



# Article On Discrete Poisson–Mirra Distribution: Regression, INAR(1) Process and Applications

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Abstract: Several pieces of research have spotlighted the importance of count data modelling and its applications in real-world phenomena. In light of this, a novel two-parameter compound-Poisson distribution is developed in this paper. Its mathematical functionalities are investigated. The two unknown parameters are estimated using both maximum likelihood and Bayesian approaches. We also offer a parametric regression model for the count datasets based on the proposed distribution. Furthermore, the first-order integer-valued autoregressive process, or INAR(1) process, is also used to demonstrate the utility of the suggested distribution in time series analysis. The unknown parameters of the proposed INAR(1) model are estimated using the conditional maximum likelihood, conditional least squares, and Yule–Walker techniques. Simulation studies for the suggested distribution and the INAR(1) model based on this innovative distribution are also undertaken as an assessment of the long-term performance of the estimators. Finally, we utilized three real datasets to depict the new model's real-world applicability.

**Keywords:** compounding; over-dispersion; Bayesian estimation; count time series; COVID-19 data; earthquake data

MSC: 60E05; 62E15; 62M10

# 1. Introduction

Many studies have underlined the importance of modelling count data as well as the time series of counts, which has sparked considerable interest in a variety of sectors, including medical science, earth science, physics, finance, and insurance. For modelling count datasets, the Poisson distribution is the most frequently used, but it has the drawback of not being able to model over-dispersed datasets, while the negative binomial distribution is used for modelling over-dispersed data. When investigating counting data, the existence of at least one overdispersion warrants extra consideration when selecting a count model (for details, see [1,2]). As a consequence, the conventional Poisson regression model is rarely used in these situations.

Furthermore, one of the most widely utilized methodologies for analyzing timedependent data are time series analysis. The corresponding time series is known as a time series of counts or, equivalently, an integer-valued time series when the data are generated by a random counting procedure. The demand for modelling count time series is seen in diverse real-life situations—for example, the monthly number of earthquakes in



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**Copyright:** © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). a specific place, the monthly number of automobile sales, the monthly number of deaths due to a specific disease, the monthly number of traffic accidents, and the number of living cells (see [3]). Refs. [4–6] proposed the class of first-order non-negative integer-valued autoregressive, shortly INAR(1), processes with Poisson innovation based on the binomial thinning operator to model time series of counts. INAR models are frequently built using the binomial thinning operator.

Moreover, the INAR(1) model can be altered by changing the innovation distributions according to various real-life situations. According to [7], the INAR(1) model with respect to negative binomial innovations (NB-INAR(1)) is effective for generating overdispersion. Since the common occurrence in overdispersion is that the incidence of zero counts is higher than expected from the Poisson distribution, Ref. [8] proposed an INAR(1) process with geometric distribution as the innovation model. Ref. [9] proposed the PL-INAR(1) model, which combines the INAR(1) model with Poisson–Lindley innovations. Then, Ref. [10] introduced the INAR(1) process with Poisson–Bilal innovations. These are a few of the most significant studies on the overdispersed INAR(1) process. Even while these methods give fine solutions for over-dispersed time series count datasets, they still have significant drawbacks that can pose computing challenges in real-world applications. Thus, discovering more INAR(1) models will provide more possibilities for superiorly fitting the real datasets by choosing those models according to the situations.

In this paper, we construct a novel discrete two-parameter distribution by mixing the Poisson and Mirra distributions, which can be used to model overdispersed count data sets. Hereafter, we call this new distribution the Poisson–Mirra distribution (PMiD). The two-parameter Mirra distribution by [11] is considered to be the generalization of the Xgamma distribution with one parameter, which is proposed by [12]. In addition, Ref. [13] considers the Poisson–Xgamma distribution to be an alternative or rival to the well-known one-parameter Poisson–Lindley distribution proposed by [14]. Interestingly, we discovered that the proposed PMiD is the generalization of the Poisson–Xgamma distribution. As a result, more research into the PMiD is unavoidable, both theoretically and in terms of applied aspects, which is the primary motivation for this work. The suggested distribution has the advantages of having a simple probability mass function (*pmf*) and cumulative distribution function (*cdf*), as well as explicit probability and moment generating functions, and can monitor over-dispersed count datasets, which are common in real-world data modelling. By using the PMiD as an innovation process, we also emphasize the significance of the PMiD in the INAR(1) process.

The following is how the rest of the article is sorted. The detailed description of the Mirra distribution is covered in Section 2. The definition of the new distribution, its factorial moments, moments about the origin, skewness, kurtosis, mode, generating functions, entropy measures, and other details are presented in Section 3. In Section 4, the maximum likelihood (ML) estimation and the Bayesian estimation technique are defined to estimate the unknown parameters of the new distribution, and the performance of the PMiD parameters for the ML estimation is also studied using simulation technique in the same section. A regression model with respect to the new distribution is discussed in Section 5. Section 6 develops the INAR(1) model construction with PMiD innovations; the PMiD-INAR(1) process is created. The methods for the estimation of parameters, which include the conditional maximum likelihood, conditional least squares, and the Yule-Walker estimation procedures for the PMiD-INAR(1) process are discussed in Section 7, and the simulation studies based on these three estimation procedures are also conducted in the same section. The applications and the empirical studies based on the new model concerning three real datasets are conducted in Section 8. A detailed discussion of the article is presented in Section 9. Then, Section 10 finishes with the decisive concluding words.

#### 2. The Mirra Distribution

An inaugural approach to the Mirra distribution (MiD) was implemented in the literature by [11]. Let *T* be a continuous random variable which follows the MiD with

parameters  $\alpha$  and  $\theta$ . Then, the probability density function (*pdf*), and the survival function (*sf*) of the random variable *T* are respectively given by

$$f(t) = \frac{\theta^3}{\theta^2 + \alpha} \left( 1 + \frac{\alpha t^2}{2} \right) e^{-\theta t}$$

and

$$S(t) = \frac{\theta^2}{\theta^2 + \alpha} \left( 1 + \frac{\alpha}{\theta^2} + \frac{\alpha t}{\theta} + \frac{\alpha t^2}{2} \right) e^{-\theta t}$$

where t > 0 and the parameters  $\alpha, \theta > 0$ . This distribution is denoted by  $Mi(\alpha, \theta)$ . The plots in Figure 1 portray the graphical representation of the pdf.



Figure 1. Plots of the *pdf* of the MiD.

Now, the hazard function (*hf*) of the MiD is given as

$$h(t) = \frac{\theta\left(1 + \frac{\alpha t^2}{2}\right)}{1 + \frac{\alpha}{\theta^2} + \frac{\alpha t}{\theta} + \frac{\alpha t^2}{2}}$$

As indicated by [11], the *hf* of the MiD is shaped as a bathtub, decreasing for  $t < \sqrt{\frac{2}{\alpha}}$ , and increasing for  $t > \sqrt{\frac{2}{\alpha}}$ .

As a special case, for  $\alpha = \theta = \lambda$ , the Xgamma distribution (XGD) is obtained with the *pdf* given by

$$f(t) = \frac{\lambda^2}{\lambda + 1} \left( 1 + \frac{\lambda t^2}{2} \right) e^{-t\lambda},$$

where t > 0 and the parameter  $\lambda > 0$ . For more information on the XGD, see [12].

# 3. The Poisson–Mirra Distribution

3.1. Presentation

Through the following proposition, a novel mixed-Poisson distribution is introduced by compounding the Poisson and Mirra distributions.

