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Cayley Graphs Defined by Systems of Equations

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Abstract: Let R be a finite ring. In this paper, we mainly explore the conditions to ensure the graph $B\Gamma_n$ defined by a system of equations $\{f_i \mid i = 2, \dots, n\}$ to be a Cayley graph or a Hamiltonian graph. More precisely, we prove that $B\Gamma_n$ is a Cayley graph with $G = \langle \phi, A \rangle$ a group of dihedral type if and only if the system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R . As an application, the well-known Lovász Conjecture, which states that any finite connected Cayley graph has a Hamilton cycle, holds for the connected $B\Gamma_n$ defined by Cayley graphic system $\mathcal{F}_n$ of dihedral type in the field $GF(p^k)$.

Keywords: Cayley graph; hamiltonian graph; dihedral type; Hamilton-laceable

MSC: 05C25; 05C45; 17B67

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1. Introduction

The graph $B\Gamma_n(R; f_2, \dots, f_n)$ defined by a system of equations over a finite ring R have been extensively studied and used since the 1990s. The construction of graph $B\Gamma_n(R; f_2, \dots, f_n)$ is motivated by the problems from extremal graph theory (see [1,2] for details), and the concrete construction was originally due to [3]. The construction of the graphs by systems of equations provides a useful tool to study graph theory, for instance, constructing the graphs $D(k, q)$ and $CD(k, q)$, to approximate the structure and behavior of incidence graphs of regular generalized polygons [4], establishing the lower bound of the Ramsey numbers of graphs associated to generalized Kac–Moody algebras of rank two [5] and hypergraphs [6], and constructing certain graphs with given properties [1,7,8]. In addition, Lazebnik and Woldar [3] established some important properties including regularity and bi-regularity, existence of special vertex colorings, and existence of covering maps of $B\Gamma_n(R; f_2, \dots, f_n)$. Additional properties of $B\Gamma_n(R; f_2, \dots, f_n)$ can be found in the survey [9] and the references therein. Moreover, the girth [8,10–12] and the classification up to isomorphisms [13] of graph $B\Gamma_n(R; f_2, \dots, f_n)$ are further considered recently.

It is natural to study other properties of graph $B\Gamma_n(R; f_2, \dots, f_n)$. For example, Lazebnik et al. [14] explored the condition to ensure the graphs to be semisymmetric or connected. They proved that $B\Gamma_n(R; p_1 l_1, \dots, p_1 l_i, \dots, p_1 l_{n-1})$ is semisymmetric if $R = GF(p^k)$ is a finite field for any $n \geq 3$ and $p \geq 3$. Moreover, some open problems of $B\Gamma_n(R; f_2, \dots, f_n)$ were raised up in [3] including when $B\Gamma_n(R; f_2, \dots, f_n)$ is a Cayley graph or Hamiltonian graph. Cayley graphs are those graphs whose vertices are identified with elements of groups and adjacency relations are defined by subsets of the groups.

The present paper aims to study the question of when $B\Gamma_n(R; f_2, \dots, f_n)$ is a Cayley graph. To be more precise, we consider a class of function systems and we call them Cayley graphic function systems of dihedral type, which ensures $B\Gamma_n$ to be a Cayley graph. The main theorem is as follows.

Theorem 1. *The graph $B\Gamma_n$ is a Cayley graph on some group $G = \langle \phi, A \rangle$ of dihedral type with $\phi : (p) \leftrightarrow [p] \in \text{Aut}(B\Gamma_n)$ and A uniform if and only if the function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type (see Definition 5) in R .*

Cayley graphs are widely studied together with Hamiltonian graphs (see [15,16] for example), where Hamiltonian graphs are those graphs having a cycle visiting every vertex exactly once. A variant of the well-known Lovász Conjecture on Hamiltonian paths states that every finite connected Cayley graph has a Hamilton cycle. Many people focus on discussing the special case of the above conjecture (see [15,17–20]). We pay attention to these results and then study the Hamiltonicity of graph $B\Gamma_n(R; f_2, \dots, f_n)$.

Furthermore, we consider the Hamiltonicity of this class of graphs and we have two following theorems; note that Theorem 3 supports the Lovász Conjecture.

Theorem 2. *Let R be a finite ring with $|R|$ an even positive integer, and $B\Gamma_n$ be a connected graph determined by the function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$, which is Cayley graphic of dihedral type. Then $B\Gamma_n$ is Hamilton-laceable, i.e., for any vertices u and v from two distinct parts in the bipartite graph $B\Gamma_n$, there is a Hamilton path whose terminal vertices are u and v .*

Theorem 3. *Let $R = GF(p^k)$ be a field, for some prime p . If $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R and $B\Gamma_n$ connected, then $B\Gamma_n$ has a Hamilton cycle.*

The paper is organized as follows: in Section 2, we recall the definition and some properties of the graph $B\Gamma_n(R; f_2, \dots, f_n)$. We prove our main theorem in Section 3, i.e., the function system $\{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R if and only if $B\Gamma_n$ is a Cayley graph with the group $G = \langle \phi, A \rangle$ is of dihedral type with A satisfying so-called uniform condition (see Definition 4). The last section of this paper mainly study the Hamiltonicity of graph $B\Gamma_n(R; f_2, \dots, f_n)$.

2. The Graphs Defined by Systems of Equations

In this section, we will recall the definition of the graph defined by systems of equations.

Throughout this paper, for a graph Γ , we always denote by $V(\Gamma)$ the vertex set and by $E(\Gamma)$ the edge set. The element of $E(\Gamma)$ connecting two vertices $u, v \in V(\Gamma)$ will be written as $u \sim v$.

