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Maximum Principle and Second-Order Optimality Conditions in Control Problems with Mixed Constraints

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Abstract: This article concerns the optimality conditions for a smooth optimal control problem with an endpoint and mixed constraints. Under the normality assumption, which corresponds to the full-rank condition of the associated controllability matrix, a simple proof of the second-order necessary optimality conditions based on the Robinson stability theorem is derived. The main novelty of this approach compared to the known results in this area is that only a local regularity with respect to the mixed constraints, that is, a regularity in an ε -tube about the minimizer, is required instead of the conventional stronger global regularity hypothesis. This affects the maximum condition. Therefore, the normal set of Lagrange multipliers in question satisfies the maximum principle, albeit along with the modified maximum condition, in which the maximum is taken over a reduced feasible set. In the second part of this work, we address the case of abnormal minimizers, that is, when the full rank of controllability matrix condition is not valid. The same type of reduced maximum condition is obtained.

Keywords: optimal control; maximum principle; mixed constraints

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1. Introduction

In this article, second-order necessary optimality conditions for an optimal control problem with mixed equality and inequality constraints are investigated. Under the normality condition, which is ensured by the full rank of the controllability matrix, a rather simple proof of the optimality conditions is proposed based on Robinson’s theorem on the metric regularity for set-valued mappings. For the case in which the normality condition is violated, the second-order conditions are derived based on the index approach. This means that some reduced cone of Lagrange multipliers is invoked, which is defined by using the index of the quadratic form of the Lagrange function; see, e.g., [1–3].

In work [4], the two notions of the strong and of weak regularity of an admissible trajectory with respect to mixed constraints have been considered. Strong regularity means that the constraint qualification, or the so-called Robinson condition, is satisfied for all time-points and for all admissible control values. This corresponds to the regularity condition in the classical sense. Weak regularity means that this condition is satisfied merely in some neighborhood of the optimal process. By their nature, these two concepts correspond, respectively, to global, and local regularity settings. Under weak regularity, a refined maximum condition of Pontryagin’s type has been obtained, in which the maximum is taken over the closure of regular points of the feasible set, but not over the entire feasible set. In this article, the results of [4] are carried over to the second-order conditions in the case of global minimum.

The literature on optimality conditions for optimal control problems with mixed constraints is extensive. In the context of this research related to the study of mixed

constraints, we note the works of [5–12]. Regarding the second-order conditions in mixed constrained problems, one may consider, e.g., [3,13,14] and the bibliography cited therein. At the same time, these selective lists of publications are far from exhaustive.

This work is organized as follows. In the next section, the problem formulation is presented, together with main definitions and notation. In Section 3, the issue of normality is discussed. In Section 4, the main result of this work—the normal maximum principle and second-order optimality conditions—is formulated and proved. In Section 5, the abnormal situation is taken into consideration, and the result of the previous section is refined. Section 6 concludes the work with a short summary.

2. Problem Formulation

Consider the following optimal control problem on the fixed time interval $[0, 1]$:

$$\begin{cases} \text{Minimize} & \varphi(p) \\ \text{subject to} & \dot{x}(t) = f(x(t), u(t), t) \text{ for a.a. } t, \\ & e_1(p) \leq 0, \quad e_2(p) = 0, \\ & u(t) \in U(x(t), t) \text{ for a.a. } t, \end{cases} \quad (1)$$

where $p = (x_0, x_1)$, $x_0 = x(0)$, $x_1 = x(1)$, $t \in [0, 1]$, and

$$U(x, t) := \{u \in \mathbb{R}^m : r_1(x, u, t) \leq 0, r_2(x, u, t) = 0\}.$$

The mappings $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $e_i : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{k_i}$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$, $r_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \rightarrow \mathbb{R}^{q_i}$, $i = 1, 2$, satisfy the following hypothesis.

Hypothesis 1 (H1). *Mappings $\varphi, e_1, e_2, f, r_1, r_2$ are twice continuously differentiable.*

The vector $p = (x_0, x_1)$ is termed the endpoint, as well as the constraints given by mappings e_1, e_2 . The scalar function $\varphi(p)$ defines the minimizing functional. Mappings r_1, r_2 define the mixed constraints which are imposed on both state and control variables.

A pair of functions (x, u) is designed by the control process; if $x(\cdot)$ is absolutely continuous, $u(\cdot)$ is measurable and essentially bounded, whereas $\dot{x}(t) = f(x(t), u(t), t)$ for a.a. $t \in [0, 1]$. A control process is feasible, provided that the endpoint, control and state constraints are satisfied. A feasible process $(\bar{x}, \bar{u})$ is termed optimal if, for any feasible process (x, u) , $\varphi(\bar{p}) \leq \varphi(p)$, where $\bar{p} = (\bar{x}(0), \bar{x}(1))$.

This concept of the minimum is known as a *global strong minimum*. The purpose of this work is to derive the second-order necessary optimality conditions for this type of minimum under the normality assumptions. That is, to find such a set of Lagrange multipliers that simultaneously satisfies the maximum principle and Legendre's condition, and for which $\lambda^0 > 0$. Such a set of multipliers must be unique upon normalization. The abnormal situation is also examined after the normal case.

Consider the reference control process $(\bar{x}, \bar{u})$, which can be optimal, extremal, regular, or normal in what follows. Denote by $r = (r_1, r_2)$ the joint mapping acting onto $\mathbb{R}^q$, where $q = q_1 + q_2$. Let $J(x, u, t) := \{j : r^j(x, u, t) = 0\}$ be the set of active indices, where the upper index specifies the vector component. Set $J(u, t) := J(\bar{x}(t), u, t)$. Let $\mathcal{U}(\cdot)$ designate the closure of function $\bar{u}(\cdot)$ w.r.t. the Lebesgue measure; that is, for a given $t \in [0, 1]$, the set $\mathcal{U}(t)$ consists of essential values of $\bar{u}(\cdot)$ at point t , [8]. Recall that the vector a is said to be the essential value of a function $u(\cdot)$ at point τ , provided that $\ell(\{t \in [\tau - \varepsilon, \tau + \varepsilon] : u(t) \in B_\varepsilon(a)\}) > 0 \forall \varepsilon > 0$, where $B_\varepsilon(a)$ is the closed ball centered at a with the radius ε , and ℓ designates the Lebesgue measure on $\mathbb{R}$.

The main regularity concept is as follows.

Definition 1. The control process $(\bar{x}, \bar{u})$ is said to be regular w.r.t. the mixed constraints, provided that, for all $t \in [0, 1]$ and for all $u \in \mathcal{U}(t)$, the active gradients $(r^j)'_u(\bar{x}(t), u, t)$, $j \in J(u, t)$ are linearly independent.

The following proposition represents an equivalent reformulation of the introduced regularity concept. For $\varepsilon \geq 0$, define the set

$$J_\varepsilon(x, u, t) := \left\{ j \in \{1, \dots, q_1\} : r^j(x, u, t) \in [-\varepsilon, 0] \right\} \cup \{q_1 + 1, \dots, q\},$$

which is subject to the same conventions as the mapping $J(x, u, t)$. It is clear that $J \subseteq J_\varepsilon$, and $J_0 = J$.

Proposition 1. Let the control process $(\bar{x}, \bar{u})$ be regular w.r.t. the mixed constraints. Then, there exists a number $\varepsilon_0 > 0$ such that, for all $t \in [0, 1]$ and for almost all $s \in [0, 1]$ such that $|s - t| \leq \varepsilon_0$, the ε_0 -active gradients $(r^j)'_u(\bar{x}(s), \bar{u}(s), s)$, $j \in J_{\varepsilon_0}(s)$ are linearly independent. Moreover, the number ε_0 can be chosen such that the modulus of surjectivity for this set of gradients is not lower than ε_0 .

The proof is based on a simple contradiction argument.