**Proposition 1.** Assume that X follows the new compound Poisson–Mirra distribution (PMiD), which has the stochastic representation as follows:

$$X|\lambda \sim Poisson(\lambda)$$
$$\lambda|\alpha, \theta \sim Mi(\alpha, \theta),$$

where  $\lambda$ ,  $\alpha$  and  $\theta > 0$ . Then, the pmf of X is given by

$$p(x;\alpha,\theta) = Pr(X=x) = \frac{\theta^3}{(\theta^2 + \alpha)(1+\theta)^{x+1}} \left\{ 1 + \frac{\alpha(x+1)(x+2)}{2(1+\theta)^2} \right\}, \quad x = 0, 1, 2, \dots$$
(1)

*This distribution is denoted as*  $PMiD(\alpha, \theta)$ *, and one can note*  $X \sim PMiD(\alpha, \theta)$  *to inform that* X *follows the* PMiD *with parameters*  $\alpha$  *and*  $\theta$ *.* 

**Proof.** The *pmf* of *X* can be derived using the general compounding formula as follows:

$$p(x;\alpha,\theta) = \int_{0}^{\infty} \Pr(X = x|\lambda) f(\lambda|\alpha,\theta) d\lambda$$
  
= 
$$\int_{0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} \frac{\theta^{3}}{\theta^{2} + \alpha} \left(1 + \frac{\alpha\lambda^{2}}{2}\right) e^{-\theta\lambda} d\lambda$$
  
= 
$$\frac{\theta^{3}}{x!(\theta^{2} + \alpha)} \left\{ \int_{0}^{\infty} e^{-\lambda(1+\theta)}\lambda^{x} d\lambda + \frac{\alpha}{2} \int_{0}^{\infty} e^{-\lambda(1+\theta)}\lambda^{x+2} d\lambda \right\}$$
  
= 
$$\frac{\theta^{3}}{x!(\theta^{2} + \alpha)} \left\{ \frac{\Gamma(x+1)}{(1+\theta)^{x+1}} + \frac{\alpha}{2} \frac{\Gamma(x+3)}{(1+\theta)^{x+3}} \right\}$$
  
= 
$$\frac{\theta^{3}}{(\theta^{2} + \alpha)(1+\theta)^{x+1}} \left\{ 1 + \frac{\alpha(x+1)(x+2)}{2(1+\theta)^{2}} \right\}.$$

We have employed the gamma function defined by  $\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt$ , with the relation  $\Gamma(m) = (m-1)!$  for any positive integer *m*. The proof is completed.  $\Box$ 

Now, for  $\alpha = \theta = \lambda$ , the *pmf* of the PMiD reduces to

$$p(x;\lambda) = \frac{\lambda^2}{2(1+\lambda)^{x+4}} \Big\{ 2(1+\lambda)^2 + \lambda(x+1)(x+2) \Big\}, \quad x = 0, 1, 2, \dots$$
(2)

The expression in Equation (2) is the *pmf* of Poisson–Xgamma distribution (PXGD), which was introduced by [13]. Thus, the PXGD is a special case of the PMiD.

Now, the possible *pmf* plots for various values of the parameters of the PMiD are portrayed in Figure 2.



Figure 2. Plots of the *pmf* of the PMiD.

The *pmf* appears to be declining, growing, and unimodal, with some fluctuation in the mode and tails.

# 3.2. Mode

The next result clarifies the mode analysis of the PMiD.

**Proposition 2.** Let X be a random variable following a PMiD. Then, if  $\alpha \geq \frac{8\theta^2(1+\theta)^2}{(\theta+2)^2}$ , the mode of X, denoted by  $x_m$ , exists, and lies in either of these two cases:

$$\begin{cases} \max\{a_1(\alpha,\theta), b_2(\alpha,\theta)\} \le x_m \le b_1(\alpha,\theta) \\ or \\ a_1(\alpha,\theta) \le x_m \le \min\{b_1(\alpha,\theta), a_2(\alpha,\theta)\}, \end{cases}$$

where

$$a_{1}(\alpha,\theta) = \frac{\alpha(2-\theta) - \sqrt{\epsilon}}{2\alpha\theta}, \quad b_{1}(\alpha,\theta) = \frac{\alpha(2-\theta) + \sqrt{\epsilon}}{2\alpha\theta},$$
  

$$a_{2}(\alpha,\theta) = \frac{\alpha(2-3\theta) - \sqrt{\epsilon}}{2\alpha\theta}, \quad b_{2}(\alpha,\theta) = \frac{\alpha(2-3\theta) + \sqrt{\epsilon}}{2\alpha\theta},$$
(3)

with  $\epsilon = \alpha^2(\theta + 2)^2 - 8\alpha\theta^2(1+\theta)^2$ . The condition  $\alpha \ge \frac{8\theta^2(1+\theta)^2}{(\theta+2)^2}$  is to ensure that  $\epsilon \ge 0$ , such that  $\sqrt{\epsilon}$  has sense.

**Proof.** We have to find the integer  $x = x_m$ , for which  $p(x; \alpha, \theta)$  takes its maximum value. That is, we aim to solve  $p(x; \alpha, \theta) \ge p(x - 1; \alpha, \theta)$  and  $p(x; \alpha, \theta) \ge p(x + 1; \alpha, \theta)$ , which is equivalent to solving:

$$\alpha\theta x^{2} + \alpha(\theta - 2)x + 2[\theta(1+\theta)^{2} - \alpha] \le 0, \tag{4}$$

and

$$\alpha\theta x^2 + \alpha(3\theta - 2)x + 2[\alpha(\theta - 2) + \theta(1 + \theta)^2] \ge 0,$$
(5)

respectively. On solving the quadratic inequality in Equation (4), we obtain  $a_1(\alpha, \theta) \le x_m \le b_1(\alpha, \theta)$ , and on solving the quadratic inequality in Equation (5), we obtain  $b_2(\alpha, \theta) \le x_m$  or  $x_m \le a_2(\alpha, \theta)$ , where  $a_1(\alpha, \theta), b_1(\alpha, \theta), a_2(\alpha, \theta)$ , and  $b_2(\alpha, \theta)$  are given in Equation (3). Combining these three inequalities, we obtain the mode  $x_m$  such that

$$\begin{cases} \max\{a_1(\alpha,\theta), b_2(\alpha,\theta)\} \le x_m \le b_1(\alpha,\theta) \\ or \\ a_1(\alpha,\theta) \le x_m \le \min\{b_1(\alpha,\theta), a_2(\alpha,\theta)\}. \end{cases}$$

This completes the proof.  $\Box$ 

3.3. Cdf and Hf

The corresponding *cdf* of the PMiD is given by

$$F(x;\alpha,\theta) = \Pr(X \le x) = \frac{(\theta+1)^{-x-3}}{2(\alpha+\theta^2)} \{ \alpha [2(\theta+1)^{x+3} - \theta(x+3)(\theta(x+2)+2) - 2] + 2\theta^2(\theta+1)^2 [(\theta+1)^{x+1} - 1] \},$$
(6)

and the hf of the PMiD is given by

$$H(x; \alpha, \theta) = \frac{p(x; \alpha, \theta)}{1 - F(x; \alpha, \theta)}$$

where  $p(x; \alpha, \theta)$  and  $F(x; \alpha, \theta)$  are respectively given in Equations (1) and (6). Furthermore, plots in Figure 3 refer to the shapes of the *hf* of the PMiD.



Figure 3. Plots of the *hf* of the PMiD.

The *hf* is found to have all of the typical shapes, such as increasing, decreasing, bathtub, and upside-down bathtub shapes.

# 3.4. Moments

Some aspects of the PMiD are now being studied using various moment measures.

**Proposition 3.** The  $r^{th}$  factorial moment of  $X \sim PMiD(\alpha, \theta)$  is given by

$$\mu_r = E(X(X-1)\dots(X-r+1)) = \frac{\theta^3}{(\theta^2 + \alpha)} \frac{r!}{\theta^{r+1}} \left[ 1 + \frac{\alpha(r+1)(r+2)}{2\theta^2} \right].$$
(7)

**Proof.** Based on the compound-Poisson theory (see [14]), the  $r^{th}$  factorial moment of *X* can be obtained as follows:

$$\mu_r = \int_0^\infty \lambda^r \frac{\theta^3}{\theta^2 + \alpha} \left( 1 + \frac{\alpha \lambda^2}{2} \right) e^{-\theta \lambda} d\lambda$$
$$= \frac{\theta^3}{\theta^2 + \alpha} \left\{ \int_0^\infty e^{-\theta \lambda} \lambda^x d\lambda + \frac{\alpha}{2} \int_0^\infty e^{-\theta \lambda} \lambda^{x+2} d\lambda \right\}$$
$$= \frac{\theta^3}{(\theta^2 + \alpha)} \frac{r!}{\theta^{r+1}} \left[ 1 + \frac{\alpha(r+1)(r+2)}{2\theta^2} \right].$$

Thus, the proof is complete.  $\Box$ 

The first four factorial moments of the PMiD can be obtained by substituting r = 1, 2, 3, and 4 in Equation (7). That is,

$$\mu_1 = \frac{\theta}{\theta^2 + \alpha} \left( 1 + \frac{3\alpha}{\theta^2} \right), \qquad \mu_2 = \frac{2}{\theta^2 + \alpha} \left( 1 + \frac{6\alpha}{\theta^2} \right),$$
$$\mu_3 = \frac{6}{\theta(\theta^2 + \alpha)} \left( 1 + \frac{10\alpha}{\theta^2} \right), \quad \text{and} \quad \mu_4 = \frac{24}{\theta^2(\theta^2 + \alpha)} \left( 1 + \frac{15\alpha}{\theta^2} \right).$$

