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Finite Time Blowup in a Fourth-Order Dispersive Wave Equation with Nonlinear Damping and a Non-Local Source

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Abstract: In this work, we consider a class of initial boundary value problems for fourth-order dispersive wave equations with superlinear damping and non-local source terms as well as time-dependent coefficients in $\Omega \times (t > 0)$, where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 2$. We prove that there exists a safe time interval of existence in the solution $[0, T]$, with T being a lower bound of the blowup time t^*. Moreover, we find an explicit lower bound of t^*, assuming the coefficients are positive constants.

Keywords: nonlinear higher-order hyperbolic equations; blowup

MSC: 35L35; 35L75; 35B44

1. Introduction

During the past few decades, the investigation of unboundedness phenomena has been one of the most developed topics. In order to examine these phenomena, different important methods have been introduced, such as Lyapunov functions ([1,2]) or the potential well theory ([3–6]).

For linear or nonlinear parabolic and hyperbolic equations, which have solutions that blow up in a finite time, the blowup time t^* cannot in general be computed exactly. As a consequence, many papers are devoted to finding the upper and lower bounds for t^* (see [7,8] and the references therein).

The aim of this paper is to obtain a lower bound of t^* for the solutions to the problem under investigation.

We consider the following problem for a fourth-order dispersive wave equation with nonlinear damping and a non-local source term and time-dependent coefficients:

$$u_{tt} - a_1(t)\Delta u - a_2(t)\Delta u_{tt} + a_3(t)\Delta^2 u + g(u_t) = f(u), \; x \in \Omega, t > 0, \tag{1}$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \; or \; u = 0, \quad \Delta u = 0, \quad x \in \partial\Omega, \; t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ such that $N \geq 2$ with a smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial n}$ as the outward normal derivative of u on the boundary $\partial\Omega$ and the superlinear damping term $g(u_t)$, and the superlinear source term $f(u)$ are defined as follows:

$$g(u_t) = k_1(t)u_t|u_t|^{m-2}, \quad m > 2 \tag{4}$$

$$f(u) =: k_2(t)\; u|u|^{p-2} \int_\Omega |u|^q dx, \quad p \geq q > 2, \tag{5}$$

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where $a_i(t),\ i = 1,2,3$ are positive differentiable functions and $k_1(t), k_2(t)$ are positive continuous functions for $t \geq 0$. All the coefficients are bounded in any time interval. Furthermore $u_0(x),\ u_1(x)$ are given functions defined in Ω. The presence of the term Δu_{tt} classifies Equation (1) as a dispersive equation. We refer to $u = 0,\ \frac{\partial u}{\partial n} = 0$ in Equation (2) as the Dirichlet boundary conditions, while $u = 0,\ \Delta u = 0$ represents the Navier boundary conditions.

We define the following:

$$t^* = \sup\{t \in \mathbb{R}^+ : \text{u exists in } \Omega \times [0, t)\}, \tag{6}$$

where t^* is called the blowup time or lifespan of the solution.

In the literature, a large part of the results concerns the global existence of the solutions. Less attention was paid to the blowup solutions, since in physical situations, the blowup phenomenon must be avoided (see, for instance, the collapse of a suspension bridge in [9,10]). For this reason, we will consider a bounded time interval $[0, T]$, where the solution is bounded and T is obtained by deriving a lower bound of the blowup time, should a blowup occur. In this sense, an upper bound is useless.

Let us mention some known results for solutions to the fourth-order hyperbolic problems with nonlinear damped and source terms.

Messaoudi in [11] gave the following for solutions to the Petrovsky equation:

$$u_{tt} + \Delta^2 u + a|u_t|^{m-2} = b|u|^{p-2}\ in\ \Omega,\ (t > 0), \tag{7}$$

Under Dirichlet boundary conditions, $a, b > 0$, $p, q > 2$, and Ω is a bounded domain in $\mathbb{R}^N$ where $N \geq 1$, proving the existence of a local weak solution, and when discussing the competition between the damping and the source terms, through suitable Lyapunov functions, he proved that if $p > m$ (with the initial energy $E(0) < 0$), the solution blows up in a finite time. However, if $p \leq m$, the solution exists globally.

