## Article

# New Formulas and Connections Involving Euler Polynomials 

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#### Abstract

The major goal of the current article is to create new formulas and connections between several well-known polynomials and the Euler polynomials. These formulas are developed using some of these polynomials' well-known fundamental characteristics as well as those of the Euler polynomials. In terms of the Euler polynomials, new formulas for the derivatives of various symmetric and non-symmetric polynomials, including the well-known classical orthogonal polynomials, are given. This leads to the deduction of several new connection formulas between various polynomials and the Euler polynomials. As an important application, new closed forms for the definite integrals for the product of various symmetric and non-symmetric polynomials with the Euler polynomials are established based on the newly derived connection formulas.


Keywords: Euler polynomials; special polynomials; hypergeometric functions; definite integrals; connection formulas

## 1. Introduction

Numerous problems in various fields, such as approximation theory and theoretical physics, depend on special functions. Considerable research has been conducted on several well-known polynomial sequences and the numbers that they are associated with. Therefore, from both theoretical and practical aspects, it is interesting to investigate various special functions. Among the essential special functions are the well-known Hermite, Laguerre, and Jacobi polynomials. These classical orthogonal polynomials were extensively studied by many authors, both theoretically and practically; see, for example, [1-4]. The Jacobi polynomials include six special polynomials. Four of these polynomials are symmetric: the ultraspherical, Legendre, and the first and second kinds of Chebyshev polynomials. The polynomials, namely, the third- and fourth-kind Chebyshev polynomials, are two celebrated non-symmetric classes of Jacobi polynomials. All six classes of Jacobi polynomials have their parts in approximation theory and numerical analysis; see, for example, [5-7]. Other types of polynomials were also studied by many authors. For example, the Lucas and Fibonacci sequences, as well as their extensions and modified polynomials, were investigated by many authors. The authors in [8,9] studied certain kinds of generalized Fibonacci and generalized Lucas polynomials and their corresponding numbers. Furthermore, they employed them to find reduction formulas for some even and odd radicals. New identities of Horadam sequences of integers with four parameters were introduced by the authors in [10]. In [11], certain Appel polynomials are treated using a matrix technique. To handle bivariate Appell polynomials, matrix calculus was used in [12]. Classical and quantum orthogonal polynomials are extensively studied in [13].

Euler polynomials and Euler numbers have been the subject of numerous contemporary and older investigations. For example, the author in [14] developed some relations between the Bernoulli and Euler polynomials. Some properties on the integral of the product of several Euler polynomials are presented in [15]. In [16], the authors discussed the
decomposition of the linear combinations of Euler polynomials with odd degrees. In [17], the authors found some identities for Euler and Bernoulli polynomials and their zeros. Other identities for the product of two Bernoulli and Euler polynomials were obtained in [18]. New types of Euler polynomials and numbers are developed in [19]. For some other classes relating to Euler polynomials, one can refer, for example, to [20,21]. From a practical point of view, Euler polynomials were utilized to treat different types of differential and integral equations. For example, in [22], certain fractional-order delay integro-differential equations were numerically treated using an operational matrix of derivatives based on the utilization of fractional-order Euler polynomials. In [23], a numerical scheme utilizing Euler wavelets was derived to handle the fractional order pantograph Volterra delay-integrodifferential equation. Two-dimensional Volterra integral equations of the fractional order were treated using two-dimensional Euler polynomials in [24].

The various formulas of special functions are important from both theoretical and practical perspectives. For example, the expressions for the high-order derivatives of different polynomials in terms of their original ones can be used to obtain some spectral solutions to different differential equations. For example, in [25], new expressions for the third- and fourth-kinds of Chebyshev polynomials were established and utilized for solving specific even-order BVPs. Some other expressions for the high-order derivatives were utilized in [26] for treating linear and non-linear BVPs of even order. The author in [27] found new derivative formulas for the sixth-kind Chebyshev polynomials and used them to provide a numerical solution to the non-linear Burgers' equation in one dimension. Additionally, among the important formulas concerned with special functions are the connection and linearization formulas. These formulas are useful in some applications (see, for example, [28]).

This paper aims to find some new formulas concerning the Euler polynomials. To be more precise, the objectives of the current paper can be listed in the following items:

- Developing new expressions for the high-order derivatives of different symmetric and non-symmetric polynomials in terms of Euler polynomials.
- Deducing connection formulas between different polynomials and Euler polynomials.
- Presenting an application to the derived connection formulas. Several new definite integral formulas of the product of different symmetric and non-symmetric polynomials with the Euler polynomials in closed forms.
The paper is organized as follows. Section 2 introduces an overview of Euler polynomials. In addition, some properties of some celebrated symmetric and non-symmetric polynomials are presented in this section. Section 3 develops new expressions for the derivatives of symmetric and non-symmetric polynomials as combinations of Euler polynomials. Section 4 is interested in deducing connection formulas between symmetric and non-symmetric polynomials with the Euler polynomials. In Section 5, an application to the connection formulas presented in Section 4 is displayed. More precisely, some new definite integral formulas of the product of different symmetric and non-symmetric polynomials with the Euler polynomials are given. Finally, Section 6 reports some conclusions.


## 2. Preliminaries and Some Essential Formulas

This section is interested in presenting an overview of the Euler polynomials and their related numbers. Furthermore, we introduce some properties of symmetric and non-symmetric polynomials. In addition, an account of some classes of polynomials that will be connected with Euler polynomials is given.

### 2.1. An Account of Euler Polynomials

The classical Euler polynomials $E_{m}(x)$ can be defined with the aid of the generating function [29]

$$
\frac{2 e^{x z}}{e^{z}+1}=\sum_{m=0}^{\infty} E_{m}(x) \frac{z^{m}}{m!}, \quad|z|<\pi
$$

The corresponding Euler number is given by

$$
E_{m}=2^{m} E_{m}\left(\frac{1}{2}\right) .
$$

This is the inversion formula of Euler polynomials:

$$
\begin{equation*}
x^{m}=\frac{1}{2} \sum_{k=0}^{m} c_{k}\binom{m}{m-k} E_{m-k}(x), \quad m \geq 0 \tag{1}
\end{equation*}
$$

where $c_{k}$ is defined as

$$
c_{k}= \begin{cases}2, & k=0 \\ 1, & k>0\end{cases}
$$

Additionally, among the famous identities of the polynomials $E_{m}(x)$ are the following identities [29]:

$$
\begin{aligned}
\frac{d}{d x} E_{m}(x) & =m E_{m-1}(x), \\
\int_{a}^{b} E_{m}(x) d x & =\frac{E_{m+1}(b)-E_{m+1}(a)}{m+1} .
\end{aligned}
$$

### 2.2. An Overview on Symmetric and Non-Symmetric Polynomials

Let us consider, respectively, the two classes of symmetric and non-symmetric polynomials, $\left\{P_{i}(x)\right\}_{i \geq 0}$ and $\left\{Q_{i}(x)\right\}_{i \geq 0}$. We can express these polynomials as:

$$
\begin{align*}
& P_{i}(x)=\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor} A_{m, i} x^{i-2 m}  \tag{2}\\
& Q_{i}(x)=\sum_{m=0}^{i} B_{m, i} x^{i-m} \tag{3}
\end{align*}
$$

where the symbol $\lfloor z\rfloor$ denotes the well-known floor function.
We give some of the celebrated symmetric and non-symmetric polynomials. We first refer to the classical normalized Jacobi polynomials $V_{m}^{(\lambda, \delta)}(x)$. These polynomials can be written in a hypergeometric form as [30]

$$
V_{m}^{(\lambda, \delta)}(x)={ }_{2} F_{1}\left(\left.\begin{array}{c}
-m, m+\lambda+\delta+1 \\
\lambda+1
\end{array} \right\rvert\, \frac{1-x}{2}\right) .
$$