Now, the first four moments about the origin of the PMiD are obtained by utilizing the general relationship between factorial moments and moments about the origin. We get

Mean = 
$$\mu'_1 = E(X) = \mu_1 = \frac{\theta}{\theta^2 + \alpha} \left( 1 + \frac{3\alpha}{\theta^2} \right)$$
, (8)

$$\mu_2' = E(X^2) = \frac{1}{\theta^2 + \alpha} \left\{ 2\left(1 + \frac{6\alpha}{\theta^2}\right) + \theta\left(1 + \frac{3\alpha}{\theta^2}\right) \right\},$$
$$\mu_3' = E(X^3) = \frac{1}{\theta^2 + \alpha} \left\{ \frac{6}{\theta} \left(1 + \frac{10\alpha}{\theta^2}\right) + 6\left(1 + \frac{6\alpha}{\theta^2}\right) + \theta\left(1 + \frac{3\alpha}{\theta^2}\right) \right\}$$

and

$$\mu'_{4} = E(X^{4})$$
  
=  $\frac{1}{\theta^{2} + \alpha} \left\{ \frac{24}{\theta^{2}} \left( 1 + \frac{15\alpha}{\theta^{2}} \right) + \frac{36}{\theta} \left( 1 + \frac{10\alpha}{\theta^{2}} \right) + 14 \left( 1 + \frac{6\alpha}{\theta^{2}} \right) + \theta \left( 1 + \frac{3\alpha}{\theta^{2}} \right) \right\}.$ 

Therefore, the variance of the PMiD is obtained as

$$Var(X) = \frac{1}{\theta^2(\theta^2 + \alpha)} \left\{ 2(\theta^2 + 6\alpha) + \theta(\theta^2 + 3\alpha) \left[ 1 - \frac{\theta}{\theta^2 + \alpha} \left( 1 + \frac{3\alpha}{\theta^2} \right) \right] \right\}.$$
 (9)

The dispersion index of the PMiD is given by

$$DI = 1 + \frac{2}{\theta} \left( \frac{\theta^2 + 6\alpha}{\theta^2 + 3\alpha} \right) - \frac{\theta^2 + 3\alpha}{\theta(\theta^2 + \alpha)}.$$
 (10)

Since  $\alpha$  and  $\theta > 0$ , one can establish that the PMiD's dispersion index is greater than one, i.e., DI > 1, indicating that the PMiD is over-dispersed. The following formulae can be used to get explicit expressions for the skewness and kurtosis of the PMiD:

$$Skewness(X) = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3}{[Var(X)]^{3/2}}$$

and

$$Kurtosis(X) = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4}{[Var(X)]^2}$$

Now, Table 1 shows some numerical values for the mean, variance, DI, skewness, and kurtosis for the PMiD for various parameter settings.

		$\alpha = 0.5$	and Various Val	ues of $\theta$	
Measures	$\theta = 1.5$	$\theta = 3.5$	$\theta = 5.5$	heta= 7.5	$\theta = 9.5$
Mean	0.9091	0.3081	0.1877	0.1357	0.1064
Variance	1.7796	0.4085	0.2240	0.1544	0.1179
DI	1.9576	1.3256	1.1931	1.1379	1.1076
Skewness	2.1407	2.5913	2.9319	3.2482	3.5398
Kurtosis	9.3872	11.8878	13.6830	15.5978	17.5592
		$\alpha = 1.5$	and Various Val	ues of θ	
Measures	$\theta = 1.5$	$\theta = 3.5$	$\theta = 5.5$	$\theta = 7.5$	$\theta = 9.5$
Mean	1.2	0.3481	0.1990	0.1403	0.1087
Variance	2.4267	0.4792	0.2411	0.1608	0.1209
DI	2.0222	1.3769	1.2117	1.1462	1.1118
Skewness	1.8289	2.5314	2.9032	3.2258	3.5212
Kurtosis	7.4713	11.5402	13.5972	15.5194	17.4747

**Table 1.** Values of some moment measures of the PMiD for various values of  $\alpha$  and  $\theta$ .

**Proposition 4.** The probability generating function (pgf) of  $X \sim PMiD(\alpha, \theta)$  is obtained as

$$G(s) = E(s^{X}) = \frac{\theta^{3}}{(\theta^{2} + \alpha)(\theta + 1 - s)} \left[ 1 + \frac{\alpha}{(\theta + 1 - s)^{2}} \right],$$
 (11)

for  $s \in (-1, 1)$ .

**Proof.** Owing to the well-known compound-Poisson theory, the *pgf* of the PMiD is obtained as follows:

$$\begin{split} G(s) &= \int_{0}^{\infty} e^{\lambda(s-1)} \frac{\theta^{3}}{\theta^{2} + \alpha} \left( 1 + \frac{\alpha \lambda^{2}}{2} \right) e^{-\theta \lambda} d\lambda \\ &= \frac{\theta^{3}}{\theta^{2} + \alpha} \left\{ \int_{0}^{\infty} e^{-\lambda(\theta + 1 - s)} d\lambda + \frac{\alpha}{2} \int_{0}^{\infty} e^{-\lambda(\theta + 1 - s)} \lambda^{2} d\lambda \right\} \\ &= \frac{\theta^{3}}{\theta^{2} + \alpha} \left[ \frac{\Gamma(1)}{\theta + 1 - s} + \frac{\alpha}{2} \frac{\Gamma(3)}{(\theta + 1 - s)^{3}} \right] \\ &= \frac{\theta^{3}}{(\theta^{2} + \alpha)(\theta + 1 - s)} \left[ 1 + \frac{\alpha}{(\theta + 1 - s)^{2}} \right]. \end{split}$$

Thus, the proof is complete.  $\Box$ 

When *s* is replaced by  $e^t$  and  $e^{it}$  in Equation (11), we obtain the moment generating function (*mgf*) and characteristic function (*cf*) of the PMiD, respectively. They are respectively given by

$$M(t) = \frac{\theta^3}{(\theta^2 + \alpha)(\theta + 1 - e^t)} \left[ 1 + \frac{\alpha}{(\theta + 1 - e^t)^2} \right]$$

for  $t \leq 0$ , and

$$\phi(t) = \frac{\theta^3}{(\theta^2 + \alpha)(\theta + 1 - e^{it})} \left[ 1 + \frac{\alpha}{(\theta + 1 - e^{it})^2} \right],$$

for  $t \in \mathbb{R}$ .

## 3.5. Rényi and Shannon Entropies

Entropy is a measure of uncertainty fluctuation in a stochastic situation, in which higher entropy indicates less information. The most popular entropy measures are Rényi entropy and Shannon entropy, which are among the most accessible in the literature.

For every discrete distribution with *pmf* p(x), the related Rényi entropy is defined by

$$H_{\gamma} = rac{1}{1-\gamma} \log \sum_{x} p^{\gamma}(x),$$

for  $\gamma > 0$  and  $\gamma \neq 1$ .

In the context of the PMiD, by using Equation (1), we obtain

$$\sum_{x=0}^{\infty} p^{\gamma}(x;\alpha,\theta) = \frac{\theta^{3\gamma}}{(\theta^2 + \alpha)^{\gamma}} \sum_{x=0}^{\infty} \left\{ \frac{1}{(1+\theta)^{x+1}} + \frac{\alpha(x+1)(x+2)}{2(1+\theta)^{x+3}} \right\}^{\gamma}.$$

Thus, the Rényi entropy of the PMiD is simplified to the following formula:

$$H_{\gamma} = \frac{1}{1-\gamma} \Biggl\{ \log \Biggl[ \frac{\theta^{3\gamma}}{(\theta^2 + \alpha)^{\gamma}} \Biggr] + \log \sum_{x=0}^{\infty} \Biggl[ \frac{1}{(1+\theta)^{x+1}} + \frac{\alpha(x+1)(x+2)}{2(1+\theta)^{x+3}} \Biggr]^{\gamma} \Biggr\}.$$

Now, the Shannon entropy for a discrete distribution with *pmf* p(x) is given by

$$H_1 = -\sum_{x=0}^{\infty} p(x) \log p(x).$$

Hence, the Shannon entropy for the PMiD can be expressed as

$$H_1 = -\log\left(\frac{\theta^3}{\theta^2 + \alpha}\right) - \sum_{x=0}^{\infty} p(x) \log\left[\frac{1}{(1+\theta)^{x+1}} + \frac{\alpha(x+1)(x+2)}{2(1+\theta)^{x+3}}\right].$$

#### 4. Estimation of the Parameters

Hereafter, we perform the estimation of parameters of the PMiD using two well-known estimation approaches: ML and Bayesian methods.