In [12], Chen and Zhou succeeded in showing that the conditions for blowup established by Messaoudi can be somewhat relaxed, establishing that it is enough to assume $E(0) \leq 0$.

In [13], Wu and Tsai improved the results in [11,12] by showing that the solution of Equation (7) is global under some conditions, but without the relation between p and m and the blowups if $p > m$ and the initial energy $E(0) \leq 0$.

For the solutions of Equation (7) with Equations (2) and (3) for the initial boundary conditions, Philippin and Vernier-Piro in [14] obtained a lower bound of the lifespan when the spatial domain $\Omega \subset \mathbb{R}^3$.

When $f(u) = u|u|^{p-2}\ log\ u^k, k > 0$ in Equation (7) , Liu in [15], by using the potential well method, derived the local and global existence and decay estimate of the solution and also proved that if the initial energy is negative, it blows up in a finite time. The logarithmic nonlinearity has attracted the interest of researchers in light of the connection with nuclear physics, optics, and geophysics, as pointed out in [15] (see also [16]).

Di and Shang in [17] investigated the existence of global solutions for the following equation:

$$u_{tt} - \Delta u - \beta\Delta u_t + \gamma\Delta^2 u - \delta\Delta u_{tt} + a|u_t|^{m-2}|u_t = b|u|^{p-2}u,\ in\ \Omega,\ (t > 0),$$

This is true under Equations (2) and (3) in presence of the dispersive term Δu_{tt}, the strong dissipation term Δu_t, the nonlinear damping term $|u_t|^{m-2}|u_t$, the nonlinear source term $|u|^{p-2}u$, and with positive constant coefficients. To prove the results, they used a combination of the Galerkin method and the monotonicity compactness method.

We recall that plate models have also been of great importance in studying the structural behavior and instability of suspension bridges (see [9,10]). If $\Omega \subset \mathbb{R}^2$, Mukiawa and Messaoudi in [18] considered the problem which comes from the modeling of the downward displacement of a suspension bridge using a thin rectangular plate with partially hinged boundary conditions:

$$u_{tt} + \Delta^2 u(x,y,t) - \int g(t-s)\Delta^2 u ds + h(u_t) = u|u|^{p-2}, \ in\ \Omega,\ (t>0),$$

They showed that the solution blows up in a finite time, proving that in the presence of a nonlinear source such as earthquake shocks, the bridge will collapse in a finite time. For more detail, see also the quoted report [19] on the Tacoma Narrows Bridge failure [9,10,20].

The study of the lower bounds of the blowup time was extended to hyperbolic systems of the fourth order (see [21–23]).

In the case of parabolic fourth-order equations, interesting results are present in [24,25] due to the presence of the determinant of the Hessian matrix. Depending on the boundary conditions and the size of the data, the existence of a finite time blowup as well as the existence of global in-time solutions are discussed.

For higher order hyperbolic problems, Autuori and Pucci in [26] treated the local asymptotic stability for different classes of polyharmonic Kirchhoff systems governed by time-dependent source forces and nonlinear damping terms. One of them is the following:

$$u_{tt} + (-\Delta)^L u - M(||\nabla u||_2^2)\Delta u + \mu u + Q(x,t,u,u_t) + f(x,t,u) = 0, \quad L \geq 1,$$

where $u = (u_1, \cdots, u_N)$ under Dirichlet boundary conditions, the nonlocal term $M(||\nabla u||_2^2)$ is the coefficient of Δu, and it is a model for vibrating beams of the Woinowsky-Krieger type when $L = 2$. For a blowup at infinity for solutions to polyharmonic Kirchhoff systems, see [27].

In this paper, we focus our attention on the possibility of establishing a time interval where the solution exists and it is bounded, providing a lower bound of the blowup time if a blowup occurs. Our approach is based on first-order differential inequalities satisfied by suitable energy functions associated to the problem in Equations (1)–(3) when the coefficients are either time-dependent or constants.