Jacobi polynomials include six important classes of polynomials. The ultraspherical, Legendre, and first-and second-kind Chebyshev polynomials are symmetric Jacobi polynomials, so they can be expressed as in (2), while the two celebrated third- and fourth-kind Chebyshev polynomials are particular polynomials of the non-symmetric Jacobi polynomials, so they can be expressed as in (3). In addition, we have the following identities [31]:

$$
\begin{array}{ll}
T_{m}(x)=V_{m}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x), & U_{m}(x)=(m+1) V_{m}^{\left(\frac{1}{2}, \frac{1}{2}\right)}(x), \\
V_{m}(x)=V_{m}^{\left(-\frac{1}{2}, \frac{1}{2}\right)}(x), & W_{m}(x)=(2 m+1) V_{m}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x), \\
P_{m}(x)=V_{m}^{(0,0)}(x), & G_{m}^{(\delta)}(x)=V_{m}^{\left(\delta-\frac{1}{2}, \delta-\frac{1}{2}\right)}(x),
\end{array}
$$

where the first-, second-, third-, and fourth kinds of Chebyshev polynomials are, respectively, denoted by the symbols $T_{m}(x), U_{m}(x), V_{m}(x)$, and $W_{m}(x)$. Additionally, the polynomials $P_{n}(x)$ and $G_{n}^{(\delta)}(x)$ denote the Legendre and ultraspherical polynomials, respectively.

The helpful books by Andrews et al. [32] and Mason and Handscomb [33] are both excellent resources for in-depth surveys of Jacobi polynomials and their celebrated classes.

Additionally, among the non-symmetric Jacobi polynomials are the shifted Jacobi polynomials on $[0,1]$. These polynomials are defined as

$$
\tilde{V}_{m}^{(\lambda, \delta)}(x)=V_{m}^{(\lambda, \delta)}(2 x-1)
$$

We comment here that all six shifted special polynomials of the shifted Jacobi polynomials are non-symmetric. The power form representation of $\tilde{V}_{m}^{(\lambda, \delta)}(x)$ is given by [34]:

$$
\begin{equation*}
\tilde{V}_{m}^{(\lambda, \delta)}(x)=\frac{m!\Gamma(1+\lambda)}{\Gamma(1+m+\lambda)} \sum_{r=0}^{m} \frac{(-1)^{r}(1+\delta)_{m}(1+\lambda+\delta)_{2 m-r}}{(m-r)!r!(1+\delta)_{m-r}(1+\lambda+\delta)_{m}} x^{m-r} \tag{4}
\end{equation*}
$$

Note that the symbol $(z)_{\ell}$ in Formula (4) represents the Pochhammer function defined as: $(z)_{\ell}=\frac{\Gamma(z+\ell)}{\Gamma(z)}$.

Among the important symmetric polynomials are the Fibonacci and Lucas polynomials and their generalizations and modifications (see, [35]). Recently, Abd-Elhameed et al. in [9] studied two polynomials generalizing Fibonacci and Lucas polynomials. These polynomials may be constructed with the aid of the following two recursive formulas:

$$
\begin{equation*}
F_{k}^{A, B}(x)=A x F_{k-1}^{A, B}(x)+B F_{k-2}^{A, B}(x), \quad F_{0}^{A, B}(x)=1, F_{1}^{A, B}(x)=A x, \quad k \geq 2, \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{k}^{R, S}(x)=R x L_{k-1}^{R, S}(x)+S L_{k-2}^{R, S}(x), \quad L_{0}^{R, S}(x)=2, L_{1}^{R, S}(x)=R x, \quad k \geq 2 \tag{6}
\end{equation*}
$$

It is to be noted that several celebrated classes of polynomials can be obtained as special cases of the two generalized classes of $F_{k}^{A, B}(x)$ and $L_{k}^{R, S}(x)$ (see, [9]). For example, the Fibonacci polynomials $F_{k+1}(x)$ and Lucas polynomials $L_{k}(x)$ can be considered as special cases of $F_{k}^{A, B}(x)$ and $L_{k}^{R, S}(x)$. In fact, we have:

$$
F_{k+1}(x)=F_{k}^{1,1}(x), \quad L_{k}(x)=L_{k}^{1,1}(x)
$$

Furthermore, the power form representations of the generalized polynomials $F_{i}^{A, B}(x)$ and $L_{i}^{R, S}(x)$ are, respectively, given as follows [9]:

$$
\begin{align*}
& F_{i}^{A, B}(x)=\sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor}\binom{i-m}{m} B^{m} A^{i-2 m} x^{i-2 m}, \quad i \geq 0  \tag{7}\\
& L_{i}^{R, S}(x)=i \sum_{m=0}^{\left\lfloor\frac{i}{2}\right\rfloor} \frac{S^{m} R^{i-2 m}\binom{i-m}{m}}{i-m} x^{i-2 m}, \quad i \geq 1
\end{align*}
$$

## 3. New Expressions for the Derivatives of Some Celebrated Polynomials in Terms of Euler Polynomials

This section is devoted to developing new expressions for the high-order derivatives of some symmetric and non-symmetric polynomials in terms of Euler polynomials.

### 3.1. Derivative Expressions for Some Symmetric Polynomials

In this section, we give the derivatives of some symmetric polynomials in terms of the Euler polynomials. To be more precise, the derivatives of the generalized Fibonacci polynomials that are defined in (5), the generalized Lucas polynomials that are defined in (6), the ultraspherical polynomials, and the Hermite polynomials will be expressed in terms of the Euler polynomials.

Theorem 1. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the generalized Fibonacci polynomials $F_{n}^{A, B}$ defined in (5) have the following expansion in terms of Euler polynomials:

$$
\begin{align*}
D^{\ell} F_{n}^{A, B}(x)= & A^{n}\left(\sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{A^{-2 r} B^{r}(n-r)!(2 r)!+n!r!}{2 r!(2 r)!(-\ell+n-2 r)!}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r+\frac{1}{2} \\
-n
\end{array} \right\rvert\,-\frac{4 B}{A^{2}}\right) E_{n-\ell-2 r}(x)\right. \\
& \left.+n!\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{{ }^{( } F_{1}\left(\left.\begin{array}{c}
-r,-r-\frac{1}{2} \\
-n
\end{array} \right\rvert\,-\frac{4 B}{A^{2}}\right)}{2(2 r+1)!(-\ell+n-2 r-1)!} E_{n-\ell-2 r-1}(x)\right) . \tag{8}
\end{align*}
$$

Proof. The power-form representation of the polynomials $F_{n}^{A, B}(x)$ in (7) allows one to write

$$
D^{\ell} F_{n}^{A, B}(x)=\sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{A^{n-2 m} B^{m}(1-2 m+n)_{m}(1-\ell-2 m+n)_{\ell}}{m!} x^{n-2 m-\ell}
$$

Inserting the inversion formula of the Euler polynomials (1) yields the following relation:

$$
\begin{aligned}
D^{\ell} F_{n}^{A, B}(x)= & \sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{A^{n-2 m} B^{m}(1-2 m+n)_{m}(1-\ell-2 m+n)_{\ell}}{2 m!} \times \\
& \sum_{s=0}^{n-2 m-\ell} c_{s}\binom{-\ell-2 m+n}{-\ell-2 m+n-s} E_{n-2 m-s-\ell}(x)
\end{aligned}
$$

After some algebraic computations, the last formula can be rewritten in the form

$$
\begin{aligned}
D^{\ell} F_{n}^{A, B}(x)= & \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{1}{2(n-2 r-\ell)!} \sum_{s=0}^{r} \frac{c_{2 r-2 s} A^{n-2 s} B^{s}(n-s)!}{s!(2 r-2 s)!} E_{n-\ell-2 r}(x) \\
& +\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{1}{2(-\ell+n-2 r-1)!} \sum_{s=0}^{r} \frac{c_{2 r-2 s-1} A^{n-2 s} B^{s}(n-s)!}{s!(2 r-2 s+1)!} E_{n-\ell-2 r-1}(x) .
\end{aligned}
$$

Based on the following two identities:

$$
\begin{aligned}
& \sum_{s=0}^{r} \frac{c_{2 r-2 s} A^{n-2 s} B^{s}(n-s)!}{s!(2 r-2 s)!}=\frac{A^{n}\left(A^{-2 r} B^{r}(n-r)!(2 r)!+n!r!{ }_{2} F_{1}\left(\begin{array}{c}
-r,-r+\frac{1}{2} \\
-n
\end{array}\right.\right.}{\left.\left.r!-\frac{4 B}{A^{2}}\right)\right)} \\
& r!(2 r)!
\end{aligned},
$$

Formula (8) can be obtained. This proves Theorem 1.
Remark 1. It is to be noted that, for the case corresponding to the choice $B=-\frac{A^{2}}{4}$, Formula (8) can be simplified due to the Chu-Vandermond identity. The following corollary exhibits this result.

