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Post-Quantum Midpoint-Type Inequalities Associated with Twice-Differentiable Functions

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Abstract: In this study, first we establish a (p,q)-integral identity involving the second (p,q)-derivative, and then, we use this result to prove some new midpoint-type inequalities for twice-(p,q)-differentiable convex functions. It is also shown that the newly established results are the refinements of the comparable results in the literature.

Keywords: Hermite–Hadamard inequality; (p,q)-calculus; convex functions

Citation: Sitthiwirattham, T.; Murtaza, G.; Ali, M.A.; Promsakon, C.; Sial, I.B.; Agarwal, P. Post-Quantum Midpoint-Type Inequalities Associated with Twice-Differentiable Functions . *Axioms* **2022**, *11*, 46. https://doi.org/10.3390/axioms11020046

Academic Editor: Christophe Chesneau

Received: 23 December 2021
Accepted: 15 January 2022
Published: 23 January 2022

1. Introduction

In convex functions theory, the Hermite–Hadamard (HH) inequality is very important, which was discovered by C. Hermite and J. Hadamard independently (see [1] (p. 137)).

$$\Pi\left(\frac{\pi_1+\pi_2}{2}\right) \leq \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \leq \frac{\Pi(\pi_1)+\Pi(\pi_2)}{2} \tag{1}$$

where Π is a convex function. In the case of concave mappings, the above inequality is satisfied in reverse order. For more recent refinements of Inequality (1), one can consult [2,3].

On the other hand, several works in the field of q-analysis are being carried out, beginning with Euler, in order to achieve mastery of the mathematics that drives quantum computing. The link between physics and mathematics is referred to as q-calculus. It has a wide range of applications in mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, relativity theory, and quantum theory [4,5]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [6,7]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton's work on infinite series, he used the q-parameter. Jackson [8,9] was the first to present the q-calculus, who knew calculus without limits in a methodical manner. In 1966, Al-Salam [10] introduced a q-analogue of the q-fractional integral and q-Riemann–Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, Tariboon introduced the ${}_{\pi_1}D_q$-difference operator and ${}_{q\pi_1}$-integral

in [11]. In 2020, Bermudo et al. introduced the notion of the ${}^{\pi_2}D_q$-derivative and q^{π_2}-integral in [12]. Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or simply (p,q)-calculus in [13]. Soontharanon et al. [14] studied the concepts of fractional (p,q)-calculus later on. In [15], Tunç and Göv gave the post-quantum variant of the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Recently, in 2021, Chu et al. introduced the notions of the ${}^{\pi_2}D_{p,q}$-derivative and $(p,q)^{\pi_2}$-integral in [16].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [17–22], the authors used ${}_{\pi_1}D_q$, ${}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals to prove HH integral inequalities and their left-right estimates for convex and coordinated convex functions. In [23], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [24]. Khan et al. proved the quantum HH inequality using the Green function in [25]. Budak et al. [26], Ali et al. [27], and Vivas-Cortez et al. [28] developed new quantum Simpson and quantum Newton-type inequalities for convex and coordinated convex functions. For quantum Ostrowski inequalities for convex and coordinated convex functions, one can consult [29,30]. Kunt et al. [31] generalized the results of [19] and proved HH-type inequalities and their left estimates using the ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, Latif et al. [32] found the right estimates of the HH-type inequalities proven by Kunt et al. [31]. To prove Ostrowski's inequalities, Chu et al. [16] used the concepts of the ${}^{\pi_2}D_{p,q}$-difference operator and $(p,q)^{\pi_2}$-integral. Recently, Vivas-Cortez et al. [33] generalized the results of [12] and proved the HH-type inequalities and their left estimates using the ${}^{\pi_2}D_{p,q}$-difference operator and $(p,q)^{\pi_2}$-integral.

Inspired by the ongoing studies, we use the (p,q)-integral to develop some new post-quantum midpoint-type inequalities for (p,q)-differentiable convex functions. We also show that the newly developed inequalities are extensions of some previously known inequalities.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of q-calculus, as well as other related studies in this field. In Section 3, we go over the concepts of (p,q)-calculus, as well as some recent research in this field. The midpoint-type inequalities for twice (p,q)-differentiable functions via (p,q)-integrals are described in Section 4. The relationship between the findings reported here and similar findings in the literature is also taken into account. Section 5 concludes with some recommendations for future research.

2. Quantum Derivatives and Integrals

In this portion, we recall a few known definitions in q-calculus. Throughout the paper, we use real numbers p and q such that $0 < q < p \leq 1$. Set the following notation (see [5]):

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \\ (1-s)_q^n &= (s,q)_n = \prod_{i=0}^{n-1}\left(1 - q^i s\right). \end{aligned}$$

The q-Jackson integral of a mapping Π from zero to π_2 was given by Jackson [9], which is defined as:

$$\int_0^{\pi_2} \Pi(\varkappa)\ d_q\varkappa\ = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n) \tag{2}$$

provided that the sum converges absolutely. Moreover, over the interval $[\pi_1, \pi_2]$, he gave the following integral of a mapping Π:

$$\int_{\pi_1}^{\pi_2} \Pi(\varkappa)\ d_q\varkappa\ = \int_0^{\pi_2} \Pi(\varkappa)\ d_q\varkappa\ - \int_0^{\pi_1} \Pi(\varkappa)\ d_q\varkappa\ .$$

Definition 1 ([11]). *The q_{π_1}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is defined as:*

$$_{\pi_1}D_q\Pi(\varkappa)\ = \frac{\Pi(\varkappa) - \Pi(q\varkappa + (1-q)\pi_1)}{(1-q)(\varkappa - \pi_1)},\ \varkappa \neq \pi_1. \tag{3}$$

For $\varkappa = \pi_1$, we state $_{\pi_1}D_q\Pi(\pi_1) = \lim_{\varkappa\to\pi_1}\ _{\pi_1}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 2 ([12]). *The q^{π_2}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$^{\pi_2}D_q\Pi(\varkappa)\ = \frac{\Pi(q\varkappa + (1-q)\pi_2) - \Pi(\varkappa)}{(1-q)(\pi_2 - \varkappa)},\ \varkappa \neq \pi_2. \tag{4}$$