The novelties for the lower bounds of t^* with respect to the cited papers are as follows: the classes of problems under investigation having time-dependent coefficients, the presence of a dispersion term Δu_{tt}, the source term being the product of a superlinear term with a term of a nonlocal type, and the existence of a safe time interval $[0,T]$ where the solution remains bounded, with T as a lower bound of t^* and T being explicit and easily computable if the coefficients are positive constants.

The scheme of this paper is as follows. In Section 2, we present some preliminary definitions and Lemmas as well as our main results. In Sections 3–5, we prove Theorem 1, Corollary 1, and Theorem 2, respectively. Appendix A is devoted to proving how the boundedness of the energy functions $E(t)$ and $\mathcal{E}(t)$ (defined in Equations (8) and (17)) in a closed time interval implies the boundedness of the L^2 norm of the solution in the same interval.

2. Preliminaries and Main Results

First, we recall the definition of a weak solution.

Definition 1. *We say that $u(x,t)$ is a weak solution to the problem in Equations (1)–(3) for $\Omega \times [0,T)$, where if $u \in L^\infty(0,T; H_0^2(\Omega) \cap L^p(\Omega))$, $u_t \in L^\infty(0,T; H_0^1(\Omega) \cap L^m(\Omega))$ satisfies the following conditions:*

$$\begin{aligned}&(1) for\ any\ \phi \in H_0^2(\Omega)\cap L^m(\Omega)\ and\ \ a.e.\ t\in[0,T)\\&(u_{tt},\phi)+a_1(\nabla u,\nabla\phi)+a_2(\nabla u_{tt},\nabla\phi)+a_3(\Delta u,\Delta\phi)+k_1(u_t|u_t|^{m-2},\phi)\\&\qquad =k_2(u|u|^{p-2}(\int_\Omega |u|^q dx),\phi);\\&(2)\ u(x,0)=u_0(x)\in H_0^2(\Omega)\cap L^p(\Omega),\ u_t(x,0)=u_1(x)\in H_0^1(\Omega)\cap L^m(\Omega).\end{aligned}$$

With the aim of deriving a lower bound of the lifespan of the solution $u(x,t)$ and to obtain an interval where the solution remains bounded, we introduce an energy function:

$$E(t)=K^d(t)\Big(||u_t||_2^2+a_1(t)||\nabla u||_2^2+a_2(t)||\nabla u_t||_2^2+a_3(t)||\Delta u||_2^2\Big) \tag{8}$$

where $K(t)$ is a derivable positive function defined for $t\in(0,\infty)$ and d is a positive constant, both yet to be chosen, and

$$E_0=:E(0)=K^d(0)\Big(||u_1||_2^2+a_1^0||\nabla u_0||_2^2+a_2^0||\nabla u_1||_2^2+a_3^0||\Delta u_0||_2^2\Big), \tag{9}$$

where $a_i^0=a_i(0), i=1,2,3$. Since we are interested in blowup solutions, now we give the definition of a blowup in the $E(t)$ norm.

Definition 2. *We assert that the solution to Equations (1)–(3) blows up at a finite time t^* in the $E(t)$ norm if*

$$\lim_{t\to t^*}E(t)=+\infty. \tag{10}$$

We point out that the boundedness of $E(t)$ in the interval $[0,T]$ implies the boundedness of $||u(x,t)||_2^2$ in the same interval (see Appendix A).

It is clear that if there exists a finite time $T<t^*$, with T being a lower bound of the blowup time, and as a consequence, the energy function $E(t)$ is bounded in the interval $[0,T]$.

We now state some lemmas to be used in the proofs of the main results. Let us recall the Sobolev embedding inequality $W^{p,m}(\Omega)\subset L^r(\Omega)$ (see Theorem 2.4 in [28] for $p=2$).

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^N$. Let $m\geq 1$, and let r be an arbitrary number with $2\leq r<+\infty$ if $N<2m$ and $2\leq r<\frac{2N}{N-2m}$ if $N>2m$. Then, for any $w\in W_0^{2,m}(\Omega)$, there exists a constant $S_r=S(r,\Omega)$ such that*

$$||w||_{L^r}\leq S_r||(-\Delta)^{\frac{m}{2}}w||_2, \tag{11}$$

where S_r denotes the best embedding constant.

