axioms

Article

Topological Transcendental Fields [†]

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Abstract: This article initiates the study of topological transcendental fields $\mathbb{F}$ which are subfields of the topological field $\mathbb{C}$ of all complex numbers such that $\mathbb{F}$ only consists of rational numbers and a nonempty set of transcendental numbers. $\mathbb{F}$, with the topology it inherits as a subspace of $\mathbb{C}$, is a topological field. Each topological transcendental field is a separable metrizable zero-dimensional space and algebraically is $\mathbb{Q}(T)$, the extension of the field of rational numbers by a set T of transcendental numbers. It is proven that there exist precisely $2^{\aleph_0}$ countably infinite topological transcendental fields and each is homeomorphic to the space $\mathbb{Q}$ of rational numbers with its usual topology. It is also shown that there is a class of $2^{2^{\aleph_0}}$ of topological transcendental fields of the form $\mathbb{Q}(T)$ with T a set of Liouville numbers, no two of which are homeomorphic.

Keywords: topological field; transcendental number; algebraic; countably infinite; homeomorphic; extension field; subfield

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1. Preliminaries

We begin by setting out our notation and making some simple preliminary observations.

Remark 1. *We shall discuss four fields: $\mathbb{C}$, the field of all complex numbers; $\mathbb{R}$, the field of all real numbers; $\mathbb{A}$, the field of all algebraic numbers; and $\mathbb{Q}$, the field of all rational numbers. Observe the following easily verified facts:*

(i) *Fields $\mathbb{C}$ and $\mathbb{R}$ have cardinality $\mathfrak{c}$, the cardinality of the continuum;*
(ii) *Fields $\mathbb{A}$ and $\mathbb{Q}$ have cardinality $\aleph_0$;*
(iii) *$\mathbb{C}$ with its Euclidean topology is homeomorphic to $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ has its Euclidean topology;*
(iv) *Each of these four fields has a natural topology; $\mathbb{C}$ and $\mathbb{R}$ have Euclidean topologies, while $\mathbb{A}$ and $\mathbb{Q}$ inherit a natural topology as a subspace of $\mathbb{C}$;*
(v) *Field $\mathbb{Q}$ is a dense subfield of the topological field $\mathbb{R}$ (that is, the closure, in the topological sense, of $\mathbb{Q}$ is $\mathbb{R}$);*
(vi) *Topological field $\mathbb{A}$ is a dense subfield of the topological field $\mathbb{C}$;*
(vii) *$\mathbb{C} \supset \mathbb{A} \supset \mathbb{A} \cap \mathbb{R} \supset \mathbb{Q}$, but $\mathbb{A}$ is not a subset of $\mathbb{R}$;*
(viii) *Field $\mathbb{C}$ is a vector space of dimension $\mathfrak{c}$ over $\mathbb{A}$ and it is also a vector space of dimension $\mathfrak{c}$ over $\mathbb{Q}$;*
(ix) *$\mathbb{A}$ is a vector space of countably infinite dimension over $\mathbb{Q}$;*
(x) *$\mathbb{N}$ denotes the set of positive integers and $\mathbb{Z}$ denotes the set of all integers, each with the discrete topology;*
(xi) *$\mathcal{T}$ is the topological space of all transcendental numbers, where $\mathcal{T} = \mathbb{C} \setminus \mathbb{A}$ and has a natural topology as a subspace of $\mathbb{C}$. The topology of $\mathcal{T}$ is separable, metrizable, and zero-dimensional. Furthermore, the cardinality of $\mathcal{T}$ is $\mathfrak{c}$ and $\mathcal{T}$ is dense in $\mathbb{C}$.*

Remark 2. *Now, we mention some not so easily verified known results:*

(i) *$\mathcal{T}$ is homeomorphic to the space $\mathbb{P}$ of all irrational real numbers. $\mathbb{P}$ is also homeomorphic to the countably infinite product $\mathbb{N}^{\mathbb{N}}$. (see ([1], §1.9));*