For $\varkappa = \pi_2$, we state $^{\pi_2}D_q\Pi(\pi_2) = \lim_{\varkappa\to\pi_2}\ ^{\pi_2}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 3 ([11]). *The q_{π_1}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is defined as:*

$$\int_{\pi_1}^{\varkappa} \Pi(\tau)\ _{\pi_1}d_q\tau\ = (1-q)(\varkappa - \pi_1)\sum_{n=0}^{\infty} q^n\Pi(q^n\varkappa + (1-q^n)\pi_1),\ \varkappa \in [\pi_1, \pi_2]. \tag{5}$$

On the other side, the following concept of the q-definite integral was stated by Bermudo et al. [12]:

Definition 4 ([12]). *The q^{π_2}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is given as:*

$$\int_{\varkappa}^{\pi_2} \Pi(\tau)\ ^{\pi_2}d_q\tau\ = (1-q)(\pi_2 - \varkappa)\sum_{n=0}^{\infty} q^n\Pi(q^n\varkappa + (1-q^n)\pi_2),\ \varkappa \in [\pi_1, \pi_2]. \tag{6}$$

3. Post-Quantum Derivatives and Integrals

In this section, we review some fundamental notions and notations of (p, q)-calculus. We set the notation (see [13]):

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

The $[n]_{p,q}!$, $\left[\begin{array}{c} n \\ k \end{array}\right]!$ and $(1-s)_{p,q}^n$ are called the (p, q)-factorial, (p, q)-binomial and (p, q)-power, respectively, and are expressed as (see [13]):

$$\begin{aligned} [n]_{p,q}! &= \prod_{k=1}^{n}[k]_{p,q},\ n \geq 1,\ [0]_{p,q}! = 1, \\ \left[\begin{array}{c} n \\ k \end{array}\right]! &= \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} \end{aligned}$$

and:

$$(1-s)_{p,q}^{n}=\prod_{i=0}^{n-1}\left(p^{i}-q^{i}s\right).$$

Definition 5 ([13]). *The (p,q)-derivative of mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$$D_{p,q}\Pi(\varkappa)=\frac{\Pi(p\varkappa)-\Pi(q\varkappa)}{(p-q)\varkappa},\ \varkappa\neq 0. \tag{7}$$

Definition 6 ([15]). *The $(p,q)_{\pi_1}$-derivative of mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$${}_{\pi_1}D_{p,q}\Pi(\varkappa)=\frac{\Pi(p\varkappa+(1-p)\pi_1)-\Pi(q\varkappa+(1-q)\pi_1)}{(p-q)(\varkappa-\pi_1)},\ \varkappa\neq\pi_1. \tag{8}$$

For $\varkappa=\pi_1$, we state ${}_{\pi_1}D_{p,q}\Pi(\pi_1)=\lim_{\varkappa\to\pi_1}\ {}_{\pi_1}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Definition 7 ([16]). *The $(p,q)^{\pi_2}$-derivative of mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ is given as:*

$${}^{\pi_2}D_{p,q}\Pi(\varkappa)=\frac{\Pi(q\varkappa+(1-q)\pi_2)-\Pi(p\varkappa+(1-p)\pi_2)}{(p-q)(\pi_2-\varkappa)},\ \varkappa\neq\pi_2. \tag{9}$$

For $\varkappa=\pi_2$, we state ${}^{\pi_2}D_{p,q}\Pi(\pi_2)=\lim_{\varkappa\to\pi_2}\ {}^{\pi_2}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Remark 1. *It is clear that if we use $p=1$ in (8) and (9), then the equalities (8) and (9) reduce to (3) and (4), respectively.*

Definition 8 ([13]). *The definite (p,q)-integral of mapping Π on $[0,\pi_2]$ is stated as:*

$$\int_0^{\pi_2}\Pi(\tau)\,d_{p,q}\tau=(p-q)\pi_2\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}\pi_2\right). \tag{10}$$

Moreover, the definite (p,q)-integral of mapping Π on $[\pi_1,\pi_2]$ is stated as:

$$\int_{\pi_1}^{\pi_2}\Pi(\tau)\ d_{p,q}\tau\ =\int_0^{\pi_2}\Pi(\tau)\ d_{p,q}\tau\ -\int_0^{\pi_1}\Pi(\tau)\ d_{p,q}\tau\ .$$

Definition 9 ([15]). *The definite $(p,q)_{\pi_1}$-integral of mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{\pi_1}^{\varkappa}\Pi(\tau)\ {}_{\pi_1}d_{p,q}\tau=(p-q)(\varkappa-\pi_1)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}\varkappa+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_1\right). \tag{11}$$

Definition 10 ([16]). *The definite $(p,q)^{\pi_2}$-integral of mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$ on $[\pi_1,\pi_2]$ is stated as:*

$$\int_{\varkappa}^{\pi_2}\Pi(\tau)\ {}^{\pi_2}d_{p,q}\tau=(p-q)(\pi_2-\varkappa)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}\varkappa+\left(1-\frac{q^n}{p^{n+1}}\right)\pi_2\right). \tag{12}$$

Remark 2. *It is evident that if we pick $p=1$ in (11) and (12), then the equalities (11) and (12) change into (5) and (6), respectively.*

Remark 3. *If we take $\pi_1=0$ and $\varkappa=\pi_2=1$ in (11), then we have:*

$$\int_0^1\Pi(\tau)\ {}_0d_{p,q}\tau=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n+1}}\right).$$

Similarly, by taking $\varkappa = \pi_1 = 0$ *and* $\pi_2 = 1$ *in (12), then we obtain that:*

$$\int_0^1 \Pi(\tau)\, {}^1d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(1-\frac{q^n}{p^{n+1}}\right).$$

Lemma 1 ([33]). *We have the following equalities:*

$$\int_{\pi_1}^{\pi_2}(\pi_2-\varkappa)^{\underline{\alpha}}\, {}^{\pi_2}d_{p,q}\varkappa = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(\varkappa-\pi_1)^{\underline{\alpha}}\, {}_{\pi_1}d_{p,q}\varkappa = \frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

where $\alpha \in \mathbb{R}-\{-1\}$.

4. Post-Quantum Midpoint-Type Inequalities

In this section, we prove some new midpoint-type inequalities for twice-(p,q)-differentiable mappings via the (p,q)-integral.

Let us begin with the following lemma.