Definition 1. ([3], Section 2.1) Let $f_i : R^{2i-2} \rightarrow R$ be an arbitrary function, $2 \leq i \leq n$. We define the bipartite graph $B\Gamma_n = B\Gamma_n(R; f_2, \dots, f_n)$ as follows. The set of vertices $V(B\Gamma_n)$ is the disjoint union of two copies of R^n , one denoted by P_n and the other by L_n . Elements of P_n will be called *points* and those of L_n *lines*. In order to distinguish points from lines we introduce the use of parentheses and brackets: if $a \in R^n$, then $(a) \in P_n$ and $[a] \in L_n$. We define edges of $B\Gamma_n$ by declaring point $(p) = (p_1, p_2, \dots, p_n)$ and line $[l] = [l_1, l_2, \dots, l_n]$ to be adjacent if and only if the following $n - 1$ relations on their coordinates hold:

$$\begin{aligned} p_2 + l_2 &= f_2(p_1, l_1), \\ p_3 + l_3 &= f_3(p_1, l_1, p_2, l_2), \\ &\dots \dots \\ p_n + l_n &= f_n(p_1, l_1, p_2, l_2, \dots, p_{n-1}, l_{n-1}). \end{aligned}$$

For a function $f_i : R^{2i-2} \rightarrow R$, we define $\bar{f}_i : R^{2i-2} \rightarrow R$ by the rule $\bar{f}_i(x_1, y_1, \dots, x_{i-1}, y_{i-1}) = f_i(y_1, x_1, \dots, y_{i-1}, x_{i-1})$. We call f_i symmetric if the functions f_i and $\bar{f}_i$ coincide.

Evidently, if the order of R is $|R| = r$, then the number of vertices of graph $B\Gamma_n$ is $2r^n$, i.e., $|V(B\Gamma_n)| = 2r^n$.

There is another fundamental family of graphs as follows if all functions are symmetric.

Definition 2. ([3], Section 2.2) Let $f_i : R^{2i-2} \rightarrow R$ be symmetric for all $2 \leq i \leq n$. We define $\Gamma_n = \Gamma(R; f_2, \dots, f_n)$ to be the graph with vertex set $V(\Gamma_n) = R^n$, where distinct vertices (vectors) $a = \langle a_1, a_2, \dots, a_n \rangle$ and $b = \langle b_1, b_2, \dots, b_n \rangle$ are adjacent if and only if the following $n - 1$ relations on their coordinates hold:

$$\begin{aligned} a_2 + b_2 &= f_2(a_1, b_1), \\ a_3 + b_3 &= f_3(a_1, b_1, a_2, b_2), \\ &\dots \dots \\ a_n + b_n &= f_n(a_1, b_1, a_2, b_2, \dots, a_{n-1}, b_{n-1}). \end{aligned}$$

Let $\Gamma = (V, E)$ be a graph. We say that Γ has a *loop* at vertex v provided that there is an edge $e \in E$ connecting v and itself. The following proposition can be deduced directly by the definition, see [3,9] for details and more properties of $B\Gamma_n$ and Γ_n .

Proposition 1. ([3], Proposition 2) If $\text{char} R = 2$, Γ_n has a loop at vertex $\langle a \rangle$ if and only if

$$f_i(a_1, a_1, \dots, a_{i-1}, a_{i-1}) = 0$$

for all $2 \leq i \leq n$.

Now we recall the definitions related to the automorphism group.

Definition 3. ([21], Section 15) An automorphism of a simple graph Γ is a bijection $\sigma : V(\Gamma) \rightarrow V(\Gamma)$ such that vertex u is adjacent to v is an edge of Γ if and only if $\sigma(u)$ is adjacent to $\sigma(v)$ is an edge of Γ .

The group consisting of all automorphisms of Γ is called the automorphism group of Γ , and we denote it by $\text{Aut}(\Gamma)$.

The following example appears in Section 3.5 of paper [9].

Example 1. For any $x \in R$, the map $\sigma_{n,x}$ is an automorphism of $B\Gamma_n$, where

$$\begin{aligned} \sigma_{n,x}((p)) &= (p_1, \dots, p_{n-1}, p_n + x), \\ \sigma_{n,x}([l]) &= [l_1, \dots, l_{n-1}, l_n - x], \end{aligned}$$

for any $(p) \in P_n$, $[l] \in L_n$.

For convenience, we denote by $(p)_k$ (resp. $[l]_k$) the k -th coordinate of (p) (resp. $[l]$), and by $(p(i), l(i))$ the sequence of variables $(p_1, l_1, \dots, p_{i-1}, l_{i-1})$ for any $i = 2, 3, \dots, n$, where $(p) = (p_1, \dots, p_n)$ and $[l] = [l_1, \dots, l_n]$.

Definition 4. An automorphism $\sigma \in \text{Aut}(B\Gamma_n)$ is *uniform* if

- (1) σ maps points to points and lines to lines, i.e., $\sigma((p)) = (p')$, $\sigma([l]) = [l']$;
- (2) $\sigma((p))_k = p_k + x$ and $\sigma((p))_j = p_j$ for all $(p) \in P_n$ if there is some $k \in \{1, 2, \dots, n\}$ such that $\sigma((0))_j = 0$ for all $1 \leq j < k$, and $\sigma((0))_k = x \in R$;
- (3) for all (p) , $[l]$ and $k < i \leq n$, we have

$$f_i(p'(i), l'(i)) - (p')_i - [l']_i = f_i(p(i), l(i)) - (p)_i - [l]_i, \quad (1)$$

where $(p') = \sigma((p))$, $[l'] = \sigma([l])$.

Moreover, a subset $S \subseteq \text{Aut}(B\Gamma_n)$ is called to be *uniform* if S consists of uniform automorphisms.

Remark 1. In the above definition,

- (1) for the second condition, k and x are not necessary to be unique and we can take the identity automorphism for example;
- (2) the third condition can be seen as a generalization of automorphism of graph $B\Gamma_n$, as any automorphism σ of graph $B\Gamma_n$ satisfies that both the left side and the right side of Equation (1) equal to 0 for two adjacent points (p) and $[l]$.

Now we introduce Cayley graphic functions of dihedral type.

Definition 5. We call the function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ to be *Cayley graphic of dihedral type in R* , if there is a uniform subset $X = \{\tau_{k,x} \mid x \in R, k = 1, \dots, n\} \subseteq \text{Aut}(B\Gamma_n)$ and $\phi : (p) \leftrightarrow [p]$ lies in $\text{Aut}(B\Gamma_n)$, such that

- (1) $\tau_{k,x}((p))_k = p_k + x$ and $\tau_{k,x}([l])_k = l_k - x$;
- (2) $\tau_{k,x}((p))_j = p_j$ and $\tau_{k,x}([l])_j = l_j$ for all $1 \leq j < k$;
- (3) $\tau_{k,-x}([p])_j = \tau_{k,x}(\phi([p]))_j$ for all $k + 1 \leq j \leq n$.

We have a property for Cayley graphic functions of dihedral type in R .

