decimal places).

Table 6. Interval value vector of the comprehensive attribute.

	x_1	x_2	x_3	x_4
Z	[0.0232, 0.1017]	[0.0182, 0.0693]	[0.0230, 0.0934]	[0.0156, 0.0997]

After calculating the possibility degree, we establish the possibility degree matrix P as follows:

$$P = \begin{pmatrix} 0.5000 & 0.6441 & 0.5285 & 0.5293 \\ 0.3559 & 0.5000 & 0.3812 & 0.3971 \\ 0.4715 & 0.6188 & 0.5000 & 0.5034 \\ 0.4707 & 0.6029 & 0.4966 & 0.5000 \end{pmatrix}$$

Finally, we use the formula mentioned above to get the ranking vector, as shown in the following Table 7.

Table 7. Ranking vector of the possibility degree matrix.

	x_1	x_2	x_3	x_4
V	0.2668	0.2195	0.2578	0.2559

From the above table, we can obtain the ranking of four alternatives as follows:

$$x_1 \succ x_3 \succ x_4 \succ x_2$$

Of course, the above five-dimensional interval-valued pseudo overlap function function O_5 is also not fixed, but can also be replaced by the following functions:

$$O_6(X_1, X_2, X_3, X_4, X_5) = [\underline{O_6}, \overline{O_6}]$$

$$\underline{O_6} = \sqrt{\underline{X_1 X_2 X_3 X_4 X_5}(0.2189\underline{X_1} + 0.2182\underline{X_2} + 0.1725\underline{X_3} + 0.2143\underline{X_4} + 0.1761\underline{X_5})},$$

$$\overline{O_6} = \sqrt{\overline{X_1 X_2 X_3 X_4 X_5}(0.2189\overline{X_1} + 0.2182\overline{X_2} + 0.1725\overline{X_3} + 0.2143\overline{X_4} + 0.1761\overline{X_5})}$$

$$O_7(X_1, X_2, X_3, X_4, X_5) = [\underline{O_7}, \overline{O_7}]$$

$$\underline{O_7} = \frac{1}{\frac{0.2189}{\underline{X_1}} + \frac{0.2182}{\underline{X_2}} + \frac{0.1725}{\underline{X_3}} + \frac{0.2143}{\underline{X_4}} + \frac{0.1761}{\underline{X_5}}}, \overline{O_7} = \frac{1}{\frac{0.2189}{\overline{X_1}} + \frac{0.2182}{\overline{X_2}} + \frac{0.1725}{\overline{X_3}} + \frac{0.2143}{\overline{X_4}} + \frac{0.1761}{\overline{X_5}}}$$

$$O_8(X_1, X_2, X_3, X_4, X_5) = [\underline{O_8}, \overline{O_8}]$$

$$\underline{O_8} = \frac{\underline{X_1 X_2 X_3 X_4 X_5}(1 + (1 - 0.2189\underline{X_1})(1 - 0.2182\underline{X_2})(1 - 0.1725\underline{X_3})(1 - 0.2143\underline{X_4})(1 - 0.1761\underline{X_5}))}{1 + (1 - 0.2189)(1 - 0.2182)(1 - 0.1725)(1 - 0.2143)(1 - 0.1761)},$$

$$\overline{O_8} = \frac{\overline{X_1 X_2 X_3 X_4 X_5}(1 + (1 - 0.2189\overline{X_1})(1 - 0.2182\overline{X_2})(1 - 0.1725\overline{X_3})(1 - 0.2143\overline{X_4})(1 - 0.1761\overline{X_5}))}{1 + (1 - 0.2189)(1 - 0.2182)(1 - 0.1725)(1 - 0.2143)(1 - 0.1761)}$$

Table 8 contains four rankings obtained by the given functions and the results in [42,43] for the same problem.

Table 8. Summary of the rankings obtained from the proposed method and the method in [42,43].

Method	Ranking
[42]	$x_1 \succ x_2 \succ x_3 \succ x_4$
[43]	$x_2 \succ x_1 \succ x_4 \succ x_3$
O_5	$x_1 \succ x_3 \succ x_4 \succ x_2$
O_6	$x_1 \succ x_3 \succ x_4 \succ x_2$
O_7	$x_3 \succ x_1 \succ x_2 \succ x_4$
O_8	$x_1 \succ x_3 \succ x_4 \succ x_2$

Note that all rankings in the above table are obtained by taking $W = (0.2189, 0.2182, 0.1725, 0.2143, 0.1761)^T$ as the weight vector of the attributes. In addition, six methods in Table 8 return four different results, most of which believe that the best alternative is x_1 and the worst alternative is x_2.

In summary, contrasted with the methods of [42,43], our method does not need to calculate the projection value or absolute (negative) ideal solution. Using the interval-valued pseudo overlap functions, we can directly achieve the effect of aggregation of multiple attribute information, with less calculations and easy operation. On the other hand, the above table also shows that when the weight is fixed, we can get multiple ranking results by taking different IPO, which is conducive to the comparative evaluation of decision makers. In fact, compared with IO, IPO can imply different degrees of importance to multiple attributes, which demonstrates their flexibility and superiority.

5. Conclusions

This paper mainly introduces the notion of interval-valued pseudo overlap functions and their properties including migrativity and homogeneity, as well as the residuated implications induced by them, and expounds the application of multi-dimensional interval-valued pseudo overlap functions in the I-MADM problem, which is supported by examples. In addition, where we mainly focus on the research of the representable interval-valued pseudo overlap function.

Specifically, we not only illustrate the relevance between IPO and interval-valued pseudo t-norms, give some construction theorems of IPO, but also present the interval-valued residuated implications induced by interval-valued pseudo overlap functions and state examples. Secondly, we study the equivalent conditions of representable IPO, give some examples and the properties of interval-valued residuated implications' sat-

isfaction induced by them are discussed. Then some related propositions of interval-valued pseudo overlap functions satisfying migrativity and homogeneity are elaborated and proved. Finally, we extend the interval-valued pseudo overlap function to the n-dimension, illustrate its advantages in interval-valued multi-attribute decision-making through concrete examples.

As a further work, we will study other properties of IPO, discuss the existence and related contents of its additive (multiplicative) generators, and also pay attention to the relationship between IPO and other aggregation operators and fuzzy rough sets (see [44–46]).

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