Corollary 1. For the case $B=-\frac{A^{2}}{4}$, Formula (8) reduces to the following one:

$$
\begin{align*}
D^{\ell} F_{n}^{A,-\frac{A^{2}}{4}}(x)= & \frac{1}{2} A^{n} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{\frac{\left(-\frac{1}{4}\right)^{r}(n-r)!}{r!}+\frac{n!\left(n-2 r+\frac{3}{2}\right)_{r}}{(2 r)!(n-r+1)_{r}}}{(-\ell+n-2 r)!} E_{n-\ell-2 r}(x)  \tag{9}\\
& +\frac{1}{2} A^{n} n!\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{\left(n-2 r+\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+1)_{r}} E_{n-\ell-2 r-1}(x) .
\end{align*}
$$

Proof. The substitution by $B=-\frac{A^{2}}{4}$ into Formula (8) yields

$$
\begin{aligned}
D^{\ell} F_{n}^{A,-\frac{A^{2}}{4}}(x)= & =\frac{1}{2} A^{n} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{\left(-\frac{1}{4}\right)^{r}(n-r)!(2 r)!+n!r!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r+\frac{1}{2} \\
-n
\end{array} \right\rvert\, 1\right)}{r!(2 r)!(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
& +\frac{1}{2} A^{n} n!\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{{ }_{2} F_{1}\left(\left.\begin{array}{c}
\left.-r, \left.-r-\frac{1}{2} \right\rvert\, 1\right) \\
-n
\end{array} \right\rvert\,\right.}{(2 r+1)!(-\ell+n-2 r-1)!} E_{n-\ell-2 r-1}(x) .
\end{aligned}
$$

Chu-Vandermonde identity implies the following two identities:

$$
\begin{aligned}
& { }_{2} F_{1}\left(\left.\begin{array}{c|}
-r,-r+\frac{1}{2} \\
-n
\end{array} \right\rvert\, 1\right)=\frac{\left(n-2 r+\frac{3}{2}\right)_{r}}{(n-r+1)_{r}}, \\
& { }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r-\frac{1}{2} \\
-n
\end{array} \right\rvert\, 1\right)=\frac{\left(n-2 r+\frac{1}{2}\right)_{r}}{(n-r+1)_{r}},
\end{aligned}
$$

therefore, the following formula can be obtained:

$$
\begin{aligned}
D^{\ell} F_{n}^{A,-\frac{A^{2}}{4}}(x)= & \frac{1}{2} A^{n} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{\frac{\left(-\frac{1}{4}\right)^{r}(n-r)!}{r!}+\frac{n!\left(n-2 r+\frac{3}{2}\right)_{r}}{(2 r)!(n-r+1)_{r}}}{(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
& +\frac{1}{2} A^{n} n!\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{\left(n-2 r+\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+1)_{r}} E_{n-\ell-2 r-1}(x) .
\end{aligned}
$$

Remark 2. An expression for the derivatives of Chebyshev polynomials of the first kind can be obtained as a direct special case of Formula (9). The following corollary displays this important specific result.

Corollary 2. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the Chebyshev polynomials of the second kind can be represented in terms of Euler polynomials as

$$
\begin{aligned}
D^{\ell} U_{n}(x)= & 2^{n-1} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{\frac{\left(-\frac{1}{4}\right)^{r}(n-r)!}{r!}+\frac{n!\left(n-2 r+\frac{3}{2}\right)_{r}}{(2 r)!(n-r+1)_{r}}}{(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
& +2^{n-1} n!\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{\left(n-2 r+\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+1)_{r}} E_{n-\ell-2 r-1}(x) .
\end{aligned}
$$

Theorem 2. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the ultraspherical polynomials $G_{n}^{(\delta)}(x)$ can be expanded in terms of Euler polynomials as in the following form:

$$
\begin{align*}
& D^{\ell} G_{n}^{(\delta)}(x)=\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)} \times \\
& \left\lfloor\sum_{r=0}^{\frac{n-\ell}{2}}\right\rfloor \frac{2^{n-2 r+2 \delta-2}\left((-1)^{r}(2 r)!\Gamma(n-r+\delta)+\frac{4^{r} r!\Gamma(n+\delta)\left(n-2 r+\delta+\frac{1}{2}\right)_{r}}{(n-r+\delta)_{r}}\right)}{r!(2 r)!(-\ell+n-2 r)!} E_{n-\ell-2 r}(x)  \tag{10}\\
& +\frac{n!2^{n+2 \delta-2} \Gamma\left(\delta+\frac{1}{2}\right) \Gamma(n+\delta)}{\sqrt{\pi} \Gamma(n+2 \delta)} \sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{\left(n-2 r+\delta-\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+\delta)_{r}} E_{n-\ell-2 r-1}(x) .
\end{align*}
$$

Proof. The power form representation of the ultraspherical polynomials given by

$$
G_{n}^{(\delta)}(x)=\frac{n!\Gamma(2 \delta+1)}{2 \Gamma(\delta+1) \Gamma(n+2 \delta)} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{m} 2^{n-2 m} \Gamma(n-m+\delta)}{m!(n-2 m)!} x^{n-2 m},
$$

enables one to write:

$$
D^{\ell} G_{n}^{(\delta)}(x)=\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)} \sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(-1)^{m} 2^{-2 m+n+2 \delta-1} \Gamma(-m+n+\delta)}{m!(-\ell-2 m+n)!} x^{n-2 m-\ell},
$$

which can be written again with the aid of the inversion Formula (1) into the form

$$
\begin{aligned}
D^{\ell} G_{n}^{(\delta)}(x)= & \frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{2 \sqrt{\pi} \Gamma(n+2 \delta)} \times \\
& \left\lfloor\sum_{m=0}^{\left.\frac{n-\ell}{2}\right\rfloor} \frac{(-1)^{m} 2^{-2 m+n+2 \delta-1} \Gamma(-m+n+\delta)}{m!(-\ell-2 m+n)!} \sum_{s=0}^{n-2 m-\ell} c_{s}\binom{-\ell-2 m+n}{-\ell-2 m+n-s} E_{n-2 m-s-\ell}(x) .\right.
\end{aligned}
$$

Some lengthy algebraic computations lead to

$$
\begin{align*}
& D^{\ell} G_{n}^{(\delta)}(x)=\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)}\left(\sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{1}{(-\ell+n-2 r)!} \sum_{s=0}^{r} \frac{(-1)^{s} c_{2 r-2} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s)!} E_{n-\ell-2 r}(x)\right. \\
& \left.+\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{1}{(-\ell+n-2 r-1)!} \sum_{s=0}^{r} \frac{(-1)^{s} c_{2 r-2 s+1} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s+1)!} E_{n-\ell-2 r-1}(x)\right) . \tag{11}
\end{align*}
$$