Proposition 2. Let the function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ be Cayley graphic of dihedral type in R . Then there is a uniform subset $X' = \{\sigma_{k,x} \mid x \in R, k = 1, \dots, n\} \subseteq \text{Aut}(B\Gamma_n)$, such that

- (1) $\sigma_{k,x}((p))_j = \tau_{k,x}((p))_j$ and $\sigma_{k,x}([l])_j = \tau_{k,x}([l])_j$ for all $j \leq k$;
- (2) $\sigma_{k,-x}([p])_j = \sigma_{k,x}((p))_j$ for all $k + 1 \leq j \leq n$;
- (3) when $j > k$, the sum of all odd functions $h(x)$ (i.e. $h(x) = -h(-x)$) in the additive terms of $\sigma_{k,x}((p))_j$ is 0.

Proof of Proposition 2. Evidently, $\tau_{k,x}\tau_{k',z}$ is uniform if $\tau_{k,x}$ and $\tau_{k',z}$ are uniform (by a straightforward check). When $j > k$, if the sum of all odd functions in the additive terms of $\sigma_{k,x}((p))_j$ is $h(x)$, which is nonzero, we replace the automorphism $\tau_{k,x}$ with $\sigma_{k,x} = \tau_{j,-h(x)}\tau_{k,x}$. Repeating the replacement finitely many steps, we obtain the automorphisms as required. $\square$

In what follows, for Cayley graphic function system of dihedral type, we always take uniform automorphisms $X' = \{\sigma_{k,x} \mid x \in R, k = 1, \dots, n\} \subseteq \text{Aut}(B\Gamma_n)$ in the previous proposition.

We further explain the definition and introduce the functions $b_{i,k}$ and $y_{i,k}$, associated to $\{f_i \mid i = 2, \dots, n\}$.

Remark 2. By Proposition 1 in [3], if $\{f_i \mid i = 2, \dots, n\}$ are symmetric then $\phi : (p) \leftrightarrow [p]$ is an automorphism. Moreover, $(p) \sim [l]$ if and only if $\sigma_{k,x}((p)) \sim \sigma_{k,x}([l])$ for $\sigma_{k,x} \in X'$. Now we set

$$b_{i,k}(p_1, \dots, p_{i-1}, x) := \sigma_{k,x}((p))_i - p_i,$$

$$y_{i,k}(l_1, \dots, l_{i-1}, x) := \sigma_{k,x}([l])_i - l_i,$$

for all $i = k + 1, \dots, n$. Let $(p')_m = \sigma_{k,x}((p))_m$ and $(l')_m = \sigma_{k,x}([l])_m$ for all $1 \leq m \leq n$. Since $X' = \{\sigma_{k,x} \mid x \in R, k = 1, \dots, n\}$ is uniform, by Definition 4 (3) follows

$$b_{i,k}(p_1, \dots, p_{i-1}, x) + y_{i,k}(l_1, \dots, l_{i-1}, x) = f_i(p'(i), l'(i)) - f_i(p(i), l(i))$$

for all $(p) \in P_n$ and $[l] \in L_n$.

For convenience, we write $b_{i,k}$ and $y_{i,k}$ for short unless stated otherwise. Note that the existence of functions $b_{i,k}$ and $y_{i,k}$ is equivalent to the existence of $\sigma_{k,x}$.

Example 2. If $n = 2$, $f_2(p_1, l_1) = p_1 + l_1$, then

$$b_{2,1} + y_{2,1} = f_2(p_1 + x, l_1 - x) - f_2(p_1, l_1) = 0.$$

Set $b_{2,1} = y_{2,1} = 0$. It is straightforward to check that $f_2(p_1, l_1) = p_1 + l_1$ is Cayley graphic of dihedral type in R .

By Definition 5, Proposition 2, and Remark 2, we obtain the uniqueness of $b_{i,k}$ and $y_{i,k}$ for Cayley graphic functions of dihedral type in R .

Proposition 3. If $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R , then (in terms of the above notations)

- (1) the functions $b_{i,k}$ and $y_{i,k}$ are unique;
- (2) we have $b_{i,k}(p_1, \dots, p_{i-1}, 0) = y_{i,k}(l_1, \dots, l_{i-1}, 0) = 0$;
- (3) for the case $n = 2$, the functions $b_{2,1}$ and $y_{2,1}$ are additive, i.e., if $b'_{2,1}$ and $y'_{2,1}$ are functions associated to f'_2 in another function system $\mathcal{F}'_2$ that is Cayley graphic of dihedral type, then $b'_{2,1} + b_{2,1}$ and $y'_{2,1} + y_{2,1}$ are functions associated to $f'_2 + f_2$;
- (4) the system $\mathcal{F}_j = \{f_i \mid i = 2, \dots, j\}$ is Cayley graphic of dihedral type for $j \leq n$.

Proof of Proposition 3. (1) Let us assume the opposite and let $b_{i,k}, b'_{i,k}$ and $y_{i,k}, y'_{i,k}$ be the functions associated to $\{f_i \mid i = 2, \dots, n\}$, respectively. Now we set

$$\begin{aligned}\bar{b}_{i,k} &= b'_{i,k} - b_{i,k}, \\ \bar{y}_{i,k} &= y'_{i,k} - y_{i,k}.\end{aligned}$$

By Remark 2,

$$\begin{aligned}\bar{b}_{k+1,k}(p_1, \dots, p_k, x) &= \bar{y}_{k+1,k}(p_1, \dots, p_k, -x), \\ \bar{b}_{k+1,k}(p_1, \dots, p_k, x) + \bar{y}_{k+1,k}(l_1, \dots, l_k, x) &= 0,\end{aligned}$$

for all $(p) \in P_n$ and $[l] \in L_n$. We have

$$\bar{y}_{k+1,k}(p_1, \dots, p_k, -x) + \bar{y}_{k+1,k}(l_1, \dots, l_k, x) = 0.$$

Then there is an odd function $h(x)$ such that

$$\begin{aligned}\bar{b}_{k+1,k} &= b'_{k+1,k} - b_{k+1,k} = h(x), \\ \bar{y}_{k+1,k} &= y'_{k+1,k} - y_{k+1,k} = h(-x).\end{aligned}$$

By Proposition 2 (3), we have

$$\begin{aligned}b'_{k+1,k} &= b_{k+1,k}, \\ y'_{k+1,k} &= y_{k+1,k}.\end{aligned}$$