## 4.1. Maximum Likelihood Estimation

Let  $X_1, X_2, ..., X_n$  be a random sample of size *n* from  $X \sim PMiD(\alpha, \theta)$  (so *n* independent and identically distributed (*iid*) random variables with the PMiD), with unknown  $\alpha$  and  $\theta$ , and  $x_1, x_2, ..., x_n$  be observations of  $X_1, X_2, ..., X_n$ . Then, the log-likelihood function is

$$\log L_n = 3n \log(\theta) - n \log(\theta^2 + \alpha) - \log(1+\theta) \sum_{i=1}^n (x_i + 1) + \sum_{i=1}^n \log \left[ 1 + \frac{\alpha(x_i + 1)(x_i + 2)}{2(1+\theta)^2} \right].$$

The maximization of  $\log L_n$  with respect to the parameters give their ML estimates (MLEs).

The following approach can be considered. The score function associated with this log-likelihood function is

$$\mathbf{U} = \left(\frac{\partial \log L_n}{\partial \alpha}, \frac{\partial \log L_n}{\partial \theta}\right)^T.$$

Now, by solving  $\frac{\partial \log L_n}{\partial \alpha} = 0$ , and  $\frac{\partial \log L_n}{\partial \theta} = 0$ , we obtain the associated nonlinear log-likelihood equations. They are respectively given by

$$\sum_{i=1}^{n} \frac{(x_i+1)(x_i+2)}{2(1+\theta)^2 + \alpha(x_i+1)(x_i+2)} - \frac{n}{\theta^2 + \alpha} = 0,$$

and

$$\frac{3n}{\theta} - \frac{2n\theta}{\theta^2 + \alpha} - \sum_{i=1}^n \left(\frac{x_i+1}{1+\theta}\right) \left[1 - \frac{2\alpha(x_i+2)}{2(1+\theta)^2 + \alpha(x_i+1)(x_i+2)}\right] = 0.$$

The solutions of these two equations give the MLEs. We obtained the MLEs numerically using the *fitdistrplus* package of the **R** software (see [15]). For more details on the *fitdistrplus* package, one should go through the lin k https://CRAN.R-project.org/package=fitdistrplus accessed on 14 February 2021. For the detailed R-code for finding the MLEs of the PMiD, see Appendix A of this article.

#### 4.2. Bayesian Estimation

The Bayesian estimation technique is used to estimate the PMiD parameters in this subsection. That is, each parameter of PMiD must have some prior densities. For both of the parameters  $\alpha$  and  $\theta$ , the half-Cauchy (hC) distribution is used as the prior densities. The hereunder is the *pdf* of the hC distribution with scale parameter  $\delta$ :

$$f_{hC}(u) = \frac{2\delta}{\pi(u^2 + \delta^2)}, \quad u > 0, \ \delta > 0.$$

There is no mean and variance for the hC distribution. Other than that, its mode is equal to 0. The hC distribution with the value of  $\delta$  equals 25 is the preferable alternative to the uniform distribution, if more information is needed, according to [16]. Thus, as a noninformative prior distribution for the parameters  $\alpha$  and  $\theta$ , we utilize hC distribution with its  $\delta$  value fixed to 25. That is, we use

$$\alpha, \theta \sim hC(25). \tag{12}$$

Thus, using Equation (12), the joint posterior *pdf* is given by

$$\psi_*(\alpha, \theta | x) \propto L_n \times \psi(\alpha) \times \psi(\theta),$$
 (13)

where  $L_n$  is the likelihood function for the PMiD, and  $\psi(x)$  is the *pdf* of the hC distribution with  $\delta = 25$ . It is clear from Equation (13) that there is no analytical solution for determining the Bayesian estimates. As a consequence, we adopt the Metropolis–Hastings algorithm (MHA) of the Markov Chain Monte Carlo (MCMC) technique, which is a phenomenal simulation method, using the **R** software.

## 4.3. Performance of the PMiD Parameters Using Simulation Study

For some finite sample sizes, we execute the simulation studies to test the long-run accuracy of the MLEs of the PMiD parameters. We have generated samples of sizes n = 100, 250, 500, 750, and 1000 from the PMiD using various sets of parameter values. The R-code for generating the PMiD random samples for the specified parameter settings are given in Appendix A. The iteration is conducted 1001 times in this case. As a consequence, we calculated the average of the biases, mean squared errors (MSEs), coverage probabilities (CPs), and average lengths (ALs) of each parameter estimate for all iterations in the relevant sample sizes. The results are reported in Table 2. It can be seen that, as the sample size increases, the MSEs and ALs were associated with each estimate decrease. Interestingly, the CPs of the confidence intervals (CIs) for each parameter are relatively close to the nominal 95 percent level. This illustrates the steady performances of the MLEs.

Parameters	п	MLE	Bias	MSE	СР	AL
α	100	2.8296	0.3296	5.4612	0.8302	18.9348
	250	2.9959	0.4959	4.3959	0.8701	12.1873
	500	2.9402	0.4402	2.7254	0.9131	7.6664
	750	2.8896	0.3896	2.1199	0.9171	5.9141
	1000	2.8283	0.3283	1.5774	0.9181	4.8197
θ	100	0.4922	-0.0078	0.0019	0.9570	0.1855
	250	0.4973	-0.0027	0.00075	0.9630	0.1161
	500	0.4989	-0.0011	0.00038	0.9640	0.0816
	750	0.4998	-0.00023	0.00025	0.9710	0.0666
	1000	0.5003	0.00031	0.0002	0.9610	0.0578

**Table 2.** The simulation results for ( $\alpha = 2.5, \theta = 0.5$ ).

## 5. PMiD Regression Model

In this section, we define a new count regression model based on the PMiD known as a PMiD regression model. Let  $Y \sim PMiD(\alpha, \theta)$ . By considering the re-parametrization  $\alpha = \theta^2(\mu\theta - 1)/(3 - \mu\theta)$ , the *pmf* of the PMiD can be expressed in terms of the mean  $E(Y) = \mu > 0$  and  $\theta > 0$ , and it is given by

$$P(Y = y | \mu, \theta) = \frac{\theta(3 - \theta\mu)}{2(1 + \theta)^{y+1}} \left[ 1 + \frac{\theta^2(\theta\mu - 1)(y + 1)(y + 2)}{2(3 - \theta\mu)(1 + \theta)^2} \right], y = 0, 1, 2, \dots$$
(14)

The *pmf* in Equation (14) that defines a distribution is denoted as  $PMiD(\mu, \theta)$ . Thus, we have  $Y \sim PMiD(\mu, \theta)$ . Assume that the response variable Y satisfies  $Y \sim PMiD(\mu, \theta)$ , and the covariates are in relation with the *i*<sup>th</sup> mean by the log-link function given as follows:

$$\mu_i = E(Y_i) = \exp\left(\mathbf{v}_i^T \boldsymbol{\tau}\right), \quad i = 1, 2, \dots, n,$$
(15)

where  $\mathbf{v}_i^T = (1, v_{i1}, v_{i2}, \dots, v_{ip})$  is the vector of covariates and  $\boldsymbol{\tau} = (\tau_0, \tau_1, \dots, \tau_p)^T$  is the unknown vector of regression coefficients. The log-likelihood function of the PMiD regression model is derived by inserting the log-link function of Equation (15) in Equation (14), and is given by

$$\log L(\Theta) = n \log(\theta/2) + \sum_{i=1}^{n} \log(3 - \theta\mu_i) - \log(1 + \theta) \sum_{i=1}^{n} (y_i + 1) + \sum_{i=1}^{n} \log\left[1 + \frac{\theta^2(\theta\mu_i - 1)(y_i + 1)(y_i + 2)}{2(3 - \theta\mu_i)(1 + \theta)^2}\right],$$
(16)

where  $y_i$  is the *i*<sup>th</sup> observations of *Y*,  $\mu_i$  is given in Equation (15) for i = 1, 2, ..., n, and  $\Theta = (\tau, \theta)$  are obtained by maximizing Equation (16) using *optim* function in the **R** software.