To transform (11) into a simplified formula, we will find closed forms for the two interior sums that appear in it. Regarding the first sum, we can write

$$
\begin{align*}
& \sum_{s=0}^{r} \frac{(-1)^{s} c_{2 r-2 s} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s)!} \\
& =2^{n+2 \delta-2}\left(\frac{\left(-\frac{1}{4}\right)^{r} \Gamma(n-r+\delta)}{r!}+\frac{\Gamma(n+\delta){ }_{2} F_{1}\left(\left.\begin{array}{c}
-r+\frac{1}{2},-r \\
1-n-\delta
\end{array} \right\rvert\,\right.}{(2 r)!}\right) \tag{12}
\end{align*}
$$

and accordingly, the Chu-Vandermonde identity implies the following identity:

$$
\begin{aligned}
& \sum_{s=0}^{r} \frac{c_{2 r-2 s}(-1)^{s} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s)!}= \\
& \frac{2^{n+2 \delta-2}}{r!}\left(\left(-\frac{1}{4}\right)^{r} \Gamma(n-r+\delta)+\frac{r!\Gamma(n+\delta)\left(n-2 r+\delta+\frac{1}{2}\right)_{r}}{(2 r)!(n-r+\delta)_{r}}\right)
\end{aligned}
$$

Regarding the second sum, set

$$
M_{r, n}=\sum_{s=0}^{r} \frac{c_{2 r-2 s+1}(-1)^{s} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s+1)!},
$$

and employ the important algorithm of Zeilberger [36] to show that the following recurrence relation of order one is satisfied by $M_{r, n}$ :

$$
M_{r+1, n}-\frac{(3-2 n+4 r-2 \delta)(5-2 n+4 r-2 \delta)}{4(r+1)(2 r+3)(3-2 n+2 r-2 \delta)(1-n+r-\delta)} M_{r, n}=0, \quad M_{0, n}=1,
$$

which can be immediately solved to give

$$
\begin{equation*}
\sum_{s=0}^{r} \frac{(-1)^{s} c_{2 r-2 s+1} 2^{n-2 s+2 \delta-2} \Gamma(n-s+\delta)}{s!(2 r-2 s+1)!}=\frac{2^{n+2 \delta-2} \Gamma(n+\delta)\left(n-2 r+\delta-\frac{1}{2}\right)_{r}}{(2 r+1)!(n-r+\delta)_{r}} . \tag{13}
\end{equation*}
$$

In virtue of the two Identities (12) and (13), Formula (11) can be put into the simpler formula:

$$
\begin{aligned}
& D^{\ell} G_{n}^{(\delta)}(x)=\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)} \times \\
& \left\lfloor\frac{n-\ell}{2}\right\rfloor \frac{2^{n-2 r+2 \delta-2}\left((-1)^{r}(2 r)!\Gamma(n-r+\delta)+\frac{4^{r} r!\Gamma(n+\delta)\left(n-2 r+\delta+\frac{1}{2}\right)_{r}}{(n-r+\delta)_{r}}\right)}{r!(2 r)!(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
& +\frac{n!2^{n+2 \delta-2} \Gamma\left(\delta+\frac{1}{2}\right) \Gamma(n+\delta)\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor}{\sqrt{\pi} \Gamma(n+2 \delta)} \frac{\left(n-2 r+\delta-\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+\delta)_{r}} E_{n-\ell-2 r-1}(x) .
\end{aligned}
$$

This proves Theorem 2.
Remark 3. Since the Legendre and Chebyshev polynomials of the first and second kinds are included in the ultraspherical polynomials, $G_{n}^{(\delta)}$, three specific expressions for the derivatives of these polynomials can be inferred as direct special cases of Formula (10). These expressions can be seen in the subsequent corollary.

Corollary 3. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The formulas that express the derivatives of Legendre and Chebyshev polynomials of the first and second kinds in terms of Euler polynomials are given as follows:

$$
\begin{align*}
& D^{\ell} P_{n}(x)= \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{2^{n-2 r-1}\left((-1)^{r}(2 r)!\Gamma\left(n-r+\frac{1}{2}\right)+\frac{4^{r} r!\Gamma\left(n+\frac{1}{2}\right)(n-2 r+1)_{r}}{\left(n-r+\frac{1}{2}\right)_{r}}\right)}{r!(2 r)!(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
&+\frac{2^{n-1} \Gamma\left(\frac{1}{2}+n\right)}{\sqrt{\pi}} \sum_{r=0}^{\sum_{\left.\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{(n-2 r)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!\left(n-r+\frac{1}{2}\right)_{r}} E_{n-\ell-2 r-1}(x),}  \tag{1}\\
& D^{\ell} T_{n}(x)= n!\sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{2^{n-2 r-2}\left((-1)^{r}(2 r)!+4^{r} r!\left(n-2 r+\frac{1}{2}\right)_{r}\right)}{r!(2 r)!(-\ell+n-2 r)!(n-r)_{r}} E_{n-\ell-2 r}(x) \\
&+2^{n-2} n!  \tag{15}\\
& D^{\ell} U_{n}(x)= \frac{1}{2} \sum_{r=0}^{\left.\left\lfloor\frac{n-\ell}{2}\right\rfloor(n-\ell-1)\right\rfloor} \frac{\left(n-2 r-\frac{1}{2}\right)_{r}}{\sum_{r=0}^{n-2 r}\left(2^{2 r} n!r!\left(n-2 r+\frac{3}{2}\right)_{r}+(-1)^{r}(2 r)!(n-r)!(n-r+1)_{r}\right)} \\
& r!(2 r)!(-\ell+n-2 r)!(n-r+1)_{r} E_{n-\ell-2 r}(x)  \tag{16}\\
&+2^{n-1} n! \\
& \sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{\left(n-2 r+\frac{1}{2}\right)_{r}}{(2 r+1)!(-\ell+n-2 r-1)!(n-r+1)_{r}} E_{n-\ell-2 r-1}(x) .
\end{align*}
$$

Proof. Formulas (14), (15) and (16) can be obtained as special cases of Formula (10) by setting $\delta=\frac{1}{2}, 0,1$, respectively.

Remark 4. Expressions for the derivatives of other symmetric polynomials can be derived using similar techniques to those used in the proofs of Theorems 1 and 2. Some outcomes in this regard are shown by the following two theorems:

Theorem 3. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the Hermite polynomials $H_{n}$ can be expanded in terms of Euler polynomials as

$$
\begin{align*}
D^{\ell} H_{n}(x)= & n!\sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(-1)^{r} 2^{n-2 r-1}\left(1+{ }_{1} F_{1}\left(-r ; \frac{1}{2} ; 1\right)\right)}{r!(-\ell+n-2 r)!} E_{n-\ell-2 r}(x) \\
& +\sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{U\left(-r, \frac{3}{2}, 1\right)}{(-\ell+n-2 r-1)!(2 r+1)!} E_{n-\ell-2 r-1}(x),
\end{align*}
$$

where $U(a, b ; z)$ is the well-known confluent hypergeometric [37].
Proof. Based on the power form representation of Hermite polynomials given by [37]

$$
H_{n}(x)=n!\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{m} 2^{n-2 m}}{m!(n-2 m)!} x^{n-2 m},
$$

along with the inversion formula of Euler polynomials (1), and performing similar steps that followed in the proof of Theorem 1, Formula (17) can be obtained.

Theorem 4. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the generalized Lucas polynomials that are constructed by (6) can be expanded in terms of Euler polynomials as

$$
\begin{align*}
& D^{\ell} L_{n}^{R, S}(x)= \frac{1}{2} R^{n} \sum_{r=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor R^{-2 r} S^{r} n(n-r-1)!(2 r)!+n!r!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r+\frac{1}{2} \\
1-n
\end{array} \right\rvert\,-\frac{4 S}{R^{2}}\right)} E_{n-\ell-2 r}(x) \\
&+\frac{1}{2} R^{n} n!(2 r)!(-\ell+n-2 r)!  \tag{18}\\
& \sum_{r=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor}{ }_{2} F_{1}\left(\begin{array}{c}
\left.-r,-r-\frac{1}{2} \left\lvert\,-\frac{4 S}{R^{2}}\right.\right) \\
1-n
\end{array} E_{n-\ell-2 r-1}(x) .\right.
\end{align*}
$$

Proof. Similar to the proof of Theorem 1.