Now we prove the uniqueness of $b_{i,k}$ and $y_{i,k}$ by induction on i . Suppose that it holds for all $k + 1 \leq j \leq i - 1$, i.e., $b'_{j,k} = b_{j,k}$, $y'_{j,k} = y_{j,k}$, it suffice to prove it for the case i . Similarly, we have

$$\begin{aligned}\bar{b}_{i,k}(p_1, \dots, p_{i-1}, x) &= \bar{y}_{i,k}(p_1, \dots, p_{i-1}, -x), \\ b_{i,k} + y_{i,k} &= b'_{i,k} + y'_{i,k} = f_i(p'(i), l'(i)) - f_i(p(i), l(i)),\end{aligned}$$

where $(p') = \sigma_{k,x}((p))$, $[l'] = \sigma_{k,x}([l])$. Then

$$\begin{aligned}\bar{b}_{i,k}(p_1, \dots, p_{i-1}, x) &= \bar{y}_{i,k}(p_1, \dots, p_{i-1}, -x), \\ \bar{b}_{i,k}(p_1, \dots, p_{i-1}, x) + \bar{y}_{i,k}(l_1, \dots, l_{i-1}, x) &= 0.\end{aligned}$$

Then, there is an odd function $h(x)$ such that

$$\begin{aligned}\bar{b}_{i,k} &= b'_{i,k} - b_{i,k} = h(x), \\ \bar{y}_{i,k} &= y'_{i,k} - y_{i,k} = h(-x).\end{aligned}$$

By Proposition 2 (3), we have

$$\begin{aligned}b'_{i,k} &= b_{i,k}, \\ y'_{i,k} &= y_{i,k}.\end{aligned}$$

(2) Since

$$\begin{aligned}b_{k+1,k}(p_1, \dots, p_k, 0) &= y_{k+1,k}(p_1, \dots, p_k, 0), \\ b_{k+1,k}(p_1, \dots, p_k, 0) + y_{k+1,k}(l_1, \dots, l_k, 0) &= 0,\end{aligned}$$

we have $2b_{k+1,k}(p_1, \dots, p_k, 0) = 0$. If $b_{k+1,k}(p_1, \dots, p_k, 0) \neq 0$, we let

$$b'_{k+1,k}(p_1, \dots, p_k, x) = b_{k+1,k}(p_1, \dots, p_k, x) + b_{k+1,k}(p_1, \dots, p_k, 0),$$

thus $b_{k+1,k}(p_1, \dots, p_k, 0) = 0$ by (1).

We have

$$\begin{aligned}b_{i,k}(p_1, \dots, p_{i-1}, 0) &= y_{k+1,k}(p_1, \dots, p_{i-1}, 0), \\ b_{i,k}(p_1, \dots, p_{i-1}, 0) + y_{k+1,k}(l_1, \dots, l_{i-1}, 0) &= 0,\end{aligned}$$

then $2b_{i,k}(p_1, \dots, p_{i-1}, 0) = 0$. If $b_{i,k}(p_1, \dots, p_{i-1}, 0) \neq 0$, we let

$$b'_{i,k}(p_1, \dots, p_{i-1}, x) = b_{i,k}(p_1, \dots, p_{i-1}, x) + b_{i,k}(p_1, \dots, p_{i-1}, 0).$$

By induction on i and (1), we have $b_{i,k}(p_1, \dots, p_{i-1}, 0) = y_{i,k}(p_1, \dots, p_{i-1}, 0) = 0$.

(3) We have

$$\begin{aligned}b_{2,1} + y_{2,1} &= f_2(p_1 + x, l_1 - x) - f_2(p_1, l_1), \\ b'_{2,1} + y'_{2,1} &= f'_2(p_1 + x, l_1 - x) - f'_2(p_1, l_1).\end{aligned}$$

Sum them up, and we have

$$(b'_{2,1} + b_{2,1}) + (y'_{2,1} + y_{2,1}) = (f'_2 + f_2)(p_1 + x, l_1 - x) - (f'_2 + f_2)(p_1, l_1).$$

By (1), then $b'_{2,1} + b_{2,1}$ and $y'_{2,1} + y_{2,1}$ are functions associated to Cayley graph of dihedral type $f'_2 + f_2$.

(4) The unique functions $b_{i,k}$ and $y_{i,k}$ are functions associated to Cayley graph function system $\{f_i \mid i = 2, \dots, j\}$ of dihedral type, for $j \leq n$. $\square$

3. A Condition for Graphs $B\Gamma_n$ to be Cayley Graphs

In this section, we provide a condition to ensure a graph $B\Gamma_n$ to be a Cayley graph. We first recall the definition of Cayley graph.

Definition 6. ([16], Section 1) Let G be a group and S be a subset of G . A digraph $\Gamma = \text{Cay}(G, S)$ is a *Cayley graph* with vertex set G and two vertices $u, v \in G$ are adjacent if and only if $vu^{-1} \in S$. Moreover, if for any element $s \in S$, the inverse s^{-1} also lies in S , then the adjacency is symmetric and thus the *Cayley graph* $\text{Cay}(G, S)$ may be viewed as an undirected graph.

Obviously, the subset S contains the identity of G if and only if the Cayley graph $\text{Cay}(G, S)$ contains a loop at every vertex.

Definition 7. Let Γ be a graph and H be a subgroup of $\text{Aut}(\Gamma)$. H is regular on $V(\Gamma)$ if

- (1) H is transitive on set $V(\Gamma)$,
- (2) if $\sigma \in H$ satisfies that $\sigma(v) = v$ for any $v \in V(\Gamma)$, then σ is the identity automorphism.

The following properties provides us a criterion to judge whether a graph is a Cayley graph, and we give a complete proof here for the convenience of readers.

Proposition 4. ([21], Lemma 16.3) ([16], Proposition 1.1)) A graph Γ is a Cayley graph of a group G if and only if $\text{Aut}(\Gamma)$ contains a subgroup isomorphic to G that is regular on $V(\Gamma)$.

Proof of Proposition 4. Supposed that Γ is a Cayley graph $\text{Cay}(G, S)$. For any $g \in G$, we define the assignment

$$\bar{g} : u \mapsto ug^{-1}, \forall u \in V(\Gamma).$$

It is clear that $\bar{g}$ is a permutation on $V(\Gamma)$. Note that $v\bar{g}^{-1}(u\bar{g}^{-1})^{-1} \in S$ if and only if $vu^{-1} \in S$ and thus $\bar{g} \in \text{Aut}(\Gamma)$. Let $\bar{G} = \{\bar{g} \mid g \in G\}$, then $\bar{G} \leq \text{Aut}(\Gamma)$ and $\bar{G} \cong G$. For any two vertices $u, v \in V(\Gamma)$, there exists a unique element $\bar{g} = \bar{v}^{-1}\bar{u} \in \bar{G}$ maps u to v , and thus $\bar{G}$ acts transitively on $V(\Gamma)$. The uniqueness of $\bar{g}$ implies that $\bar{G}$ is a regular subgroup of $\text{Aut}(\Gamma)$.