Lemma 2. *Let $u(x,t)$ be the solution to Equations (1)–(3). Let $E(t)$ and E_0 be defined in Equations (8) and (9) and satisfy Equation (10). Then, there exists a time $\bar{t}\in[0,t^*)$ such that*

$$E^a(t)\leq E^b(t)E_0^{a-b},\quad \forall t\in[\bar{t},t^*), \tag{12}$$

for any $1<a<b$.

Proof. If $E(t)$ is non-decreasing for $t\in[0,t^*)$, then $E(t)\geq E_0,\ t\in[0,t^*)$, which implies $\frac{E(t)}{E_0}\geq 1$, and the Lemma is proven.

If $E(t)$ is non-increasing, there exists a time $\bar{t}\in(0,t^*)$ such that $E(\bar{t})=E_0$. Then, $E(t)\geq E_0$ for $t\in[\bar{t},t^*)$, and Equation (12) holds.

We can have a third possibility: some kind of oscillations may appear, but in this case there also exists a time $\bar{t}\in(0,t^*)$ such that $E(\bar{t})=E_0$. Then, $E(t)\geq E(\bar{t})=E_0$ for $t\in[\bar{t},t^*)$. Additionally, in this case, (12) holds.

Our aim is to seek a lower bound of the blowup time t^* for the solution $u(x,t)$ to Equations (1)–(3).

Now, we state the main results in this paper. First, we consider the case where in Equation (1), the coefficients $a_i(t), i = 1,2,3$ are assumed to be positive, time-dependent, and differentiable functions, while $k_i(t), i = 1,2$ are assumed to be positive, continuous, and time-dependent functions, each of them bounded in any time interval. □

Theorem 1. *Let $u(x,t)$ be a solution to the problem in Equations (1)–(3) and $E(t)$ and E_0 be defined in Equations (8) and (9), satisfying Equation (10). Let q and $2(p-1)$ satisfy Lemma 1. Assume that there exist positive functions $\eta(t)$ and $\delta_i(t), i = 1,2,3$ such that*

$$\frac{K'(t)}{K(t)} \leq \eta(t), \quad \forall t \in [0, t^*), \tag{13}$$

and

$$\frac{a_i'(t)}{a_i(t)} \leq \delta_i(t), \quad \forall t \in [0, t^*). \tag{14}$$

Then, E satisfies the following differential inequality:

$$E'(t) \leq \gamma_1 E(t) + \gamma_2 E^{\frac{q}{2}+1} + \gamma_3 E^{\frac{q}{2}+p-1}, \tag{15}$$

with $\gamma_i(t)$ positive functions, depending on $p, q.K(t), d, \eta, \delta_i$, and the Sobolev constant defined in Lemma 1.

Corollary 1. *Lower Bound*

Under the hypotheses of Theorem 1, $E(t)$ remains bounded in $[0, T_0]$ with

$$T_0 = H^{-1}\Big(\frac{E_0^{2-\frac{q}{2}-p}}{\frac{q}{2}+p-2}\Big), \tag{16}$$

where $H^{-1}(t)$ is the inverse of $H(t) = \int_0^t \omega(\tau)d\tau$ and $\omega(t)$ is a positive function depending on $\gamma_i(t), E_0$, and some other positive constants. T_0 provides a lower bound for the blowup time.