(ii) *$\mathcal{T}Q$ denotes the set $\mathcal{T} \cup \mathbb{Q}$. It is also homeomorphic to $\mathbb{P}$;*

(iii) *In 1932, Kurt Mahler classified the set of all transcendental numbers $\mathcal{T}$ into three disjoint classes: S, T, and U. For a discussion of this important classification, see ([2], Chapter 8). It has been proven that each of these sets has cardinality $\mathfrak{c}$. Furthermore, the Lebesgue measure of T and U are each zero. Thus, S has full measure, that is its complement has a measure of zero;*

(iv) *We introduce the classes $SQ = S \cup \mathbb{Q}$, $TQ = T \cup \mathbb{Q}$, $UQ = U \cup \mathbb{Q}$. Clearly SQ, TQ, and UQ each have cardinality $\mathfrak{c}$, TQ, and UQ have measure zero, and SQ has full measure;*

(v) *In 1844, Joseph Liouville showed that all members of a certain class of numbers, now known as the Liouville numbers, are transcendental. A real number x is said to be a* Liouville number *if, for every positive integer n, there exists a pair (p, q) of integers with $q > 1$, such that $0 < |x - \frac{p}{q}| < \frac{1}{q^n}$ (see [3]). The Liouville numbers are a subset of the Mahler class U. We denote the set of Liouville numbers by L and the set $L \cup \mathbb{Q}$ by LQ.*

Recall the following definitions from [4]. While Weintraub stated the definitions and propositions using countably infinite sets, there is no problem to state these using the sets of any cardinality.

Definition 1. *Let $\mathbb{E}$ be an extension field of $\mathbb{F}$. Then, $\alpha \in \mathbb{E}$ is said to be transcendental over $\mathbb{F}$ if α is not a root of any nonzero polynomial $p(X) \in \mathbb{F}[X]$, the ring of polynomials over $\mathbb{F}$ in the variable X with coefficients in $\mathbb{F}$. The quantity $\alpha \in \mathbb{E} \setminus \mathbb{F}$ is said to be algebraic over $\mathbb{F}$ if it is not transcendental over $\mathbb{F}$.*

Definition 2. *An extension field $\mathbb{E}$ of a field $\mathbb{F}$ is said to be a completely transcendental extension of $\mathbb{F}$ if α is transcendental over $\mathbb{F}$, for every $\alpha \in \mathbb{E} \setminus \mathbb{F}$.*

Definition 3. *Let $\mathbb{E}$ be an extension field of field $\mathbb{F}$. Then, $\mathbb{E}$ is a purely transcendental extension of $\mathbb{F}$ if $\mathbb{E}$ is isomorphic to the field of rational functions $\mathbb{Q}(\{X_i : i \in I|\})$ of variables $\{X_i : i \in I\}$, where I is a finite or infinite index set.*

Definition 4. *Let field $\mathbb{E}$ be an extension of the field $\mathbb{F}$. If I is any index set, the subset $S = \{s_i : i \in I\}$ of $\mathbb{E}$ is said to be algebraically independent over $\mathbb{F}$ if for all finite subsets $\{i_1, \ldots, i_n\}$ of I, all nonzero polynomials $p \in \mathbb{F}[X_{i_1}, \ldots, X_{i_n}]$, $p(s_{i_1}, \ldots, s_{i_n}) \neq 0$. By convention, if $S = \emptyset$, then S is said to be algebraically independent over $\mathbb{F}$.*

Remark 3. *Observe that, if a set S is algebraically independent over $\mathbb{Q}$, then it is algebraically independent over $\mathbb{A}$. Furthermore, algebraic independence implies linear independence.*

Remark 4. *Central to their definition of the classes S, T, and U, was the feature that Mahler wanted, namely that any two algebraically dependent transcendental numbers lie in the same class—S, T, or U.*

We shall use ([4], Lemmas 6.1.5 and 6.1.8) which are stated here as Proposition 2 and Proposition 1. In this context, it is useful to recall the classical result of Jacob Lüroth, proven in 1876, that every field that lies between any field $\mathbb{F}$ and an extension field $\mathbb{F}(\alpha)$ is itself an extension field of $\mathbb{F}$ by a single element of the field $\mathbb{F}(\alpha)$.

