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Rectangular G_b -Metric Spaces and Some Fixed Point Theorems

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Abstract: In this paper, we first introduce the concept of rectangular G_b -metric space which generalizes the notion of rectangular metric space and G_b -metric space. Then, some fixed point results connected with certain contractions are obtained in the setting of rectangular G_b -metric spaces. Additionally, we also introduce the concept of convex rectangular G_b -metric space by means of the convex structure and study the fixed points of enriched type contractions in this space.

Keywords: G_b -metric space; rectangular G_b -metric space; fixed points

MSC: 47H09; 47H10

1. Introduction

In recent decades, fixed point theory has had a rapid development. On the one hand, the study of new spaces have been an interesting topic among the mathematical research community. In 1993, Czerwik [1] introduced the concept of b -metric space as a generalization of metric space, and generalized Banach contraction principle [2] to this space. Since then, Branciari [3] developed the notion of rectangular metric space via substituting the triangle inequality with the quadrilateral inequality. Recently, George et al. [4] introduced the notion of rectangular b -metric space as a generalization of rectangular metric space and they also proved some well-known fixed point results. Meanwhile, Mustafa et al. [5,6] introduced a new class of generalized metric spaces aim to recovered the flaws of Dhage's theory, called G -metric spaces. Later, Aghajani et al. [7,8] generalized the concept of G -metric space to G_b -metric space through b -metric and proved some fixed point theorems in this spaces. Many fixed point results in metric spaces see [9–18].

On the other hand, Takahashi [19] introduced a notion of convexity structure in a metric space which offers the minimal tools for constructing various fixed point iterative methods for approximating fixed points of nonlinear operators. Very recently, Chen et al. [20] introduced the notion of convex b -metric space, and extend Mann's algorithms directly to b -metric spaces. Enriched contraction was introduced by Berinde et al. [21] as follows: Let $(E, \|\cdot\|)$ be a linear normed space. A mapping T is said to be an enriched contraction if

$$\|k(u - v) + Tu - Tv\| \leq \theta \|u - v\|, \text{ for all } u, v \in E,$$

where $k \in [0, \infty)$ and $\theta \in [0, k)$. For more results on enriched type contraction, the reader may see [22–25].

In this paper, we aim to extend the concept of G_b -metric space by defining the notion of rectangular G_b -metric space which is not necessarily Hausdorff. Then we obtain some well known fixed point theorems such as the Banach contraction principle, the Ćirić type fixed point theorem and the Reich type theorem. Furthermore, we also first introduce the concept of a convex rectangular G_b -metric space by means of the convex structure and generalize enriched type contraction in this space.

Let us give some basic notations and fundamental results.

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2. Preliminaries

Definition 1 ([1]). Let E be a nonempty set, and for all $u, v, z \in E$, let a function $d : E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $d(u, v) = 0$ if and only if $u = v$;
- (2) $d(u, v) = d(v, u)$;
- (3) there exists a real number $s \geq 1$ such that $d(u, v) \leq s[d(u, z) + d(z, v)]$.

Then d is called a b -metric on E and the pair (E, d) is called a b -metric space (bMS) with coefficient $s \geq 1$.

Definition 2 ([3]). Let E be a nonempty set, and for all $u, v \in E$, let a function $d : E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $d(u, v) = 0$ if and only if $u = v$;
- (2) $d(u, v) = d(v, u)$;
- (3) $d(u, v) \leq d(u, z) + d(z, h) + d(h, v)$ for all distinct points $z, h \in E \setminus \{u, v\}$.

Then d is called a rectangular metric on E and (E, d) is called a rectangular metric space (RMS).

Definition 3 ([4]). Let E be a nonempty set, and for all $u, v \in E$, let a function $d : E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $d(u, v) = 0$ if and only if $u = v$;
- (2) $d(u, v) = d(v, u)$;
- (3) there exists a real number $s \geq 1$ such that $d(u, v) \leq s[d(u, z) + d(z, h) + d(h, v)]$ for all distinct points $z, h \in E \setminus \{u, v\}$.

Then d is called a rectangular b -metric on E and (E, d) is called a rectangular b -metric space (R_b MS) with coefficient $s \geq 1$.

Example 1 ([4]). Let $E = \Gamma \cup H$, where $\Gamma = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ and H is the set of all positive integers. Let $d_b : E \times E \rightarrow [0, \infty)$ such that $d_b(u, v) = d_b(v, u)$ for all $u, v \in E$ and

$$d_b(u, v) = \begin{cases} 0, & \text{if } u = v; \\ 2a, & \text{if } u, v \in \Gamma; \\ \frac{a}{2n}, & \text{if } u \in \Gamma \text{ and } v \in \{2, 3\}; \\ a, & \text{otherwise,} \end{cases}$$

where $a > 0$ is a constant. Then (E, d_b) is a R_b MS with coefficient $s = 2 > 1$.

Definition 4 ([5]). Let E be a nonempty and for all $u, v, o \in E$, let a function $G : E \times E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $G(u, v, o) = 0$ if $u = v = o$;
- (2) $0 < G(u, u, v)$ with $u \neq v$;
- (3) $G(u, u, v) \leq G(u, v, z)$ for all $z \in E$ with $v \neq z$;
- (4) $G(u, v, o) = G(u, o, v) = G(v, o, u) = \dots$, (symmetry in all three variables);
- (5) $G(u, v, o) \leq G(u, z, z) + G(z, v, o)$ for all $z \in E$.

Then G is called a G -metric on E and the pair (E, G) is called a G -metric space (GMS).

Example 2 ([6]). Let $E = \mathbb{R} \setminus \{0\}$ and define $G : E \times E \times E \rightarrow [0, \infty)$ by

$$G(u, v, o) = \begin{cases} 1 + |u - v| + |v - o| + |u - o| & \text{if } u, v, o \text{ all have the same sign;} \\ |u - v| + |v - o| + |u - o| & \text{otherwise,} \end{cases}$$

for all $u, v, o \in E$. Then (E, G) is a GMS.

Definition 5 ([7]). Let E be a nonempty and for all $u, v, o \in E$, let a function $G : E \times E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (1) $G(u, v, o) = 0$ if $u = v = o$;
- (2) $0 < G(u, u, v)$ with $u \neq v$;
- (3) $G(u, u, v) \leq G(u, v, z)$ for all $z \in E$ with $v \neq z$;
- (4) $G(u, v, o) = G(u, o, v) = G(v, o, u) = \dots$, (symmetry in all three variables);
- (5) there exists a real number $s \geq 1$ such that $G(u, v, o) \leq s[G(u, z, z) + G(z, v, o)]$.

Then G is called a G_b -metric on E and the pair (E, G) is called a G_b -metric space (G_bMS) .

Example 3 ([7]). Let (E, G) be a GMS and G_b -metric defined by $G_b(u, v, o) = (G(u, v, o))^p$. Then (E, G_b) is a G_bMS with $s = 2^{p-1}$.

Each GMS is a G_bMS with $s = 1$, and a G_bMS is not necessarily a GMS (see [7]).

Definition 6 ([5]). A G -metric is said to be symmetric if $G(u, v, v) = G(v, u, u)$ for all $u, v \in E$.

3. Rectangular G_b -Metric Spaces

Now, we define the notion of rectangular G_b -metric spaces.

Definition 7. Let E be a nonempty and for all $u, v, o \in E$, let a function $G : E \times E \times E \rightarrow [0, \infty)$ satisfies the following conditions:

- (RGb1) $G(u, v, o) = 0$ if $u = v = o$;
- (RGb2) $0 < G(u, u, v)$ with $u \neq v$;
- (RGb3) $G(u, u, v) \leq G(u, v, z)$ for all $z \in E$ with $v \neq z$;
- (RGb4) $G(u, v, o) = G(u, o, v) = G(v, o, u) = \dots$, (symmetry in all three variables);
- (RGb5) there exists a real number $s \geq 1$ such that $G(u, v, o) \leq s[G(u, z, z) + G(z, h, h) + G(h, v, o)]$ for all distinct points $z, h \in E \setminus \{u, v\}$.

Then G is called a rectangular G_b -metric on E and the pair (E, G) is called a rectangular G_b -metric space (RG_bMS) .

Remark 1.

- (1) It is worth mentioning that RG_bMS generalizes rectangular G -metric space $(RGMS)$, because a RG_bMS reduces a $RGMS$ for $s = 1$.
- (2) Every G_bMS with coefficient s is a RG_bMS with coefficient s^2 . Indeed, let (E, G_b) is a G_bMS , for all distinct $z, h \in E \setminus \{u, v, o\}$, we have

$$\begin{aligned} G_b(u, v, o) &\leq s[G_b(u, z, z) + G_b(z, v, o)] \\ &\leq s[G_b(u, z, z) + sG_b(z, h, h) + sG_b(h, v, o)] \\ &\leq s^2[G_b(u, z, z) + G_b(z, h, h) + G_b(h, v, o)], \end{aligned}$$

which shows that (E, G_b) is a RG_bMS with coefficient s^2 .

Let us present some specific examples of RG_bMS .