The point $u \in U(x, t)$ is termed regular provided that the gradients $(r^j)'_u(x, u, t)$, $j \in J(x, u, t)$ are positively linearly independent. The subset of all regular points of $U(x, t)$ is denoted as $U_R(x, t)$. Denote

$$\Theta(x, t) := \text{clos } U_R(x, t).$$

It is clear that, for the regular process $(\bar{x}, \bar{u})$, one has $\mathcal{U}(t) \subseteq \Theta(\bar{x}(t), t) \neq \emptyset \forall t \in [0, 1]$. Consider the Hamilton–Pontryagin function

$$\mathcal{H}(x, u, \psi, t) := \langle \psi, f(x, u, t) \rangle,$$

and the Lagrangian

$$\mathcal{L}(p, \lambda) := \lambda^0 \varphi(p) + \langle \lambda_1, e_1(p) \rangle + \langle \lambda_2, e_2(p) \rangle.$$

Here, $\psi \in (\mathbb{R}^n)^*$, $\lambda = (\lambda^0, \lambda_1, \lambda_2) \in (\mathbb{R}^{1+k_1+k_2})^*$ are the conjugate variables.

Definition 2. The control process $(\bar{x}, \bar{u})$ is said to satisfy the maximum principle provided that there exists a vector $\lambda = (\lambda^0, \lambda_1, \lambda_2) \in (\mathbb{R}^{1+k_1+k_2})^*$, where $\lambda^0 \geq 0$ and $\lambda_1 \geq 0$, an absolutely continuous vector-valued function $\psi \in \mathbb{W}_{1,\infty}([0, 1]; (\mathbb{R}^n)^*)$, and a measurable, essentially bounded, vector-valued function $v \in \mathbb{L}_\infty([0, 1]; (\mathbb{R}^q)^*)$ of which the j -th component is nonnegative for $j = 1, \dots, q_1$, such that $\lambda \neq 0$, and, on $[0, 1]$, it holds that:

$$\dot{\psi}(t) = -\mathcal{H}'_x(\bar{x}(t), \bar{u}(t), \psi(t), t) + v(t)r'_x(\bar{x}(t), \bar{u}(t), t) \text{ for a.a. } t, \quad (2)$$

$$\psi(\alpha) = (-1)^\alpha \mathcal{L}'_{x_\alpha}(\bar{p}, \lambda) \text{ for } \alpha = 0, 1, \quad (3)$$

$$\max_{u \in \Theta(\bar{x}(t), t)} \mathcal{H}(\bar{x}(t), u, \psi(t), t) = \mathcal{H}(\bar{x}(t), \bar{u}(t), \psi(t), t) \text{ for a.a. } t, \quad (4)$$

$$\mathcal{H}'_u(\bar{x}(t), \bar{u}(t), \psi(t), t) - v(t)r'_u(\bar{x}(t), \bar{u}(t), t) = 0 \text{ for a.a. } t, \quad (5)$$

$$\langle \lambda_1, e_1(\bar{p}) \rangle = 0, \text{ and } \int_0^1 \langle v(t), r(\bar{x}(t), \bar{u}(t), t) \rangle dt = 0. \quad (6)$$

Here, Condition (2) is the co-state equation, that is, the differential equation for the conjugate variable ψ . Equalities (3) are the transversality conditions. Equality (4) is the maximum condition. Equality (5) is the so-called Euler–Lagrange equation. Equalities (6)

are known as the complementary slackness condition. Furthermore, (λ, ψ, ν) are known as the Lagrange multipliers.

Under the regularity condition given in Definition 3, the multipliers ψ , and ν are uniquely defined by the vector λ , where (λ, ψ, ν) is the set of Lagrange multipliers corresponding to $(\bar{x}, \bar{u})$ in view of the maximum principle. This assertion simply follows from the Euler–Lagrange equation. Then, denote by $\Lambda = \Lambda(\bar{x}, \bar{u})$ the set of vectors $\lambda \in (\mathbb{R}^{1+k_1+k_2})^*$ for which there exist (ψ, ν) such that the corresponding set of Lagrange multipliers (λ, ψ, ν) generated by λ satisfies the maximum principle.

3. Normality Condition

Let us introduce the notion of normality. This notion is based on the concept of linearization of the control problem and the corresponding variational differential system. Consider the reference control process $(\bar{x}, \bar{u})$, and a pair $(\delta x_0, \delta u) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{L}_2([0, 1]; \mathbb{R}^m)$. Denote by $\delta x(\cdot)$ the solution to the variational differential equation on the time interval $[0, 1]$, which corresponds to $(\delta x_0, \delta u)$, that is,

$$\delta x(t) = f'_x(\bar{x}(t), \bar{u}(t), t)\delta x(t) + f'_u(\bar{x}(t), \bar{u}(t), t)\delta u(t), \quad (7)$$

where $\delta x(0) = \delta x_0$. Such a solution exists on the entire time interval $[0, 1]$ and, as soon as δu is an L_2 -function, one finds that $\delta x \in \mathbb{W}_{1,2}([0, 1]; \mathbb{R}^n)$.

In what follows, it is not restrictive to set $e_1(\bar{p}) = 0$. Thus, all the endpoint constraints of the inequality type are assumed to be active. Consider the two following subspaces in $\mathcal{X}$:

$$\begin{aligned} \mathcal{N}_e &:= \{(\delta x_0, \delta u) \in \mathcal{X} : e'(\bar{p})\delta p = 0\}, \\ \mathcal{N}_r &:= \{(\delta x_0, \delta u) \in \mathcal{X} : D(t)\left[r'_x(\bar{x}(t), \bar{u}(t), t)\delta x(t) + r'_u(\bar{x}(t), \bar{u}(t), t)\delta u(t)\right] = 0\}. \end{aligned}$$

Here, e is the joint mapping of e_1, e_2 ; $\delta p = (\delta x_0, \delta x_1)$, where $\delta x_1 = \delta x(1)$, and $D(t)$ is the diagonal $q \times q$ -matrix which has 1 in the position (j, j) iff $j \in J(t)$ and 0 otherwise.

Consider the matrix

$$R(t) = r'_u(\bar{x}(t), \bar{u}(t), t)^* D(t) r'_u(\bar{x}(t), \bar{u}(t), t).$$

Set

$$M(t) := R(t)^+ r'_u(\bar{x}(t), \bar{u}(t), t)^* D(t) r'_x(\bar{x}(t), \bar{u}(t), t).$$

Here, A^+ stands for the generalized inverse [15]. Here, the generalized inverse $R(t)^+$ can be computed as follows. Let $T(t)$ be a non-singular orthogonal linear transform which maps the subspace $\ker R(t)^\perp = \text{im } R(t)$ onto the subspace of $\mathbb{R}^m$ with the first $m - q(t)$ coordinates vanished, where $q(t) = |J(t)|$ is the number of active indices. Then, $R(t)^+ = T^{-1}(t)[T(t)R(t)T^{-1}(t)]^{-1}T(t)$, where the pseudo-inverse of the block $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}^{-1}$ is understood as $\begin{pmatrix} 0 & 0 \\ 0 & A^{-1} \end{pmatrix}$.

Consider the matrix differential system

$$\dot{\Phi}(t) = f'_x(\bar{x}(t), \bar{u}(t), t)\Phi(t) - f'_u(\bar{x}(t), \bar{u}(t), t)M(t)\Phi(t), \quad (8)$$

where $\Phi(0) = I$. Let $\Phi(t)$ be the solution to (8), and $P(t)$ be the matrix of orthogonal projection onto $\ker R(t)$.