Proposition 1. *Let $\mathbb{E}$ be a purely transcendental extension of a field $\mathbb{F}$. Then, $\mathbb{E}$ is a completely transcendental extension of $\mathbb{F}$.*

Proposition 2. *Let $\mathbb{E}$ be an extension field of the field $\mathbb{F}$ and let $S = \{s_i : i \in I\}$ be algebraically independent over $\mathbb{F}$, where I is an index set. Then, the extension field $\mathbb{F}(S)$ is a purely transcendental extension.*

Recall the following definition from, for example, [5,6]:

Definition 5. *A field $\mathbb{F}$ with a topology τ is said to be a* topological field *if the field operations:*

(i) $(x, y) \to x + y$ *from* $\mathbb{F} \times \mathbb{F}$ *to* $\mathbb{F}$;
(ii) $x \to -x$ *from* $\mathbb{F} \setminus \{0\}$ *to* $\mathbb{F} \setminus \{0\}$;
(iii) $(x, y) \to xy$ *from* $\mathbb{F} \times \mathbb{F}$ *to* $\mathbb{F}$; *and*
(iv) $x \to x^{-1}$ *from* $\mathbb{F}$ *to* $\mathbb{F}$

are all continuous.

The standard examples of topological fields of characteristic 0 are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ with the usual Euclidean topologies. Indeed, by ([7], §27, Theorem 22), the only connected locally compact Hausdorff fields are $\mathbb{R}$ and $\mathbb{C}$. However, Shakhmatov in [8] proved the following beautiful result:

Theorem 1. *On every field $\mathbb{F}$ of infinite cardinality $\aleph$, there exist precisely $2^{2^{\aleph}}$ distinct topologies which make $\mathbb{F}$ a topological field.*

Motivated by the definition of a transcendental group introduced in [9], we define here the notion of a *topological transcendental field*.

Definition 6. *The topological field $\mathbb{F}$ is said to be a* topological transcendental field *if algebraically it is a subfield of $\mathbb{C}$, is a subset of $\mathbb{Q} \cup \mathcal{T}$, and has the topology it inherits as a subspace of $\mathbb{C}$.*

Remark 5. *Of course, the underlying field of a topological transcendental field is a a completely transcendental extension of $\mathbb{Q}$.*

2. Countably Infinite Transcendental Fields

Proposition 3. *If t is any transcendental number, then $\mathbb{Q}(t)$ is a topological transcendental field.*

Proof. This proposition is an immediate consequence of Propositions 1 and 2. □

Remark 6. *Of course it is not true that if t_1 and t_2 are transcendental, then $\mathbb{Q}(t_1, t_2)$ is necessarily a transcendental field. For example, if $t_1 = \pi$ and $t_2 = \pi + \sqrt{2}$, then $\mathbb{Q}(t_1, t_2)$ is not a topological transcendental field as $\sqrt{2} \in \mathbb{Q}(t_1, t_2)$. In fact, Paul Erdős [10] proved that for every real number r there exist Liouville numbers t_3, t_4, t_5, t_6 such that $t_3 \cdot t_4 = r$ and $t_5 + t_6 = r$. Indeed, he proved that for each real number r, there are uncountably many Liouville numbers t_3, t_4 and t_5, t_6 with these properties. As a consequence, we see that if L is the set of all Liouville numbers, then $\mathbb{Q}(L)$ is not a topological transcendental field.*

Having established the existence of countably infinite topological transcendental fields, we now describe a very concrete example. However, first we state a well-known theorem on transcendental numbers—please see Theorem 1.4 and the comments following it, in [2].