Conversely, we assume that G is a subgroup of $\text{Aut}(\Gamma)$ and regular on $V(\Gamma)$. We will show that Γ is a Cayley graph of G . Let $V(\Gamma) = \{v_i \mid i = 1, \dots, n\}$, then G has a unique element g_i such that $g_i(v_1) = v_i$ since G is regular. Set

$$S = \{g_j \in G \mid v_1 \text{ is adjacent to } v_j\}.$$

Now we define a map $f : \Gamma \rightarrow \text{Cay}(G, S)$ sending v_i to g_i . Then v_j is adjacent to v_k in Γ if and only if $g_j(v_1)$ is adjacent to $g_k(v_1)$, if and only if v_1 is adjacent to $g_j^{-1}g_k(v_1)$, if and only if $g_j^{-1}g_k \in S$, and, finally, if and only if $g_j = f(v_j)$ is adjacent to $g_k = f(v_k)$ in $\text{Cay}(G, S)$. So f is an isomorphism. $\square$

Let $\mathbb{F}_q$ be a finite field and q be the power of odd prime numbers. By Definition 4 of uniform isomorphism in Section 2, it is straightforward to check that $B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2)$ without uniform automorphism (except identity automorphism) but ϕ is an automorphism.

Lemma 1. Let $\mathbb{F}_q$ be a finite field and q be the power of odd prime numbers. Then $B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2)$ is not a Cayley graph.

Proof of Lemma 1. For any $x \in \mathbb{F}_q$, we consider the 3-path in $B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2)$: $(1, 0) \sim [x, x^2] \sim (-1, 0)$. There are at least q paths between two points $(1, 0)$ and $(-1, 0)$. Suppose $B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2)$ is a Cayley graph, then it is a vertex transitive graph by Proposition 4. Then there is $\sigma \in \text{Aut}(B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2))$ such that $\sigma((1, 0)) = (0, 0)$. We get 3-path: $(0, 0) \sim [y, 0] \sim (p_1, p_2)$, where $(p_1, p_2) = \sigma((-1, 0))$ and $p_1 \neq 0$. Then $y^2 = \frac{p_2^2}{p_1^2}$ and the number of 3-paths between $(0, 0)$ and (p_1, p_2) is strictly less than q , which contradicts that σ is an automorphism. $\square$

We recall the definition of group of dihedral type.

Definition 8. ([15], Definition 2.21) A group G is of *dihedral type* if it has an abelian subgroup A of index 2, and an element $\pi \in G \setminus A$ of order 2, such that $\pi g \pi = g^{-1}$ for all $g \in A$.

In this case, we write $G = \langle \pi, A \rangle$. Note that if A is the cyclic subgroup of order n , then G is precisely the dihedral group D_{2n} .

Lemma 2. Let $|R| = r$. If $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R , then the automorphism group of $B\Gamma_n(R; f_2, \dots, f_n)$ admits a subgroup is of dihedral type with $2r^n$ elements.

Proof of Lemma 2. For the graph $B\Gamma_n$, we have the automorphism $\phi : (p) \leftrightarrow [p]$, and the following automorphisms

$$\sigma_{k,x} : B\Gamma_n \longrightarrow B\Gamma_n$$

$$\begin{aligned} (p_1, \dots, p_k, \dots, p_n) &\mapsto (p_1, \dots, p_k + x, p_{k+1} + b_{k+1,k}, \dots, p_n + b_{n,k}), \\ [l_1, \dots, l_k, \dots, l_n] &\mapsto [l_1, \dots, l_k - x, l_{k+1} + y_{k+1,k}, \dots, l_n + y_{n,k}], \end{aligned}$$

where $x \in R$ and $k = 1, \dots, n$.

Next, we will prove that $A = \langle \sigma_{k,x} \rangle$ is an abelian group. We first show the commutativity of these generators, i.e., $\sigma_{k,x}\sigma_{k',z} = \sigma_{k',z}\sigma_{k,x}$ for any $k \leq k'$ and $x, z \in R$.

Let $(p) = (p_1, \dots, p_n)$ and $[l] = [l_1, \dots, l_n]$. We denote $b_{k,k',i} + p_i, y_{k,k',i} + l_i, b_{k',k,i} + p_i, y_{k',k,i} + l_i$ for the i -th component of $\sigma_{k,x}\sigma_{k',z}((p)), \sigma_{k,x}\sigma_{k',z}([l]), \sigma_{k',z}\sigma_{k,x}((p)), \sigma_{k',z}\sigma_{k,x}([l])$ respectively, for $1 \leq i \leq n$.

It is straightforward to verify that $b_{k,k',i} = b_{k',k,i}$ and $y_{k,k',i} = y_{k',k,i}$ for $i \leq k'$. For the case $k' < i \leq n$, suppose that

$$\begin{aligned} F &= (p_1, l_1, \dots, p_{k-1}, l_{k-1}, p_k + x, l_k - x), \\ G &= (p_{k+1} + b_{k+1,k}, l_{k+1} + y_{k+1,k}, \dots, p_{k'-1} + b_{k'-1,k}, l_{k'-1} + y_{k'-1,k}), \\ H &= (p_{k'+1} + b_{k,k',k'+1}, l_{k'+1} + y_{k,k',k'+1}, \dots, p_{i-1} + b_{k,k',i-1}, l_{i-1} + y_{k,k',i-1}), \\ I &= (p_{k'} + b_{k',k}, l_{k'} + y_{k',k}, p_{k'+1} + b_{k'+1,k}, l_{k'+1} + y_{k'+1,k}, \dots, p_{i-1} + b_{i-1,k}, l_{i-1} + y_{i-1,k}). \end{aligned}$$

By induction on i , we have

$$\begin{aligned} b_{k',k,i} - b_{i,k} + y_{k',k,i} - y_{i,k} &= b_{k,k',i} - b_{i,k} + y_{k,k',i} - y_{i,k}, \\ &= f_i(F, G, p_{k'} + b_{k',k} + z, l_{k'} + y_{k',k} - z, H) - f_i(F, G, I). \end{aligned}$$