It is clear that by virtue of the construction, any element $(\delta x_0, \delta u) \in \mathcal{N}_r$ can be represented as

$$(\delta x_0, \delta u) = (\delta x_0, P\delta u + \mathcal{V}[\delta x_0, \delta u]), \quad (9)$$

where $\mathcal{V}[\delta x_0, \delta u] = M(t)\delta x(t)$, whereas

$$\delta x(t) = \Phi(t)\left(\delta x_0 + \int_0^t \Phi^{-1}(s)f'_u(\bar{x}(s), \bar{u}(s), s)P(s)\delta u(s)ds\right). \quad (10)$$

Conversely, any $\delta x_0 \in \mathbb{R}^n$ and $\delta v \in \mathbb{L}_2([0, 1]; \mathbb{R}^m)$: $\delta v(t) \in \ker R(t)$ a.e. yields an element of $\mathcal{N}_r$ as $(\delta x_0, \delta v + \mathcal{V}[\delta x_0, \delta v]) \in \mathcal{N}_r$. Therefore, there is a one-to-one correspondence between $\mathcal{N}_r$ and the space of the above-specified elements $(\delta x_0, \delta v)$. At the same time, the formula for the solution δx in $\mathcal{N}_r$ is given by (10).

Let us proceed with the construction of the controllability matrix. Define the $\mathbb{R}^{n \times k}$ -matrix A as

$$A = e'_{x_0}(\bar{p}) + e'_{x_1}(\bar{p})\Phi(1),$$

with the $\mathbb{R}^{m \times k}$ -matrix $B(t)$ given as

$$B(t) = e'_{x_1}(\bar{p})\Phi(1)\Phi^{-1}(t)f'_u(\bar{x}(t), \bar{u}(t), t).$$

Now, the controllability matrix Q is introduced as the $\mathbb{R}^{k \times k}$ -matrix:

$$Q = AA^* + \int_0^1 B(t)P(t)B^*(t)dt.$$

Definition 3. The regular control process $(\bar{x}, \bar{u})$ is said to be normal, provided that $Q > 0$, or equivalently, $\text{rank } Q = k$.

4. Main Result

In this section, the second-order necessary optimality conditions are addressed. Consider the two cones

$$C_e := \{y \in \mathbb{R}^k : y^j \leq 0 \text{ for } j = 1, \dots, k_1, \text{ and } y^j = 0 \text{ for } j = k_1 + 1, \dots, k\},$$

$$C_r := \{y \in \mathbb{R}^q : y^j \leq 0 \text{ for } j = 1, \dots, q_1, \text{ and } y^j = 0 \text{ for } j = q_1 + 1, \dots, q\}.$$

Define the cone

$$\mathcal{K} := \left\{ (\delta x_0, \delta u) \in \mathcal{X} : e'(\bar{p})\delta p \in C_e, \right. \\ \left. D(t)[r'_x(\bar{x}(t), \bar{u}(t), t)\delta x(t) + r'_u(\bar{x}(t), \bar{u}(t), t)\delta u(t)] \in C_r \right\}.$$

On the space $\mathcal{X}$, consider the quadratic form

$$\Omega_\lambda[(\delta x_0, \delta u)]^2 = \mathcal{L}''_{pp}(\bar{p}, \lambda)[\delta p]^2 - \int_0^1 \mathcal{H}''_{ww}(\bar{x}(t), \bar{u}(t), \psi(t), t)[\delta w(t)]^2 dt \\ + \int_0^1 \left\langle \nu(t), r''_{ww}(\bar{x}(t), \bar{u}(t), t)[\delta w(t)]^2 \right\rangle dt.$$

Here and further, for convenience of notation: $w = (x, u)$, $\delta w(t) = (\delta x(t), \delta u(t))$.

The main result of this section consists in the following theorem.

Theorem 1. Let $(\bar{x}, \bar{u})$ be an optimal control process in Problem (1). Suppose that this process is normal.

Then, $\Lambda \neq \emptyset$. Moreover, $\dim \text{span}(\Lambda) = 1$, and, for $\lambda = (\lambda^0, \lambda_1, \lambda_2) \in \Lambda$, it holds that $\lambda^0 > 0$, and

$$\Omega_\lambda[(\delta x_0, \delta u)]^2 \geq 0 \quad \forall (\delta x_0, \delta u) \in \mathcal{K}. \quad (11)$$

The proof is preceded with the following auxiliary assertion.

Lemma 1. Consider linear bounded operators A and A_i , $i = 1, 2, \dots$, acting in a given Hilbert space X , such that $A_i \rightarrow A$ pointwise. Assume that the spaces $\text{im } A_i$ and $\text{im } A_i^*$ are closed and that $\text{im } A \subseteq \text{im } A_i$ for all i . Assume also that the sequence of norms $\|(A_i A_i^*)^{-1}\|_{\text{im } A_i}$ is uniformly bounded. Let $C \subseteq X$ be a closed and convex set.

Then,

$$A^{-1}(C) = \limsup_{i \rightarrow \infty} A_i^{-1}(C). \quad (12)$$

Proof. Let $\xi_i \in A_i^{-1}(C)$ and $\xi_i \rightarrow \xi_0$ as $i \rightarrow \infty$. By virtue of the uniform boundedness principle, $A_i \xi_i \rightarrow A \xi_0$. Thus, $A \xi_0 \in C$ and the embedding ' $\supseteq$ ' is proven.

Let us confirm the inverse embedding. Given $\xi_0 \in A^{-1}(C)$, it is necessary to indicate a sequence of elements $\xi_i \in A_i^{-1}(C)$, such that $\xi_i \rightarrow \xi_0$.

Consider the extremal problem

$$\|\xi - \xi_0\|^2 \rightarrow \min, \quad A_i \xi = A \xi_0.$$

Denote the solution to this problem as ξ_i . The solution exists since $\text{im } A \subseteq \text{im } A_i$ and since the quadratic functional is weakly lower semi-continuous, whereas the closed convex set $A_i^{-1}(A \xi_0)$ is weakly closed. Since the image of A_i is closed, one can apply the Lagrange multiplier rule as follows. There exists a non-zero vector $\lambda_i \in \text{im } A_i$ such that

$$\xi_i - \xi_0 + A_i^* \lambda_i = 0.$$

Applying A_i , the multiplier is expressed as follows

$$\lambda_i = (A_i A_i^*)^{-1} (A_i \xi_0 - A_i \xi_i).$$

Therefore,

$$\xi_i = \left(I - A_i^* (A_i A_i^*)^{-1} (A_i - A) \right) \xi_0.$$

Note that $\|A_i^* (A_i A_i^*)^{-1}\| \leq \|A_i\| \cdot \|(A_i A_i^*)^{-1}\| \leq \text{const}$ by the assumption of the lemma. However, $A_i \xi_0 \rightarrow A \xi_0$, and thus, $\xi_i \rightarrow \xi_0$. $\square$

Proof to Theorem 1. By virtue of Theorem 3.5 in [4] and the regularity of the process $(\bar{x}, \bar{u})$, there exists a set of multipliers (λ, ψ, ν) satisfying the maximum principle, such that $\lambda \neq 0$.

Firstly, we prove that the given λ satisfies the following Lagrange multipliers rule:

$$\lambda^0 (\langle \varphi'_{x_0}(\bar{p}), \delta x_0 \rangle + \langle \varphi'_{x_1}(\bar{p}), \delta x_1 \rangle) + (\lambda_1, \lambda_2) (e'_{x_0}(\bar{p}) \delta x_0 + e'_{x_1}(\bar{p}) \delta x_1) = 0 \quad (13)$$

for all $(\delta x_0, \delta u) \in \mathcal{N}_r$.