Example 4. Let (E, G_*) be a G -metric space. Define $G : E \times E \times E \rightarrow [0, \infty)$ by $G(u, v, o) = [G_*(u, v, o)]^r$ for all $u, v, o \in E$ and $r \geq 1$. Then, (E, G) is a RG_bMS with $s = 3^{r-1}$. Obviously, G satisfies conditions (RGb1) – (RGb4) of the Definition 7. Next, we shall prove that (RGb5) is hold. If $1 \leq r < \infty$, from the convexity of function $f(u) = u^r$ for $u \geq 0$, we have

$$(a + b + c)^r \leq 3^{r-1}(a^r + b^r + c^r),$$

for nonnegative real numbers a, b, c . Thus for distinct points $z, h \in E \setminus \{u, v, o\}$, we obtain

$$\begin{aligned} G(u, v, o) &= [G_*(u, v, o)]^r \\ &\leq [G_*(u, z, z) + G_*(z, h, h) + G_*(h, v, o)]^r \\ &\leq 3^{r-1}([G_*(u, z, z)^r + G_*(z, h, h)^r + G_*(h, v, o)^r]) \\ &= 3^{r-1}[G(u, z, z) + G(z, h, h) + G(h, v, o)]. \end{aligned}$$

Therefore, (E, G) is a RG_bMS with $s = 3^{r-1}$.

Example 5. Let $E = \mathbb{R}$ and we define $G : E \times E \times E \rightarrow [0, \infty)$ by the formula

$$G(u, v, o) = (|u - v| + |v - o| + |u - o|)^r, (r \geq 1)$$

Then (E, G) is a RG_bMS with $s = 3^{r-1}$. Obviously, G satisfies conditions $(RGb1) - (RGb4)$ of the Definition 7. Next, we shall verify that G satisfies $(RGb5)$. If $1 \leq r < \infty$, from the convexity of function $f(u) = u^r$ for $u \geq 0$, we have

$$(a + b + c)^r \leq 3^{r-1}(a^r + b^r + c^r),$$

for nonnegative real numbers a, b, c . Thus for distinct points $z, h \in E \setminus \{u, v, o\}$, we obtain

$$\begin{aligned} G(u, v, o) &= (|u - v| + |v - o| + |u - o|)^r \\ &\leq (|u - z| + |z - v| + |v - o| + |u - z| + |z - o|)^r \\ &\leq (|u - z| + |z - h| + |h - v| + |v - o| + |u - z| + |z - h| + |h - o|)^r \\ &= [(|u - z| + |u - z|) + (|z - h| + |z - h|) + (|h - v| + |v - o| + |h - o|)]^r \\ &\leq 3^{r-1}[(|u - z| + |u - z|)^r + (|z - h| + |z - h|)^r + (|h - v| + |v - o| + |h - o|)^r] \\ &= 3^{r-1}[G(u, z, z) + G(z, h, h) + G(h, v, o)]. \end{aligned}$$

Hence, (E, G) is a RG_bMS with $s = 3^{r-1}$.

The following example shows that a RG_bMS does not a $RGMS$ and a G_bMS .

Example 6. Let $E = \Gamma \cup \{1, 2\} \cup H$, where $\Gamma = \{\frac{1}{n} : n = 2, 3, 4, \dots\}$ and $H = \{3, 4, 5, \dots\}$. Define $G : E \times E \times E \rightarrow [0, \infty)$ such that $G(u, v, o) = G(u, o, v) = G(v, o, u) = \dots$ for all $u, v, o \in E$ and

$$G(u, v, o) = \begin{cases} 0 & \text{if } u = v = o; \\ 4a & \text{if } u, v, o \in \Gamma \text{ and } u \neq v \neq o; \\ \frac{a}{2^n} & \text{if } u \in \Gamma \text{ and } v = o \in \{1, 2\}; \\ a & \text{otherwise,} \end{cases}$$

where $a > 0$ is a constant. Then (E, G) is a RG_bMS with coefficient $s = 2 > 1$. However, (E, G) is not a $RGMS$, as $G(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}) = 4a > \frac{9a}{4} = G(\frac{1}{2}, 1, 1) + G(1, 2, 2) + G(2, \frac{1}{3}, \frac{1}{4})$ and not a G_bMS , as there does not exists $s \geq 1$ such that $G(u, v, o) \leq s[G(u, z, z) + G(z, v, o)]$ for all $u, v, o, z \in E$.

Proposition 1. Let (E, G) be a RG_bMS and $\varepsilon > 0$, then for each $u, v, o \in E$, we have:

- (1) if $G(u, v, o) = 0$, then $u = v = o$;
- (2) if $G(u_n, u_m, u_l) < \varepsilon$ for any $n, m, l \in \mathbb{N}$, then $G(u_n, u_m, u_m) < \varepsilon$ and $G(u_n, u_n, u_m) < \varepsilon$.

Proof.

- (1) By $(RGb2)$ and $(RGb3)$, we get a contradiction. Indeed,

$$0 = G(u, v, o) \geq G(u, u, v) > 0,$$

when $u \neq v$ and $v \neq o$. In addition,

$$0 = G(u, v, o) = G(v, v, o) > 0$$

when $u = v$ and $v \neq o$.

- (2) It can be obtain from (RGb3) of the Definition 7.

□

Proposition 2. Let (E, G) be a RG_bMS . Then, the function given as $d_G(u, v) = G(u, v, v) + G(v, u, u)$, defines a rectangular b -metric on E . We call it a rectangular b -metric d_G induced by the rectangular G_b -metric G .

Proof. Let us prove that conditions of Definition 3 are fulfilled for $d_G(u, v)$.

- (1) If $d_G(u, v) = 0$, then $G(u, v, v) = G(v, u, u) = 0$ and by Proposition 1, it follows that $u = v$. If $u = v$, then $G(u, v, v) = G(v, u, u) = 0$ and $d_G(u, v) = 0$;
(2) Property (RGb4) of the Definition 7 implies that

$$d_G(u, v) = G(u, v, v) + G(v, u, u) = G(v, u, u) + G(u, v, v) = d_G(v, u);$$

- (3) By (RGb5) of the Definition 7, for distinct points $z, h \in E \setminus \{u, v\}$, it follows that

$$\begin{aligned} d_G(u, v) &= G(u, v, v) + G(u, u, v) \\ &\leq s[G(u, z, z) + G(z, h, h) + G(h, v, v)] + s[G(v, h, h) + G(h, z, z) + G(z, u, u)] \\ &= s[G(u, z, z) + G(z, u, u) + G(z, h, h) + G(h, z, z) + G(h, v, v) + G(v, h, h)] \\ &= s[d_G(u, z) + d_G(z, h) + d_G(h, v)]. \end{aligned}$$

□

We now define G -convergent and G -Cauchy sequence in RG_bMS and completeness of RG_bMS .

Definition 8. Let (E, G) be a RG_bMS . The sequence $\{u_n\}$ in E and $u \in E$, then

- (1) the sequence $\{u_n\}$ is said to be G -convergent in E to u , if for any $\varepsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that $G(u_n, u_m, u) < \varepsilon$ for all $m, n \geq n_0$, and this fact is represented by $\lim_{n, m \rightarrow \infty} G(u_n, u_m, u) = 0$;
(2) the sequence $\{u_n\}$ is said to be G -Cauchy sequence in E , if for any $\varepsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that $G(u_n, u_m, u_l) < \varepsilon$ for all $n, m, l \geq n_0$, and this fact is represented by $\lim_{n, m, l \rightarrow \infty} G(u_n, u_m, u_l) = 0$;
(3) (E, G) is said to be a complete RG_bMS , if every G -Cauchy sequence in E converges to some $u \in E$.

Definition 9. A rectangular G_b -metric G is said to be symmetric if $G(u, v, v) = G(u, u, v)$ for all $u, v \in E$.

Remark 2. It follows Example 4 that rectangular G_b -metric G is symmetric if G -metric G_* is symmetric.

Definition 10. Let (E, G) be a RG_bMS , for $u_0 \in E$, $r > 0$, then G -ball with center u_0 and radius r is

$$B_G(u_0, r) = \{v \in E : G(u_0, v, v) < r\}.$$

Remark 3. Open balls in RG_bMS are not necessarily open. In Example 6, $B_G(\frac{1}{2}, \frac{a}{2}) = \{\frac{1}{2}, 1, 2\}$ and there does not exists any open ball with center 1 and contained in $B_G(\frac{1}{2}, \frac{a}{2})$. So $B_G(\frac{1}{2}, \frac{a}{2})$ is not an open set.

Proposition 3. Let (E, G) be a RG_bMS , then for any $u_0 \in E$, $r > 0$, we have

- (1) if $G(u_0, v, o) < r$, then $v, o \in B_G(u_0, r)$.
- (2) if G is symmetric, then

$$B_G(u_0, \frac{r}{2}) \subseteq B_{d_G}(u_0, r) \subseteq B_G(u_0, r).$$

Thus, if rectangular G_b -metric is symmetric, then the RG_bMS is topologically equivalent to a R_bMS . This allows us to readily transport many concepts and results from R_bMS into RG_bMS setting.

It should be noted that RG_bMS is not necessarily Hausdorff.

Example 7. Let (E, d_b) is a R_bMS with $s = 2$ as shows in Example 1. For all $u, v, o \in E$, we define $G : E \times E \times E \rightarrow [0, \infty)$ by the formula $G(u, v, o) = \max\{d_b(u, v), d_b(u, o), d_b(o, u)\}$. Obviously, G satisfies conditions $(RGb1) - (RGb4)$ of the Definition 7, so we only need to verify that $(RGb5)$ is hold. Indeed,

- (1) if $u \neq v \neq o$, then $G(u, v, o) = \max\{d_b(u, v), d_b(u, o), d_b(o, u)\}$;
- (2) if $u = v \neq o$, then $G(u, v, o) = G(u, u, o) = d_b(u, o)$.