### 3.2. Derivative Expressions for Some Non-Symmetric Polynomials

This section is confined to developing new expressions for the derivatives of some non-symmetric polynomials in terms of Euler polynomials. To be more precise, the expressions for the derivatives of the shifted Jacobi, Laguerre, and Schröder polynomials will be presented.

Theorem 5. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the shifted Jacobi polynomials can be written in terms of the Euler polynomials as

$$
\begin{align*}
D^{\ell} \tilde{V}_{n}^{(\lambda, \delta)}(x)= & \frac{n!\Gamma(\lambda+1)}{2 \Gamma(n+\lambda+1) \Gamma(n+\lambda+1+\delta)} \times \\
& \sum_{m=0}^{n-\ell} \frac{\Gamma(2 n-m+\lambda+\delta+1)}{m!(n-m-\ell)!\Gamma(n-m+\lambda+1) \Gamma(n-m+\delta+1)} \times  \tag{19}\\
& \left((-1)^{m} \Gamma(n-m+\lambda+1) \Gamma(n+\delta+1)+\Gamma(n+\lambda+1) \Gamma(n-m+\delta+1)\right) E_{n-\ell-m}(x) .
\end{align*}
$$

Proof. The representation of the shifted Jacobi polynomials in (4) serves to obtain the following formula:

$$
D^{\ell} \tilde{V}_{n}^{(\lambda, \delta)}(x)=\frac{n!\Gamma(n+\delta+1) \Gamma(\lambda+1)}{\Gamma(n+\lambda+1) \Gamma(n+\delta+1+\lambda)} \sum_{r=0}^{n+m-\ell} \frac{(-1)^{r} \Gamma(2 n-r+\delta+\lambda+1)}{r!(-\ell+n-r)!\Gamma(n-r+\delta+1)} x^{n-r-\ell},
$$

hence, when the inversion Formula (1) is applied, it yields the following formula:

$$
\begin{aligned}
D^{\ell} \tilde{V}_{n}^{(\lambda, \delta)}(x)= & \frac{n!\Gamma(n+\delta+1) \Gamma(\lambda+1)}{2 \Gamma(n+\lambda+1) \Gamma(n+\delta+1+\lambda)} \times \\
& \sum_{r=0}^{n+m-\ell} \frac{(-1)^{r} \Gamma(2 n-r+\delta+\lambda+1)}{r!(-\ell+n-r)!\Gamma(n-r+\delta+1)} \sum_{t=0}^{n-\ell-r} c_{t}\binom{-\ell+n-r}{-\ell+n-r-t} E_{n-r-\ell-t}(x) .
\end{aligned}
$$

Rearranging the terms in the last formula turns it into the following form:

$$
\begin{align*}
D^{\ell} \tilde{V}_{n}^{(\lambda, \delta)}(x)= & \frac{n!\Gamma(n+\delta+1) \Gamma(\lambda+1)}{2 \Gamma(n+\lambda+1) \Gamma(n+\delta+1+\lambda)} \sum_{m=0}^{n-\ell} \frac{1}{(-\ell+n-m)!} \times  \tag{20}\\
& \sum_{r=0}^{m} \frac{(-1)^{r} c_{m-r} \Gamma(2 n-r+\delta+\lambda+1)}{(m-r)!r!\Gamma(n-r+\delta+1)} E_{n-\ell-m}(x) .
\end{align*}
$$

The second sum that appears on the right-hand side of (20) can be rewritten in the following form:

$$
\begin{aligned}
& \sum_{r=0}^{m} \frac{(-1)^{r} c_{m-r} \Gamma(2 n-r+\delta+\lambda+1)}{r!(m-r)!\Gamma(n-r+\delta+1)}= \\
& \frac{(-1)^{m} \Gamma(n+\delta+1) \Gamma(2 n-m+\delta+\lambda+1)+\Gamma(n-m+\delta+1) \Gamma(2 n+\delta+\lambda+1) H_{m, n}}{m!\Gamma(n+\delta+1) \Gamma(n-m+\delta+1)},
\end{aligned}
$$

where $H_{m, n}$ is given by

$$
H_{m, n}={ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-n-\delta \\
-2 n-\delta-\lambda
\end{array} \right\rvert\, 1\right) .
$$

Chu-Vandermond identity implies that

$$
{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-n-\delta \\
-2 n-\delta-\lambda
\end{array} \right\rvert\, 1\right)=\frac{(n-m+\lambda+1)_{m}}{(2 n-m+\delta+\lambda+1)_{m}},
$$

thus, the following identity can be obtained:

$$
\sum_{r=0}^{m} \frac{(-1)^{r} c_{m-r} \Gamma(2 n-r+\delta+\lambda+1)}{(m-r)!r!\Gamma(n-r+\delta+1)}=\frac{\left(\frac{(-1)^{m}}{\Gamma(n-m+\delta+1)}+\frac{\Gamma(n+\lambda+1)}{\Gamma(n+\delta+1) \Gamma(n-m+\lambda+1)}\right) \Gamma(2 n-m+\delta+\lambda+1)}{m!} .
$$

The reduction of the last sum enables one to reduce Formula (20) in the following simpler form:

$$
\begin{aligned}
D^{\ell} \tilde{V}_{n}^{(\lambda, \delta)}(x)= & \frac{n!\Gamma(\lambda+1)}{2 \Gamma(n+\lambda+1) \Gamma(n+\lambda+1+\delta)} \times \\
& \sum_{m=0}^{n-\ell} \frac{\Gamma(2 n-m+\lambda+\delta+1)}{m!(n-m-\ell)!\Gamma(n-m+\lambda+1) \Gamma(n-m+\delta+1)} \times \\
& \left((-1)^{m} \Gamma(n-m+\lambda+1) \Gamma(n+\delta+1)+\Gamma(n+\lambda+1) \Gamma(n-m+\delta+1)\right) E_{n-\ell-m}(x) .
\end{aligned}
$$

This finalizes the proof of Theorem 5.
Taking into consideration the six special polynomials of the shifted Jacobi polynomials, six special formulas of Formula (19) can be obtained. The following two corollaries present these formulas.

Corollary 4. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The following expressions give the derivatives of the shifted ultraspherical, shifted Legendre, and shifted Chebyshev polynomials of the first and second kinds:

$$
\begin{aligned}
D^{\ell} \tilde{G}_{n}^{(\delta)}(x) & =\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\Gamma(n+2 \delta)} \sum_{m=0}^{\left.\frac{n-\ell}{2}\right\rfloor} \frac{\Gamma(2(n-m+\delta))}{(2 m)!(-\ell+n-2 m)!\Gamma\left(\frac{1}{2}+n-2 m+\delta\right)} E_{n-\ell-2 m}(x), \\
D^{\ell} \tilde{P}_{n}(x) & =\sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(2 n-2 m)!}{(2 m)!(n-2 m)!(-\ell+n-2 m)!} E_{n-\ell-2 m}(x), \\
D^{\ell} \tilde{T}_{n}(x) & =n \sqrt{\pi} \sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(-\ell+n-2 m)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-\ell-2 m}(x), \\
D^{\ell} \tilde{U}_{n}(x) & =\frac{1}{2} \sqrt{\pi} \sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(-\ell+n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-\ell-2 m}(x) .
\end{aligned}
$$