Due to the maximum principle, one has

$$\begin{aligned} \langle \psi(1), \delta x(1) \rangle &= \langle \psi(0), \delta x(0) \rangle + \int_0^1 (\langle \dot{\psi}(t), \delta x(t) \rangle + \langle \psi(t), \dot{\delta x}(t) \rangle) dt \\ &= \langle \psi(0), \delta x(0) \rangle + \int_0^1 (\langle -\psi(t) f'_x(\bar{x}(t), \bar{u}(t), t) + \nu(t) r'_x(\bar{x}(t), \bar{u}(t), t), \delta x(t) \rangle \\ &\quad \langle \psi(t), f'_x(\bar{x}(t), \bar{u}(t), t) \delta x(t) + f'_u(\bar{x}(t), \bar{u}(t), t) \delta u(t) \rangle) dt \\ &= \langle \mathcal{L}'_{x_0}(\bar{p}, \lambda), \delta x_0 \rangle \\ &\quad + \int_0^1 (\nu(t) r'_x(\bar{x}(t), \bar{u}(t), t) \delta x(t) + \psi(t) f'_u(\bar{x}(t), \bar{u}(t), t) \delta u(t)) dt. \end{aligned}$$

Let us add and subtract the term $\nu(t) r'_u(\bar{x}(t), \bar{u}(t), t) \delta u(t)$ under the integral. Then, by virtue of (5), and also by taking into account that

$$\nu(t) (r'_x(t) \delta x(t) + r'_u(t) \delta u(t)) = 0 \quad \text{for a.a. } t$$

when $(\delta x_0, \delta u) \in \mathcal{N}_r$, we derive $\langle \psi(1), \delta x(1) \rangle = \langle \mathcal{L}'_{x_0}(\bar{p}, \lambda), \delta x_0 \rangle$. Then, from (3),

$$\langle -\mathcal{L}'_{x_1}(\bar{p}, \lambda), \delta x_1 \rangle = \langle \psi(1), \delta x(1) \rangle = \langle \mathcal{L}'_{x_0}(\bar{p}, \lambda), \delta x_0 \rangle.$$

Hence,

$$\langle \mathcal{L}'_{x_0}(\bar{p}, \lambda), \delta x_0 \rangle + \langle \mathcal{L}'_{x_1}(\bar{p}, \lambda), \delta x_1 \rangle = 0,$$

and therefore, Condition (13) is proven.

Consider the endpoint constraint operator

$$\mathcal{E}(\delta x_0, \delta u) := e'(\bar{p})\delta p$$

acting from $\mathcal{X}$ to $\mathbb{R}^k$. It is a straightforward task to derive that condition $Q > 0$ implies that $\mathcal{E}(\mathcal{N}_r) = \mathbb{R}^k$. Then, using $\lambda \neq 0$, Equation (13) yields that $\lambda^0 > 0$. Moreover, the multiplier λ is unique, upon normalization.

Let us proceed to the proof of the second-order condition (11). Take a number $\varepsilon > 0$. Let $D_\varepsilon(t)$ designate the diagonal $q \times q$ -matrix defined as $D(t)$ but, now, with the set $J(t)$ replaced by $J_\varepsilon(t)$. Define the cone $\mathcal{K}_\varepsilon$ in the same way as $\mathcal{K}$, but with the matrix $D(t)$ replaced by $D_\varepsilon(t)$. It is clear that $\mathcal{K}_\varepsilon \subseteq \mathcal{K}$ for all $\varepsilon > 0$. Firstly, we prove (11) for the reduced cone $\mathcal{K}_\varepsilon$. Consider the space $\mathcal{X}_\infty = \mathbb{R}^n \times \mathbb{L}_\infty([0, 1]; \mathbb{R}^n)$ as the L_∞ -analogue of $\mathcal{X}$, and the following image-space $\mathcal{Y}_\varepsilon := \prod_{j=1}^q \mathbb{L}_\infty(T_j^\varepsilon; \mathbb{R}) \times \mathbb{R}^k$, where $T_j^\varepsilon = \{t \in [0, 1] : j \in J_\varepsilon(t)\}$, $j = 1, \dots, q$.

Define the mapping $\mathcal{F}_\varepsilon : \mathcal{X}_\infty \rightarrow \mathcal{Y}_\varepsilon$ as follows

$$\mathcal{F}_\varepsilon(x_0, u(\cdot)) = (r^1(x(\cdot), u(\cdot), \cdot)|_{T_1^\varepsilon}, \dots, r^q(x(\cdot), u(\cdot), \cdot)|_{T_q^\varepsilon}, e^1(p), e^2(p), \dots, e^k(p)).$$

Set $\bar{y}_\varepsilon = \mathcal{F}_\varepsilon(\bar{x}_0, \bar{u}(\cdot))$. Let $\mathcal{C}_\varepsilon \subset \mathcal{Y}_\varepsilon$ be the closed cone such that, for all pairs $(\xi(\cdot), \gamma) \in \mathcal{C}_\varepsilon$, one has $\xi(t) \in C_r$ for a.a. $t \in [0, 1]$, and $\gamma \in C_e$. Here, $\xi^j(t) = 0$ when $t \notin T_j^\varepsilon$.

Consider the inclusion

$$\mathcal{F}_\varepsilon(x_0, u(\cdot)) \in \bar{y}_\varepsilon + \mathcal{C}_\varepsilon, \quad (x_0, u(\cdot)) \in \mathcal{X}_\infty. \quad (14)$$

The Fréchet derivative $\mathcal{F}'_\varepsilon(\bar{x}_0, \bar{u}(\cdot))$ is the linear mapping $(\mathcal{A}_\varepsilon, \mathcal{B}) : \mathcal{X}_\infty \rightarrow \mathcal{Y}_\varepsilon$, where $\mathcal{A}_\varepsilon = (\mathcal{A}_\varepsilon^1, \dots, \mathcal{A}_\varepsilon^q)$,

$$\mathcal{A}_\varepsilon^j(\delta x_0, \delta u) := (r^j)'_x(\bar{x}(t), \bar{u}(t), t)\delta x(t) + (r^j)'_u(\bar{x}(t), \bar{u}(t), t)\delta u(t), \quad t \in T_j^\varepsilon,$$

and $\mathcal{B}(\delta x_0, \delta u) := e'(\bar{p})\delta p$. The proof of this fact involves a standard argument.

Firstly, consider this derivative as the extended linear mapping acting from $\mathcal{X}$ to $\prod_{j=1}^q \mathbb{L}_2(T_j^\varepsilon; \mathbb{R}) \times \mathbb{R}^k$, that is, in Hilbert spaces. Let us prove its surjection. Since the linear mapping $\mathcal{A}_\varepsilon$ is surjective due to regularity w.r.t. the mixed and state constraints (this is a simple task to ensure by solving the corresponding Volterra equation and using Proposition 1 in this way), it is sufficient to show that the linear mapping $\mathcal{B}$ is surjective on $\ker \mathcal{A}_\varepsilon$. Let Q_ε be the matrix constructed as Q ; however, with the matrix $D(t)$ replaced by $D_\varepsilon(t)$. It is clear that $Q_\varepsilon \rightarrow Q$ as $\varepsilon \rightarrow 0$. Therefore, one has that $Q_\varepsilon > 0$ for all sufficiently small ε . At the same time, this condition implies that $\mathcal{E}(\ker \mathcal{A}_\varepsilon) = \mathbb{R}^k$. Therefore, it is simple to conclude that $(\mathcal{A}_\varepsilon, \mathcal{B})$ is a surjective linear mapping for all sufficiently small ε .

The surjection of $(\mathcal{A}_\varepsilon, \mathcal{B})$ as the linear mapping from $\mathcal{X}_\infty$ to $\mathcal{Y}_\varepsilon$ results from the following simple argument. Firstly, notice that, in space $\mathcal{X}$, one has the relation

$$\text{clos}(\ker \mathcal{A}_\varepsilon \cap \mathcal{X}_\infty) = \ker \mathcal{A}_\varepsilon, \quad (15)$$

which is clear due to Formulas (9) and (10), as these still hold when $D(t)$ is replaced by $D_\varepsilon(t)$ for a sufficiently small ε . Then, simply, $\mathcal{N}_r = \mathcal{N}_r(\varepsilon) = \ker \mathcal{A}_\varepsilon$. At the same time, the linear operator $\mathcal{A}_\varepsilon$ is surjective as the mapping from $\mathcal{X}_\infty$ to $\prod_{j=1}^q \mathbb{L}_\infty(T_j^\varepsilon; \mathbb{R})$ by virtue of the same arguments involving the solution to a Volterra equation. However, the image of $\mathcal{B}$ is finite-dimensional, whereas, as has already been confirmed, $\mathcal{B}$ is surjective on the space $\ker \mathcal{A}_\varepsilon$. Therefore, by virtue of (15), one finds that $\mathcal{B}$ is surjective on the subspace $\ker \mathcal{A}_\varepsilon \cap \mathcal{X}_\infty$. Thus, the derivative $\mathcal{F}'_\varepsilon(\bar{x}_0, \bar{u}(\cdot))$ is surjective and, thereby, $(\bar{x}_0, \bar{u}(\cdot))$ is a normal point for the mapping $\mathcal{F}_\varepsilon$.