Lemma 2. *Let* $\Pi : I = [\pi_1, \pi_2] \to R$ *be a* (p,q)*-differentiable function on the interior of* I*. If* ${}^{\pi_2}D^2_{p,q}\Pi$ *is continuous and integrable on* I*, then the following equality holds:*

$$\begin{aligned}
&\frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^{\frac{1}{[2]_{p,q}}}\left(p^2q^3t^2-(1-pqt)^2_{p,q}+p(1-qt)^2_{p,q}\right){}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t\right. \qquad (13)\\
&\left.+\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)^2_{p,q}\,{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t\right]\\
=\;&\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right).
\end{aligned}$$

Proof. Consider from Definition 7:

$$\begin{aligned}
&{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)\\
=\;&{}^{\pi_2}D_{p,q}\left[\frac{\Pi(qt\pi_1+(1-qt)\pi_2)-\Pi(pt\pi_1+(1-pt)\pi_2)}{(p-q)(\pi_2-\pi_1)t}\right]\\
=\;&\frac{p\Pi\left(q^2t\pi_1+(1-q^2t)\pi_2\right)-([2]_{p,q})\Pi(pqt\pi_1+(1-pqt)\pi_2)+q\Pi\left(p^2t\pi_1+(1-p^2t)\pi_2\right)}{pq(p-q)^2(\pi_2-\pi_1)^2t^2}.
\end{aligned}$$

From Definition 8, we have:

$$\begin{aligned}
&\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)^2_{p,q}\,{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t\\
&+\int_0^{\frac{1}{[2]_{p,q}}}\left(p^2q^3t^2-(1-pqt)^2_{p,q}+p(1-qt)^2_{p,q}\right){}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 p(1-qt)_{p,q}^2 \, {}^{\pi_2}D_{p,q}^2\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t \\
&\quad + \int_0^{\frac{1}{[2]_{p,q}}} (p^2q^3t^2-(1-pqt)_{p,q}^2) \, {}^{\pi_2}D_{p,q}^2\Pi(t\pi_1+(1-t)\pi_2)d_{p,q}t \\
&= \frac{1}{q(p-q)^2(\pi_2-\pi_1)^2} \\
&\quad \times \int_0^1 \frac{(1-qt)_{p,q}^2}{t^2} \begin{pmatrix} p\Pi(q^2t\pi_1+(1-q^2t)\pi_2)-[2]_{p,q}\Pi(pqt\pi_1+(1-pqt)\pi_2) \\ +q\Pi(p^2t\pi_1+(1-p^2t)\pi_2) \end{pmatrix} d_{p,q}t \\
&\quad + \frac{1}{pq(p-q)^2(\pi_2-\pi_1)^2} \int_0^{\frac{1}{[2]_{p,q}}} \frac{(p^2q^3t^2-(1-pqt)_{p,q}^2)}{t^2} \\
&\quad \times \begin{pmatrix} p\Pi(q^2t\pi_1+(1-q^2t)\pi_2)-[2]_{p,q}\Pi(pqt\pi_1+(1-pqt)\pi_2) \\ +q\Pi(p^2t\pi_1+(1-p^2t)\pi_2) \end{pmatrix} d_{p,q}t \\
&= \frac{1}{q(p-q)^2(\pi_2-\pi_1)^2}\left(I_1+\frac{1}{p}I_2\right).
\end{aligned}
$$

We calculate the integrals I_1 and I_2 by using the Definition 10. Let us consider:

$$
\begin{aligned}
&I_1 \\
&= \int_0^1 \frac{(1-qt)_{p,q}^2}{t^2} \begin{pmatrix} p\Pi(q^2t\pi_1+(1-q^2t)\pi_2)-[2]_{p,q}\Pi(pqt\pi_1+(1-pqt)\pi_2) \\ +q\Pi(p^2t\pi_1+(1-p^2t)\pi_2) \end{pmatrix} d_{p,q}t \\
&= (p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{(1-q\frac{q^n}{p^{n+1}})_{p,q}^2}{\frac{q^{2n}}{p^{2(n+1)}}} \left[p\Pi\left(q^2\frac{q^n}{p^{n+1}}\pi_1+(1-q^2\frac{q^n}{p^{n+1}})\pi_2\right) \right. \\
&\quad -[2]_{p,q}\Pi\left(pq\frac{q^n}{p^{n+1}}\pi_1+(1-pq\frac{q^n}{p^{n+1}})\pi_2\right) \\
&\quad \left. +q\Pi\left(p^2\frac{q^n}{p^{n+1}}\pi_1+(1-p^2\frac{q^n}{p^{n+1}})\pi_2\right)\right] \\
&= (p-q)\sum_{n=0}^{\infty} \frac{(1-q\frac{q^n}{p^{n+1}})_{p,q}^2}{\frac{q^n}{p^{n+1}}} \left[p\Pi\left(\frac{q^{n+2}}{p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{p^{n+1}})\pi_2\right) \right. \\