# 6. INAR(1) Model with PMiD Innovations

The novel distribution is particularly well suited for modelling integer-valued time series data.

The stochastic process,  $\{X_t\}_{t \in \mathbb{Z}}$ , is an INAR(1) process if it is given by

$$X_t = p \circ X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z},$$

where  $p \in [0, 1)$  and  $\varepsilon_t$  is represented as the innovation process, thus composed of *iid* integer-valued random variables, with mean  $E(\varepsilon_t) = \mu_{\varepsilon}$  and variance  $Var(\varepsilon_t) = \sigma_{\varepsilon}^2$ . The operator symbol ' $\circ$ ' represents the binomial thinning operator, which is defined as

$$p \circ X_{t-1} = \sum_{j=1}^{X_{t-1}} B_j,$$

where  $\{B_j\}_{j\geq 1}$  is a sequence of iid Bernoulli random variables with success probability, p. For  $p \in [0, 1)$ , the INAR(1) process is stationary, while, for the case p = 1, the process is non-stationary. The INAR(1) process, according to [4,5], is a homogeneous Markov chain with 1-step transition probabilities given by

$$P(X_t = k | X_{t-1} = l) = \sum_{i=0}^{\min(k,l)} {l \choose i} p^i (1-p)^{l-i} \Pr(\varepsilon_t = k-i), \quad k, l \ge 0,$$

where  $p \in (0, 1)$ . Therefore, in general, the mean, variance, and dispersion index of the INAR(1) process are, respectively, given by

$$E(X_t) = \frac{\mu_{\varepsilon}}{1 - p'},\tag{17}$$

$$\operatorname{Var}(X_t) = \frac{p\mu_{\varepsilon} + \sigma_{\varepsilon}^2}{1 - p^2},\tag{18}$$

and

$$DI_{X_t} = \frac{DI_{\varepsilon} + p}{1 + p},\tag{19}$$

where  $\mu_{\varepsilon}$ ,  $\sigma_{\varepsilon}^2$ , and  $DI_{\varepsilon}$  are given in Equations (8)–(10), respectively. For the detailed information on the INAR process, one can go through [17].

Now, in this section, based on the PMiD innovations, we present a new over-dispersed INAR(1) model, and we call it the PMiD-INAR(1) process.

**Definition 1.** Assume that the innovation process  $\{\varepsilon_t\}_{t \in \mathbb{Z}}$  follows a PMiD. Then, the (one-step) transition probability of the PMiD-INAR(1) process is given by

$$P_{k,l} = Pr(X_t = k | X_{t-1} = l)$$
  
=  $\sum_{i=0}^{\min(k,l)} {l \choose i} p^i (1-p)^{l-i} \frac{\theta^3}{(\theta^2 + \alpha)(1+\theta)^{k-i+1}} \left\{ 1 + \frac{\alpha(k-i+1)(k-i+2)}{2(1+\theta)^2} \right\}.$ 

Now, using Equations (17)–(19), the mean, variance, and dispersion index of the PMiD-INAR(1) process are respectively obtained as

$$\mu_{X_t} = \frac{\theta}{(\theta^2 + \alpha)(1 - p)} \left( 1 + \frac{3\alpha}{\theta^2} \right),$$
  
$$\sigma_{X_t}^2 = \frac{\theta}{(\theta^2 + \alpha)(1 - p^2)} \left\{ \left( 1 + \frac{3\alpha}{\theta^2} \right) \left[ p + 1 - \frac{1}{(\theta^2 + \alpha)} \left( 1 + \frac{3\alpha}{\theta^2} \right) \right] + 2 \left( 1 + \frac{6\alpha}{\theta^2} \right) \right\},$$

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and

$$DI_{X_t} = 1 + \frac{2}{(1+p)\theta} \left[ \frac{\theta^2 + 6\alpha}{\theta^2 + 3\alpha} - \frac{\theta^2 + 3\alpha}{2\theta(\theta^2 + \alpha)} \right].$$
 (20)

Since  $DI_{X_t}$  is clearly greater than 1, the process is appropriate for over-dispersed integer valued autoregressive time series data. The conditional expectation and variance of the PMiD-INAR(1) process are given by

$$E(X_t|X_{t-1}) = E[p \circ X_{t-1}|X_{t-1}] + E[\varepsilon_t|X_{t-1}]$$
  
=  $pX_{t-1} + \frac{\theta}{\theta^2 + \alpha} \left(1 + \frac{3\alpha}{\theta^2}\right),$  (21)

and

$$Var(X_t|X_{t-1}) = Var[p \circ X_{t-1}|X_{t-1}] + Var[\varepsilon_t|X_{t-1}] = p(1-p)X_{t-1} + \sigma_{\varepsilon}^2,$$
(22)

where  $\sigma_{\epsilon}^2$  is given in Equation (9) (see [17,18]).

# 7. Estimation of the Parameters: PMiD-INAR(1) Process

In this section, the conditional maximum likelihood (CML), conditional least squares (CLS), and Yule–Walker (YW) methods are used to investigate the inference of the PMiD-INAR(1) process. To examine the efficiency of these three strategies, a simulation study is also carried out.

# 7.1. Conditional Maximum Likelihood (CML) Estimation

Let  $X_1, X_2, ..., X_T$  be a random sample of the stationary PMiD-INAR(1) process, and  $x_1, x_2, ..., x_T$  be observations of  $X_1, X_2, ..., X_T$ . Then, the conditional log-likelihood function is given by

$$L(p, \alpha, \theta) = \sum_{t=2}^{T} \log P(X_t = x_t | X_{t-1} = x_{t-1})$$

$$= \sum_{t=2}^{T} \log \left\{ \sum_{i=0}^{\min(x_t, x_{t-1})} {x_{t-1} \choose i} \frac{p^i (1-p)^{x_{t-1}-i} \theta^3}{(\theta^2 + \alpha)(1+\theta)^{x_t-i+1}} \left[ 1 + \frac{\alpha(x_t - i + 1)(x_t - i + 2)}{2(1+\theta)^2} \right] \right\}.$$
(23)

By applying the direct maximization technique on Equation (23), the respective CML estimates, say  $\hat{p}_{cls}$ ,  $\hat{\alpha}_{cls}$ , and  $\hat{\theta}_{cls}$  corresponding to the parameters of the PMiD-INAR(1) process, p,  $\alpha$ , and  $\theta$  can be obtained using the *optim* function of the **R** software.

## 7.2. Conditional Least Squares (CLS) Estimation

By using Equation (21), the CLS estimates of the PMiD-INAR(1) process parameters  $\zeta = (p, \alpha, \theta)$  are obtained by minimizing the function given below:

$$S(\zeta) = \sum_{t=2}^{I} [x_t - E(X_t | X_{t-1} = x_{t-1})]^2$$
  
= 
$$\sum_{t=2}^{T} \left[ x_t - p x_{t-1} - \frac{\theta}{\theta^2 + \alpha} \left( 1 + \frac{3\alpha}{\theta^2} \right) \right]^2.$$

As a result, by solving  $\frac{\partial}{\partial \zeta} S(\zeta) = 0$ , the system of equations based on the estimates can be derived. The CLS estimate of  $\alpha$  is obtained as

$$\hat{p}_{CLS} = \frac{(T-1)\sum_{t=2}^{T} x_t x_{t-1} - \sum_{t=2}^{T} x_t \sum_{t=2}^{T} x_{t-1}}{(T-1)\sum_{t=2}^{T} x_{t-1}^2 - \left(\sum_{t=2}^{T} x_{t-1}\right)^2}.$$

Hence, in  $S(\zeta)$ , p is switched with  $\hat{p}_{CLS}$ , and the resultant function  $S(\alpha, \theta)$  should be minimized by concerning  $\alpha$  and  $\theta$  to obtain their CLS estimates. Since  $\hat{\alpha}_{CLS}$  and  $\hat{\theta}_{CLS}$  do not have the closed-form expression, we utilize *optim* function of the **R** software to obtain it numerically by minimizing  $S(\alpha, \theta)$ .