The Robinson theorem (see Theorem 1 in [16]) asserts the existence of a neighbourhood O_ε of point $((\bar{x}_0, \bar{u}(\cdot)), \bar{y}_\varepsilon) \in \mathcal{X}_\infty \times \mathcal{Y}_\varepsilon$ such that

$$\text{dist}\left((x_0, u(\cdot)), \mathcal{F}_\varepsilon^{-1}(y + \mathcal{C}_\varepsilon)\right) \leq c \text{dist}(\mathcal{F}_\varepsilon(x_0, u(\cdot)), y + \mathcal{C}_\varepsilon) \quad \forall ((x_0, u(\cdot)), y) \in O_\varepsilon. \quad (16)$$

Consider an arbitrary element $h = (\delta x_0, \delta u) \in \mathcal{K}_\varepsilon \cap \mathcal{X}_\infty$, and a number $\tau > 0$. By putting in (16) the value $\xi(\tau) = (\bar{x}_0, \bar{u}(\cdot)) + \tau h$ for $(x_0, u(\cdot))$, and $\bar{y}_\varepsilon$ for y , one obtains

$$\text{dist}\left(\xi(\tau), \mathcal{F}_\varepsilon^{-1}(\bar{y}_\varepsilon + \mathcal{C}_\varepsilon)\right) \leq o(\tau),$$

and, thus, the set $\mathcal{K}_\varepsilon \cap \mathcal{X}_\infty$ is tangent to the solution set $\mathcal{F}_\varepsilon^{-1}(\bar{y}_\varepsilon + \mathcal{C}_\varepsilon)$; see Corollary 2 in [16]. This means that, for every small τ , there exists a vector $\omega(\tau) = (a(\tau), v(\cdot; \tau)) \in \mathcal{X}_\infty$ such that $\frac{\|\omega(\tau)\|}{\tau} \rightarrow 0$ as $\tau \rightarrow 0$, whereas $\mathcal{F}_\varepsilon(\xi(\tau) + \omega(\tau)) \in \bar{y}_\varepsilon + \mathcal{C}_\varepsilon$.

Let us set $x_0(\tau) := \bar{x}_0 + \tau \delta x_0 + a(\tau)$, and $u(t; \tau) := \bar{u}(t) + \tau \delta u(t) + v(t; \tau)$. Then, one may verify that the control pair $(x_0(\tau), u(\cdot; \tau)) \in \mathcal{X}_\infty$ is admissible to Problem (1) for all sufficiently small τ due to the construction of the mapping $\mathcal{F}_\varepsilon$. At this point, it is essential that $\varepsilon > 0$. Let $x(\cdot; \tau)$ be the trajectory corresponding to the pair $(x_0(\tau), u(\cdot; \tau))$, and $p(\tau) = (x(0; \tau), x(1; \tau))$. Take the multiplier $\lambda = (1, \lambda_1, \lambda_2) \in \Lambda(\bar{x}, \bar{u})$, and the corresponding multiplier ν entailed by λ . Let $\varphi(\bar{p}) = 0$.

Consider the inequality

$$\mathcal{L}(p(\tau), \lambda) + \int_0^1 \langle \nu(t), r(x(t; \tau), u(t; \tau), t) \rangle dt \geq 0, \quad (17)$$

which results from the condition of minimum and from the fact that the control process $(x(\cdot; \tau), u(\cdot; \tau))$ is admissible.

Consider the second-order variational system

$$\begin{cases} \dot{\delta x}_{(1)}(t; \tau) = f'_x(\bar{x}(t), \bar{u}(t), t) \delta x_{(1)}(t; \tau) + f'_u(\bar{x}(t), \bar{u}(t), t) \delta u(t; \tau), \\ \dot{\delta x}_{(2)}(t; \tau) = f'_x(\bar{x}(t), \bar{u}(t), t) \delta x_{(2)}(t; \tau) + \frac{1}{2} f''_{ww}(\bar{x}(t), \bar{u}(t), t) [(\delta x_{(1)}(t; \tau), \delta u(t; \tau))]^2, \\ \delta x_{(1)}(0; \tau) = \delta x_0(\tau), \quad \delta x_{(2)}(0; \tau) = 0. \end{cases}$$

Here, $\delta x_0(\tau) = \delta x_0 + \frac{a(\tau)}{\tau}$, and $\delta u(t; \tau) = \delta u(t) + \frac{v(t; \tau)}{\tau}$. It is clear that

$$x(t; \tau) = \bar{x}(t) + \tau \delta x_{(1)}(t; \tau) + \tau^2 \delta x_{(2)}(t; \tau) + o(\tau^2).$$

Therefore, by expanding in the Taylor series in (17), one has

$$\begin{aligned} o(\tau^2) \leq & \langle \mathcal{L}'_{x_0}(\bar{p}, \lambda), \tau \delta x_0(\tau) \rangle + \langle \mathcal{L}'_{x_1}(\bar{p}, \lambda), \tau \delta x_{(1)}(1; \tau) + \tau^2 \delta x_{(2)}(1; \tau) \rangle \\ & + \frac{\tau^2}{2} \mathcal{L}''_{pp}(\bar{p}, \lambda) [(\delta x_0(\tau), \delta x_{(1)}(1; \tau))]^2 + \int_0^1 \langle \nu(t), r'_x(\bar{x}(t), \bar{u}(t), t) \\ & [\tau \delta x_{(1)}(t; \tau) + \tau^2 \delta x_{(2)}(t; \tau)] + \tau r'_u(\bar{x}(t), \bar{u}(t), t) \delta u(t; \tau) \\ & + \frac{\tau^2}{2} r''_{ww}(\bar{x}(t), \bar{u}(t), t) [(\delta x_{(1)}(t; \tau), \delta u(t; \tau))]^2 \rangle dt. \end{aligned}$$

Using the adjoint equation, one has

$$\begin{aligned} \frac{d}{dt} \langle \psi(t), \delta x_{(1)}(t; \tau) \rangle &= \nu(t) r'_x(t) \delta x_{(1)}(t; \tau) + \psi(t) f'_u(t) \delta u(t; \tau), \\ \frac{d}{dt} \langle \psi(t), \delta x_{(2)}(t; \tau) \rangle &= \nu(t) r'_x(t) \delta x_{(2)}(t; \tau) + \frac{1}{2} \psi(t) f''_{ww}(t) [(\delta x_{(1)}(t; \tau), \delta u(t; \tau))]^2. \end{aligned}$$

Here, and from now on, the dependence on the optimal process is, for simplicity, omitted. Therefore, using these relations and the transversality conditions (3), and by gathering the terms with τ , and τ^2 in two different groups, we obtain

$$o(\tau^2) \leq -\tau \int_0^1 [\mathcal{H}'_u(t) - v(t)r'_u(t)] \delta u(t; \tau) dt + \frac{\tau^2}{2} \left(\mathcal{L}''_{pp}(\bar{p}, \lambda) [(\delta x_0(\tau), \delta x_{(1)}(1; \tau))]^2 - \int_0^1 (\psi(t)f''_{ww}(t) + v(t)r''_{ww}(t)) [(\delta x_{(1)}(t; \tau), \delta u(t; \tau))]^2 dt \right).$$

Now, as the implication of (5), we obtain (11) for the given $(\delta x_0, \delta u) \in \mathcal{K}_\varepsilon \cap \mathcal{X}_\infty$. Then, Estimate (11) is proven on the cone $\mathcal{K}_\varepsilon$ by a simple passage to the limit in $\mathcal{X}$.