Corollary 5. Let $n$ and $\ell$ be two non-negative integers with $n \geq \ell$. The derivatives of the shifted third- and fourth-kind Chebyshev polynomials are, respectively, given by the following expressions:

$$
\begin{align*}
D^{\ell} \tilde{V}_{n}(x)= & \frac{1}{2} \sqrt{\pi}\left(\sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(-\ell+n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-\ell-2 m}(x)\right.  \tag{21}\\
& -\left\lfloor\sum_{m=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{(2 n-2 m-1)!}{(2 m)!(-\ell+n-2 m-1)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-\ell-2 m-1}(x)\right), \\
D^{\ell} \tilde{W}_{n}(x)= & \frac{1}{2} \sqrt{\pi}\left(\sum_{m=0}^{\left\lfloor\frac{n-\ell}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(-\ell+n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-\ell-2 m}(x)\right. \\
& \left.+\sum_{m=0}^{\left\lfloor\frac{1}{2}(n-\ell-1)\right\rfloor} \frac{(2 n-2 m-1)!}{(2 m)!(-\ell+n-2 m-1)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-\ell-2 m-1}(x)\right) . \tag{22}
\end{align*}
$$

Theorem 6. For non-negative integers $n$ and $q$ with $n \geq q$, the derivatives of the generalized Laguerre polynomials $L_{n}^{(\lambda)}(x)$ can be expanded in terms of the Euler polynomials as

$$
\begin{equation*}
D^{\ell} L_{n}^{(\lambda)}(x)=\frac{1}{2} \Gamma(n+\lambda+1) \sum_{m=0}^{n-\ell} \frac{(-1)^{n+m}\left(1+{ }_{1} F_{1}(-m ; n-m+\lambda+1 ; 1)\right)}{m!(-\ell+n-m)!\Gamma(n-m+\lambda+1)} E_{n-\ell-m}(x) . \tag{23}
\end{equation*}
$$

Proof. The proof can be done with the aid of the following formula [37]:

$$
L_{n}^{(\lambda)}(x)=\frac{\Gamma(n+\lambda+1)}{n!} \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{n}{k}}{\Gamma(n+\lambda-k+1)} x^{n-k},
$$

along with Formula (1).
Theorem 7. For non-negative integers $n$ and $q$ with $n \geq q$, the derivatives of the Schröder polynomials can be expanded in terms of Euler polynomials as

$$
\begin{align*}
D^{\ell} S_{n}(x)= & \left.\frac{1}{2(n+1)!} \sum_{m=0}^{n-\ell} \frac{(n+1)!(2 n-m)!+(2 n)!(n-m+1)!{ }_{2} F_{1}\left(\begin{array}{c}
-m,-n-1 \\
-2 n
\end{array}\right.}{} \begin{array}{l}
E_{n-\ell-m}(x) .
\end{array} . \begin{array}{l}
m!(n-m+1)!(-\ell+n-m)!
\end{array}\right] \tag{24}
\end{align*}
$$

Proof. The proof can be done with the aid of the following representation of Schröder polynomials [38]

$$
S_{n}(x)=\sum_{r=0}^{n} \frac{\binom{2 r}{r}\binom{n+r}{n-r}}{j+1} x^{r},
$$

along with Formula (1).

## 4. Connection Formulas of Different Polynomials with Euler Polynomials

In this section, the connection formulas between some symmetric and non-symmetric polynomials and the Euler polynomials are given. In fact, since all the derivative formulas developed in Section 3 are valid for $\ell=0$, it is an easy matter to deduce the connection formulas as special cases of these formulas.

### 4.1. Connection Formulas between Some Symmetric Polynomials and Euler Polynomials

In this section, we present new connection formulas between some symmetric polynomials and Euler polynomials. More precisely, the connection formulas between the ultraspherical, generalized Fibonacci, generalized Lucas, and Hermite polynomials and Euler polynomials will be presented.

Corollary 6. For every non-negative integer $n$, the following connection formulas hold:

$$
\begin{align*}
U_{n}^{(\delta)}(x)= & \frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)} \times \\
& \left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{2^{n-2 r+2 \delta-2}\left((-1)^{r}(2 r)!\Gamma(n-r+\delta)+\frac{4^{r} r!\Gamma(n+\delta)\left(-n+r-\delta+\frac{1}{2}\right)_{r}}{(1-n-\delta)_{r}}\right)}{r!(2 r)!(n-2 r)!} E_{n-2 r}(x)\right.  \tag{25}\\
& \left.+\Gamma(n+\delta) \sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{2^{n+2 \delta-2}\left(-n+r-\delta+\frac{3}{2}\right)_{r}}{(2 r+1)!(n-2 r-1)!(1-n-\delta)_{r}} E_{n-2 r-1}(x)\right), \\
P_{n}(x)= & \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{2^{n-1}\left(\frac{\left(-\frac{1}{4}\right)^{r} \Gamma\left(n-r+\frac{1}{2}\right)}{r!}+\frac{\Gamma\left(n+\frac{1}{2}\right)(n-2 r+1)_{r}}{(2 r)!\left(n-r+\frac{1}{2}\right)_{r}}\right)}{(n-2 r)!} E_{n-2 r}(x) \\
& +\frac{2^{n-1} \Gamma\left(n+\frac{1}{2}\right)}{\left.\sum_{n} \frac{n-1}{2}\right\rfloor} \frac{\sum_{r=0}}{(2 r+1)!(n-2 r-1)!\left(n-r+\frac{1}{2}\right)_{r}} E_{n-2 r-1}(x), \\
T_{n}(x)= & n!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{2^{n-2 r-2}\left((-1)^{r}(2 r)!+4^{r} r!\left(n-2 r+\frac{1}{2}\right)_{r}\right)}{r!(2 r)!(n-2 r)!(n-r)_{r}} E_{n-2 r}(x) \\
& +2^{n-2} n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{\left(n-2 r-\frac{1}{2}\right)_{r}}{(2 r+1)!(n-2 r-1)!(n-r)_{r}} E_{n-2 r-1}(x), \\
U_{n}(x)= & \frac{1}{2} n!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{2^{n}\left(\left(-\frac{1}{4}\right)^{r}(2 r)!+r!\left(n-2 r+\frac{3}{2}\right)_{r}\right)}{r!(2 r)!(n-2 r)!(n-r+1)_{r}} E_{n-2 r}(x) \\
& +2^{n-1} n!\frac{\left.\sum_{r=0}^{2}\right\rfloor}{(2 r+1)!(n-2 r-1)!(n-r+1)_{r}} E_{n-2 r-1}(x) .
\end{align*}
$$

Proof. All formulas listed in Corollary 6 are direct consequences of Theorem 2 and Corollary 3 with the same arrangement of their equations. They can be deduced by setting $\ell=0$.

Corollary 7. Let $n$ be any positive integer. The following are the generalized Fibonacci-Euler, the generalized Lucas-Euler, and the Hermite-Euler connection formulas.

$$
\begin{align*}
& F_{n}^{A, B}(x)= A^{n}\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{A^{-2 r} B^{r}(n-r)!(2 r)!+n!r!{ }_{2} F_{1}\left(\begin{array}{c}
-r,-r+\frac{1}{2} \\
-n
\end{array}\right.}{\substack{-\frac{4 B}{A^{2}}}} E_{n-2 r}(x)\right.  \tag{26}\\
& 2 r!(2 r)!(n-2 r)! \\
&\left.+n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r-\frac{1}{2} \\
-n
\end{array} \right\rvert\,-\frac{4 B}{A^{2}}\right)}{2(2 r+1)!(n-2 r-1)!} E_{n-2 r-1}(x)\right),
\end{align*}
$$

$$
\begin{align*}
L_{n}^{R, S}(x)= & \frac{1}{2} R^{n} n!\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{R^{-2 r} S^{r} n(n-r-1)!(2 r)!+n!r!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r+\frac{1}{2} \\
1-n
\end{array} \right\rvert\,-\frac{4 S}{R^{2}}\right)}{r!(2 r)!(n-2 r)!} E_{n-2 r}(x)\right. \\
& \left.+n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{{ }_{2} F_{1}\left(\left.\begin{array}{c}
\left.-r,-r-\frac{1}{2} \left\lvert\,-\frac{4 S}{R^{2}}\right.\right) \\
1-n
\end{array} \right\rvert\,\right.}{(2 r+1)!(n-2 r-1)!} E_{n-2 r-1}(x)\right),  \tag{27}\\
H_{n}(x)= & n!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{r} 2^{n-2 r-1}\left(1+{ }_{1} F_{1}\left(-r ; \frac{1}{2} ; 1\right)\right)}{r!(n-2 r)!} E_{n-2 r}(x) \\
& +2^{n-1} n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{1}{(2 r+1)!(n-2 r-1)!} U\left(-r, \frac{3}{2}, 1\right) E_{n-2 r-1}(x) . \tag{28}
\end{align*}
$$

Proof. Formulas (26), (27) and (28) are, respectively, special cases of Formulas (8), (18) and (17) for the case $\ell=0$.