&\quad -[2]_{p,q}\Pi\left(\frac{q^{n+1}}{p^n}\pi_1+(1-\frac{q^{n+1}}{p^n})\pi_2\right) \\
&\quad \left. +q\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right] \\
&= (p-q)\left[p\sum_{n=0}^{\infty} \frac{(1-q\frac{q^n}{p^{n+1}})_{p,q}^2}{\frac{q^n}{p^{n+1}}} \Pi\left(\frac{q^{n+2}}{p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{p^{n+1}})\pi_2\right) \right. \\
&\quad -[2]_{p,q}\sum_{n=0}^{\infty} \frac{(1-q\frac{q^n}{p^{n+1}})_{p,q}^2}{\frac{q^n}{p^{n+1}}} \Pi\left(\frac{q^{n+1}}{p^n}\pi_1+(1-\frac{q^{n+1}}{p^n})\pi_2\right) \\
&\quad \left. +q\sum_{n=0}^{\infty} \frac{(1-q\frac{q^n}{p^{n+1}})_{p,q}^2}{\frac{q^n}{p^{n+1}}} \Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (p-q)\left[p\sum_{n=2}^{\infty}\frac{(1-q\frac{q^{n-2}}{p^{n-1}})^2_{p,q}}{\frac{q^{n-2}}{p^{n-1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right.\\
&\quad -[2]_{p,q}\sum_{n=1}^{\infty}\frac{(1-q\frac{q^{n-1}}{p^n})^2_{p,q}}{\frac{q^{n-1}}{p^n}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\\
&\quad \left.+q\sum_{n=0}^{\infty}\frac{(1-q\frac{q^n}{p^{n+1}})^2_{p,q}}{\frac{q^n}{p^{n+1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right]\\
&= (p-q)\left[p\sum_{n=0}^{\infty}\frac{(1-q\frac{q^{n-2}}{p^{n-1}})^2_{p,q}}{\frac{q^{n-2}}{p^{n-1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right.\\
&\quad -p\frac{(1-q\frac{q^{-2}}{p^{-1}})^2_{p,q}}{\frac{q^{-2}}{p^{-1}}}\Pi(p\pi_1+(1-p)\pi_2)-p\frac{(1-qq^{-1})^2_{p,q}}{q^{-1}}\Pi(q\pi_1+(1-q)\pi_2)\\
&\quad -[2]_{p,q}\sum_{n=0}^{\infty}\frac{(1-q\frac{q^{n-1}}{p^n})^2_{p,q}}{\frac{q^{n-1}}{p^n}}\Pi\left(\frac{q^n}{p^{n-1}}+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\\
&\quad +[2]_{p,q}\frac{(1-qq^{-1})^2_{p,q}}{q^{-1}}\Pi(p\pi_1+(1-p)\pi_2)\\
&\quad \left.+q\sum_{n=0}^{\infty}\frac{(1-q\frac{q^n}{p^{n+1}})^2_{p,q}}{\frac{q^n}{p^{n+1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right]\\
&= (p-q)\left[p\sum_{n=0}^{\infty}\frac{(1-q\frac{q^{n-2}}{p^{n-1}})^2_{p,q}}{\frac{q^{n-2}}{p^{n-1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right.\\
&\quad -[2]_{p,q}\sum_{n=0}^{\infty}\frac{(1-q\frac{q^{n-1}}{p^n})^2_{p,q}}{\frac{q^{n-1}}{p^n}}\Pi\left(\frac{q^n}{p^{n-1}}+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\\
&\quad \left.+q\sum_{n=0}^{\infty}\frac{(1-q\frac{q^n}{p^{n+1}})^2_{p,q}}{\frac{q^n}{p^{n+1}}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right]\\
&= (p-q)\sum_{n=0}^{\infty}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\\
&\quad \times\left(p\frac{(1-q\frac{q^{n-2}}{p^{n-1}})^2_{p,q}}{\frac{q^{n-2}}{p^{n-1}}}-[2]_{p,q}\frac{(1-q\frac{q^{n-1}}{p^n})^2_{p,q}}{\frac{q^{n-1}}{p^n}}+q\frac{(1-q\frac{q^n}{p^{n+1}})^2_{p,q}}{\frac{q^n}{p^{n+1}}}\right)\\
&= q(p-q)(p-q)^2[2]_{p,q}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^n}{p^{n-1}}\pi_1+(1-\frac{q^n}{p^{n-1}})\pi_2\right)\right]\\
&= \frac{q(p-q)^2[2]_{p,q}}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa
\end{aligned}
$$