Theorem 2. *(Lindemann–Weierstrass Theorem) For any $m \in \mathbb{N}$ and any algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ which are linearly independent over $\mathbb{Q}$, the numbers $\mathrm{e}^{\alpha_1}, \mathrm{e}^{\alpha_2}, \ldots, \mathrm{e}^{\alpha_m}$ are algebraically independent.*

Theorem 3. *Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$ be a countably infinite set of algebraic numbers which are linearly independent over $\mathbb{Q}$. If $T = \{e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}, \dots\}$, then $\mathbb{Q}(T)$, is a topological transcendental field.*

Proof. By Propositions 1 and 2, $\mathbb{Q}(T)$ is a topological transcendental field. □

Theorem 4. *There exist precisely $2^{\aleph_0}$ countably infinite topological transcendental fields, each of which is homeomorphic to $\mathbb{Q}$.*

Proof. Using the notation of Theorem 3, there are $2^{\aleph_0}$ subsets of T and, due to algebraic independence, any two such subsets V, W, $V \neq W$, are such that $\mathbb{Q}(V) \neq \mathbb{Q}(W)$.

Furthermore, there are only $2^{\aleph_0}$ countably infinite subsets of $\mathbb{C}$. Thus, there exist precisely $2^{\aleph_0}$ countably infinite topological transcendental groups.

By ([1], Theorem 1.9.6), the space $\mathbb{Q}$ of all rational numbers up to homeomorphism is the unique non-empty countably infinite separable metrizable space without isolated points. In a topological field (indeed in a topological group), there are isolated points if and only if the topological field has a discrete topology. However, by ([11], Theorem 6), the only discrete subgroups of $\mathbb{C}$ are isomorphic to $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$, neither of which has the algebraic structure of a field. Thus, every countably infinite topological transcendental field is homeomorphic to $\mathbb{Q}$. □

3. Topological Transcendental Fields of Continuum Cardinality

Theorem 5. *Let $\mathbb{K}$ be a topological transcendental field of cardinality card$(\mathbb{K})$. Then, the extension field $\mathbb{K}(t)$ is a topological transcendental field for all but a set of cardinality card$(\mathbb{K})$ of $t \in \mathbb{C}$.*

Proof. The extension field $\mathbb{K}(t)$ consists of elements z of the form

$$z = \frac{c_0 + c_1 t + c_2 t^2 + \dots c_n t^n}{d_0 + d_1 t + d_2 t^2 + \dots d_m t^m},$$

for $c_0, c_1, \dots, c_n, d_0, d_1, \dots, d_m \in \mathbb{K}$, $n, m \in \mathbb{N}$. If z is an algebraic number a, then

$$c_0 + c_1 t + c_2 t_2 + \dots c_n t^n - a d_0 - a d_1 t - a d_2 t^2 - \dots - a d_m t^m = 0. \qquad (*)$$

For any given $n, m \in \mathbb{N}$, given $c_0, c_1, \dots, c_n, d_0, d_1, \dots, d_m \in \mathbb{K}$, and given $a \in \mathbb{A}$, the Fundamental Theorem of Algebra says that there at most $\max(n, m)$ algebraic number solutions of $(*)$ for t. As there are only a countably infinite number of algebraic numbers a, we see that for given $c_0, c_1, \dots, c_n, d_0, d_1, \dots, d_m \in \mathbb{K}$, there are a countable number of solutions of $(*)$ for t. Noting that the number of choices of $c_0, c_1, \dots, c_n, d_0, d_1, \dots, d_m \in \mathbb{K}$ is card $\mathbb{K}$, for each $n, m \in \mathbb{N}$, we obtain that z is a transcendental number except for at most $\aleph_0 \times \text{card}(\mathbb{K}) = \text{card}(\mathbb{K})$ values of t, which proves the theorem. □

Noting our Remark 6, Corollary 1 is of interest.