The next theorem examines the case when, in Equation (1), all the coefficients a_i, $i = 1,2,3$ and $k_j, j = 1,2$ are positive constants. We introduce a new energy function:

$$\mathcal{E}(t) = ||u_t||_2^2 + a_1||\nabla u||_2^2 + a_2||\nabla u_t||_2^2 + a_3||\Delta u||_2^2 \tag{17}$$

with

$$\mathcal{E}_0 =: \mathcal{E}(0) = ||u_1||_2^2 + a_1||\nabla u_0||_2^2 + a_2||\nabla u_1||_2^2 + a_3||\Delta u_0||_2^2. \tag{18}$$

Theorem 2 (Constant Coefficients). *Let $u(x,t)$ be a solution to the problem in Equations (1)–(3) with constant coefficients and $\mathcal{E}(t)$ and $\mathcal{E}_0$ defined in Equations (17) and (18), satisfying Equation (10). Then, the following is true:*

$$\mathcal{E}'(t) \leq \sigma_1 \mathcal{E}^{\frac{q}{2}+1} + \sigma_2 \mathcal{E}^{\frac{q}{2}+p-1}, \tag{19}$$

with σ_1, σ_2 depending on a_3, k_2, and the Sobolev constant defined in Lemma 1. Moreover, a lower bound T_1 for the lifespan t^* is given by

$$T_1 =: \frac{\mathcal{E}_0^{2-\frac{q}{2}-p}}{c(\frac{q}{2}+p-2)}, \tag{20}$$

where $c = \sigma_1 \mathcal{E}_0^{2-p} + \sigma_2$.

3. Proof of Theorem 1

Proof. First of all, we point out that the function $K^d(t)$ in Equation (8) can be fixed to be equal to one or chosen so that $\gamma_i(t)$ in Equation (15) can be simplified. In the computations below, $K^d(t)$ is present.

By the definition of $E(t)$ in Equation (8), it follows that

$$\begin{aligned} E'(t) &= dK^{d-1}K'\Big(||u_t||_2^2 + a_1(t)||\nabla u||_2^2 + a_2(t)||\nabla u_t||_2^2 + a_3(t)||\Delta u||_2^2\Big) \\ &+ 2K^d\Big((u_t, u_{tt}) - a_1(t)(u_t, \Delta u) - a_2(t)(u_t, \Delta u_{tt}) + a_3(t)(u_t, \Delta^2 u)\Big) \\ &+ K^d\Big(a_1'(t)||\nabla u||_2^2 + a_2'(t)||\nabla u_t||_2^2 + a_3'(t)||\Delta u||_2^2\Big) \\ &= J_1 + J_2 + J_3. \end{aligned} \tag{21}$$

Let us estimate J_1, J_2, J_3 in terms of $E(t)$. By using Equation (14), we find that J_1 satisfies the inequality

$$J_1 \leq d\,\eta(t)E(t). \tag{22}$$

$$J_2 = 2K^d\Big(k_2(t)\int_\Omega [u_t\, u|u|^{p-2}\int_\Omega |u|^q dx]dx - k_1(t)\int_\Omega |u_t|^m dx\Big). \tag{23}$$

Now, in J_2, we estimate the term containing the source with the nonlocal term. By using the Schwarz inequality and the following arithmetic-geometric inequality $A^\theta\, B^{1-\theta} \leq \theta A + (1-\theta)B, A, B > 0, 0 < \theta < 1$, we obtain

$$\begin{aligned} &||u||_q^q(u_t, u|u|^{p-2}) \leq ||u||_q^q(||u_t||_2||u||_{2(p-1)}^{(p-1)}) \\ &\leq ||u||_q^q\Big(\frac{1}{2}||u_t||_2^2 + \frac{1}{2}||u||_{2(p-1)}^{2(p-1)}\Big). \end{aligned} \tag{24}$$

By using Lemma 1 with $m = 2, r = q$, and $r = 2(p-1)$, we have

$$||u||_q^q \leq S_q^q||\Delta u||_2^q \tag{25}$$

and

$$||u||_{2(p-1)}^{2(p-1)} \leq S_p^{2(p-1)}||\Delta u||_2^{2(p-1)}. \tag{26}$$

By inserting the inequalities in Equations (25) and (26) in (24), the following estimate holds:

$$\begin{aligned} &||u||_q^q(u_t, u|u|^{p-2}) \\ \leq &\frac{1}{2}S_q^q||\Delta u||_2^q\Big(||u_t||_2^2 + S_p^{2(p-1)}\Big(||\Delta u||_2^{2(p-1)}\Big) \\ &\leq \frac{1}{2}S_q^q K^{-d\frac{q}{2}} a_3^{-\frac{q}{2}} E^{\frac{q}{2}}\Big(K^{-d}E + S_p^{2(p-1)}K^{-d(p-1)}a_3^{1-p}E^{p-1}\Big). \end{aligned} \tag{27}$$