Now, consider:

$$
\begin{aligned}
I_2 &= \int_0^{\frac{1}{[2]_{p,q}}} \frac{(p^2q^3t^2-(1-pqt)^2_{p,q})}{t^2} \begin{pmatrix} p\Pi(q^2t\pi_1+(1-q^2t)\pi_2) \\ -[2]_{p,q}\Pi(pqt\pi_1+(1-pqt)\pi_2) \\ +q\Pi(p^2t\pi_1+(1-p^2t)\pi_2) \end{pmatrix} d_{p,q}t \\
&= \frac{(p-q)}{[2]_{p,q}} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{\left(p^2q^3\left(\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2-(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}})^2_{p,q}\right)}{\left(\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2} \\
&\quad \times \left(p\Pi\left(q^2\frac{q^n}{[2]_{p,q}p^{n+1}}\pi_1+(1-q^2\frac{q^n}{[2]_{p,q}p^{n+1}})\pi_2\right)\right. \\
&\quad -[2]_{p,q}\Pi\left(pq\frac{q^n}{[2]_{p,q}p^{n+1}}\pi_1+(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}})\pi_2\right)\Big) \\
&\quad +q\Pi\left(p^2\frac{q^n}{[2]_{p,q}p^{n+1}}\pi_1+(1-p^2\frac{q^n}{[2]_{p,q}p^{n+1}})\pi_2\right) \\
&= \frac{(p-q)}{[2]_{p,q}} \sum_{n=0}^{\infty} \frac{\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]^2_{p,q}\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2_{p,q}\right)}{\frac{q^n}{p^{n+1}}} \\
&\quad \times \left(p\Pi\left(\frac{q^{n+2}}{[2]_{p,q}p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{[2]_{p,q}p^{n+1}})\pi_2\right)\right. \\
&\quad -[2]_{p,q}\Pi\left(\frac{q^{n+1}}{[2]_{p,q}p^{n}}\pi_1+(1-\frac{q^{n+1}}{[2]_{p,q}p^{n}})\pi_2\right) \\
&\quad \left.+q\Pi\left(\frac{q^{n}}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^{n}}{[2]_{p,q}p^{n-1}})\pi_2\right)\right) \\
&= \frac{(p-q)}{[2]_{p,q}} \left[\sum_{n=0}^{\infty} \frac{p\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]^2_{p,q}\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2_{p,q}\right)}{\frac{q^n}{p^{n+1}}}\right. \\
&\quad \times\Pi\left(\frac{q^{n+2}}{[2]_{p,q}p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{[2]_{p,q}p^{n+1}})\pi_2\right) \\
&\quad -\sum_{n=0}^{\infty} \frac{[2]_{p,q}\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]^2_{p,q}\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2_{p,q}\right)}{\frac{q^n}{p^{n+1}}} \\
&\quad \times\Pi\left(\frac{q^{n+1}}{[2]_{p,q}p^{n}}\pi_1+(1-\frac{q^{n+1}}{[2]_{p,q}p^{n}})\pi_2\right) \\
&\quad +\sum_{n=0}^{\infty} \frac{q\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]^2_{p,q}\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)^2_{p,q}\right)}{\frac{q^n}{p^{n+1}}} \\
&\quad \left.\times\Pi\left(\frac{q^{n}}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^{n}}{[2]_{p,q}p^{n-1}})\pi_2\right)\right] \\
&= \frac{(p-q)}{[2]_{p,q}} \left[\sum_{n=0}^{\infty} \frac{p\left(p^2q^3\left(\frac{q^{n-2}}{p^{n-1}}\right)^2-[2]^2_{p,q}\left(1-pq\frac{q^{n-2}}{[2]_{p,q}p^{n-1}}\right)^2_{p,q}\right)}{\frac{q^{n-2}}{p^{n-1}}}\right. \\
&\quad \left.\times\Pi\left(\frac{q^{n}}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^{n}}{[2]_{p,q}p^{n-1}})\pi_2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{p\left(p^2q^3\left(\frac{q^{-2}}{p^{-1}}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{-2}}{[2]_{p,q}p^{-1}}\right)_{p,q}^2\right)}{\frac{q^{-2}}{p^{-1}}}\\
&\times\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\\
&-\frac{p\left(p^2q^3\left(q^{-1}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{-1}}{[2]_{p,q}}\right)_{p,q}^2\right)}{q^{-1}}\\
&\times\Pi\left(\frac{q}{[2]_{p,q}}\pi_1+(1-\frac{q}{[2]_{p,q}})\pi_2\right)\\
&-\sum_{n=0}^{\infty}\frac{[2]_{p,q}\left(p^2q^3\left(\frac{q^{n-1}}{p^n}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{n-1}}{[2]_{p,q}p^n}\right)_{p,q}^2\right)}{\frac{q^{n-1}}{p^n}}\\
&\times\Pi\left(\frac{q^n}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^n}{[2]_{p,q}p^{n-1}})\pi_2\right)\\
&+\frac{[2]_{p,q}\left(p^2q^3\left(q^{-1}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{-1}}{[2]_{p,q}}\right)_{p,q}^2\right)}{q^{-1}}\\
&\times\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\\
&+\sum_{n=0}^{\infty}\frac{q\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)_{p,q}^2\right)}{\frac{q^n}{p^{n+1}}}\\
&\times\Pi\left(\frac{q^n}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^n}{[2]_{p,q}p^{n-1}})\pi_2\right)\Bigg]\\
=\;&\frac{(p-q)}{[2]_{p,q}}\sum_{n=0}^{\infty}\Pi\left(\frac{q^n}{[2]_{p,q}p^{n-1}}\pi_1+(1-\frac{q^n}{[2]_{p,q}p^{n-1}})\pi_2\right)\\
&\times\Bigg(\frac{p\left(p^2q^3\left(\frac{q^{n-2}}{p^{n-1}}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{n-2}}{[2]_{p,q}p^{n-1}}\right)_{p,q}^2\right)}{\frac{q^{n-2}}{p^{n-1}}}\\
&-\frac{[2]_{p,q}\left(p^2q^3\left(\frac{q^{n-1}}{p^n}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{n-1}}{[2]_{p,q}p^n}\right)_{p,q}^2\right)}{\frac{q^{n-1}}{p^n}}\\
&+\frac{q\left(p^2q^3\left(\frac{q^n}{p^{n+1}}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^n}{[2]_{p,q}p^{n+1}}\right)_{p,q}^2\right)}{\frac{q^n}{p^{n+1}}}\Bigg)\\
&+\frac{(p-q)}{[2]_{p,q}}\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\\
&\times\Bigg(-\frac{p\left(p^2q^3\left(\frac{q^{-2}}{p^{-1}}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{-2}}{[2]_{p,q}p^{-1}}\right)_{p,q}^2\right)}{\frac{q^{-2}}{p^{-1}}}\\
&+\frac{[2]_{p,q}\left(p^2q^3\left(q^{-1}\right)^2-[2]_{p,q}^2\left(1-pq\frac{q^{-1}}{[2]_{p,q}}\right)_{p,q}^2\right)}{q^{-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\left. -\frac{p\left(p^2q^3(q^{-1})^2-[2]_{p,q}^2\left(1-pq\frac{q^{-1}}{[2]_{p,q}}\right)_{p,q}^2\right)}{q^{-1}}\Pi\left(\frac{q}{[2]_{p,q}}\pi_1+(1-\frac{q}{[2]_{p,q}})\pi_2\right)\right)\\
&= \frac{(p-q)}{[2]_{p,q}}\left[-pq(p-q)[2]_{p,q}^2\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\right]\\
&= -pq(p-q)^2[2]_{p,q}\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\\
&\quad \frac{1}{q(p-q)^2(\pi_2-\pi_1)^2}\left(I_1+\frac{1}{p}I_2\right)\\
&= \frac{1}{q(p-q)^2(\pi_2-\pi_1)^2}\left[\frac{q(p-q)^2[2]_{p,q}}{\pi_2-\pi_1}\int\limits_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa\right.\\
&\quad \left.-q(p-q)^2[2]_{p,q}\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\right]\\
&= \frac{[2]_{p,q}}{(\pi_2-\pi_1)^2}\left(\frac{1}{\pi_2-\pi_1}\int\limits_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p}{[2]_{p,q}}\pi_1+(1-\frac{p}{[2]_{p,q}})\pi_2\right)\right).
\end{aligned}
$$

Thus, the proof is complete. □

Remark 4. *In Lemma 2, if we set $p=1$, then we obtain the following identity:*

$$
\begin{aligned}
&\frac{(\pi_2-\pi_1)^2}{[2]_q}\left[\int\limits_0^{\frac{1}{[2]_q}}q^3t^2\ {}^{\pi_2}D_q^2\Pi(t\pi_1+(1-t)\pi_2)\ d_qt\right.\\
&\left.+\int\limits_{\frac{1}{[2]_q}}^1(1-qt)_q^2\ {}^{\pi_2}D_q^2\Pi(t\pi_1+(1-t)\pi_2)\ d_qt\right]\\
&= \frac{1}{\pi_2-\pi_1}\int\limits_{\pi_1}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_q\varkappa\ -\Pi\left(\frac{\pi_1+q\pi_2}{[2]_q}\right).
\end{aligned}
$$

This was proven by Ali et al. in [18] (Lemma 5).