# 7.3. Yule–Walker (YW) Estimation

The concept of the YW approach is to synchronously solve the theoretical moments with the empirical ones. Because of the autocorrelation function (ACF) of the INAR(1) process at lag  $\tau$  is  $\rho_x(\tau) = p^{\tau}$ , the YW estimate of p is given by

$$\hat{p}_{YW} = \frac{\sum\limits_{t=2}^{T} (x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum\limits_{t=1}^{T} (x_t - \bar{x})^2}.$$

Now, the theoretical mean and dispersion index of the PMiD-INAR(1) process is then solved with their empirical equivalents to derive the YW estimates of  $\alpha$  and  $\theta$ . When the theoretical mean is equated to the empirical mean, the YW estimate  $\hat{\alpha}_{YW}$  of  $\alpha$  is calculated as follows:

$$\hat{\alpha}_{YW} = \frac{\theta_{YW}^2 \left[ \bar{X} \theta_{YW} (1 - \hat{p}_{YW}) - 1 \right]}{3 - \bar{X} \hat{\theta}_{YW} (1 - \hat{p}_{YW})},$$
(24)

where  $\bar{x} = \frac{1}{N} \sum_{t=1}^{T} x_t$ . Then, substituting  $\hat{\alpha}_{YW}$  with Equation (24) in Equation (20) and equating Equation (20) to the empirical value of the dispersion index  $(\widehat{DI}_{X_t})$ , we obtain the YW estimate  $\hat{\theta}_{YW}$  of  $\theta$ . Since  $\hat{\theta}_{YW}$  does not have the closed-form expression, we utilize the *uniroot* function available in the **R** software to obtain it numerically.

#### 7.4. Simulation: PMiD-INAR(1) Process

In this section, a simulation study is conducted for the comprehensive assessment of the long-standing performances of CML, YW, and CLS estimates of the PMiD-INAR(1) process parameters. We have generated samples of sizes n = 100, 250, 500, 750, and 1000 using values of the parameter setting (p = 0.5,  $\alpha = 0.6$ ,  $\theta = 0.7$ ), and the iteration process is conducted 1001 times. The simulation results are analyzed using estimated biases, MSEs, and mean relative errors (MREs), and are presented in Table 3. The following formulae are used to determine these values:

Average bias 
$$=\frac{1}{1001} \sum_{i=1}^{1001} (\hat{\zeta}_i - \zeta)$$
, Average MSE  $=\frac{1}{1001} \sum_{i=1}^{1001} (\hat{\zeta}_i - \zeta)^2$ ,  
Average MRE  $=\frac{1}{1001} \sum_{i=1}^{1001} \frac{\hat{\zeta}_i}{\zeta}$ ,

where  $\boldsymbol{\zeta} = (p, \alpha, \theta)$ .

7			CML			CLS			YW	
ς	n	Bias	MSE	MRE	Bias	MSE	MRE	Bias	MSE	MRE
р	100	-0.0029	0.0021	0.9943	0.0366	0.0079	0.9269	-0.0674	0.0181	0.8653
	250	-0.0017	0.00095	0.9966	-0.0137	0.0030	0.9725	-0.0246	0.0047	0.9508
	500	-0.0015	0.00054	0.9984	-0.0069	0.0016	0.9862	-0.0101	0.0019	0.9799
	750	-0.0012	0.00035	0.9966	-0.0056	0.0011	0.9888	-0.0070	0.0013	0.9859
	1000	-0.0011	0.00029	0.9979	-0.0027	0.0008	0.9946	-0.0037	0.0008	0.9927
α	100	0.0701	0.1251	1.1169	0.0826	0.0248	1.1377	1.8696	20.5181	4.1161
	250	0.0628	0.0957	1.1047	0.0517	0.0107	1.0862	0.9482	7.7421	2.5804
	500	0.0483	0.0740	1.0805	0.0456	0.0071	1.0760	0.4234	1.8246	1.7056
	750	0.0456	0.0617	1.0760	0.0427	0.0053	1.0712	0.2460	0.7204	1.4099
	1000	0.0421	0.0566	1.0785	0.0384	0.0044	1.0641	0.1495	0.2579	1.2492
θ	100	-0.0203	0.0120	0.9710	-0.0151	0.0049	0.9784	0.0291	0.1533	1.0416
	250	-0.0092	0.0058	0.9868	-0.0016	0.0020	0.9979	-0.0083	0.0236	0.9881
	500	-0.0057	0.0038	0.9918	0.0014	0.0011	1.0022	-0.0047	0.0091	0.9933
	750	-0.0032	0.0021	0.9954	0.0011	0.0008	1.0038	-0.0040	0.0044	0.9943
	1000	-0.0011	0.0018	0.9984	0.0009	0.0006	1.0067	-0.0021	0.0035	0.9970

**Table 3.** Simulation results for (p = 0.5,  $\alpha = 0.6$ ,  $\theta = 0.7$ ).

#### 8. Applications and Empirical Study

To show the usage of the PMiD model, we utilize three real datasets in this section: the first is COVID-19 data, the second is hospital length of stay, and the third is the number of earthquakes data.

## 8.1. COVID-19 Data: Armenia

First, we utilize the dataset of the daily new deaths in Armenia due to the COVID-19 disease. The data are available at https://www.worldometers.info/coronavirus/country/armenia/ accessed on the 10 January 2021 and are also studied by [19]. They contain the daily new COVID cases between 15 February 2020 and 4 October 2020. To demonstrate the PMiD's potential benefit, the distributions given in Table 4 are considered for comparison.

Table 4. The considered competitive distributions.

Distribution	Abbreviation	Reference
Discrete generalized Lindley	DGLi	[19]
Poisson–Xgamma	PXGD	[13]
Discrete Poisson-Lindley	DPLi	[9]
Discrete Lindley	DLi	[20]
New Poisson-weighted exponential	NPWE	[21]
Poisson-transmuted exponential	PTE	[22]
Poisson	Р	-

We compare the competitive distributions to the suggested distribution using the statistical techniques provided, namely the negative log-likelihood ( $-\log L$ ), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC),  $\chi^2$  statistic, and *p*-value. Tables 5 and 6 display the corresponding MLEs (with standard errors (SEs) and CIs) and goodness-of-fit (GOF) results, respectively. The PMiD's GOF statistics values are less than the other examined distributions, as shown in these tables. As a result, the suggested model is the most appropriate for modelling the given COVID-19 data. Now, the empirical mean, variance, and DI of this Armenia dataset are 4.1931, 18.7944, and 4.4822, respectively, and the theoretical values for the mean, variance, and DI measures of the PMiD are 4.1931, 19.6659, and 4.6900. It is observed that the empirical and the theoretical means are exactly the same, and the empirical and the theoretical values of variances, and DIs are the closest to each other. The empirical *cdf*, *pmf*, and P-P plots for the Armenia dataset are respectively given in Figures 4–6. It again gives some better-shaped curves for those fitted in the PMiD.

Distribution			α	θ			
Distribution	MLE	SE	CI	MLE	SE	CI	
PMiD	0.1029	0.0586	(-0.0121, 0.2178)	0.4162	0.0463	(0.3254, 0.5070)	
PXGD	0.5431	0.0275	(0.4892, 0.5969)	-	-	-	
DGLi	0.2477	0.0843	(0.0825, 0.4128)	0.7763	0.0316	(0.7144, 0.8382)	
DPLi	0.41001	0.0227	(0.3656, 0.4545)	-	-	-	
DLi	0.6914	0.0121	(0.6677, 0.7151)	-	-	-	
NPWE	0.2167	3.1158	(-5.8903, 6.3236)	0.1008	15.8322	(-30.9297, 31.1313)	
PTE	0.0001	0.2144	(-0.4202, 0.4202)	0.2385	0.0303	(0.1790, 0.2980)	
Р	4.1931	0.1342	(-3.9302, 4.4561)	-	-	-	

 Table 5. Armenia dataset: MLEs of the parameters.