Then, by neglecting the negative term $-k_1(t)\int_\Omega |u_t|^m dx$ and inserting Equation (27) in (23), we obtain

$$J_2 \le \gamma_2(t)E^{\frac{q}{2}+1} + \gamma_3(t)E^{\frac{q}{2}+p-1} \tag{28}$$

where

$$\begin{aligned} \gamma_2(t) &= k_2(t)S_q^q K^{-d\frac{q}{2}} a_3^{-\frac{q}{2}}, \\ \gamma_3(t) &= k_2(t)S_q^q S_p^{2(p-1)} K^{d(2-\frac{q}{2}-p)} a_3^{1-\frac{q}{2}-p}. \end{aligned} \tag{29}$$

We can also estimate J_3 by using Equation (14)

$$J_3 \le \delta(t)E(t), \tag{30}$$

where $\delta(t) =: \delta_1(t) + \delta_2(t) + \delta_3(t)$. When plugging Equations (22), (28) and (30) into Equation (21), we obtain the differential inequality in Equation (15) satisfied by $E(t)$ such that

$$\begin{aligned} E'(t) &\le \gamma_1(t)E(t) + \gamma_2(t)E^{\frac{q}{2}+1} + \gamma_3(t)E^{\frac{q}{2}+p-1}, \\ \gamma_1(t) &= \eta(t)d + \delta(t) \end{aligned} \tag{31}$$

where $\gamma_2(t)$ and $\gamma_3(t)$ in Equation (29), and Theorem 1 is proven. □

4. Proof of Corollary 1

Proof. We note that it is possible to obtain from Equation (31) an inequality that can be integrated explicitly and, as a consequence, find an explicit lower bound of the lifespan.

Note that in Equation (31), the relation between the powers of the energy function $E(t)$ is the following:

$$1 < \frac{q}{2} + 1 < \frac{q}{2} + p - 1,$$

since it was supposed that $p > 2$ and $q > 2$. From Lemma 2, we have

$$\frac{E(t)}{E_0} \le \frac{E(t)^{\frac{q}{2}+1}}{E_0^{\frac{q}{2}+1}} \le \frac{E(t)^{\frac{q}{2}+p-1}}{E_0^{\frac{q}{2}+p-1}} \tag{32}$$

By inserting the last inequalities in Equation (32) in (31), we obtain a simpler differential inequality:

$$E'(t) \le \omega(t)E^{\frac{q}{2}+p-1}, \quad t \in (\bar{t}, t^*) \tag{33}$$

where $\omega(t) = \gamma_1(t)E_0^{2-\frac{q}{2}-p} + \gamma_2(t)E_0^{2-p} + \gamma_3(t)$.

When integrating Equation (33) between $\bar{t}$ to t, taking into account that $E(t) \to +\infty$ as $t \to t^*$, it follows that

$$\frac{E_0^{2-\frac{q}{2}-p}}{\frac{q}{2}+p-2} = \frac{E_0^{2-\frac{q}{2}-p}(\bar{t})}{\frac{q}{2}+p-2} \le \int_{\bar{t}}^{t^*} \omega(\tau)d\tau \le \int_0^{t^*} \omega(\tau)d\tau, \quad t \in (0, t^*). \tag{34}$$

Denoted by $H(t^*) = \int_0^{t^*} \omega(\tau)d\tau$, and with H^{-1} as its inverse, the inequality in Equation (34) provides a lower bound T_0 of t^* with T_0 in Equation (16). We conclude that the solution $E(t)$ remains bounded in the interval $[0, T_0]$. □