### 4.2. Connection Formulas between Some Non-Symmetric Polynomials with Euler Polynomials

In this section, we introduce new connection formulas between some non-symmetric polynomials and Euler polynomials. The shifted Jacobi-Euler, generalized Laguerre-Euler, and Schröder-Euler connection formulas will be displayed.

Corollary 8. Let n be a non-negative integer. The shifted Jacobi-Euler connection formula is

$$
\begin{align*}
\tilde{V}_{n}^{(\lambda, \delta)}(x)= & \frac{n!\Gamma(\lambda+1)}{2 \Gamma(n+\lambda+1) \Gamma(n+\lambda+1+\delta)} \times \\
& \sum_{m=0}^{n} \frac{\left(\Gamma(-m+n+\delta+1) \Gamma(n+\lambda+1)+(-1)^{m} \Gamma(n+\delta+1) \Gamma(1-m+n+\lambda)\right)}{m!(n-m)!\Gamma(-m+n+\delta+1) \Gamma(1-m+n+\lambda)} \times  \tag{29}\\
& \Gamma(-m+2 n+\delta+\lambda+1) E_{n-m}(x) .
\end{align*}
$$

Proof. Formula (29) can be immediately deduced for Formula (19) by setting $q=0$.
Corollary 9. Let n be a non-negative integer. The following are the ultraspherical-Euler, LegendreEuler, first-kind-Euler, and second-kind-Euler connection formulas

$$
\begin{aligned}
\tilde{G}_{n}^{(\delta)}(x) & =\frac{n!\Gamma\left(\delta+\frac{1}{2}\right)}{\Gamma(n+2 \delta)} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\Gamma(2(n-m+\delta))}{(2 m)!(n-2 m)!\Gamma\left(\frac{1}{2}+n-2 m+\delta\right)} E_{n-2 m}(x), \\
\tilde{P}_{n}(x) & =\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(2 n-2 m)!}{(2 m)!((n-2 m)!)^{2}} E_{n-2 m}(x), \\
\tilde{T}_{n}(x) & =n \sqrt{\pi} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(2 n-2 m-1)!}{(2 m)!(n-2 m)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-2 m}(x), \\
\tilde{U}_{n}(x) & =\frac{1}{2} \sqrt{\pi} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-2 j}(x) .
\end{aligned}
$$

Proof. Corollary 9 is a special case of Corollary 4 for $\ell=0$.

Corollary 10. The following are the shifted third-kind Chebyshev-Euler and shifted fourth-kind Chebyshev-Euler connection formulas.

$$
\begin{align*}
\tilde{V}_{n}(x)= & \frac{1}{2} \sqrt{\pi}\left(\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-2 m}(x)\right. \\
& \left.-\sum_{m=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(2 n-2 m-1)!}{(2 m)!(n-2 m-1)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-2 m-1}(x)\right),  \tag{30}\\
\tilde{W}_{n}(x)= & \frac{1}{2} \sqrt{\pi}\left(\sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(2 n-2 m+1)!}{(2 m)!(n-2 m)!\Gamma\left(n-2 m+\frac{3}{2}\right)} E_{n-2 m}(x)\right. \\
& \left.+\sum_{m=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(2 n-2 m-1)!}{(2 m)!(n-2 m-1)!\Gamma\left(n-2 m+\frac{1}{2}\right)} E_{n-2 m-1}(x)\right) . \tag{31}
\end{align*}
$$

Proof. Formulas (30) and (31) are, respectively, special ones of Formulas (21) and (22) only by setting $\ell=0$.

Corollary 11. The following are the generalized Laguerre-Euler and Schröder-Euler connection formulas:

$$
\begin{align*}
L_{n}^{(\lambda)}(x) & =\frac{1}{2} \Gamma(n+\lambda+1) \sum_{m=0}^{n} \frac{(-1)^{n+m}\left(1+{ }_{1} F_{1}(-m ; n-m+\lambda+1 ; 1)\right)}{m!(n-m)!\Gamma(n-m+\lambda+1)} E_{n-m}(x),  \tag{32}\\
S_{n}(x) & =\frac{1}{2(n+1)!} \sum_{m=0}^{n} \frac{(n+1)!(2 n-m)!+(2 n)!(n-m+1)!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-n-1 \\
-2 n
\end{array} \right\rvert\,-1\right)}{m!(n-m)!(n-m+1)!} E_{n-m}(x) . \tag{33}
\end{align*}
$$

Proof. Formulas (32) and (33) are, respectively, special ones of Formula (23) and (24) only by setting $\ell=0$.

## 5. Application to Compute Some New Integrals

This section is confined to developing an application to the connection formulas between different polynomials and the Euler polynomials. In this regard, new formulas are developed for computing some definite integrals of the products of different symmetric and non-symmetric polynomials with Euler polynomials. In fact, the connection coefficients aid in the evaluation of the desired definite integrals.

### 5.1. Definite Integrals for the Product of Euler Polynomials with Symmetric Polynomials

This section is interested in introducing a new explicit formula for evaluating a definite integral for the product of the Euler polynomial of any degree with a symmetric polynomial of any degree. After that, we apply this general formula to evaluate the definite integral for the product of Euler polynomials with some celebrated symmetric polynomials.

Theorem 8. Let $\phi_{n}(x)$ be any symmetric polynomial that can be expressed as in (2), and let it have the following connection formula with Euler polynomials:

$$
\begin{equation*}
\phi_{n}(x)=\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} R_{r, n} E_{n-2 r}(x)+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \bar{R}_{r, n} E_{n-2 r-1}(x) . \tag{34}
\end{equation*}
$$

The following integral formula is valid:

$$
\begin{align*}
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x= & 4 m!\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{n}\left(2^{m+n-2 r+2}-1\right)(n-2 r)!}{(m+n-2 r+2)!} B_{m+n-2 r+2} R_{r, n}\right. \\
& \left.+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-1)^{n+1}\left(2^{m+n-2 r+1}-1\right)(n-2 r-1)!}{(m+n-2 r+1)!} B_{m+n-2 r+1} \bar{R}_{r, n}\right), \tag{35}
\end{align*}
$$

and $B_{n}$ are the well-known Bernoulli numbers.
Proof. The connection Formula (34) immediately yields

$$
\begin{equation*}
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x=\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} R_{r, n} \int_{0}^{1} E_{m}(x) E_{n-2 r}(x) d x+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \bar{R}_{r, n} \int_{0}^{1} E_{m}(x) E_{n-2 r-1}(x) d x . \tag{36}
\end{equation*}
$$

In virtue of the well-known formula [29]:

$$
\begin{equation*}
\int_{0}^{1} E_{m}(x) E_{n}(x) d x=F_{m, n}=\frac{4(-1)^{n}\left(2^{m+n+2}-1\right) n!m!}{(m+n+2)!} B_{m+n+2} \tag{37}
\end{equation*}
$$

Formula (36) can be transformed into the following formula:

$$
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x=\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} R_{r, n} F_{m, n-2 r}+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \bar{R}_{r, n} F_{m, n-2 r-1},
$$

and this leads to the following integral formula:

$$
\begin{aligned}
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x= & 4 m!\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{n}\left(2^{m+n-2 r+2}-1\right)(n-2 r)!}{(m+n-2 r+2)!} B_{m+n-2 r+2} R_{r, n}\right. \\
& \left.+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-1)^{n+1}\left(2^{m+n-2 r+1}-1\right)(n-2 r-1)!}{(m+n-2 r+1)!} B_{m+n-2 r+1} \bar{R}_{r, n}\right) .
\end{aligned}
$$

This proves Theorem 8.
Remark 5. As a consequence of Theorem 8 along with the connection formulas stated in Section 4, several new definite integral formulas of the product of some symmetric polynomials with the Euler polynomials can be obtained. The following corollaries exhibit these formulas.