Corollary 1. *If t_1, t_2 are transcendental numbers, then $\mathbb{Q}(t_1, t_2)$ is a topological transcendental field for all but a countably infinite number of pairs (t_1, t_2). Indeed, if W is a countable set of transcendental numbers, then $\mathbb{Q}(W)$ is a topological transcendental field for all but a countably infinite number of sets W.* □

Corollary 2. *Let $\mathbb{K}$ be a topological transcendental field of cardinality $\aleph < 2^{\aleph_0}$. Then, there exists a $t \in \mathbb{C}$ such that $\mathbb{K}(t)$ is a topological transcendental field which properly contains $\mathbb{K}$.* □

Theorem 6. *Let E be any set of cardinality $\mathfrak{c}$ of transcendental numbers. Then, there exists a topological transcendental field $\mathbb{Q}(T)$ of cardinality $\mathfrak{c}$, where $T \subseteq E$. Further, $\mathbb{Q}(T)$ has $2^{\mathfrak{c}}$ distinct topological transcendental subfields.*

Proof. Consider the set $\mathcal{F}$ of all topological transcendental fields $\mathbb{Q}(F)$, where F is a subset of E, with the property that for each pair $W, V \subset F$ such that $W \neq V$, $\mathbb{Q}(V) \neq \mathbb{Q}(W)$.

By Corollary 1 and the fact that E is uncountable, there exist $s, t \in E, t \notin \mathbb{Q}(s), s \notin \mathbb{Q}(t)$, and $\mathbb{Q}(s,t)$ is a topological transcendental field. Then, $\mathbb{Q}(s,t) \in \mathcal{F}$.

Put a partial order on the members of $\mathcal{F}$ by set theory containment. Consider any totally ordered subset $\mathcal{S}$ of members of $\mathcal{F}$. Let $\mathbb{K}$ be the union of members of $\mathcal{S}$. Clearly it is a member of $\mathcal{F}$ and is an upper bound of $\mathcal{S}$. Therefore, by Zorn's Lemma, $\mathcal{F}$ has a maximal member $\mathbb{Q}(T)$, where $T \subseteq E$.

Suppose that T has cardinality $\aleph < \mathfrak{c}$, then by the proof of Theorem 4, there exists an $e \in E$, such that $\mathbb{Q}(T)(e) = \mathbb{Q}(T, \{e\})$ is a topological transcendental field which is easily seen to be a member of $\mathcal{F}$. This contradicts the maximality of $\mathbb{Q}(T)$. Thus, T has cardinality $\mathfrak{c}$.

Furthermore, by the definition of $\mathcal{F}$, $\mathbb{Q}(T)$ has $2^{\mathfrak{c}}$ distinct topological transcendental subfields. □

Theorem 7. *Let E be a set of transcendental numbers of cardinality $\mathfrak{c}$. Then, there exist $2^{\mathfrak{c}}$ topological transcendental fields $\mathbb{Q}(T)$, where $T \subseteq E$, no two of which are homeomorphic.*

Proof. By the Laverentieff Theorem, Theorem A8.5 of [1], there are at most $\mathfrak{c}$ subspaces of $\mathbb{C}$ which are homeomorphic. Thus, from Theorem 6 there are $2^{\mathfrak{c}}$ topological transcendental fields, no two of which are homeomorphic. □

Corollary 3. *Let E be the set L of Liouville numbers or the Mahler set U or the Mahler set T or the Mahler set S. Then, there exist $2^{\mathfrak{c}}$ topological transcendental fields $\mathbb{Q}(T)$, where $T \subseteq E$, no two of which are homeomorphic.* □

As noted in Remark 3, the Mahler sets T and U and the set of Liouville numbers, being a subset of U, have Lebesgue measure zero, while the Mahler set S has full measure; we thus conclude by asking whether there are any topological transcendental fields of nonzero Lebesgue measure.

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