5. Proof of Theorem 2

Proof. By the definition of $\mathcal{E}(t)$ in Equation (17), it follows that

$$\begin{aligned}\mathcal{E}'(t) &= 2\Big((u_t, u_{tt}) - a_1(u_t, \Delta u) - a_2(u_t, \Delta u_{tt}) + a_3(u_t, \Delta^2 u)\Big) \\ &\leq 2\Big(k_2(||u||_q^q(u_t,\ u|u|^{p-2}) - k_1||u_t||_m^m.\Big)\end{aligned} \tag{35}$$

When neglecting the negative term $-k_1||u_t||_m^m$, plugging Equations (25) and (26) into Equation (35) yields

$$\mathcal{E}'(t) \leq \frac{k_2}{a_3^{\frac{q}{2}}} S_q^q\, \mathcal{E}^{\frac{q}{2}+1} + \frac{k_2}{a_3^{\frac{q}{2}+p-1}} S_q^q S_p^{2(p-1)} \mathcal{E}^{\frac{q}{2}+p-1} := \sigma_1 \mathcal{E}^{\frac{q}{2}+1} + \sigma_2 \mathcal{E}^{\frac{q}{2}+p-1} \tag{36}$$

where $\sigma_1 = k_2\, a_3^{-\frac{q}{2}}\, S_q^q$ and $\sigma_2 = k_2\, a_3^{1-\frac{q}{2}-p} S_q^q S_p^{2(p-1)}$. From Lemma 2, we have

$$\mathcal{E}^{\frac{q}{2}+1} \leq \mathcal{E}(t)^{\frac{q}{2}+p-1} \mathcal{E}_0{}^{2-p}. \tag{37}$$

By replacing Equation (37) into Equation (36), we have

$$\mathcal{E}'(t) \leq \{\sigma_1 \mathcal{E}_0^{2-p} + \sigma_2\} \mathcal{E}^{\frac{q}{2}+p-1}. \tag{38}$$

By integrating Equation (38) from 0 up to t and then letting $t \to t^*$, it then follows, arguing as in the proof of Corollary 1, that $\mathcal{E}(t)$ remains bounded for $t \in [0, T_1]$ with T_1 in Equation (20). T_1 provides a lower bound for the blowup time t^*. □

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Appendix A

The existence of a lower bound T_0 for the blowup time to the energy function $E(t)$ has a consequence that the interval $[0, T_0]$ is a safe interval of existence for the L^2 norm of the solution $u(x,t)$. Indeed, let us consider the biharmonic eigenvalue problem with Dirichlet boundary conditions:

$$\Delta\Delta\phi = \Lambda\phi, \quad x \in \Omega \subset \mathbb{R}^N,\ N \geq 2, \tag{A1}$$

$$\phi = 0, \quad \frac{\partial\phi}{\partial n} = 0, \quad x \in \partial\Omega, \tag{A2}$$

where ϕ is normalized by $||\phi||_2^2 = 1$.

Let Λ_1 be the first eigenvalue of the problem in Equations (A1) and (A2). For all $\phi \neq 0$, Λ_1 satisfies the following inequality (see [29]):

$$||\phi||_2^2 \leq \Lambda_1^{-1} ||\Delta\phi||_2^2, \tag{A3}$$

The problem in Equation (A1) is closed, being related to the biharmonic differential equation

$$\Delta\Delta\phi = f$$

with the same boundary conditions from Equation (A2), which describes the characteristic vibrations of a clamped plate. For this reason, the biharmonic eigenvalue problem is also known as the clamped plate eigenvalue problem. Now, when applying Equation (A3) to the solutions u, the boundedness of $E(t)$ in the time interval $[0, T_0]$ implies the boundedness of the L^2 norm of u in the same interval, since

$$||u||_2^2 \leq \Lambda_1^{-1}||\Delta u||_2^2.$$

The same remark holds for $\mathcal{E}(t)$ in the time interval $[0, T_1]$. Clearly, these bounds T_0 and T_1 are not optimal. Moreover, the boundedness of $E(t)$ in $[0, T_0]$ and of $\mathcal{E}(t)$ in $[0, T_1]$ also implies the boundedness of $||u_t||_2^2$, $||\nabla u||_2^2$, $||\nabla u_t||_2^2$, and $||\Delta u||_2^2$ in the same intervals.

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