Let the sum of all odd functions in the additive terms of $b_{k',k,i}$ be $h(x, z)(h(x, z) = -h(-x, -z))$. If $h(x, z) \neq 0$, by Definition 5 and Proposition 2, we have $\sigma_{k,x}((p))_j = \sigma_{k,-x}([p])_j$ for all $1 \leq j \leq n$. Moreover, set $p_m = l_m$ for all $1 \leq m \leq n$, and

$$\begin{aligned} O_b(p_1, \dots, p_k + x, p_{k+1} + b_{k+1,k}, \dots, p_{i-1} + b_{i-1,k}, z) &= b_{k',k,i} - b_{i,k} - h(x, z), \\ O_y(l_1, \dots, l_k - x, l_{k+1} + y_{k+1,k}, \dots, l_{i-1} + y_{i-1,k}, z) &= y_{k',k,i} - y_{i,k} + h(x, z). \end{aligned}$$

We have $O_b + O_y = b_{k',k,i} - b_{i,k} + y_{k',k,i} - y_{i,k}$, which contradicts Proposition 3 (1) where

$$(F, G, I)_{2j-1} = (F, G, I)_{2j} \text{ for all } j = 1, \dots, i-1.$$

We have $h(x, z) = 0$. Similarly, let the sum of all odd functions in the additive terms of $b_{k,k',i}$ be $h'(x, z)$, then $h'(x, z) = 0$. With a similar discussion as in the proof of Proposition 3, we have $\sigma_{k,x}\sigma_{k',z} = \sigma_{k',z}\sigma_{k,x}$ for any $x, z \in R$.

By Proposition 3 (1)(2), we have $\sigma_{k,0} = \text{id}$ and $\sigma_{k,x}^{-1} = \sigma_{k,-x}$. By a straightforward check, we have $\phi\sigma_{k,x}\phi = \sigma_{k,-x}$, and $A = \langle \sigma_{k,x} \rangle$ is an abelian group, and thus $G = \langle \phi, A \rangle$ is of dihedral type. Moreover, it is clear that $|\phi| = 2$, and for any $1 \leq k \leq n$, $\{\sigma_{k,x} \mid x \in R\}$ has r elements. By the commutativity and $\sigma_{k,x}\sigma_{k,y} = \sigma_{k,x+y}$, the abelian group A has r^n automorphisms and thus $G = \langle \phi, A \rangle$ is a subgroup of $\text{Aut}(B\Gamma_n)$ is of dihedral type with $2r^n$ elements. $\square$

Now we are ready to prove our main theorem.

Theorem 4. The function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R if and only if $B\Gamma_n$ is a Cayley graph $\text{Cay}(G, S)$ on some group $G = \langle \phi, A \rangle$ of dihedral type with $\phi : (p) \leftrightarrow [p]$ lies in $\text{Aut}(B\Gamma_n)$ and A uniform.

Proof of Theorem 1. By Lemma 2 and Proposition 4, we can verify that G is regular on $V(B\Gamma_n)$. We have $B\Gamma_n(R; f_2, \dots, f_n)$ is a Cayley graph $\text{Cay}(G, S)$ where $G = \langle \phi, A \rangle$ is of dihedral type and A is uniform.

If $B\Gamma_n$ is a Cayley graph $\text{Cay}(G, S)$, by Definition 4, then there is $\sigma_{k,x} \in A$, such that

$$\sigma_{k,x}((0))_k = x \text{ and } \sigma_{k,x}((0))_j = 0 \text{ for all } 1 \leq j < k \leq n,$$

$$\sigma_{k,x}((p))_k = p_k + x \text{ and } \sigma_{k,x}((p))_j = p_j \text{ for all } (p) \in P_n.$$

Since $\phi\sigma_{k,x}\phi = \sigma_{k,x}^{-1}$, we have

$$\sigma_{k,x}([p])_k = p_k - x \text{ and } \sigma_{k,x}([p])_j = p_j \text{ for all } [p] \in L_n.$$

By Proposition 2 and 3 (1), we have $\sigma_{k,x}^{-1} = \sigma_{k,-x}$ and then the theorem follows. $\square$

By Lemma 1, there is a family of non-Cayley graphs $B\Gamma_2(\mathbb{F}_q; p_1^2 l_1^2)$ with q the power of odd prime number that have no uniform automorphism (except identity automorphism) but satisfy $\phi \in \text{Aut}(B\Gamma_2)$; therefore the above conditions in Definition 5 are necessary. By the previous theorem, we have two corollaries as follows.

Corollary 1. The graph $B\Gamma_n(R; f_2, \dots, f_n)$ is a Cayley graph in the following cases

- (1) $f_i = \sum_{s=1}^{i-1} a_s(p_s + l_s)^{b_s} + c$, where the constants $a_s, c \in R$ and integers b_s ;
- (2) $n = 2$ and $f_2 = cp_1 l_1$ with constants $c, 2^{-1} \in R$, where R is a commutative ring;
- (3) $n = 2$ and $f_2 = c(p_1 l_1)^{kp}$ with constant $c \in R$, where R is a commutative ring of characteristic p with kp odd and $2^{-1} \in R$;
- (4) $n = 2$ and $f_2 = a_1 p_1 l_1 + b_1 (p_1 + l_1)^{c_1} + c$ with constants $a_1, b_1, c, 2^{-1} \in R$, where R is a commutative ring and c_1 is an integer;
- (5) $n = 2$ and $f_2 = a_1 (p_1 l_1)^{kp} + b_1 (p_1 + l_1)^{c_1} + c$ with constants $a_1, b_1, c, 2^{-1} \in R$, where R is a commutative ring of characteristic p with kp odd and c_1 an integer;
- (6) $f_n = \sum_{s=1}^{n-1} b_s(p_s l_s) + \sum_{s=1}^{n-1} c_s(p_s + l_s)^{d_s} + \sum_{s=1}^{n-1} e_s(p_s l_s)^{mp} + c$, and $f_i = \sum_{s=1}^{i-1} a_{i,s}(p_s + l_s)^{b_{i,s}} + d_i$ for $i = 2, \dots, n-1$, where R is a commutative ring of characteristic p with mp odd, the constants $a_{i,s}, b_s, c_s, e_s, c, d_i, 2^{-1} \in R$, and $b_{i,s}, d_s$ integers.