Table 6. Armenia dataset: Goodness-of-fit test.

v	Y OF		Expected Frequency								
λ	OF	PMiD	PXGD	DGLi	DPLi	DLi	NPWE	РТЕ	Р		
0	56	45.1654	35.4419	43.6875	33.6739	28.4819	44.8674	44.8672	3.5181		
1	31	35.0039	31.4999	35.8015	33.7917	33.0940	36.2276	36.2274	14.7516		
2	22	28.0128	28.7071	29.2575	30.9936	32.1465	29.2515	29.2514	30.9278		
3	25	22.8835	25.7701	23.8497	26.9655	28.6318	23.6187	23.6186	43.2281		
4	11	18.8974	22.5056	19.3972	22.6594	24.2248	19.0706	19.0705	45.3153		
5	14	15.6646	19.0995	15.7433	18.5775	19.8108	15.3983	15.3982	38.0026		
6	14	12.9730	15.7909	12.7535	14.9535	15.8141	12.4331	12.4331	26.5583		
7	10	10.7033	12.7614	10.3135	11.8663	12.3974	10.0389	10.0390	15.9089		
8	11	8.7834	10.1133	8.3269	9.3101	9.5833	8.1058	8.1059	8.3385		
9	3	7.1637	7.8812	6.7131	7.2372	7.3255	6.5449	6.5450	3.8850		
10	10	5.8053	6.0535	5.4045	5.5826	5.5485	5.2846	5.2846	1.6290		
11	7	4.6746	4.5919	4.3455	4.2783	4.1706	4.2670	4.2670	0.6210		
12	4	3.7409	3.4454	3.4898	3.2605	3.1147	3.4453	3.4453	0.2170		
13	5	2.9762	2.5607	2.7995	2.4729	2.3133	2.7819	2.7818	0.0700		
14	2	2.3547	1.8870	2.2434	1.8676	1.7099	2.2462	2.2462	0.0210		
15	2	1.8534	1.3802	1.7960	1.4053	1.2586	1.8136	1.8140	0.0059		
$\geq 16$	6	6.3439	3.5106	7.0775	4.1044	3.3744	7.6048	7.6049	0.0020		
Total	233	233	233	233	233	233	233	233			
$-\log L$		590.3751	596.7075	592.6174	598.9318	605.3913	592.7991	592.7991	827.4472		
AIC		1184.750	1195.415	1189.235	1199.864	1212.783	1189.598	1189.598	1656.894		
BIC		1191.652	1198.866	1196.137	1203.315	1216.234	1196.500	1196.500	1660.345		
$\chi^2$		9.3187	25.1615	12.3710	28.1328	45.7236	12.0550	12.0549	483.1912		
df		6	7	6	8	7	6	6	7		
<i>p</i> value		0.1564	< 0.001	0.0542	0.0004	< 0.001	0.0608	0.0607	< 0.001		



Figure 4. Empirical *cdfs* of the fitted distributions for the Armenia dataset.



Figure 5. Empirical *pmfs* of the fitted distributions for the Armenia dataset.



Figure 6. Empirical PP plots of the fitted distributions for the Armenia dataset.

The next goal is to use the Bayesian technique to estimate the parameters of the PMiD using the above-mentioned COVID-19 dataset. The analysis was carried out using the MHA of the MCMC technique with 1000 iterations in this perspective. For comparison purposes, both the Bayesian and MLE estimates of the PMiD parameters for the real dataset are given in Table 7. **R** programming is used to perform the numerical computations.

Table 7. MLEs and Bayes estimates of the PMiD parameters on the COVID-19 dataset.

Parameter	ML	Bayes
lpha  heta  het	0.1029 0.4162	0.1688 0.4515

# 8.2. Length of Hospital Stay

By applying the PMiD regression model to an actual dataset, we can confirm its prominence. The dataset is about the 1991 Arizona cardiovascular patients (AZPRO data), which is available in **COUNT** package of the **R** software. The goal of this study is to investigate the combined relationship between *the length of hospital stay* ( $y_i$ ) with the covariates  $x_{i1}$ - the cardiovascular procedures (*CABG* = 1, *PTCA* = 0),  $x_{i2}$ - the sex of the patients (male = 1, female = 0),  $x_{i3}$ - the type of the admission (urgent = 1, elective = 0), and  $x_{i4}$ - the age of the patients ((age > 75) = 1, (age  $\leq$  75) = 0). The fitted nonlinear regression model is given by

$$\mu_i = \exp(\tau_0 + \tau_1 x_{i1} + \tau_2 x_{i2} + \tau_3 x_{i3} + \tau_4 x_{i4}).$$

In Table 8, we compare the performance of the PMiD regression model with the Poisson regression model, the NPWE regression model, and the PXGD regression model and also display the summaries due to the real dataset, which include the SEs, *p*-value, negative log-likelihood, AIC, and BIC values. Table 8 points out that the PMiD regression model has the lowest values across all the model selection criteria, indicating that it is the better count regression model than all the Poisson, NPWE, and PXGD regression models.

Table 8. Regression results on the length of the hospital stay dataset (Standard errors in brackets).

Constitutes	Poisson		NPWE		PXGD		PMiD	
Covariates	Estimate (SE)	<i>p</i> -Value	Estimate (SE)	<i>p</i> -Value	Estimate (SE)	<i>p</i> -Value	Estimate (SE)	<i>p</i> -Value
$\tau_0$	1.4560 (0.0159)	< 0.001	1.3871 (0.0458)	< 0.001	1.3996 (0.0349)	< 0.001	2.1007 (0.0228)	< 0.001
$ au_1$	0.9604 (0.0122)	< 0.001	0.9982 (0.0363)	< 0.001	0.9721 (0.0270)	< 0.001	0.3693 (0.0789)	< 0.001
$ au_2$	-0.1240 (0.0118)	< 0.001	-0.1276 (0.0380)	< 0.001	-0.1269 (0.0280)	< 0.001	-0.0553 (0.0130)	< 0.001
$ au_3$	0.3266 (0.0121)	< 0.001	0.4047 (0.0376)	< 0.001	0.1201 (0.0298)	< 0.001	0.2087 (0.0220)	< 0.001
$ au_4$	0.1222 (0.0125)	< 0.001	0.1189 (0.0405)	< 0.001	0.3732 (0.0280)	< 0.001	0.0309 (0.0108)	< 0.001
$\theta$	_		$0.5311 (1.4829 \times 10^3)$	0.9997	_		0.3296 (0.0054)	< 0.001
– log L	11,189.90		11,194.01		10,569.80		10,382.38	
AIC	22,389.80		22,400.02		21,149.60		20,776.76	
BIC	22,420.72		22,437.13		21,180.60		20,813.88	

#### 8.3. Japan Earthquake Data

We use data from the ETAS package of **R** software to calculate annual counts of earthquakes with a magnitude of 4.5 or higher that occurred in Japan between the years 1926 and 2007. The data comprise 82 observations. For more details on this package, one can go through [23]. In addition, Figure 7 depicts the Japan earthquake catalog. The spatial distribution of earthquakes in the study area is depicted in the top-left picture. The three figures in the right half of Figure 7 depict variations in the latitude, longitude, and magnitude of the earthquakes over time. The two figures in the bottom-left corner of Figure 7 depict the earthquake catalog's completeness and time stationarity. Here, the number of earthquakes having a magnitude higher than or equal to *m* is represented by  $N_m$ . If the plot of  $log(N_m)$  versus *m* exhibits a linear trend, it reflects the completeness of the earthquake catalog. Furthermore, the time stationarity of the catalog can be determined by looking at the plot of  $N_t$  versus *t*, where  $N_t$  is the total number of occurrences in the catalog up to time *t*. The earthquake time series is stationary if the plotted points of  $N_t$ versus *t* have a functional form in such a way that  $N_t = \lambda_0 t$ , where  $\lambda > 0$ .

We calculated the CML estimates of parameters, as well as the negative log-likelihood  $(-\log L)$ , AIC, and BIC of the PMiD-INAR(1) as well as the INAR(1) with the innovations, new Poisson-weighted exponential (INAR-NPWE(1)), Poisson-transmuted exponential (INAR-PTE(1)), discrete Poisson–Lindley (INAR-DPLi(1)), and Poisson (PINAR(1)), for the comparison. For more details of these innovations, see Table 4. Now, Table 9 displays the results corresponding to the earthquake data. We see that the  $-\log L$ , AIC, and BIC values of the PMiD-INAR(1) model is smaller than those of the other compared INAR(1) models,



and we conclude that the PMiD-INAR(1) is the most suitable model for the earthquake data compared to that of the other models.

Figure 7. The plots of the earthquake catalog of Japan.

Model	Parameters	Estimate	SE	- log L	AIC	BIC	
	р	0.2813	0.0293				
INAR-PMiD(1)	α	0.6869	2.8522	446.0982	898.1965	905.4166	
	$\theta$	0.0247	0.0019				
	р	0.3503	0.0204				
INAR-NPWE(1)	α	0.0068	0.0043	465.8715	937.7431	944.9632	
	heta	0.3408	0.8622				
	р	0.0320	0.0293				
INAR-PTE(1)	α	-0.9999	0.2543	451.1948	908.3896	915.6098	
	$\theta$	0.0130	0.00124				
	р	0.3179	0.0238	450 902	905 804	010 6174	
IINAK-DPLI(1)	α	0.0172	0.0015	430.902	203.004	210.0174	
PINAP(1)	р	0.0592	0.0145	1/18 018	28/1 836	2846.65	
$\operatorname{FINAR}(1)$	α	158.6002	2.7801	1410.910	2041.030	2040.03	

Table 9. The CML estimates of the fitted INAR(1) models and corresponding GOF statistics.