For distinct points $z, h \in E \setminus \{u, v\}$, we have

$$\begin{aligned} G(u, v, o) &\leq \max\{s[d_b(u, z) + d_b(z, h) + d_b(h, v)], d_b(v, o), s[d_b(o, h) + d_b(h, z) + d_b(z, u)]\} \\ &\leq s[d_b(u, z) + d_b(z, h)] + s \max\{d_b(h, v), d_b(v, o), d_b(o, h)\} \\ &= s[G(u, z, z) + G(z, h, h) + G(h, v, o)]. \end{aligned}$$

Hence, (E, G) is a RG_bMS with $s = 2$. However, there does not exists any $r_1, r_2 > 0$ such that $B_G(2, r_1) \cap B_G(3, r_2) = \emptyset$, thus (E, G) is not Hausdorff.

4. Fixed Point Results

In this section, we show some results of fixed points for several contraction mappings in a RG_bMS under suitable hypotheses.

Lemma 1 ([26]). Let (E, G) be a R_bMS with coefficient $s \geq 1$, and $\{u_n\}$ be a sequence in E such that

$$d(u_{n+1}, u_n) \leq \lambda d(u_n, u_{n-1})$$

for all $n \in \mathbb{N}$ and $\lambda \in [0, 1)$. Then sequence $\{u_n\}$ is Cauchy sequence in E .

Lemma 2. Let (E, G) be a RG_bMS with coefficient $s \geq 1$. Suppose that there exists a symmetric rectangular G_b -metric G on E and $\{u_n\}$ is a sequence in E induced by $u_{n+1} = Tu_n$ such that

$$G(u_n, u_{n+1}, u_{n+1}) + G(u_n, u_n, u_{n+1}) \leq \lambda(G(u_{n-1}, u_n, u_n) + G(u_{n-1}, u_{n-1}, u_n)), \quad (1)$$

for all $n \in \mathbb{N}$ and $\lambda \in [0, 1)$. Then $\{u_n\}$ is a G -Cauchy sequence in E .

Proof. The condition in Equation (1), together with Proposition 2, implies that

$$d_G(u_n, u_{n+1}) \leq \lambda d_G(u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$. Utilizing Lemma 1, it shows that $\{u_n\}$ is a Cauchy sequence in R_bMS . Since (E, d_G) and (E, G) are topologically equivalent while rectangular G_b -metric is symmetric, we have that $\{u_n\}$ is also a G -Cauchy sequence in RG_bMS . $\square$

Lemma 3. Let (E, G) be a RG_bMS with coefficient $s \geq 1$ and $\{u_n\}$ be a sequence in E induced by $u_{n+1} = Tu_n$ such that

$$G(u_n, u_{n+1}, u_{n+2}) \leq \lambda G(u_{n-1}, u_n, u_{n+1}) \quad (2)$$

where $\lambda \in [0, 1)$ and $n \in \mathbb{N}$. Then $\{u_n\}$ is a G -Cauchy sequence in E .

Proof. Without loss of generality, we assume that $u_n \neq u_m$ for all $n \neq m$. By the inequality (2), we obtain

$$G(u_n, u_{n+1}, u_{n+2}) \leq \lambda^n G(u_0, u_1, u_2)$$

and

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+1}) &\leq \lambda^n G(u_0, u_1, u_2) \\ G(u_n, u_n, u_{n+1}) &\leq \lambda^n G(u_0, u_1, u_2). \end{aligned}$$

Setting $G(u_n, u_{n+1}, u_{n+1}) = G_n$, $G(u_{n+1}, u_n, u_n) = G'_n$ and $G(u_n, u_{n+1}, u_{n+2}) = G_n^*$ for all $n \in \mathbb{N}$. Obviously, $G_n \leq G_n^*$ and $G'_n \leq G_n^*$ for all $n \in \mathbb{N}$. In order to prove our conclusion, we divide the proof into three cases.

Case 1 Let $\lambda \in [0, \frac{1}{s})$ ($s > 1$). For the sequence $\{u_n\}$, we consider $G(u_n, u_{n+p}, u_{n+p+q})$ in four subcases.

Subcase 1 If p is even say $2m$ ($m \geq 1$), and q is even say $2l$ ($l \geq 1$). From Definition 7, it follows that

$$\begin{aligned} &G(u_n, u_{n+2m}, u_{n+2m+2l}) \\ &= G(u_{n+2m+2l}, u_{n+2m}, u_n) \\ &\leq s[G(u_{n+2m+2l}, u_{n+2m+2l-1}, u_{n+2m+2l-1}) + G(u_{n+2m+2l-1}, u_{n+2m-1}, u_{n+2m-1}) \\ &\quad + G(u_{n+2m-1}, u_{n+2m}, u_n)] \\ &\leq s[G'_{n+2m+2l-1} + G(u_{n+2m+2l-1}, u_{n+2m+2l-2}, u_{n+2m-1}) + G(u_{n+2m-1}, u_{n+2m}, u_n)] \\ &= sG(u_{n+2m-1}, u_{n+2m+2l-2}, u_{n+2m+2l-1}) + sG'_{n+2m+2l-1} + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\ &\leq s^2[G_{n+2m-1} + G_{n+2m} + G(u_{n+2m+1}, u_{n+2m+2l-2}, u_{n+2m+2l-1})] + sG'_{n+2m+2l-1} \\ &\quad + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\ &\leq s^2[G_{n+2m-1} + G_{n+2m}] + s^3[G_{n+2m+1} + G_{n+2m+2}] + \dots + s^l[G_{n+2m+2l-5} + G_{n+2m+2l-4} \\ &\quad + G_{n+2m+2l-3}] + sG'_{n+2m+2l-1} + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\ &\leq s^2[\lambda^{n+2m-1}G_0^* + \lambda^{n+2m}G_0^*] + s^3[\lambda^{n+2m+1}G_0^* + \lambda^{n+2m+2}G_0^*] + \dots + s^l[\lambda^{n+2m+2l-5}G_0^* \\ &\quad + \lambda^{n+2m+2l-4}G_0^* + \lambda^{n+2m+2l-3}G_0^*] + s\lambda^{n+2m+2l-1}G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\ &\leq s^2\lambda^{n+2m-1}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* + s^2\lambda^{n+2m}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* \\ &\quad + s\lambda^{n+2m+2l-1}G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\ &= \frac{1+\lambda}{1-s\lambda^2}s^2\lambda^{n+2m-1}G_0^* + s\lambda^{n+2m+2l-1}G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}) \end{aligned}$$

and

$$\begin{aligned}
& G(u_n, u_{n+2m-1}, u_{n+2m}) \\
& \leq s[G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2}) + G(u_{n+2}, u_{n+2m-1}, u_{n+2m})] \\
& \leq s[G_n + G_{n+1}] + s^2[G_{n+2} + G_{n+3}] + \dots + s^{m-1}[G_{n+2m-4} + G_{n+2m-3} + G_{n+2m-2}^*] \\
& \leq s[\lambda^n G_0^* + \lambda^{n+1} G_0^*] + s^2[\lambda^{n+2} G_0^* + \lambda^{n+3} G_0^*] + \dots + s^{m-1}[\lambda^{n+2m-3} G_0^* + \lambda^{n+2m-2} G_0^* \\
& \quad + \lambda^{n+2m-1} G_{n+2m-2}^*] \\
& \leq s\lambda^n(1 + s\lambda^2 + s^2\lambda^4 + \dots)G_0^* + s\lambda^{n+1}(1 + s\lambda^2 + s^2\lambda^4 + \dots)G_0^* \\
& \leq \frac{1 + \lambda}{1 - s\lambda^2} s\lambda^n G_0^*.
\end{aligned}$$

Therefore, we have

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{1 + \lambda}{1 - s\lambda^2} s^2 \lambda^{n+2m-1} G_0^* + s\lambda^{n+2m+2l-1} G_0^* + \frac{1 + \lambda}{1 - s\lambda^2} s^2 \lambda^n G_0^*.$$

Since $\lambda < \frac{1}{s}$, we can deduce that

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{3 + \lambda}{1 - s\lambda^2} s^2 \lambda^n G_0^*;$$

Subcase 2 If p is even say $2m(m \geq 1)$, and q is odd say $2l + 1(l \geq 0)$. From Definition 7, it follows that

$$\begin{aligned}
& G(u_n, u_{n+2m}, u_{n+2m+2l+1}) \\
& = G(u_{n+2m+2l+1}, u_{n+2m}, u_n) \\
& \leq s[G(u_{n+2m+2l+1}, u_{n+2m+2l+2}, u_{n+2m+2l+2}) + G(u_{n+2m+2l+2}, u_{n+2m-1}, u_{n+2m-1}) \\
& \quad + G(u_{n+2m-1}, u_{n+2m}, u_n)] \\
& \leq s[G_{n+2m+2l+1} + G(u_{n+2m+2l+2}, u_{n+2m+2l+3}, u_{n+2m-1}) + G(u_{n+2m-1}, u_{n+2m}, u_n)] \\
& = sG(u_{n+2m-1}, u_{n+2m+2l+2}, u_{n+2m+2l+3}) + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\
& \leq s^2[G_{n+2m-1} + G_{n+2m} + G(u_{n+2m+1}, u_{n+2m+2l+2}, u_{n+2m+2l+3})] + sG_{n+2m+2l+1} \\
& \quad + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\
& \leq s^2[G_{n+2m-1} + G_{n+2m}] + s^3[G_{n+2m+1} + G_{n+2m+2}] + \dots + s^{l+2}[G_{n+2m+2l-1} + G_{n+2m+2l} \\
& \quad + G_{n+2m+2l+1}^*] + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\
& \leq s^2[\lambda^{n+2m-1} G_0^* + \lambda^{n+2m} G_0^*] + s^3[\lambda^{n+2m+1} G_0^* + \lambda^{n+2m+2} G_0^*] + \dots + s^{l+2}[\lambda^{n+2m+2l-1} G_0^* \\
& \quad + \lambda^{n+2m+2l} G_0^* + \lambda^{n+2m+2l-3} G_0^*] + s\lambda^{n+2m+2l+1} G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\
& \leq s^2 \lambda^{n+2m-1} [1 + s\lambda^2 + s^2\lambda^4 + \dots] G_0^* + s^2 \lambda^{n+2m} [1 + s\lambda^2 + s^2\lambda^4 + \dots] G_0^* \\
& \quad + s\lambda^{n+2m+2l+1} G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}) \\
& = \frac{1 + \lambda}{1 - s\lambda^2} s^2 \lambda^{n+2m-1} G_0^* + s\lambda^{n+2m+2l+1} G_0^* + sG(u_n, u_{n+2m-1}, u_{n+2m}),
\end{aligned}$$