Let us pass to the limit as $\varepsilon \rightarrow 0$, and prove (11) on the entire cone $\mathcal{K}$. Take $h \in \mathcal{K}$. One needs to justify that, for all $\varepsilon > 0$, there exists $h_\varepsilon \in \mathcal{K}_\varepsilon$ such that $h_\varepsilon \rightarrow h$ in $\mathcal{X}$. Indeed, then, (11) is proven due to a simple passage to the limit. However, the existence of such h_ε is yielded by Lemma 1. Indeed, it is a straightforward task to verify that the derivative operator $\mathcal{F}'_\varepsilon$ satisfies all the assumptions of this assertion; it obviously converges pointwise to $\mathcal{F}'_0$ as $\varepsilon \rightarrow 0$, while its image is closed, as was confirmed above. The image of the conjugate operator is closed due to the regularity of the reference control process with respect to the mixed constraints which is merely a technical step to ensure. Another technical step is to assert that $\mathcal{F}'_\varepsilon[\mathcal{F}'_\varepsilon]^*$ is positive due to normality. Moreover, the constant of covering does not depend on ε . It is also a straightforward task to verify that the rest of the assumptions hold if we consider $\mathcal{C}_0$ as $\mathcal{C}$.

The proof is complete. $\square$

5. Abnormal Case

In this section, we consider the case when $\text{rank } Q < k$. This case, when the normality condition is not satisfied, is called abnormal. Then, as a simple example can show that Theorem 1 fails to hold. Firstly, the normalized multiplier λ is no longer unique, and moreover, there may not exist such a multiplier from the cone Λ for which Estimate (11) is still valid everywhere on $\mathcal{K}$. Therefore, Theorem 1 requires a certain refinement in the abnormal case. Let us formulate the “abnormal” version of this statement. In this enterprise, we follow the method based on the so-called index approach.

Consider the reduced cone of Lagrange multipliers $\Lambda_a = \Lambda_a(\bar{x}, \bar{u})$, which contains multipliers $\lambda \in \Lambda$ such that

$$\text{ind}_N \Omega_\lambda = k - \text{rank } Q.$$

Here, the notation ind_X stands for the index of a quadratic form over the space X , and $\mathcal{N} = \mathcal{N}_e \cap \mathcal{N}_r$. Consider also the following extra hypotheses.

Hypothesis 2 (H2). Mappings f, r_2 are affine w.r.t. u , while r_1 is convex w.r.t. u .

Hypothesis 3 (H3). Mixed constraints are globally regular, that is, $U(x, t) = U_R(x, t)$ for all x and t . Moreover, the set-valued mapping $U(x, t)$ is uniformly bounded.

The main result of this section is as follows.

Theorem 2. Let $(\bar{x}, \bar{u})$ be an optimal control process in Problem (1). Suppose that this process is regular with respect to the mixed constraints.

Then, $\Lambda_a \neq \emptyset$. Moreover, under (H2) and (H3), one has

$$\max_{\lambda \in \Lambda_a} \Omega_\lambda [(\delta x_0, \delta u)]^2 \geq 0 \quad \forall (\delta x_0, \delta u) \in \mathcal{K}. \quad (18)$$

In the case of local weak minimum, Estimate (18) has been proven in [14]. Here, our task is to prove it in the case of global strong, or Pontryagin’s type of the minimum. In [3],

the condition that the cone Λ_a is non-empty has been proven in the class of generalized controls. Note that, under the normality condition of the optimal control process, Estimate (18) implies (11) since the normalized multiplier is unique. Thus, Theorem 2, in essence, represents a stronger assertion than Theorem 1, albeit under some extra assumptions such as (H2) and (H3). These two assumptions are meant to simplify the presentation. Note that (H3) is sufficient to suppose on the optimal trajectory only.

Proof. The proof of theorem is divided into the two stages.

STAGE 1. In this stage, we prove that $\Lambda_a \neq \emptyset$. In the beginning, suppose that Hypothesis (H3) is valid. Under (H1) and (H3), it is convenient to assume that $f(x, u, t)$, and $r(x, u, t)$ are constant with respect to (x, u) , and t , outside of some sufficiently large ball. This can be obtained due to a simple problem reduction. In what follows, it will not also be restrictive to consider that $\varphi(p^*) = 0$, and, for the simplicity of exposition, to consider that all the constraints are scalar-valued, i.e., $k_1 = k_2 = q_1 = 1$, while $q_2 = 0$.

Let a, b be non-negative numbers. Consider the mapping

$$\Delta(a, b) := \begin{cases} ab^{-4} & \text{if } b > 0, \\ 1 & \text{if } a > 0, b = 0, \\ 0 & \text{if } a = b = 0. \end{cases}$$

This function is lower semi-continuous. It will serve as a penalty function in the applied method below.

Take a pair $(x_0, u) \in \mathcal{X}$, and consider the unique solution to the Cauchy problem $\dot{x}(t) = f(x(t), u(t), t)$, $x(0) = x_0$, which exists on the entire time interval $[0, 1]$ due to the above assumptions. Set $p = (x_0, x_1)$, where $x_1 = x(1)$. Note that p depends on (x_0, u) . Let $\{\varepsilon_i\}$ be an arbitrary sequence of positive numbers converging to zero. Consider the mapping $\varphi_i^+(p) = (\varphi(p) + \varepsilon_i)^+$, where $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Thus, the following functional over the space $\mathcal{X}$ is well-defined:

$$F_i(x_0, u) := \varphi_i^+(p) + \Delta\left((e_1^+(p))^2 + |e_2(p)|^2 + \int_0^1 (r(x, u, t)^+)^2 dt, \varphi_i^+(p)\right).$$

Functional F_i is lower semi-continuous which is a straightforward exercise to verify due to the assumptions made above regarding the mappings f and r . At the same time, this functional is positive everywhere: $F_i > 0$.

Consider the following problem

$$\text{Minimize } F_i(x_0, u), \quad (x_0, u) \in \mathcal{X}.$$

Note that $F_i(\bar{x}_0, \bar{u}) = \varepsilon_i$. By applying the smooth variational principle, see, e.g., in [17], for each i , there exists an element $(x_{0,i}, u_i) \in \mathcal{X}$ and a sequence of elements $(\tilde{x}_j, \tilde{u}_j) \in \mathcal{X}$, $j = 1, 2, \dots$, converging to $(x_{0,i}, u_i)$ such that

$$F_i(x_{0,i}, u_i) \leq F_i(\bar{x}_0, \bar{u}) = \varepsilon_i, \quad (19)$$

$$|x_{0,i} - \bar{x}_0|^2 + \int_0^1 |u_i(t) - \bar{u}(t)|^2 dt \leq \sqrt[3]{\varepsilon_i^2}, \quad (20)$$

and the pair $(x_{0,i}, u_i)$ is the unique solution to the following problem:

$$\text{Minimize } F_i(x_0, u) + \sqrt[3]{\varepsilon_i} \sum_{j=1}^{\infty} 2^{-j} \left(|x_0 - \tilde{x}_j|^2 + \int_0^1 |u - \tilde{u}_j(t)|^2 dt \right), \quad (x_0, u) \in \mathcal{X}.$$

Suppose that $\varphi_i^+(p_i) = 0$. Then, $\varphi(p_i) < 0$ and, in view of optimality, taking into account that $\varphi(\bar{p}) = 0$, it follows that some of constraints in (1): e_1 , or e_2 , or r , are violated. Therefore, by definition of Δ , one has $F_i(x_{0,i}, u_i) \geq 1$. However, this contradicts (19) for

$i > 1$. Thus, $\varphi_i^+(p_i) > 0$. Consider a number $\delta_i > 0$ such that $\varphi_i^+(p) > 0 \forall p: |p - p_i| \leq \delta_i$. Then, by virtue of, again, the definition of Δ , the pair $(x_{0,i}, u_i)$ is the unique global minimum to the following control problem:

$$\begin{aligned} \text{Minimize} \quad & z_0 + z_0^{-4} \left((e_1(p)^+)^2 + |e_2(p)|^2 \right) + \int_0^1 z^{-4} (r(x, u, t)^+)^2 dt \\ & + \sqrt[3]{\varepsilon_i} \sum_{j=1}^{\infty} 2^{-j} \left(|x_0 - \tilde{x}_j|^2 + \int_0^1 |u - \tilde{u}_j(t)|^2 dt \right), \\ \text{subject to} \quad & \dot{x} = f(x, u, t), \\ & \dot{z} = 0, \text{ for a.a. } t \in [0, 1], \\ & |p - p_i| \leq \delta_i, z_0 = \varphi_i^+(p). \end{aligned} \quad (21)$$

Denote by x_i, z_i the solution to (21), that is, the trajectory corresponding to the pair $(x_{0,i}, u_i(\cdot))$. Note that function $z_i(\cdot)$ is constant, and thus it can be treated simply as number $z_i \in \mathbb{R}$.