Theorem 1. *Suppose that the assumptions of Lemma 2 hold. If $|{}^{\pi_2}D_{p,q}\Pi|$ is convex on $[\pi_1,\pi_2]$, then we have the inequality:*

$$
\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int\limits_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right| \qquad (14)\\
&\leq \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\Theta_1(p,q)|{}^{\pi_2}D_{p,q}^2\Pi(\pi_1)|+\Theta_2(p,q)|{}^{\pi_2}D_{p,q}^2\Pi(\pi_2)|\right],
\end{aligned}
$$

where:

$$
\Theta_1(p,q) = \frac{p^3[2]_{p,q}\left(p^3[2]_{p,q}^2-p^2-q^2\right)}{[2]_{p,q}^3[3]_{p,q}[4]_{p,q}},
$$

$$\Theta_2(p,q) = \frac{[3]_{p,q}[4]_{p,q}\Big((1-2p)([2]_{p,q}^2-1)+q[2]_{p,q}\Big)}{[2]_{p,q}^3[3]_{p,q}[4]_{p,q}} + \frac{\Big(p^3q^2[2]_{p,q}^2+p^3[2]_{p,q}^2[4]_{p,q}-p[2]_{p,q}[4]_{p,q}-q[4]_{p,q}\Big)}{[2]_{p,q}^2[3]_{p,q}[4]_{p,q}}.$$

Proof. By taking the modulus of (13), we have:

$$\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right| \\
\leq\ & \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)\left|{}^{\pi_2}D_{p,q}^2\Pi(t\pi_1+(1-t)\pi_2)\right|d_{p,q}t\right. \\
&\left.+\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)_{p,q}^2\left|{}^{\pi_2}D_{p,q}^2\Pi(t\pi_1+(1-t)\pi_2)\right|d_{p,q}t\right].
\end{aligned}$$

Now, by using the convexity of $|{}^{\pi_2}D_{p,q}\Pi|$, we have:

$$\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right| \qquad (15)\\
\leq\ & \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^{\frac{1}{[2]_{p,q}}}\left(\begin{array}{c}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)t|{}^{\pi_2}D_{p,q}^2\Pi(\pi_1)| \\ +(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)(1-t)|{}^{\pi_2}D_{p,q}^2\Pi(\pi_2)|\end{array}\right)d_{p,q}t\right. \\
&\left.+\int_{\frac{1}{[2]_{p,q}}}^1\Big(p(1-qt)_{p,q}^2t|{}^{\pi_2}D_{p,q}^2\Pi(\pi_1)|+p(1-qt)_{p,q}^2(1-t)|{}^{\pi_2}D_{p,q}^2\Pi(\pi_2)|\Big)d_{p,q}t\right].
\end{aligned}$$

Now, we compute the integrals appearing on the right side of (15) using Lemma 1. We have:

$$\begin{aligned}
&\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)td_{p,q}t \qquad (16)\\
=\ &\int_0^{\frac{1}{[2]_{p,q}}}\Big((p^2-p)t+pq^3t^3\Big)d_{p,q}t=\frac{[2]_{p,q}[4]_{p,q}(p^2-p)+pq^3}{[2]_{p,q}^4[4]_{p,q}},\\
&\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)(1-t)d_{p,q}t \qquad (17)\\
=\ &\int_0^{\frac{1}{[2]_{p,q}}}\Big(-pq^3t^3+pq^3t^2-(p^2-p)t+(p^2-p)\Big)d_{p,q}t
\end{aligned}$$

$$= \frac{-pq^3[3]_{p,q} + pq^3[2]_{p,q}[4]_{p,q} - (p^2-p)([2]^2_{p,q}-1)[2]_{p,q}[3]_{p,q}[4]_{p,q}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}},$$

$$\int_{\frac{1}{[2]_{p,q}}}^{1} p(1-qt)^2_{p,q} t d_{p,q}t \tag{18}$$

$$= \int_{\frac{1}{[2]_{p,q}}}^{1} \Big(pq^3t^3 - pq[2]_{p,q}t^2 + p^2t\Big) d_{p,q}t$$

$$= \frac{\begin{array}{c} pq^3[3]_{p,q}([2]^4_{p,q}-1) - pq[2]^2_{p,q}[4]_{p,q}([2]^3_{p,q}-1) \\ +p^2[2]_{p,q}[3]_{p,q}[4]_{p,q}([2]^2_{p,q}-1) \end{array}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}},$$

$$\int_{\frac{1}{[2]_{p,q}}}^{1} p(1-qt)^2_{p,q}(1-t) d_{p,q}t \tag{19}$$

$$= p\int_{\frac{1}{[2]_{p,q}}}^{1} \Big(-q^3t^3 + \Big(pq+q^2+q^3\Big)t^2 - \Big(pq+p+q^2\Big)t + p\Big) d_{p,q}t$$

$$= p\left[\frac{\begin{array}{c} -q^3\Big([2]^4_{p,q}-1\Big)[3]_{p,q} + (pq+q^2+q^3)[2]_{p,q}[4]_{p,q}\Big([2]^3_{p,q}-1\Big) \\ -(pq+p+q^2)[2]_{p,q}[3]_{p,q}[4]_{p,q}\Big([2]^2_{p,q}-1\Big) + p[2]^3_{p,q}[3]_{p,q}[4]_{p,q}([2]_{p,q}-1) \end{array}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}}\right].$$

Thus, we obtain the required inequality (14) by using the calculated integrals (16)–(19) in (15). □

Remark 5. *In Theorem 1, if we set $p = 1$, then we obtain the following inequality:*

$$\left|\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_q\varkappa\ - \Pi\left(\frac{\pi_1+q\pi_2}{[2]_q}\right)\right|$$

$$\leq \frac{(\pi_2-\pi_1)^2}{[2]^5_q[3]_q[4]_q}\Big[\Big(2q+4q^2+2q^3\Big)\Big|\ {}^{\pi_2}D^2_q\Pi(\pi_1)\ \Big|$$

$$+\Big(-q-q^2+2q^3+4q^4+3q^5+q^6\Big)\Big|\ {}^{\pi_2}D^2_q\Pi(\pi_2)\ \Big|\Big].$$

This was proven by Ali et al. [18] (Theorem 3).