combining with $G(u_n, u_{n+2m-1}, u_{n+2m}) \leq \frac{1 + \lambda}{1 - s\lambda^2} s\lambda^n G_0^*$, we obtain that

$$G(u_n, u_{n+2m}, u_{n+2m+2l+1}) \leq \frac{1 + \lambda}{1 - s\lambda^2} s^2 \lambda^{n+2m-1} G_0^* + s\lambda^{n+2m+2l} G_0^* + \frac{1 + \lambda}{1 - s\lambda^2} s^2 \lambda^n G_0^*.$$

Since $\lambda < \frac{1}{s}$, we can deduce that

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{3 + \lambda}{1 - s\lambda^2} s^2 \lambda^n G_0^*;$$

Subcase 3 If p is odd say $2m + 1 (m \geq 0)$, and q is even say $2l (l \geq 1)$. From Definition 7, it follows that

$$\begin{aligned}
 & G(u_n, u_{n+2m+1}, u_{n+2m+2l+1}) \\
 &= G(u_{n+2m+2l+1}, u_{n+2m+1}, u_n) \\
 &\leq s[G(u_{n+2m+2l+1}, u_{n+2m+2l+2}, u_{n+2m+2l+2}) + G(u_{n+2m+2l+2}, u_{n+2m+2}, u_{n+2m+2}) \\
 &\quad + G(u_{n+2m+2}, u_{n+2m+1}, u_n)] \\
 &\leq sG(u_{n+2m+2}, u_{n+2m+2l+1}, u_{n+2m+2l+2}) + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\
 &\leq s^2[G_{n+2m+2} + G_{n+2m+3}] + s^3[G_{n+2m+4} + G_{n+2m+5}] + \dots + s^l[G_{n+2m+2l-2} + G_{n+2m+2l-1} \\
 &\quad + G_{n+2m+2l}^*] + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\
 &\leq s^2[\lambda^{n+2m+2}G_0^* + \lambda^{n+2m+3}G_0^*] + s^3[\lambda^{n+2m+4}G_0^* + \lambda^{n+2m+5}G_0^*] + \dots + s^l[\lambda^{n+2m+2l-2}G_0^* \\
 &\quad + \lambda^{n+2m+2l-1}G_0^* + \lambda^{n+2m+2l}G_0^*] + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\
 &\leq s^2\lambda^{n+2m+2}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* + s^2\lambda^{n+2m+3}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* \\
 &\quad + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\
 &= \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^{n+2m+2}G_0^* + sG_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2})
 \end{aligned}$$

and

$$\begin{aligned}
 & G(u_n, u_{n+2m+1}, u_{n+2m+2}) \\
 &\leq s[G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2}) + G(u_{n+2}, u_{n+2m+1}, u_{n+2m+2})] \\
 &\leq s[G_n + G_{n+1}] + s^2[G_{n+2} + G_{n+3}] + \dots + s^m[G_{n+2m-2} + G_{n+2m-1} + G_{n+2m}^*] \\
 &\leq s[\lambda^n G_0^* + \lambda^{n+1}G_0^*] + s^2[\lambda^{n+2}G_0^* + \lambda^{n+3}G_0^*] + \dots + s^m[\lambda^{n+2m-2}G_0^* + \lambda^{n+2m-1}G_0^* \\
 &\quad + \lambda^{n+2m}G_0^*] \\
 &\leq s\lambda^n[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* + s\lambda^{n+1}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* \\
 &\leq \frac{1 + \lambda}{1 - s\lambda^2}s\lambda^n G_0^*.
 \end{aligned}$$

Therefore, we have

$$G(u_n, u_{n+2m+1}, u_{n+2m+2l+1}) \leq \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^{n+2m}G_0^* + s\lambda^{n+2m+2l}G_0^* + \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^n G_0^*.$$

Since $\lambda < \frac{1}{s}$, we can deduce that

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{3 + \lambda}{1 - s\lambda^2}s^2\lambda^n G_0^*;$$

Subcase 4 If p is odd say $2m + 1 (m \geq 0)$, and q is odd say $2l + 1 (l \geq 0)$. From Definition 7, it follows that

$$\begin{aligned} & G(u_n, u_{n+2m+1}, u_{n+2m+2l+2}) \\ &= G(u_{n+2m+2l+2}, u_{n+2m+1}, u_n) \\ &\leq s[G(u_{n+2m+2l+2}, u_{n+2m+2l+1}, u_{n+2m+2l+1}) + G(u_{n+2m+2l+1}, u_{n+2m+2}, u_{n+2m+2}) \\ &\quad + G(u_{n+2m+2}, u_{n+2m+1}, u_n)] \\ &\leq s[G(u_{n+2m+2}, u_{n+2m+2l+1}, u_{n+2m+2l+2}) + sG'_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2})] \\ &\leq s^2[G_{n+2m+2} + G_{n+2m+3}] + s^3[G_{n+2m+4} + G_{n+2m+5}] + \dots + s^l[G_{n+2m+2l-2} + G_{n+2m+2l-1} \\ &\quad + G_{n+2m+2l}] + sG'_{n+2m+2l+1} + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\ &\leq s^2[\lambda^{n+2m+2}G_0^* + \lambda^{n+2m+3}G_0^*] + s^3[\lambda^{n+2m+4}G_0^* + \lambda^{n+2m+5}G_0^*] + \dots + s^l[\lambda^{n+2m+2l-2}G_0^* \\ &\quad + \lambda^{n+2m+2l-1}G_0^* + \lambda^{n+2m+2l}G_0^*] + s\lambda^{n+2m+2l+1}G_0^* + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\ &\leq s^2\lambda^{n+2m+2}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* + s^2\lambda^{n+2m+3}[1 + s\lambda^2 + s^2\lambda^4 + \dots]G_0^* \\ &\quad + s\lambda^{n+2m+2l+1}G_0^* + sG(u_n, u_{n+2m+1}, u_{n+2m+2}) \\ &\leq \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^{n+2m+2}G_0^* + s\lambda^{n+2m+2l+1}G_0^* + sG(u_n, u_{n+2m+1}, u_{n+2m+2}), \end{aligned}$$

combining with $G(u_n, u_{n+2m+1}, u_{n+2m+2}) \leq \frac{1+\lambda}{1-s\lambda^2}s\lambda^n G_0^*$, we obtain that

$$G(u_n, u_{n+2m}, u_{n+2m+2l+1}) \leq \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^{n+2m}G_0^* + s\lambda^{n+2m+2l+1}G_0^* + \frac{1 + \lambda}{1 - s\lambda^2}s^2\lambda^n G_0^*.$$

Since $\lambda < \frac{1}{s}$, we can deduce that

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{3 + \lambda}{1 - s\lambda^2}s^2\lambda^n G_0^*.$$

In view of the all above subcases, we obtain that

$$G(u_n, u_{n+2m}, u_{n+2m+2l}) \leq \frac{3 + \lambda}{1 - s\lambda^2}s^2\lambda^n G_0^*,$$

for all $n \in \mathbb{N}$. We reach Case1, thus $\{u_n\}$ is a G -Cauchy sequence in E .

Case 2 Let $\lambda \in [\frac{1}{s}, 1) (s > 1)$. Since $\lim_{n \rightarrow \infty} \lambda^n = 0$, there exists $N_0 \in \mathbb{N}$ such that $\lambda^{N_0} < \frac{1}{s}$. Thus, by Case 1, we claim that

$$\left\{ (T^{N_0})^n u_0 \right\}_{n=0}^{\infty} := \{u_{N_0}, u_{N_0+1}, \dots, u_{N_0+n}, \dots\}$$

is G -Cauchy sequence. Since

$$\{u_n\}_{n=0}^{\infty} = \{u_1, u_2, \dots, u_{N_0}\} \cup \{u_{N_0+1}, u_{N_0+2}, \dots, u_{N_0+n}, \dots\},$$

we conclude that the sequence $\{u_n\}$ is a G -Cauchy sequence.

Case 3 Let $s = 1$. Similar to the process of Case 1, the claim holds. $\square$

Next, we will give some fixed point theorems. Firstly, we now state and prove the Banach contraction principle in RG_bMS .

Theorem 1. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$ and the mapping $T : E \rightarrow E$ satisfies the contractive condition:

$$G(Tu, Tv, To) \leq \lambda G(u, v, o), \text{ for all } u, v, o \in E, \quad (3)$$

where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in E .