The residual analysis is done to make sure the fitted model is accurate for the earthquake data. To do so, the Pearson residuals for the PMiD-INAR(1) process are computed using the following formula:

$$r_t = \frac{x_t - E(X_t | X_{t-1} = x_{t-1})}{Var(X_t | X_{t-1} = x_{t-1})^{1/2}},$$

where  $E(X_t|X_{t-1} = x_{t-1})$  and  $Var(X_t|X_{t-1} = x_{t-1})$  can be found in Equations (21) and (22), respectively. In general, the mean and variance of Pearson residuals should be near zero and one, respectively, and the computed Pearson residuals should not have any autocorrelation issues if the fitted INAR(1) process is statistically accurate. Here, the obtained Pearson residuals of the PMiD-INAR(1) process have a mean and variance of 0.0012 and 1.1612, respectively, which were quite close to the anticipated values. Therefore, the PMiD-INAR(1) process is well judged for the given earthquake data. Now, the fitted PMiD-INAR(1) process is obtained as follows:

$$X_t = 0.2813 \circ X_{t-1} + \varepsilon_t,$$

where the innovation process  $\varepsilon_t \sim \text{PMiD}(0.6869, 0.0247)$ . Now, we can calculate the predicted number of earthquakes in Japan via

$$\hat{X}_{1} = \frac{\hat{\theta}}{(\hat{\theta}^{2} + \hat{\alpha})(1 - \hat{p})} \left(1 + \frac{3\hat{\alpha}}{\hat{\theta}^{2}}\right) = 168.8961$$
$$\hat{X}_{t} = \hat{p}x_{t-1} + \frac{\hat{\theta}}{\hat{\theta}^{2} + \hat{\alpha}} \left(1 + \frac{3\hat{\alpha}}{\hat{\theta}^{2}}\right) = 0.2813x_{t-1} + 121.3856$$

# 9. Discussion

## 9.1. Context

Discovering new count models is inevitable in the scenario of overdispersion, which will provide more possibilities for superiorly fitting the real datasets by choosing the right models according to the situations. In that sense, we proposed a new overdispersed count model, discussed its regression aspects, and constructed an INAR(1) process based on them. The main motivation behind the construction of this model is also discussed. In all the aspects, we found that the proposed model outperforms the compared models.

# 9.2. This Work

The Poisson-Mirra distribution (PMiD), a novel two-parameter discrete distribution, is introduced and thoroughly investigated. We delivered specific expressions for the factorial moments, mean, variance, dispersion index, skewness, kurtosis, mode, probability generating function, moment generating function, characteristic function, and the entropy measures. The distribution parameters were estimated by using the classical maximum likelihood and Bayesian estimation methods and were also studied in their real-data analysis. A simulation study on MLE was also conducted. A new regression model for count data based on the PMiD is proposed and compared with its competitive regression models based on a real dataset. More importantly, we introduced the integer-valued autoregressive model based on the PMiD, known as the PMiD-INAR(1) process. The parameter estimation problems, which include the conditional maximum likelihood, conditional least squares, and the Yule–Walker estimation procedures for the PMiD-INAR(1) process, are discussed, and simulation studies based on these three estimation procedures are also conducted. In total, three real-world datasets were used to demonstrate the use of the novel model, the first regarding COVID-19 data, the second regarding the length of hospital stay, and the third concerning earthquake data.

## 9.3. Contributions and Limitations

The article thus developed the new distribution PMiD from which we derived a new count model, its regression model, and the first-order integer-valued autoregressive model.

The proposed distribution, we believe, is ideal for researching data from COVID-19 related areas, biology, and earthquake-related fields, and we hope the PMiD applies to other fields of study also. The absence of a bimodal feature is the possible limitation of the proposed distribution.

# 9.4. Future Work

This research could take another route if the bivariate version of the PMiD and the associated BINAR(1) process are built. This work needs considerable modifications and studies, which we shall delegate to future research.

# 10. Conclusions

In this article, we fitted three real datasets and concluded that the PMiD model outperforms all the compared models in all aspects. We anticipate that the proposed model will increase its prevalence and have a wider variety of applications in the modelling of positive integer-valued real-world datasets from different study areas such as physics, astronomy, survival analysis, and so on.

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#### Abbreviations

The following abbreviations are used in this manuscript:

INAR(1)	First-order Integer-valued Autoregressive
NB	Negative Binomial
PL	Poisson–Lindley
PMiD	Poisson-Mirra Distribution
MiD	Mirra Distribution
XGD	Xgamma Distribution
pdf	Probability Density Function
cdf	Cumulative Distribution Function
sf	Survival Function
hf	Hazard Function
pmf	Probability Mass Function
PXGD	Poisson-Xgamma Distribution
DI	Dispersion Index
pgf	Probability Generating Function
mgf	Moment Generating Function
cf	Characteristic Function
MLE	Maximum Likelihood Estimate
hC	half-Cauchy
MHA	Metropolis-Hastings Algorithm
MCMC	Markov Chain Monte Carlo
MSE	Mean Squared Error
СР	Coverage Probability
AL	Average Length
CML	Conditional Maximum Likelihood

CLS	Conditional Least Squares
YW	Yule–Walker
ACF	Autocorrelation Function
MRE	Mean Relative Error
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
GOF	Goodness-of-Fit
DGLi	Discrete Generalized Lindley
DPLi	Discrete Poisson–Lindley
DLi	Discrete Lindley
NPWE	New Poisson-Weighted Exponential
PTE	Poisson-Transmuted Exponential
SE	Standard Error
CI	Confidence Interval
df	Degrees of Freedom

# Appendix A

R-code for generating the PMiD random samples is given by

```
ppmid <- function(q, alpha, theta){</pre>
p <- (1/(2*(alpha+theta^2))) * ((theta+1)^(-q-3)) *</pre>
(alpha*(2*((theta+1)^(q+3)) - theta*(q+3)*(theta*(q+2)+2)-2) +
2*(theta^2)*((theta+1)^2)* (((theta+1)^(q+1)) - 1))
return(p)
}
rpmid <- function(n, alpha, theta)</pre>
{
U <- runif(n)</pre>
X <- rep(0,n)
for(i in 1:n)
{
if(U[i] < ppmid(0, alpha, theta))</pre>
{
X[i] <- 0
} else
{
B = FALSE
I = 0
while(B == FALSE)
{
int <- c( ppmid(I, alpha, theta), ppmid(I+1, alpha, theta) )</pre>
if( (U[i] > int[1]) & (U[i] < int[2]) )
{
X[i] <- I+1
B = TRUE
} else
{
# If not, continue the while loop and increase I by 1
I=I+1
}
}
}
}
return(X)
}.
```

R-code for finding the MLEs and GOF statistics values of the PMiD is given by

```
library(fitdistrplus)
dpmid <- function(x, alpha, theta){</pre>
d <- (theta^3)/((theta^2)+alpha) * 1/((1+theta)^(x+1)) *</pre>
(1 + (alpha*(x+1)*(x+2))/(2*(1+theta)^2))
return(d)
3
ppmid <- function(q, alpha, theta){</pre>
p <- (1/(2*(alpha+theta^2))) * ((theta+1)^(-q-3)) *</pre>
(alpha*(2*((theta+1)^(q+3)) - theta*(q+3)*(theta*(q+2)+2)-2) +
2*(theta<sup>2</sup>)*((theta+1)<sup>2</sup>)* (((theta+1)<sup>(q+1)</sup>) - 1))
return(p)
}
pre <- prefit(x, "pmid", "mle", list(alpha=0.3, theta=0.3),</pre>
lower=c(0, 0), upper = c(Inf, Inf))
fit.pmid <- fitdist(x, "pmid",</pre>
start = list(alpha = pre$alpha, theta = pre$theta),
optim.method = "L-BFGS-B", lower=c(0, 0), upper = c(Inf, Inf),
discrete = TRUE)
summary(fit.pmid)
gofstat(fit.pmid).
```

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