Theorem 2. *Suppose that the assumptions of Lemma 2 hold. If $|{}^{\pi_2}D_{p,q}\Pi|^r$, $r \geq 1$ is convex on $[\pi_1, \pi_2]$, then we have the inequality:*

$$\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa - \Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right|$$

$$\leq \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\Big[\Theta_7^{1-\frac{1}{r}}(p,q)\Big(\Theta_3(p,q)|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)|^r + \Theta_4(p,q)|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)|^r\Big)^{\frac{1}{r}}$$

$$+\Theta_8^{1-\frac{1}{r}}(p,q)\Big(\Theta_5(p,q)|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)|^r + \Theta_6(p,q)|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)|^r\Big)^{\frac{1}{r}}\Big],$$

where:

$$\begin{aligned}
\Theta_3(p,q) &= \frac{[2]_{p,q}[4]_{p,q}(p^2-p)+pq^3}{[2]^4_{p,q}[4]_{p,q}},\\
\Theta_4(p,q) &= \frac{-pq^3[3]_{p,q}+pq^3[2]_{p,q}[4]_{p,q}-(p^2-p)([2]^2_{p,q}-1)[2]_{p,q}[3]_{p,q}[4]_{p,q}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}},\\
\Theta_5(p,q) &= \frac{\begin{array}{c}pq^3[3]_{p,q}([2]^4_{p,q}-1)-pq[2]^2_{p,q}[4]_{p,q}([2]^3_{p,q}-1)\\+p^2[2]_{p,q}[3]_{p,q}[4]_{p,q}([2]^2_{p,q}-1)\end{array}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}},\\
\Theta_6(p,q) &= p\left[\frac{\begin{array}{c}-q^3\left([2]^4_{p,q}-1\right)[3]_{p,q}+(pq+q^2+q^3)[2]_{p,q}[4]_{p,q}\left([2]^3_{p,q}-1\right)\\-(pq+p+q^2)[2]_{p,q}[3]_{p,q}[4]_{p,q}\left([2]^2_{p,q}-1\right)+p[2]^3_{p,q}[3]_{p,q}[4]_{p,q}([2]_{p,q}-1)\end{array}}{[2]^4_{p,q}[3]_{p,q}[4]_{p,q}}\right],\\
\Theta_7(p,q) &= \frac{[2]^2_{p,q}[3]_{p,q}(p^2-p)+pq^3}{[2]^3_{p,q}[3]_{p,q}},\\
\Theta_8(p,q) &= \frac{pq^3\left([2]^3_{p,q}-1\right)}{[3]_{p,q}[2]^3_{p,q}}+\frac{pq\left(1-[2]^2_{p,q}\right)}{[2]^2_{p,q}}+\frac{p^2\left([2]_{p,q}-1\right)}{[2]_{p,q}}.
\end{aligned}$$

Proof. By taking the modulus of (13) and applying the power mean inequality, we have:

$$\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right|\\
\leq\ &\frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)^2_{p,q}+p(1-qt)^2_{p,q})\left|{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)\right|d_{p,q}t\right.\\
&\left.+\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)^2_{p,q}\left|{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)\right|d_{p,q}t\right]\\
\leq\ &\frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\left(\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)^2_{p,q}+p(1-qt)^2_{p,q})d_{p,q}t\right)^{1-\frac{1}{r}}\right.\\
&\times\left(\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)^2_{p,q}+p(1-qt)^2_{p,q})\left|{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)\right|^r d_{p,q}t\right)^{\frac{1}{r}}\\
&\left.+\left(\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)^2_{p,q}d_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)^2_{p,q}\left|{}^{\pi_2}D^2_{p,q}\Pi(t\pi_1+(1-t)\pi_2)\right|^r d_{p,q}t\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Applying the convexity of $|{}^{\pi_2}D_{p,q}\Pi|^r$, $r\geq 1$, we have:

$$
\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right| \\
\leq\ & \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\left(\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)d_{p,q}t\right)^{1-\frac{1}{r}}\right. \\
&\times\left(\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)\left[t|^{\pi_2}D_{p,q}\Pi(\pi_1)|^r+(1-t)|^{\pi_2}D_{p,q}\Pi(\pi_2)|^r\right]d_{p,q}t\right)^{\frac{1}{r}} \\
&+\left(\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)_{p,q}^2d_{p,q}t\right)^{1-\frac{1}{r}} \\
&\left.\times\left(\int_{\frac{1}{[2]_{p,q}}}^1 p(1-qt)_{p,q}^2\left[t|^{\pi_2}D_{p,q}\Pi(\pi_1)|^r+(1-t)|^{\pi_2}D_{p,q}\Pi(\pi_2)|^r\right]d_{p,q}t\right)^{\frac{1}{r}}\right] \\
=\ & \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\Theta_7^{1-\frac{1}{r}}(p,q)\left(\Theta_3(p,q)|^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r+\Theta_4(p,q)|^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\right)^{\frac{1}{r}}\right. \\
&\left.+\Theta_8^{1-\frac{1}{r}}(p,q)\left(\Theta_5(p,q)|^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r+\Theta_6(p,q)|^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

Thus, the proof is complete. □

Remark 6. *In Theorem 2, if we set $p=1$, then Theorem 2 becomes Theorem 5 in [18].*

Theorem 3. *Suppose that the assumptions of Lemma 2 hold. If $|^{\pi_2}D_{p,q}\Pi|^r$, $r>1$ is convex on $[\pi_1,\pi_2]$, then we have the inequality:*

$$
\begin{aligned}
&\left|\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)^{\pi_2}d_{p,q}\varkappa-\Pi\left(\frac{p\pi_1+q\pi_2}{[2]_{p,q}}\right)\right| \\
\leq\ & \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\Theta_9^{\frac{1}{s}}(p,q)\left(\frac{|^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r}{[2]_{p,q}^3}+\frac{|^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\left([2]_{p,q}^2-1\right)}{[2]_{p,q}^3}\right)^{\frac{1}{r}}\right. \\
&\left.+\Theta_{10}^{\frac{1}{s}}(p,q)\left(\frac{|^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r\left([2]_{p,q}^2-1\right)}{[2]_{p,q}^3}+\frac{|^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\left([2]_{p,q}^3-2[2]_{p,q}^2+1\right)}{[2]_{p,q}^3}\right)^{\frac{1}{r}}\right],
\end{aligned}
$$

where $\frac{1}{s}+\frac{1}{r}=1$,

$$
\Theta_9(p,q)=\int_0^{\frac{1}{[2]_{p,q}}}(p^2q^3t^2-(1-pqt)_{p,q}^2+p(1-qt)_{p,q}^2)^sd_{p,q}t
$$

and:

$$\Theta_{10}(p,q) = \int\limits_{\frac{1}{[2]_{p,q}}}^{1} p^s (1-qt)_{p,q}^{2s} d_{p,q}t.$$