Problem (21) is, as a matter of fact, unconstrained. Consider the first and second-order necessary optimality conditions for this problem.

The first-order conditions are stated as follows. There exist a number $\lambda_i^0 > 0$, and absolutely continuous conjugate functions ψ_i and σ_i which correspond to x_i , and z_i , respectively, such that, for a.a. $t \in [0, 1]$,

$$\begin{aligned} \dot{\psi}_i(t) &= -\mathcal{H}'_x(x_i(t), u_i(t), \psi_i(t), t) + 2\lambda_i^0 z_i^{-4} r^+(x_i(t), u_i(t), t) r'_x(x_i(t), u_i(t), t), \\ \dot{\sigma}_i(t) &= -4\lambda_i^0 z_i^{-5} (r(x_i(t), u_i(t), t)^+)^2, \end{aligned} \quad (22)$$

$$\begin{aligned} \psi_i(s) &= (-1)^s \lambda_i^0 \left(2z_i^{-4} \left(e_1^+(p_i) \frac{\partial e_1}{\partial x_s}(p_i) + e_2(p_i) \frac{\partial e_2}{\partial x_s}(p_i) \right) + (1-s) \sqrt[3]{\varepsilon_i} \omega'_{1,i}(x_{0,i}) \right) \\ &\quad - (-1)^s \rho_i \frac{\partial \varphi}{\partial x_s}(p_i), \quad s = 0, 1, \end{aligned} \quad (23)$$

$$\begin{aligned} \sigma_i(0) &= \lambda_i^0 \left(1 - 4z_i^{-5} (e_1^+(p_i))^2 - 4z_i^{-5} |e_2(p_i)|^2 \right) + \rho_i, \\ \sigma_i(1) &= 0, \end{aligned}$$

$$\begin{aligned} \max_{u \in \mathbb{R}^m} & \left(\mathcal{H}(x_i(t), u, \psi_i(t), t) - \lambda_i^0 z_i^{-4} (r(x_i(t), u, t)^+)^2 - \lambda_i^0 \sqrt[3]{\varepsilon_i} \cdot \omega_{2,i}(u, t) \right) \\ &= \mathcal{H}(x_i(t), u_i(t), \psi_i(t), t) - \lambda_i^0 z_i^{-4} (r(x_i(t), u_i(t), t)^+)^2 - \lambda_i^0 \sqrt[3]{\varepsilon_i} \cdot \omega_{2,i}(u_i(t), t), \end{aligned} \quad (24)$$

Here, $\rho_i \in \mathbb{R}$ is the multiplier corresponding to the constraint $z_0 = \varphi_i^+(p)$,

$$\omega_{1,i}(x) := \sum_{j=1}^{\infty} 2^{-j} |x - \tilde{x}_j|^2, \text{ and } \omega_{2,i}(u, t) := \sum_{j=1}^{\infty} 2^{-j} |u - \tilde{u}_j(t)|^2.$$

Conditions (22)–(24) are the first-order optimality conditions in the form of the maximum principle. Consider the second-order optimality conditions for Problem (21).

Take an element $(\delta x_0, \delta u) \in \mathcal{X}$. Consider the variational differential equation related to (21), that is,

$$\begin{aligned} \delta x_i(t) &= \frac{\partial f}{\partial x}(x_i(t), u_i(t), t) \delta x_i(t) + \frac{\partial f}{\partial u}(x_i(t), u_i(t), t) \delta u(t), \\ \delta z_i(t) &= 0, \end{aligned} \quad (25)$$

for a.a. $t \in [0, 1]$, where

$$\begin{aligned} \delta x_i(0) &= \delta x_0, \\ \delta z_i(0) &= \varphi'(p_i) \delta p_i. \end{aligned}$$

Here, $\delta p_i := (\delta x_0, \delta x_i(1))$.

The solution to (25) exists, and it is unique on the entire time interval $[0, 1]$ due to the assumptions made above. The function $\delta z_i(\cdot)$ is obviously constant, thus, it is treated just as number δz_i in what follows.

On the space $\mathcal{X}$, consider the quadratic form

$$\begin{aligned}\Omega_i[(\delta x_0, \delta u)]^2 &= \lambda_i^0 2z_i^{-4} e_1(p_i)^+ e_1''(p_i) [\delta p_i]^2 + 2z_i^{-4} e_2(p_i) e_2''(p_i) [\delta p_i]^2 \\ &\quad + 20z_i^{-6} \left((e_1(p_i)^+)^2 + |e_2(p_i)|^2 \right) \delta z_i^2 - \left(\rho_i \varphi''(p_i) + \lambda_i^0 \sqrt[3]{\varepsilon_i} \omega_{1,i}''(p_i) \right) [\delta p_i]^2 \\ &\quad - \int_0^1 \mathcal{H}_{ww}''(x_i(t), u_i(t), \psi_i(t), t) [(\delta x_i(t), \delta u(t))]^2 dt \\ &\quad + \lambda_i^0 \int_0^1 2z_i^{-4} r(x_i(t), u_i(t), t)^+ r_{ww}''(x_i(t), u_i(t), t) [(\delta x_i(t), \delta u(t))]^2 dt \\ &\quad + \int_0^1 20z_i^{-6} (r(x_i(t), u_i(t), t)^+)^2 \delta z_i^2 dt + \lambda_i^0 \sqrt[3]{\varepsilon_i} \int_0^1 \omega_{2,i}''(u_i(t), t) [\delta u(t)]^2 dt.\end{aligned}$$

Consider the closed subspace $\mathcal{N}_i \subseteq \mathcal{X}$ of pairs $(\delta x_0, \delta u)$ such that

- (a) $e'(p_i) \delta p_i = 0$;
- (b) $r'_x(x_i(t), u_i(t), t) \delta x_i(t) + r'_u(x_i(t), u_i(t), t) \delta u(t) = 0$ for a.a. $t \in T_i$.

Here, $T_i := \{t \in [0, 1] : r(x_i(t), u_i(t), t) \geq 0\}$.

Then, the second-order necessary optimality condition is given by the inequality

$$\Omega_i[(\delta x_0, \delta u)]^2 \geq 0 \quad \forall (\delta x_0, \delta u) \in \mathcal{N}_i. \quad (26)$$

(Note that functional $F(x_0, u)$ is not twice continuously differentiable. At the same time, the scalar function $F(x_{0,i} + \tau \delta x_0, u_i + \tau \delta u)$ of τ possesses the second derivative w.r.t. τ at $\tau = 0$, provided that $(\delta x_0, \delta u) \in \mathcal{N}_i$. Using this fact, and the fact that Problem (21) is unconstrained, it is simple to derive (26) by applying direct variations arguments.)