Proof. Let $u_0 \in E$ and define a sequence $\{u_n\}$ in E by $u_n = Tu_{n-1}$ for all $n \in \mathbb{N}$. Using the inequality (3), we have

$$G(u_n, u_{n+1}, u_{n+2}) = G(Tu_{n-1}, Tu_n, Tu_{n+1}) \leq \lambda G(u_{n-1}, u_n, u_{n+1}).$$

If $u_n = u_{n+1}$ then u_n is fixed point of T and the proof is completed. So, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Also, we can assume that $u_n \neq u_{n+p}$ for all $p \geq 2$. Indeed, if $u_n = u_{n+p}$, then using the inequality (3), we get

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+1}) &= G(u_n, Tu_n, Tu_n) \\ &= G(Tu_{n+p-1}, Tu_{n+p}, Tu_{n+p}) \\ &\leq \lambda^{p-1} G(u_n, u_{n+1}, u_{n+1}), \end{aligned}$$

a contradiction. Therefore, in what follows, we assume that $u_n \neq u_m$ for all $n \neq m$. Thus, similarly to the proof of Lemma 3, we conclude that $\{u_n\}$ is a G -Cauchy sequence in E . Since (E, G) is a complete RG_bMS , hence, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $u^* \in E$ such that $G(u_n, u_m, u^*) < \frac{\varepsilon}{3s}$ for all $n, m \geq n_0$. Moreover, it follows from Proposition 1 that $G(u^*, u_n, u_n) < \frac{\varepsilon}{3s}$, $G(u_n, u_{n-1}, u_{n-1}) < \frac{\varepsilon}{3s}$ and $G(u_n, u^*, u^*) < \frac{\varepsilon}{3s}$, for all $n \geq n_0$. Then, we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, Tu^*, Tu^*)] \\ &\leq s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + \lambda G(u_n, u^*, u^*)] \\ &< s\left[\left(\frac{\varepsilon}{3s}\right) + \left(\frac{\varepsilon}{3s}\right) + \lambda\left(\frac{\varepsilon}{3s}\right)\right] \\ &< \varepsilon \end{aligned}$$

which implies $G(u^*, Tu^*, Tu^*) = 0$. Therefore, $Tu^* = u^*$ and u^* is a fixed point of T . Now we show that the fixed point u^* is unique. If there is another fixed point $v^* \neq u^*$, by the given condition (3), we have

$$G(u^*, v^*, v^*) = G(Tu^*, Tv^*, Tv^*) \leq \lambda G(u^*, v^*, v^*),$$

which is a contradiction. Thus, u^* is a unique fixed point of T . $\square$

The following theorem is a Ćirić type fixed point result in a RG_bMS .

Theorem 2. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$ and the mapping $T : E \rightarrow E$ satisfies the contractive condition:

$$G(Tu, Tv, To) \leq \lambda M_{u,v,o} \quad (4)$$

where

$$M_{u,v,o} = \max\{G(u, v, o), G(u, Tu, Tu), G(v, Tv, Tv), G(o, To, To), G(u, Tv, Tv), G(u, To, To), G(v, Tu, Tu), G(v, To, To), G(o, Tu, Tu), G(o, Tv, Tv)\}$$

for any $u, v, o \in E$, where $0 \leq \lambda < \frac{1}{3s^2}$. Then T has a unique fixed point in E .

Proof. Let $u_0 \in E$ and define a sequence $\{u_n\}$ in E by $u_n = Tu_{n-1}$ for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$, then u_n is a fixed point of T and the proof is completed. So, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. It follows from the inequality (4) that

$$G(u_n, u_{n+1}, u_{n+2}) = G(Tu_{n-1}, Tu_n, Tu_{n+1}) \leq \lambda M_{u_{n-1}, u_n, u_{n+1}}.$$

Noting that

$$\begin{aligned} M_{u_{n-1}, u_n, u_{n+1}} &= \max\{G(u_{n-1}, u_n, u_{n+1}), G(u_{n-1}, Tu_{n-1}, Tu_{n-1}), G(u_n, Tu_n, Tu_n), \\ &\quad G(u_{n+1}, Tu_{n+1}, Tu_{n+1}), G(u_{n-1}, Tu_n, Tu_n), G(u_{n-1}, Tu_{n+1}, Tu_{n+1}), \\ &\quad G(u_n, Tu_{n+1}, Tu_{n+1}), G(u_n, Tu_{n-1}, Tu_{n-1}), G(u_{n+1}, Tu_{n-1}, Tu_{n-1}), \\ &\quad G(u_{n+1}, Tu_n, Tu_n)\} \\ &= \max\{G(u_{n-1}, u_n, u_{n+1}), G(u_{n-1}, u_n, u_n), G(u_n, u_{n+1}, u_{n+1}), \\ &\quad G(u_{n+1}, u_{n+2}, u_{n+2}), G(u_{n-1}, u_{n+1}, u_{n+1}), G(u_{n-1}, u_{n+2}, u_{n+2}), \\ &\quad G(u_n, u_{n+2}, u_{n+2}), G(u_{n+1}, u_n, u_n)\}. \end{aligned}$$

Using (RGb3) and (RGb5) in the above inequality, we obtain

$$\begin{aligned} M_{u_{n-1}, u_n, u_{n+1}} &= \max\{G(u_{n-1}, u_n, u_{n+1}), G(u_{n+1}, u_{n+2}, u_{n+2}), G(u_{n-1}, u_{n+2}, u_{n+2}), \\ &\quad G(u_n, u_{n+2}, u_{n+2})\} \\ &\leq \max\{G(u_{n-1}, u_n, u_{n+1}), G(u_{n-1}, u_n, u_n), G(u_{n+1}, u_{n+2}, u_{n+2}), \\ &\quad s[G(u_{n-1}, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2})], \\ &\quad G(u_n, u_{n+2}, u_{n+2})\} \\ &\leq \max\{G(u_{n-1}, u_n, u_{n+1}), G(u_{n-1}, u_n, u_n), G(u_{n+1}, u_{n+2}, u_{n+2}), \\ &\quad 2sG(u_{n-1}, u_n, u_{n+1}) + sG(u_n, u_{n+1}, u_{n+2}), G(u_n, u_{n+2}, u_{n+2})\} \\ &= 2sG(u_{n-1}, u_n, u_{n+1}) + sG(u_n, u_{n+1}, u_{n+2}). \end{aligned}$$

Therefore, we get

$$G(u_n, u_{n+1}, u_{n+2}) \leq 2s\lambda G(u_{n-1}, u_n, u_{n+1}) + s\lambda G(u_n, u_{n+1}, u_{n+2}).$$

As $\lambda \in [0, \frac{1}{3s^2})$, it follows from the above inequality that

$$G(u_n, u_{n+1}, u_{n+2}) \leq \frac{2s\lambda}{1-s\lambda} G(u_{n-1}, u_n, u_{n+1}). \quad (5)$$

Let $k = \frac{2s\lambda}{1-s\lambda}$, it is clear that $k < 1$. We also can assume that $u_n \neq u_{n+p}$ for all $n \in \mathbb{N}$ and $p \geq 2$. Indeed, if $u_n = u_{n+p}$, using the inequality (5), we have

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+2}) &= G(u_{n+p}, u_{n+1}, u_{n+2}) \\ &\leq kG(u_{n+p-1}, u_n, u_{n+1}) \\ &= kG(u_{n+p-1}, u_{n+p}, u_{n+1}) \\ &\leq k^2G(u_{n+p-2}, u_{n+p-1}, u_n) \\ &\leq \dots \leq k^pG(u_n, u_{n+1}, u_{n+2}), \end{aligned}$$

a contradiction. Thus, similarly to the proof of Lemma 3, we conclude that $\{u_n\}$ is a G -Cauchy sequence in E . Since (E, d) is a complete RG_bMS , hence, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $u^* \in E$ such that $G(u_n, u_m, u^*) < \frac{1-s^2\lambda}{4s}\varepsilon$ for all $n, m \geq n_0$. Moreover, it follows from Proposition 1 that $G(u^*, u_n, u_n) < \frac{1-s^2\lambda}{4s}\varepsilon$, $G(u_n, u^*, u^*) < \frac{1-s^2\lambda}{4s}\varepsilon$, $G(u_n, u_{n+1}, u_{n+1}) < \frac{1-s^2\lambda}{4s}\varepsilon$ and $G(u_{n+1}, u_n, u_n) < \frac{1-s^2\lambda}{4s}\varepsilon$ for all $n \geq n_0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq s[G(u^*, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2}) + G(u_{n+2}, Tu^*, Tu^*)] \\ &\leq sG(u^*, u_{n+1}, u_{n+1}) + sG(u_{n+1}, u_{n+2}, u_{n+2}) + s\lambda M_{u_{n+1}, u^*, u^*} \end{aligned}$$

and

$$\begin{aligned} M_{u_{n+1}, u^*, u^*} &= \max\{G(u_{n+1}, u^*, u^*), G(u_{n+1}, Tu_{n+1}, Tu_{n+1}), G(u^*, Tu^*, Tu^*), \\ &\quad G(u_{n+1}, Tu^*, Tu^*), G(u^*, Tu_{n+1}, Tu_{n+1})\} \\ &= \max\{G(u_{n+1}, u^*, u^*), G(u_{n+1}, Tu_{n+1}, Tu_{n+1}), G(u^*, Tu^*, Tu^*), \\ &\quad G(u_{n+1}, Tu^*, Tu^*), G(u^*, u_{n+2}, u_{n+2})\}. \end{aligned}$$