Proof. By taking the modulus of (13) and applying Hölder's inequality, we have:

$$\begin{aligned}
&\left| \frac{1}{\pi_2 - \pi_1} \int\limits_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)^{\pi_2} d_{p,q}\varkappa - \Pi\left(\frac{p\pi_1 + q\pi_2}{[2]_{p,q}} \right) \right| \\
\leq\ & \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left[\int\limits_0^{\frac{1}{[2]_{p,q}}} \left(p^2 q^3 t^2 - (1-pqt)_{p,q}^2 + p(1-qt)_{p,q}^2 \right) \left| {}^{\pi_2} D_{p,q}^2 \Pi(t\pi_1 + (1-t)\pi_2) \right| d_{p,q}t \right. \\
& \left. + \int\limits_{\frac{1}{[2]_{p,q}}}^{1} p(1-qt)_{p,q}^2 \left| {}^{\pi_2} D_{p,q}^2 \Pi(t\pi_1 + (1-t)\pi_2) \right| d_{p,q}t \right] \\
\leq\ & \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left[\left(\int\limits_0^{\frac{1}{[2]_{p,q}}} (p^2 q^3 t^2 - (1-pqt)_{p,q}^2 + p(1-qt)_{p,q}^2)^s d_{p,q}t \right)^{\frac{1}{s}} \right. \\
& \times \left(\int\limits_0^{\frac{1}{[2]_{p,q}}} \left| {}^{\pi_2} D_{p,q}^2 \Pi(t\pi_1 + (1-t)\pi_2) \right|^r d_{p,q}t \right)^{\frac{1}{r}} \\
& \left. + \left(\int\limits_{\frac{1}{[2]_{p,q}}}^{1} p^s (1-qt)_{p,q}^{2s} d_{p,q}t \right)^{\frac{1}{s}} \left(\int\limits_{\frac{1}{[2]_{p,q}}}^{1} \left| {}^{\pi_2} D_{p,q}^2 \Pi(t\pi_1 + (1-t)\pi_2) \right|^r d_{p,q}t \right)^{\frac{1}{r}} \right].
\end{aligned}$$

Applying the convexity of $|{}^{\pi_2} D_{p,q}\Pi|^r$, $r > 1$, we have:

$$\begin{aligned}
&\left| \frac{1}{\pi_2 - \pi_1} \int\limits_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)^{\pi_2} d_{p,q}\varkappa - \Pi\left(\frac{p\pi_1 + q\pi_2}{[2]_{p,q}} \right) \right| \\
\leq\ & \frac{(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left[\left(\int\limits_0^{\frac{1}{[2]_{p,q}}} (p^2 q^3 t^2 - (1-pqt)_{p,q}^2 + p(1-qt)_{p,q}^2)^s d_{p,q}t \right)^{\frac{1}{s}} \right. \\
& \times \left(\int\limits_0^{\frac{1}{[2]_{p,q}}} \left[t \left| {}^{\pi_2} D_{p,q}\Pi(\pi_1) \right|^r + (1-t) \left| {}^{\pi_2} D_{p,q}\Pi(\pi_2) \right|^r \right] d_{p,q}t \right)^{\frac{1}{r}} \\
& \left. + \left(\int\limits_{\frac{1}{[2]_{p,q}}}^{1} p^s (1-qt)_{p,q}^{2s} d_{p,q}t \right)^{\frac{1}{s}} \left(\int\limits_{\frac{1}{[2]_{p,q}}}^{1} \left[t \left| {}^{\pi_2} D_{p,q}\Pi(\pi_1) \right|^r + (1-t) \left| {}^{\pi_2} D_{p,q}\Pi(\pi_2) \right|^r \right] d_{p,q}t \right)^{\frac{1}{r}} \right]
\end{aligned}$$

$$= \frac{(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\Theta_9^{\frac{1}{s}}(p,q)\left(\frac{|{}^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r}{[2]_{p,q}^3}+\frac{|{}^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\left([2]_{p,q}^2-1\right)}{[2]_{p,q}^3}\right)^{\frac{1}{r}}\right.$$

$$\left.+\Theta_{10}^{\frac{1}{s}}(p,q)\left(\frac{|{}^{\pi_2}D_{p,q}^2\Pi(\pi_1)|^r\left([2]_{p,q}^2-1\right)}{[2]_{p,q}^3}+\frac{|{}^{\pi_2}D_{p,q}^2\Pi(\pi_2)|^r\left([2]_{p,q}^3-2[2]_{p,q}^2+1\right)}{[2]_{p,q}^3}\right)^{\frac{1}{r}}\right].$$

Thus, the proof is complete. □

Remark 7. *In Theorem 3, if we set $p=1$, then Theorem 3 becomes Theorem 4 in [18].*

5. Concluding Remarks

In this work, we established some new midpoint-type (p,q)-integral inequalities for twice-(p,q)-differentiable convex functions. In the last twenty-five years, quantum calculus (or q-calculus) has served as a link between mechanics and physics. Physicists make up the majority of scientists who utilize q-calculus today. Many scientists, in fact, employ q-calculus as a mathematical model in their studies. In the theory of quantum gravity, q can be thought of as a parameter connected to the exponential of the cosmological constant. Quantum gravity can be divided into two types:

(i) There is no gravity when $q=1$, which is the situation of classical quantum mechanics;
(ii) When $q\neq 0$, we have quantum mechanics, in which the energy density of the vacuum is non-zero.

Other areas of study in physics and related fields where q-calculus is being employed include: (1) the q-Coulomb problem and the q-hydrogen atom; (2) quantum hydrodynamics; (3) the Wess–Zumino model; (4) string theory; (5) Electroweak interaction; (6) Knot theory; (7) quantum cardiodynamics; (8) special relativity; (9) Newtonian quantum gravity; (10) quantum field theory; (11) elementary particle physics and chemical physics; (12) molecular and nuclear spectroscopy; (13) general relativity.

For several other applications of q-calculus in physics, one can read [4]. A good book on "Quantum Gravity" [34] is recommended for those interested in these topics. The fact that fluid dynamics resemble quantum physics is no coincidence. In truth, quantum physics and classical physics can both reflect various parts of the same physical process; for example, see [35].

Author Contributions: Conceptualization, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; methodology, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; validation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; investigation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; writing—original draft preparation, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; writing—review and editing, T.S., G.M., M.A.A., C.P., I.B.S. and P.A.; supervision, T.S. and M.A.A.; funding acquisition, T.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-63-KNOW-22.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the Referees and Editor for their valuable comments. The third author is thankful to Ch. Sohail Tariq (Project Director GCUF Sahiwal Campus) and Ch. Faraz Hanif (Project Director GCUF Sahiwal Campus) for providing good research environment at GCUF Sahiwal Campus.

Conflicts of Interest: The authors declare no conflict of interest.

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