Proof of Corollary 1. (1) Take $b_{i,k} = 0, y_{i,k} = 0$ and the statement follows.

(2) Define

$$b_{2,1} = c(-p_1 x - \frac{1}{2}x^2), y_{2,1} = c(l_1 x - \frac{1}{2}x^2).$$

Then $\mathcal{F}_2$ is Cayley graphic of dihedral type, and $B\Gamma_2$ is a Cayley graph.

(3) Define

$$b_{2,1} = c(-(p_1 x)^{kp} - \frac{1}{2}x^{2kp}), y_{2,1} = c((l_1 x)^{kp} - \frac{1}{2}x^{2kp}).$$

Then $\mathcal{F}_2$ is Cayley graphic of dihedral type, and $B\Gamma_2$ is a Cayley graph.

(4) and (5) follow by (1), (2), (3), and Proposition 3 (3).

(6) Set

$$b_{n,k} = \sum_{s=1}^{n-1} b_s(-p_s x - \frac{1}{2}x^2) + \sum_{s=1}^{n-1} e_s(-(p_s x)^{mp} - \frac{1}{2}x^{2mp}),$$

$$y_{n,k} = \sum_{s=1}^{n-1} b_s(l_s x - \frac{1}{2}x^2) + \sum_{s=1}^{n-1} e_s((l_s x)^{mp} - \frac{1}{2}x^{2mp}),$$

and $b_{i,k} = y_{i,k} = 0$ for $k < i < n$. Then $B\Gamma_n$ is a Cayley graph. $\square$

Corollary 2. Let R be a finite ring of characteristic 2 and $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ be a Cayley graphic function system of dihedral type in R . If Γ_n has a loop at each vertex or no-loop at all vertices, then Γ_n is a Cayley graph $\text{Cay}(A, S)$ with A an abelian group.

Proof of Corollary 2. By Definition 5 and Proposition 2, $b_{i,k} = y_{i,k}$ associated to $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R .

For any $x \in R$ and $k = 1, \dots, n$, we define

$$\begin{aligned} \sigma_{k,x} : \Gamma_n &\longrightarrow \Gamma_n \\ \langle a_1, \dots, a_k, \dots, a_n \rangle &\longmapsto \langle a_1, \dots, a_k + x, a_{k+1} + b_{k+1,k}, \dots, a_n + b_{n,k} \rangle. \end{aligned}$$

The proof is similar to Lemma 2 and Theorem 4. $\square$

We conclude this section with two concrete examples.

Example 3. Assume $n = 2$ and the function system $f_2 = p_1 + l_1$, we set

$$\begin{aligned} \phi : (p_1, p_2) &\mapsto [p_1, p_2] \\ [l_1, l_2] &\mapsto (l_1, l_2) \\ \sigma_{2,x} : (p_1, p_2) &\mapsto (p_1, p_2 + x) \\ [l_1, l_2] &\mapsto [l_1, l_2 - x] \\ \sigma_{1,x} : (p_1, p_2) &\mapsto (p_1 + x, p_2) \\ [l_1, l_2] &\mapsto [l_1 - x, l_2] \end{aligned}$$

Then for any $i, j = 1$ or 2 and $x \in R$, we have

$$\sigma_{i,x}\sigma_{j,y} = \sigma_{j,y}\sigma_{i,x}, \quad \phi\sigma_{i,x}\phi = \sigma_{i,-x}.$$

Thus $B\Gamma_2(R; p_1 + l_1)$ is a Cayley graph.

If $\text{char} R = 2$, by Proposition 1, Γ_2 has loop at all vertices, we have $\Gamma_2(R; p_1 + l_1)$ is a Cayley graph.

Example 4. Let $n = 2$, $f_2 = p_1 l_1$ and R be a commutative ring with $2^{-1} \in R$, take

$$\begin{aligned} \phi : (p_1, p_2) &\mapsto [p_1, p_2] \\ [l_1, l_2] &\mapsto (l_1, l_2) \\ \sigma_{2,x} : (p_1, p_2) &\mapsto (p_1, p_2 + x) \\ [l_1, l_2] &\mapsto [l_1, l_2 - x] \\ \sigma_{1,x} : (p_1, p_2) &\mapsto (p_1 + x, p_2 - p_1 x - \frac{1}{2}x^2) \\ [l_1, l_2] &\mapsto [l_1 - x, l_2 + l_1 x - \frac{1}{2}x^2] \end{aligned}$$

For any $i, j = 1$ or 2 and $x \in R$, we have

$$\sigma_{i,x}\sigma_{j,y} = \sigma_{j,y}\sigma_{i,x}, \quad \phi\sigma_{i,x}\phi = \sigma_{i,-x}.$$

Then $B\Gamma_2(R; p_1 l_1)$ is a Cayley graph.

4. The Hamiltonicity of Graph $B\Gamma_n$

Based on the previous section, this section is devoted to considering the Hamiltonicity of graph $B\Gamma_n$.

We recall two definitions related to the Hamiltonicity of graphs.

Definition 9. ([22], Section 4.2) A path that contains every vertex of graph Γ is called a *Hamilton path* of Γ ; similarly, a *Hamilton cycle* of Γ is a cycle that contains every vertex of Γ . A graph is a Hamiltonian graph or Hamiltonian if it contains a Hamilton cycle.

Definition 10. ([23,24]) A bipartite graph with parts A and B is *Hamilton-laceable* if, for all vertices $u \in A$ and $v \in B$, there is a Hamilton path whose terminal vertices are u and v .

The following two lemmas, originally from [17,25], study the group on which the Cayley graph is Hamiltonian.

Lemma 3. ([17], Corollary 2) If G is a abelian group, then every connected Cayley graph on G has a Hamilton cycle.

Lemma 4. ([25], Theorem 6.1) If G is p -group, then every connected Cayley graph on G has a Hamilton cycle.

As a generalization of Lemma 4, the proof of the following lemma can be reduced to Lemma 4.

Lemma 5. ([19], Corollary 3.3) Let G be a finite group with S a subset of G . If, moreover,

- (1) S is a generating set of G ,
- (2) A is a normal p -subgroup of G , for some prime p , and
- (3) $st^{-1} \in A$, for all $s, t \in S$,

then $\text{Cay}(G, S)$ has a Hamilton cycle.

The following lemma is useful to our discussion on the Hamilton-laceability of the graph.