If $M_{u_{n+1}, u^*, u^*} = G(u^*, Tu^*, Tu^*)$, then we have

$$G(u^*, Tu^*, Tu^*) \leq sG(u^*, u_{n+1}, u_{n+1}) + sG(u_{n+1}, u_{n+2}, u_{n+2}) + s\lambda G(u^*, Tu^*, Tu^*),$$

since $s\lambda < 1$, the above inequality yields

$$G(u^*, Tu^*, Tu^*) \leq \frac{s}{1-s\lambda} [G(u^*, u_{n+1}, u_{n+1}) + G(u_{n+1}, u_{n+2}, u_{n+2})] < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon;$$

If $M_{u_{n+1}, u^*, u^*} = G(u_{n+1}, Tu^*, Tu^*)$, we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq sG(u^*, u_{n+1}, u_{n+1}) + sG(u_{n+1}, u_{n+2}, u_{n+2}) + s\lambda G(u_{n+1}, Tu^*, Tu^*) \\ &\leq sG(u^*, u_{n+1}, u_{n+1}) + sG(u_{n+1}, u_{n+2}, u_{n+2}) + s^2\lambda [G(u_{n+1}, u_n, u_n) \\ &\quad + G(u_n, u^*, u^*) + G(u^*, Tu^*, Tu^*)], \end{aligned}$$

since $s^2\lambda < 1$, the above inequality yields

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq \frac{1}{1-s^2\lambda} [sG(u^*, u_{n+1}, u_{n+1}) + sG(u_{n+1}, u_{n+2}, u_{n+2}) \\ &\quad + s^2\lambda G(u_{n+1}, u_n, u_n) + s^2\lambda G(u_n, u^*, u^*)] \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon; \end{aligned}$$

If $M_{u_{n+1}, u^*, u^*} = \max\{G(u_{n+1}, u^*, u^*), G(u_{n+1}, Tu_{n+1}, Tu_{n+1}), G(u^*, u_{n+2}, u_{n+2})\}$, which implies that $M_{u_{n+1}, u^*, u^*} < \frac{1-s^2\lambda}{4s}\varepsilon$. Hence, we have

$$G(u^*, Tu^*, Tu^*) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

Consequently, we deduce that $G(u^*, Tu^*, Tu^*) = 0$, hence, $Tu^* = u^*$ and u^* is a fixed point of T . To prove the uniqueness of the fixed point, suppose that v^* is another fixed point of T . We have

$$\begin{aligned} G(u^*, v^*, v^*) &= G(Tu^*, Tv^*, Tv^*) \\ &\leq \lambda \max\{G(u^*, v^*, v^*), G(u^*, Tv^*, Tv^*), G(v^*, Tu^*, Tu^*)\} \\ &\leq \lambda G(u^*, v^*, v^*), \end{aligned}$$

a contradiction. Hence, $u^* = v^*$ which completes the proof. $\square$

As a similar consequence of Theorem 2, we obtain the following result.

Corollary 1. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$ and the mapping $T : E \rightarrow E$ satisfies the contractive condition:

$$G(Tu, Tv, To) \leq \lambda M_{u,v,o}$$

where

$$M_{u,v,o} = \max\{G(u, v, o), G(u, u, Tu), G(v, v, Tv), G(o, o, To), G(u, u, Tv), \\ G(u, u, To), G(v, v, Tu), G(v, v, To), G(o, o, Tu), G(o, o, Tv)\}$$

for any $u, v, o \in E$, where $0 \leq \lambda < \frac{1}{3s^2}$. Then T has a unique fixed point in E .

The following theorem is a Reich type fixed point result in a RG_bMS .

Theorem 3. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$ and the mapping $T : E \rightarrow E$ satisfies the contractive condition:

$$G(Tu, Tv, To) \leq a_1 G(u, v, o) + a_2 G(u, Tu, Tu) + a_3 G(v, Tv, Tv) + a_4 G(o, To, To) \quad (6)$$

for all $u, v, o \in E$, where $a_1 + a_2 + a_3 + a_4 < 1$ and $s(a_3 + a_4) < 1$. Then T has a unique fixed point in E .

Proof. Let $u_0 \in E$ and define a sequence $\{u_n\}$ in E by $u_n = Tu_{n-1}$ for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$ then u_n is fixed point of T and the proof is completed. So, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. It follows from the inequality (6) that

$$\begin{aligned} & G(u_n, u_{n+1}, u_{n+2}) \\ &= G(Tu_{n-1}, Tu_n, Tu_{n+1}) \\ &\leq a_1 G(u_{n-1}, u_n, u_{n+1}) + a_2 G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + a_3 G(u_n, Tu_n, Tu_n) \\ &\quad + a_4 G(u_{n+1}, Tu_{n+1}, Tu_{n+1}) \\ &= a_1 G(u_{n-1}, u_n, u_{n+1}) + a_2 G(u_{n-1}, u_n, u_n) + a_3 G(u_n, u_{n+1}, u_{n+1}) + a_4 G(u_{n+1}, u_{n+2}, u_{n+2}) \\ &\leq a_1 G(u_{n-1}, u_n, u_{n+1}) + a_2 G(u_{n-1}, u_n, u_{n+1}) + a_3 G(u_n, u_{n+1}, u_{n+2}) + a_4 G(u_n, u_{n+1}, u_{n+2}). \end{aligned}$$

Hence, we get

$$G(u_n, u_{n+1}, u_{n+2}) \leq \frac{a_1 + a_2}{1 - a_3 - a_4} G(u_{n-1}, u_n, u_{n+1}).$$

Let $k = \frac{a_1 + a_2}{1 - a_3 - a_4}$, where $0 \leq k < 1$. Also, we can assume that $u_n \neq u_{n+p}$ for all $p \geq 2$. Indeed, if $u_n = u_{n+p}$, then using the inequality (6), we get

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+1}) &= G(u_n, Tu_n, Tu_n) \\ &= G(Tu_{n+p-1}, Tu_{n+p}, Tu_{n+p}) \\ &\leq k^{p-1} G(u_n, u_{n+1}, u_{n+1}), \end{aligned}$$

a contradiction. In what follows, we can assume that $u_n \neq u_m$ for all $n \neq m$. Thus, similarly to the proof of Lemma 3, we conclude that $\{u_n\}$ is a G -Cauchy sequence in E . Since (E, G) is a complete RG_bMS , hence, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $u^* \in E$ such that $G(u^*, u_n, u_m) < \frac{\varepsilon}{3(1+a_1+a_2)s}$ for all $n, m \geq n_0$. Moreover, it follows from Proposition 1 that $G(u^*, u_n, u_n) < \frac{\varepsilon}{3s}$, $G(u_n, u_{n+1}, u_{n+1}) < \frac{\varepsilon}{3(1+a_2)s}$ and $G(u_n, u^*, u^*) < \frac{\varepsilon}{3a_1s}$ for all $n \geq n_0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} G(u^*, Tu^*, Tu^*) &\leq s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, Tu^*, Tu^*)] \\ &\leq sG(u^*, u_n, u_n) + sG(u_n, u_{n+1}, u_{n+1}) + sa_1 G(u_n, u^*, u^*) \\ &\quad + sa_2 G(u_n, Tu_n, Tu_n) + sa_3 G(u^*, Tu^*, Tu^*) + sa_4 G(u^*, Tu^*, Tu^*). \end{aligned}$$

Since $s(a_3 + a_4) < 1$, the above inequality yields

$$G(u^*, Tu^*, Tu^*) \leq \frac{s}{1 - s(a_3 + a_4)} [G(u^*, u_n, u_n) + (1 + a_2)G(u_n, u_{n+1}, u_{n+1}) + a_1G(u_n, u^*, u^*)],$$

we obtain

$$G(u^*, Tu^*, Tu^*) < \frac{s(\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3})}{(1 - s(a_3 + a_4))} = \frac{s\varepsilon}{(1 - s(a_3 + a_4))}.$$

We deduce that $G(u^*, Tu^*, Tu^*) = 0$, that is, $Tu^* = u^*$. So u^* is a fixed point of T . Now we show that the fixed point u^* is unique. If there is another fixed point $v^* \neq u^*$, by the given condition (6), we have

$$\begin{aligned} G(u^*, v^*, v^*) &= G(Tu^*, Tv^*, Tv^*) \\ &\leq a_1G(u^*, v^*, v^*) + a_2G(u^*, Tu^*, Tu^*) + a_3G(v^*, Tv^*, Tv^*) \\ &\quad + a_4G(v^*, Tv^*, Tv^*) \\ &= a_1G(u^*, v^*, v^*), \end{aligned}$$

which is a contradiction. Hence, $u^* = v^*$ which completes the proof. $\square$

As a similar consequence of Theorem 3, we obtain the following result.

Corollary 2. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$ and the mapping $T : E \rightarrow E$ satisfies the contractive condition:

$$G(Tu, Tv, To) \leq a_1G(u, v, o) + a_2G(u, u, Tu) + a_3G(v, v, Tv) + a_4G(o, o, To)$$

for all $u, v, o \in E$, where $a_1 + a_2 + a_3 + a_4 < 1$ and $s(a_3 + a_4) < 1$. Then T has a unique fixed point in E .

5. Convex Rectangular G_b -Metric Spaces

In this section, we will give the notion of convex rectangular G_b -metric space and prove several fixed point theorems for enriched type contractions in this space.