The next step is to pass to the limit as $i \rightarrow \infty$ in the obtained optimality conditions. Firstly, it follows from (20) that $x_{0,i} \rightarrow \bar{x}_0$, and $u_i(t) \rightarrow \bar{u}(t)$ strongly in L_2 , and, thereby, $x_i(t) \rightrightarrows \bar{x}(t)$ uniformly on $[0, 1]$. Then, $z_i \rightarrow 0$. Define

$$\begin{aligned}\lambda_i^1 &:= 2\lambda_i^0 z_i^{-4} e_1(p_i)^+; \\ \lambda_i^2 &:= 2\lambda_i^0 z_i^{-4} e_2(p_i); \\ v_i(t) &:= 2\lambda_i^0 z_i^{-4} r(x_i(t), u_i(t), t)^+, \end{aligned}$$

and consider the following normalization for the multipliers

$$|\lambda_i| + |\psi_i(0)| + \|v_i\|_{L_2} = 1, \quad (27)$$

where $\lambda_i = (\lambda_i^0, \lambda_i^1, \lambda_i^2)$.

Let us show that $\sigma_i(0) \rightarrow 0$. Indeed, one has

$$\sigma_i(0) = 4 \int_0^1 \lambda_i^0 z_i^{-5} r(x_i(t), u_i(t), t)^+)^2 dt = 2 \int_0^1 v_i(t) z_i^{-1} r(x_i(t), u_i(t), t)^+ dt.$$

However, due to (19), one has $\|z_i^{-1} r(x_i(t), u_i(t), t)^+\|_{L_2} \rightarrow 0$. This, together with (27), implies that $\sigma_i(0) \rightarrow 0$. Then, the transversality condition and, again, (19) and (27) simply yield that $\lambda_i^0 - \rho_i \rightarrow 0$.

By passing to a subsequence, in view of the compactness argument, one may assume from (27) that $\lambda_i \rightarrow \lambda$, $\psi_i(t) \rightrightarrows \psi(t)$ and $v_i \xrightarrow{w} v$ weakly in L_2 as $i \rightarrow \infty$ for some multipliers $\lambda = (\lambda^0, \lambda_1, \lambda_2)$, ψ and v . Then, $\rho_i \rightarrow \lambda^0$. It is also clear that by passing to a subsequence, one can assert that $\lambda_i^0 z_i^{-4} \rightarrow \infty$. Indeed, otherwise all the multipliers converge to zero, contradicting (27). By virtue of the regularity of the optimal control process with respect to mixed constraints, for each i , there exists a control function ζ_i such

that $\zeta_i(t) \in U(x_i(t), t)$ a.e., and $\zeta_i \rightarrow \bar{u}$ in L_∞ . Thus, from the maximum condition (24), it follows that $r(x_i(t), u_i(t), t)^+ \rightarrow 0$ uniformly. Since the set $U(\bar{x}(t), t)$ is uniformly bounded, this implies, again due to regularity, that the control function u_i is essentially bounded uniformly with respect to i , that is $\|u_i\|_{L_\infty} \leq \text{const}$.

From (24), one derives that

$$v_i(t) \cdot r'_u(x_i(t), u_i(t), t) = \mathcal{H}'_u(x_i(t), u_i(t), \psi_i(t), t) - \lambda_i^0 \sqrt[3]{\varepsilon_i} \cdot \frac{\partial \omega_{2,i}}{\partial u}(u_i(t), t).$$

Whence, using regularity and the above obtained facts, one has

$$|v_i(t)| \leq \text{const}(|\psi_i(t)| + \lambda_i^0) \quad \forall i, \quad t \in [0, 1]. \quad (28)$$

Using the facts and estimates obtained above, one can simply pass to the limit in (22)–(24) and prove that the set of multipliers (λ, ψ, ν) satisfies the maximum principle. At the same time, the fact that $\lambda \neq 0$ follows from (28).

Now, let us pass to the limit in (26). Take numbers $\varepsilon > 0$ and $\sigma > 0$. Restricting to a subsequence, one can state that $u_i(t) \rightarrow \bar{u}(t)$ for a.a. $t \in [0, 1]$. Due to Egorov's theorem, there is a subset E_σ , the measure of which equals $1 - \sigma$, such that $u_i(t) \rightrightarrows \bar{u}(t)$ uniformly on E_σ . Denote $T_\varepsilon := \{t \in [0, 1] : r(\bar{x}(t), \bar{u}(t), t) \geq -\varepsilon\}$, $T_\varepsilon(\sigma) := T_\varepsilon \cap E_\sigma$.

Consider the bounded linear operator $\mathcal{A}_i : \mathcal{X}(E_\sigma) \rightarrow \mathbb{L}_2(T_\varepsilon(\sigma); \mathbb{R})$ such that

$$\mathcal{A}_i(\delta x_0, \delta u) = r'_x(x_i(t), u_i(t), t)\delta x_i(t) + r'_u(x_i(t), u_i(t), t)\delta u(t)|_{T_\varepsilon(\sigma)},$$

where $\mathcal{X}(E_\sigma)$ is defined as $\mathcal{X}$, but $\delta u(t) = 0$ for a.a. $t \notin E_\sigma$. It is a simple matter to show that, due to the uniform convergence, one has $\mathcal{A}_i \rightarrow \mathcal{A}$ strongly, where

$$\mathcal{A}(\delta x_0, \delta u) = r'_x(\bar{x}(t), \bar{u}(t), t)\delta x(t) + r'_u(\bar{x}(t), \bar{u}(t), t)\delta u(t)|_{T_\varepsilon(\sigma)}.$$

Then, as is known, $\ker \mathcal{A}_i \rightarrow \mathcal{X}_\varepsilon(\sigma) := \ker \mathcal{A} \subseteq \mathcal{X}(E_\sigma)$. It is clear that $\mathcal{X}_\varepsilon(\sigma) \rightarrow \mathcal{X}_\varepsilon := \mathcal{X}_\varepsilon(0)$ as $\sigma \rightarrow 0$ by virtue of its definition and the regularity condition. One needs to use the solution to a corresponding Volterra equation to prove this simple fact. Then, Lemma 1 yields that $\mathcal{X}_\varepsilon \rightarrow \mathcal{N}_r$ as $\varepsilon \rightarrow 0$ if we consider $C = \{0\}$. Here, when treating the convergence of spaces, the symbol ' $\rightarrow$ ' stands for Limsup.

Let $\Pi_i \subseteq \mathcal{X}$ denote the kernel of the endpoint operator $e'(p_i)\delta p_i$. It is clear that $\text{codim } \Pi_i \leq k$. Then, this is a simple exercise to ensure the existence of a subspace $\Pi \subseteq \mathcal{N}_e$ such that $\text{codim } \Pi \leq k$ and

$$\Pi \cap \mathcal{N}_r \subseteq \text{Limsup } \Pi_i \cap \ker \mathcal{A}_i,$$

where Limsup is total: firstly as $i \rightarrow \infty$, then, as $\sigma \rightarrow 0$ and finally, as $\varepsilon \rightarrow 0$. At the same time, note that $T_i \cap E_\sigma \subseteq T_\varepsilon(\sigma)$ for all large i . Therefore, one has the embedding $\ker \mathcal{A}_i \cap \Pi_i \subseteq \mathcal{N}_i \cap \mathcal{X}(E_\sigma)$, and then, the passage to the limit in (26) gives the condition $\Lambda_a \neq \emptyset$. In the latter deduction, Proposition 1 of [14] has been used and also the fact that the terms with δz_i^2 in Ω_i converge to zero in view of (19) and (27).

Now, it is necessary to remove the extra assumptions imposed in (H3) regarding the boundedness and global regularity. However, this can be done following precisely the same method as presented in [4]. Take $c > 0$, and consider the additional control constraint $|u| \leq c$. For each $\varepsilon > 0$, there will be N specifically constructed regular selectors of $U(x, t)$ which are surrounded with N ε -tubes as in the above-cited source. Then, the passage to the limit, firstly as $\varepsilon \rightarrow 0$, then, as $N \rightarrow \infty$, and, at the end, as $c \rightarrow \infty$ will complete the proof for Stage 1.

The full proof for the next stage is rather lengthy. Therefore, let us present it schematically, in a sketch-form, exposing the main idea.

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