Lemma 6. ([23], Theorem 1.7) Let G be a finite group of dihedral type and Cayley graph Γ on group G be connected bipartite. If the order of G is divisible by 4 and Γ of valency at least 3, then Γ is Hamilton-laceable.

Based on our discussion in the previous section, we establish two Hamiltonian properties about graph $B\Gamma_n$ as follows.

Theorem 5. Let R be a finite ring with $|R|$ an even positive integer, and $B\Gamma_n$ be a connected graph determined by the function system $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$, which is Cayley graphic of dihedral type. Then $B\Gamma_n$ is Hamilton-laceable.

Proof of Theorem 2. If $r = |R| = 2$, the graph $B\Gamma_n$ is a cycle since $B\Gamma_n$ is a 2-regular graph and connected. For other cases of r , by Lemma 2 and Theorem 4, the graph $B\Gamma_n$ is Cayley graph on group G of dihedral type with order $2r^n$. Since $r = |R| > 3$ is an even integer, the order of G can be divided by 4 and the valency of $B\Gamma_n$ is at least 3. Thus $B\Gamma_n$ is Hamilton-laceable by Lemma 6. $\square$

Theorem 6. Let $R = GF(p^k)$ be a field, for some prime p . If $\mathcal{F}_n = \{f_i \mid i = 2, \dots, n\}$ is Cayley graphic of dihedral type in R and $B\Gamma_n$ is connected, then $B\Gamma_n$ has a Hamilton cycle.

Proof of Theorem 3. Note that the graph $B\Gamma_n$ is Cayley graph on group G of dihedral type by Theorem 4. To prove the theorem, it suffices to check the conditions in Lemma 5 one by one.

- (1) Note that S is a generating set of G if and only if $\text{Cay}(G, S)$ is connected.
- (2) By Lemma 2, we have $B\Gamma_n = \text{Cay}(G, S)$, $G = \langle \phi, A \rangle$ is of dihedral type, and $|A| = r^n = p^{kn}$. Hence, A is a normal p -subgroup of G .
- (3) By Proposition 4 and $B\Gamma_n = \text{Cay}(G, S)$ is a bipartite graph. According to the structure of S and G , we have the element in S is $s = a\phi$ or ϕa , where $a \in A$. Hence, $st^{-1} \in A$ for all $s, t \in S$. $\square$

A variant of the well-known Lovász Conjecture on Hamiltonian paths states that every finite connected Cayley graph has a Hamilton cycle. Note that both Theorem 6 and the following two corollaries support this conjecture.

The following example shows that $B\Gamma_2(GF(2); p_1 l_1)$ is a Hamiltonian graph and Cayley graph $\text{Cay}(A, S)$ where A is an abelian group.

Example 5. Let $R = GF(2)$ be a field. Then $B\Gamma_2(R; p_1 l_1)$ is a Cayley graph and has a Hamilton cycle (see Figure 1).

Figure 1. The graph $B\Gamma_2(R; p_1 l_1)$.

Note that $B\Gamma_2(\mathbb{Z}_2; p_1 l_1)$ is precisely a Cayley graph $\text{Cay}(A, S)$ with A is a cyclic group generated by the permutation $(12345678) \in S_8$ and $S = \{(12345678), (12345678)^{-1}\}$.

By Theorem 2.1 (3) in [14], if $R = GF(q)$ is a finite field with q a prime power, then $B\Gamma_2(R; p_1 l_1)$ is connected. The following corollary implies that graph $B\Gamma_2(R; p_1 l_1)$ is Hamiltonian where q is an odd prime power, which supports the conjecture that Wenger graph has pancyclicity in [26].

Corollary 3. Let p be an odd prime and $R = GF(p^k)$ be a finite field. Then graphs $B\Gamma_2(R; p_1 l_1)$ and $B\Gamma_2(GF(2); p_1 l_1)$ have a Hamilton cycle.

Proof of Corollary 3. The graph $B\Gamma_2(GF(2); p_1 l_1)$ has a Hamilton cycle by Figure 1. If p is an odd prime, by Corollary 1, $B\Gamma_2(R; p_1 l_1)$ is a Cayley graph. To prove the connectivity, we need to find out the connection set S of Cayley graph $B\Gamma_2(R; p_1 l_1) \cong \text{Cay}(G, S)$. Indeed, we have $\phi\sigma_{2, \frac{x^2}{2}}\sigma_{1,x} \in S$ since $(0,0) \sim [x,0]$ for all $x \in GF(p^k)$. Take $x = 0$ and we have $\phi \in \langle S \rangle$. Therefore σ_{2,x^2} and $\sigma_{2,-x^2}$ lie in $\langle S \rangle$, which implies $\sigma_{2, -\frac{x^2}{2}} \in \langle S \rangle$ and $\sigma_{1,x} \in \langle S \rangle$ for all $x \in GF(p^k)$. Since the equation $x^2 + y^2 = z$ has a solution (x,y) for any z in $GF(p^k)$, the automorphism $\sigma_{2,z}$ can be written as the composition of σ_{2,x^2} and σ_{2,y^2} in $\langle S \rangle$, thus $\sigma_{2,z} \in \langle S \rangle$, which implies $G = \langle S \rangle$, and then $B\Gamma_2(R; p_1 l_1)$ is connected, so $B\Gamma_2(R; p_1 l_1)$ has a Hamilton cycle by Theorem 6. $\square$

For the graph Γ_n , we have the following corollary.

Corollary 4. Let R be a finite ring of characteristic 2 and Γ_n be a connected graph. If $\mathcal{F}_n$ is Cayley graph of dihedral type and Γ_n has loop or no-loop at all vertices, then Γ_n has a Hamilton cycle.

Proof of Corollary 4. It follows directly by Corollary 2 and Lemma 3. $\square$

5. Conclusions

In Theorem 4 of this paper, under the conditions (see Definition 5) proposed in this paper, we obtain the necessary and sufficient condition that the graph $B\Gamma_n$ is Cayley graph on some group $G = \langle \phi, A \rangle$ of dihedral type with $\phi : (p) \leftrightarrow [p]$ lies in $\text{Aut}(B\Gamma_n)$ and A uniform. Moreover in this case, we discuss the Hamiltonicity of $B\Gamma_n$. However, it has not been completely described when $B\Gamma_n$ is a Cayley graph or Hamiltonian graph. It will be interesting to consider the pancyclicity of a Wenger graph for our future work (see [26]).

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