Definition 11 ([19]). Let (E, d) be a metric space and $I = [0, 1]$. A continuous function $w : E \times E \times [0, 1] \rightarrow E$ is said to be a convex structure on E if for each $u, v, o \in E$ and $\alpha \in I$,

$$d(o, w(u, v; \alpha)) \leq \alpha d(o, u) + (1 - \alpha)d(o, v)$$

holds. A metric space (E, d) with a convex structure w is called a convex metric space.

Next, we introduce the notion of convex rectangular G_b -metric space.

Definition 12. Let (E, G) be a complete RG_bMS with coefficient $s \geq 1$. A mapping $w : E \times E \times [0, 1] \rightarrow E$ is said to be a convex structure on E if for each $u, v, z, h \in E$ and $\alpha \in I$,

$$G(u, v, w(z, h; \alpha)) \leq \alpha G(u, v, z) + (1 - \alpha)G(u, v, h) \quad (7)$$

holds. Then the triplet (E, G, w) is called a convex rectangular G_b -metric space (CRG_bMS) with coefficient $s \geq 1$.

Let us present now a specific example of CRG_bMS .

Example 8. Let $E = \mathbb{R}$ and for any $u, v, o \in E$, let us defined G by

$$G(u, v, o) = (|u - v| + |v - o| + |u - o|)^2,$$

as well as the mapping $w : E \times E \times [0, 1] \rightarrow E$ by the formula

$$w(u, v; \alpha) = \alpha u + (1 - \alpha)v.$$

Then (E, G, w) is a CRG_bMS with $s = 3$. Indeed, Example 5 shows us that (E, G) is a RG_bMS with $s = 3$. We only need to verify that G satisfies inequality (7). For any $u, v, z, h \in E$, we get

$$\begin{aligned} & G_b(u, v, w(z, h; \alpha)) \\ &= [|u - v| + |v - \alpha z - (1 - \alpha)h| + |u - \alpha z - (1 - \alpha)h|]^2 \\ &\leq [\alpha|u - v| + (1 - \alpha)|u - v| + \alpha|v - z| + (1 - \alpha)|v - h| + \alpha|u - z| + (1 - \alpha)|u - h|]^2 \\ &= [\alpha(|u - v| + |v - z| + |u - z|) + (1 - \alpha)(|u - v| + |v - h| + |u - h|)]^2 \\ &= \alpha^2 G(u, v, z) + 2\alpha(1 - \alpha)(|u - v| + |v - z| + |u - z|)(|u - v| + |v - h| + |u - h|) \\ &\quad + (1 - \alpha)^2 G(u, v, h) \\ &\leq \alpha^2 G(u, v, z) + 2\alpha(1 - \alpha) \frac{G(u, v, z) + G(u, v, h)}{2} + (1 - \alpha)^2 G(u, v, h) \\ &= \alpha G(u, v, z) + (1 - \alpha)G(u, v, h) \end{aligned}$$

Therefore, (E, G, w) is a CRG_bMS with $s = 3$.

In what follows, we denote the set of all fixed points of T by $F(T)$, that is,

$$F(T) = \{u \in E : Tu = u\}.$$

The next Lemma is a partial extension of a result given in [27] from Banach spaces to rectangular G_b -metric spaces.

Lemma 4. Let (E, G, w) be a CRG_bMS and $T : E \rightarrow E$ be a mapping. Define the mapping T_α by

$$T_\alpha u = w(u, Tu; \alpha)$$

for all $u \in E$. Then, for any $\alpha \in [0, 1)$,

$$F(T) = F(T_\alpha).$$

Proof. If $\alpha = 0$, then

$$G(Tu, Tu, T_\alpha u) = G(Tu, Tu, w(u, Tu; 0)) \leq G(Tu, Tu, Tu) = 0$$

which implies that $Tu = T_\alpha u$. If $\alpha \in (0, 1)$, let $u^* \in F(T)$, that is, $u^* = Tu^*$ and we get

$$G(u^*, u^*, T_\alpha u^*) = G(u^*, u^*, w(u^*, Tu^*; \alpha)) \leq (1 - \alpha)G(u^*, u^*, Tu^*) = 0$$

which means $u^* \in F(T_\alpha u^*)$. Conversely, let $u^* \in F(T_\alpha)$, that is, $u^* = T_\alpha u^*$ and we obtain

$$G(Tu^*, Tu^*, u^*) = G(Tu^*, Tu^*, w(u^*, Tu^*; \alpha)) \leq \alpha G(Tu^*, Tu^*, u^*) = 0$$

which means $u^* \in F(Tu^*)$. $\square$

Let Θ be the set of non-decreasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for all $t \geq 0$. If $\varphi \in \Theta$, it follows from Berinde [28] that

- (1) $\varphi(0) = 0$;
- (2) $\varphi(t) < t$ for all $t > 0$.

Now we give the notion of enriched φ -contraction in a CRG_bMS .

Definition 13. Let (E, G, w) be a CRG_bMS with $s \geq 1$. A mapping $T : E \rightarrow E$ is said to be an enriched φ -contraction if there exists $\varphi \in \Theta$ and $\alpha \in [0, 1)$ such that

$$G(w(u, Tu; \alpha), w(v, Tv; \alpha), w(o, To; \alpha)) \leq \frac{1}{s^2} \varphi(G(u, v, o)) \quad (8)$$

for all $u, v, o \in E$.

Theorem 4. Let (E, G, w) be a CRG_bMS with $s \geq 1$ and let $T : E \rightarrow E$ be an enriched φ -contraction. Then

- (1) $F(T) = \{u^*\}$, for some $u^* \in E$;
- (2) the sequence $\{u_n\}$ obtained from the iterative process

$$u_{n+1} = w(u_n, Tu_n; \alpha_n)$$

converges to u^* , for any $u_0 \in E$.

Proof. Let us define the mapping $T_\alpha : E \rightarrow E$ by $T_\alpha u := w(u, Tu; \alpha)$. Then the inequality (8) can be rewritten as

$$G(T_\alpha u, T_\alpha v, T_\alpha o) \leq \frac{1}{s^2} \varphi(G(u, v, o)). \quad (9)$$

Let $u_0 \in E$ and $u_n = T_{\alpha_{n-1}} u_{n-1}$ where $T_{\alpha_n} u_n = w(u_n, Tu_n; \alpha_n)$ for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$, then u_n is a fixed point of T_α . So, suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then, it follows from inequality (9) that

$$\begin{aligned} G(u_n, u_{n+1}, u_{n+2}) &\leq \frac{1}{s^2} \varphi(G(u_{n-1}, u_n, u_{n+1})) \\ &\leq \varphi(G(u_{n-1}, u_n, u_{n+1})) \\ &\leq \dots \leq \varphi^n(G(u_0, u_1, u_2)), \end{aligned}$$

then we have

$$\lim_{n \rightarrow \infty} G(u_n, u_{n+1}, u_{n+2}) = 0.$$

Also, we can assume $u_n \neq u_{n+p}$ for any $p > 1$. Indeed, if $u_n = u_{n+p}$, then using the inequality (9), we have

$$\begin{aligned} G(u_n, u_n, u_{n+1}) &= G(u_{n+p}, u_{n+p}, u_{n+1}) \\ &\leq \frac{1}{s^2} \varphi(G(u_{n+p-1}, u_{n+p-1}, u_n)) \\ &\leq \varphi(G(u_{n+p-1}, u_{n+p-1}, u_{n+p})) \\ &\leq \varphi^2(G(u_{n+p-2}, u_{n+p-2}, u_{n+p-1})) \\ &\leq \dots \\ &\leq \varphi^p(G(u_n, u_n, u_{n+1})) \\ &< G(u_n, u_n, u_{n+1}), \end{aligned}$$

a contradiction. Notice that

$$\begin{aligned} G(u_n, u_{n+2}, u_{n+2}) &= G(T_{\alpha_{n-1}} u_{n-1}, T_{\alpha_{n+1}} u_{n+1}, T_{\alpha_{n+1}} u_{n+1}) \\ &\leq \frac{1}{s^2} \varphi(G(u_{n-1}, u_{n+1}, u_{n+1})) \\ &\leq \varphi(G(u_{n-1}, u_{n+1}, u_{n+1})) \\ &\leq \dots \leq \varphi^n(G(u_0, u_2, u_2)). \end{aligned}$$

It follows that

$$G(u_n, u_{n+2}, u_{n+2}) \leq \varphi^n(G(u_0, u_2, u_2)).$$

Similary, we can conclude that

$$G(u_n, u_n, u_{n+2}) \leq \varphi^n(G(u_0, u_0, u_2)).$$

Therefore, we obtain

$$\lim_{n \rightarrow \infty} G(u_n, u_n, u_{n+2}) = \lim_{n \rightarrow \infty} G(u_n, u_{n+2}, u_{n+2}) = 0.$$

Next, we will show that $\{u_n\}$ is a G -Cauchy sequence in E by contradiction. If $\{u_n\}$ is not a G -Cauchy sequence, then there exists $\varepsilon > 0$ and the subsequences $u_{n(k)}$, $u_{m(k)}$ and $u_{l(k)}$ of $\{u_n\}$ such for $n(k) > m(k) > l(k) > k$ with $G(u_{n(k)}, u_{m(k)}, u_{l(k)}) \geq \varepsilon$ and $G(u_{n(k)-1}, u_{m(k)}, u_{l(k)}) < \varepsilon$. On the one hand,

$$\begin{aligned} \varepsilon &\leq G(u_{n(k)}, u_{m(k)}, u_{l(k)}) \\ &\leq s[G(u_{m(k)}, u_{m(k)+2}, u_{m(k)+2}) + G(u_{m(k)+2}, u_{m(k)+1}, u_{m(k)+1}) \\ &\quad + G(u_{m(k)+1}, u_{n(k)}, u_{l(k)})] \\ &\leq s[G(u_{m(k)}, u_{m(k)+2}, u_{m(k)+2}) + G(u_{m(k)+2}, u_{m(k)+1}, u_{m(k)+1})] \\ &\quad + s^2[G(u_{l(k)}, u_{l(k)+2}, u_{l(k)+2}) + G(u_{l(k)+2}, u_{l(k)+1}, u_{l(k)+1}) + G(u_{l(k)+1}, u_{m(k)+1}, u_{n(k)})] \\ &\leq s[G(u_{m(k)}, u_{m(k)+2}, u_{m(k)+3}) + G(u_{m(k)+2}, u_{m(k)+1}, u_{m(k)})] + s^2[G(u_{l(k)}, u_{l(k)+1}, u_{l(k)+2}) \\ &\quad + G(u_{l(k)+2}, u_{l(k)+1}, u_{l(k)}) + G(u_{l(k)+1}, u_{m(k)+1}, u_{n(k)})], \end{aligned}$$

taking the limits as $k \rightarrow \infty$ in above inequality, we conclude

$$\frac{\varepsilon}{s^2} \leq \lim_{k \rightarrow \infty} G(u_{l(k)+1}, u_{m(k)+1}, u_{n(k)}).$$

On the other hand,

$$\begin{aligned} G(u_{l(k)+1}, u_{m(k)+1}, u_{n(k)}) &= G(T_{\alpha_{l(k)}} u_{l(k)}, T_{\alpha_{m(k)}} u_{m(k)}, T_{\alpha_{n(k)-1}} u_{n(k)-1}) \\ &\leq \frac{1}{s^2} \varphi(G(u_{l(k)}, u_{m(k)}, u_{n(k)-1})), \end{aligned}$$

let $k \rightarrow \infty$ in above inequality, we obtain

$$\frac{\varepsilon}{s^2} \leq \frac{1}{s^2} \varphi(\varepsilon) < \frac{\varepsilon}{s^2},$$

a contradiction. Thus, $\{u_n\}$ is a G -Cauchy sequence. By completeness of (E, d, w) , there exists $u^* \in E$ such that $\lim_{n \rightarrow \infty} u_n = u^*$. Let us show that u^* is a fixed point of T_α . Applying the rectangular inequality, we obtain that

$$\begin{aligned} G(u^*, T_\alpha u^*, T_\alpha u^*) &\leq s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + G(u_{n+1}, T_\alpha u^*, T_\alpha u^*)] \\ &= s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + G(T_\alpha u_n, T_\alpha u^*, T_\alpha u^*)] \\ &\leq s[G(u^*, u_n, u_n) + G(u_n, u_{n+1}, u_{n+1}) + \frac{1}{s^2} \varphi(G(u_n, u^*, u^*))], \end{aligned}$$

letting $n \rightarrow \infty$, we deduce $G(u^*, T_\alpha u^*, T_\alpha u^*) = 0$ which implies $u^* \in F(T_\alpha)$. Suppose that $u^*, v^* \in E$ are two distinct fixed points of T_α . That is, $G(u^*, u^*, v^*) > 0$. Then

$$G(u^*, u^*, v^*) = G(T_\alpha u^*, T_\alpha u^*, T_\alpha v^*) \leq \frac{1}{s^2} G(u^*, u^*, v^*) < G(u^*, u^*, v^*),$$

which is a contradiction. Therefore, we must have $G(u^*, u^*, v^*) = 0$, that is, $u^* = v^*$. Thus T_α has a unique fixed point in E . To finish the proof, we apply Lemma 4. $\square$

Taking $\varphi(t) = \lambda t$ in Theorem 4, we obtain the following result.

Corollary 3. Let (E, G, w) be a CRG_bMS with $s \geq 1$ and let $T : E \rightarrow E$ be an enriched contraction or (α, λ) -enriched contraction, that is, there exists $\alpha \in [0, 1)$ and $\lambda \in [0, \frac{1}{s^2})$ such that

$$G(w(u, Tu; \alpha), w(v, Tv; \alpha), w(o, To; \alpha)) \leq \lambda G(u, v, o) \quad (10)$$

for all $u, v, o \in E$. Then,

- (1) $F(T) = \{u^*\}$, for some $u^* \in E$;
- (2) the sequence $\{u_n\}$ obtained from the iterative process

$$u_{n+1} = w(u_n, Tu_n; \alpha_n)$$

converges to u^* , for any $u_0 \in E$.

The result of the local variant of enriched contractions in CRG_bMS as following.

Theorem 5. Let (E, G, w) be a CRG_bMS with $s \geq 1$ and $B = B_G(u_0, r)$, $u_0 \in E$, $r > 0$. Assume that $T : B \rightarrow E$ be an enriched contraction and $0 < \alpha < 1$. If

$$\max\{G(u_0, u_0, Tu_0), G(u_0, Tu_0, Tu_0)\} \leq (\frac{1}{2s} - \frac{\lambda}{2})\varepsilon,$$

then T has a fixed point in E .

Proof. We can choose $\varepsilon < r$ such that

$$\max\{G(u_0, u_0, Tu_0), G(u_0, Tu_0, Tu_0)\} \leq (\frac{1}{2s} - \frac{\lambda}{2})\varepsilon < (\frac{1}{2s} - \frac{\lambda}{2})r.$$

Since T is a enriched contraction, for all $u, v, o \in E$ and any $\alpha \in [0, 1)$, we can apply the contractive condition (10) with $T_\alpha u = w(u, Tu; \alpha)$, then we obtain

$$G(T_\alpha u, T_\alpha v, T_\alpha o) \leq \lambda G(u, v, o).$$

We shall prove that the closed ball

$$\bar{B}_G(u_0, \varepsilon) = \{u \in E : G(u_0, u, u) \leq \varepsilon\}$$

is invariant with respect to T_α . Suppose that u_0 is not a fixed point of T , that is, $u_0 \neq Tu_0$ and $u_0 \neq T_\alpha u_0$. Due to $\alpha > 0$, it is not difficult to see that $Tu_0 \neq T_\alpha u_0$, indeed, if not,

$$\begin{aligned} G(u_0, u_0, Tu_0) &= G(u_0, u_0, T_\alpha u_0) \\ &= G(u_0, u_0, w(u_0, Tu_0; \alpha)) \\ &\leq (1 - \alpha)G(u_0, u_0, Tu_0), \end{aligned}$$

a contradiction. For any $u \in \bar{B}_G(u_0, \varepsilon)$, we will consider the following three cases:

Case 1: if $Tu_0 = T_\alpha u$, we have

$$\begin{aligned} G(u_0, T_\alpha u, T_\alpha u) &= G(u_0, Tu_0, Tu_0) \\ &\leq G(u_0, Tu_0, T_\alpha u_0) \\ &= G(u_0, Tu_0, w(u_0, Tu_0; \alpha)) \\ &\leq \alpha G(u_0, u_0, Tu_0) + (1 - \alpha)G(u_0, Tu_0, Tu_0) \\ &\leq \left(\frac{1}{2s} - \frac{\lambda}{2}\right)\varepsilon < \varepsilon; \end{aligned}$$

Case 2: if $T_\alpha u_0 = T_\alpha u$, we have

$$\begin{aligned} G(u_0, T_\alpha u, T_\alpha u) &= G(u_0, T_\alpha u_0, T_\alpha u_0) \\ &\leq G(u_0, Tu_0, T_\alpha u_0) \\ &= G(u_0, Tu_0, w(u_0, Tu_0; \alpha)) \\ &\leq \alpha G(u_0, u_0, Tu_0) + (1 - \alpha)G(u_0, Tu_0, Tu_0) \\ &\leq \left(\frac{1}{2s} - \frac{\lambda}{2}\right)\varepsilon < \varepsilon; \end{aligned}$$

Case 3: if $Tu_0 \neq T_\alpha u$ and $T_\alpha u_0 \neq T_\alpha u$, we have

$$\begin{aligned} G(u_0, T_\alpha u, T_\alpha u) &\leq s[G(u_0, Tu_0, Tu_0) + G(Tu_0, T_\alpha u_0, T_\alpha u_0) + G(T_\alpha u_0, T_\alpha u, T_\alpha u)] \\ &\leq s[G(u_0, Tu_0, Tu_0) + G(Tu_0, u_0, T_\alpha u_0) + G(T_\alpha u_0, T_\alpha u, T_\alpha u)] \\ &= s[G(u_0, Tu_0, Tu_0) + G(Tu_0, u_0, w(u_0, Tu_0; \alpha)) + G(T_\alpha u_0, T_\alpha u, T_\alpha u)] \\ &\leq s[G(u_0, Tu_0, Tu_0) + \alpha G(Tu_0, u_0, u_0) + (1 - \alpha)G(Tu_0, u_0, Tu_0) \\ &\quad + \lambda G(u_0, u, u)] \\ &\leq s\left[\left(\frac{1}{s} - \lambda\right)\varepsilon + \lambda\varepsilon\right] = \varepsilon. \end{aligned}$$

Finally, by above cases, we prove that $G(u_0, T_\alpha u, T_\alpha u) \leq \varepsilon$, which implies $T_\alpha u \in \bar{B}_G(u_0, \varepsilon)$. Since $\bar{B}_G(u_0, \varepsilon)$ is complete, the result follows from Corollary 3. $\square$

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