Corollary 12. For all non-negative integers $m$ and $n$, the following definite integral formula holds:

$$
\begin{align*}
& \int_{0}^{1} G_{n}^{(\delta)}(x) E_{m}(x) d x=\frac{(-1)^{n} m!n!\Gamma\left(\delta+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma(n+2 \delta)} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{2^{n-2 r+2 \delta}\left(2^{m+n-2 r+2}-1\right)}{r!(2 r)!(m+n-2 r+2)!} \times \\
& \left((-1)^{r}(2 r)!\Gamma(n-r+\delta)+\frac{4^{r} r!\Gamma(n+\delta)\left(-n+r-\delta+\frac{1}{2}\right)_{r}}{(1-n-\delta)_{r}}\right) B_{m+n-2 r+2}  \tag{38}\\
& +\frac{2^{n+2 \delta} m!n!\Gamma\left(\delta+\frac{1}{2}\right) \Gamma(n+\delta)}{\sqrt{\pi} \Gamma(n+2 \delta)} \sum_{r=0}^{\left.\frac{n-1}{2}\right\rfloor} \frac{(-1)^{n+1}\left(2^{m+n-2 r+1}-1\right)\left(-n+r-\delta+\frac{3}{2}\right)_{r}}{(2 r+1)!(m+n-2 r+1)!(1-n-\delta)_{r}} B_{m+n-2 r+1} .
\end{align*}
$$

Proof. This result is a direct consequence of the connection Formula (25) along with the integral Formula (35).

The following three specific formulas of Formula (38) are concerned with the definite integral formulas for the products of Legendre and Chebyshev polynomials of the first and second kinds with Euler polynomials.

Corollary 13. Let $m$ and $n$ be any non-negative integers. The following definite integral formulas apply:

$$
\begin{align*}
& \int_{0}^{1} P_{n}(x) E_{m}(x) d x= \\
& \frac{(-1)^{n} 2^{n+1} m!}{\sqrt{\pi}} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+2}-1\right)\left(\frac{\left(-\frac{1}{4}\right)^{r} \Gamma\left(n-r+\frac{1}{2}\right)}{r!}+\frac{\Gamma\left(n+\frac{1}{2}\right)(n-2 r+1)_{r}}{(2 r)!\left(n-r+\frac{1}{2}\right)_{r}}\right)}{(m+n-2 r+2)!} B_{m+n-2 r+2}  \tag{39}\\
& +(-1)^{n+1} 2^{2-n} m!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{\left(2^{m+n+1}-4^{r}\right)(2 n-2 r-1)!}{(2 r+1)!(n-2 r-1)!(m+n-2 r+1)!} B_{m+n-2 r+1,} \\
& \int_{0}^{1} T_{n}(x) E_{m}(x) d x= \\
& m!n!\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-2)^{n-2 r}\left(2^{m+n-2 r+2}-1\right)\left((-1)^{r}(2 r)!+4^{r} r!\left(\frac{1}{2}+n-2 r\right)_{r}\right)}{r!(2 r)!(m+n-2 r+2)!(n-r)_{r}} B_{m+n-2 r+2}\right.  \tag{40}\\
& +\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-2)^{n-2 r}\left(-2^{m+n+1}+4^{r}\right)\left(-\frac{1}{2}+n-2 r\right)_{r}}{(2 r+1)!(m+n+1-2 r)!(n-r)_{r}} B_{m+n-2 r+1), \quad n \geq 1,} \\
& \int_{0}^{1} U_{n}(x) E_{m}(x) d x= \\
& m!\left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-2)^{n-2 r+1}\left(-2^{m+n+2}+4^{r}\right) n!\left(\left(-\frac{1}{4}\right)^{r}(2 r)!+r!\left(n-2 r+\frac{3}{2}\right)_{r}\right)}{r!(2 r)!(m+n-2 r+2)!(n-r+1)_{r}} B_{m+n-2 r+2}\right.  \tag{41}\\
& \left.+n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-2)^{n-2 r+1}\left(2^{m+n+1}-4^{r}\right)\left(\frac{1}{2}+n-2 r\right)_{r}}{(2 r+1)!(m+n+1-2 r)!(n-r+1)_{r}} B_{m+n-2 r+1}\right) .
\end{align*}
$$

Proof. Formulas (39), (40) and (41) can be obtained as special cases of Formula (38) by setting $\delta=\frac{1}{2}, 0,1$, respectively.

The following corollary is concerned with the definite integrals of the two generalized Fibonacci and generalized Lucas polynomials with the Euler polynomials.

Corollary 14. For all non-negative integers $m$ and $n$, the following definite integral formulas apply:

$$
\begin{align*}
& \int_{0}^{1} F_{n}^{A, B}(x) E_{m}(x) d x=2 A^{n} m!\times \\
& \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{n}\left(2^{m+n-2 r+2}-1\right)\left(A^{-2 r} B^{r}(n-r)!(2 r)!+n!r!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r+\frac{1}{2} \\
-n
\end{array} \right\rvert\,-\frac{4 B}{A^{2}}\right)\right)}{r!(2 r)!(m+n-2 r+2)!} \times  \tag{42}\\
& B_{m+n-2 r+2}+2 A^{n} m!n!\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-1)^{n+1}\left(2^{m+n-2 r+1}-1\right){ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r-\frac{1}{2} \\
-n
\end{array} \right\rvert\,-\frac{4 B}{A^{2}}\right)}{(2 r+1)!(m+n-2 r+1)!} B_{m+n-2 r+1},
\end{align*}
$$

$$
\begin{align*}
& \int_{0}^{1} L_{n}^{R, S}(x) E_{m}(x) d x=2(-1)^{n} m!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+2}-1\right) R^{n}}{r!(2 r)!(m+n-2 r+2)!} \times \\
& \left(n R^{-2 r} S^{r}(n-r-1)!(2 r)!+n!r!{ }_{2} F_{1}\left(\begin{array}{c}
\left.\left.-r,-r+\frac{1}{2} \left\lvert\,-\frac{4 S}{R^{2}}\right.\right)\right) B_{m+n-2 r+2} \\
1-n
\end{array}\right.\right.  \tag{43}\\
& +2(-1)^{n+1} m!n!R^{n} \sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+1}-1\right){ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-r-\frac{1}{2} \\
1-n
\end{array} \right\rvert\,-\frac{4 S}{R^{2}}\right)}{(2 r+1)!(m+n-2 r+1)!} B_{m+n-2 r+1}, \quad n \geq 1 .
\end{align*}
$$

Proof. Formulas (42) and (43) can be obtained, respectively, as by the application to Theorem 8 along with the two connection Formulas (26) and (27).

Corollary 15. For all non-negative integers $m$ and $n$, the following definite integral formula applies:

$$
\begin{aligned}
& \int_{0}^{1} H_{n}(x) E_{m}(x) d x=m!n!\times \\
& \left(\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{n-r} 2^{n-2 r+1}\left(2^{m+n-2 r+2}-1\right)\left(1+{ }_{1} F_{1}\left(-r ; \frac{1}{2} ; 1\right)\right)}{r!(m+n-2 r+2)!} B_{m+n-2 r+2}\right. \\
& \left.+\sum_{r=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-2)^{n-2 r+1}\left(2^{m+n+1}-4^{r}\right)}{(2 r+1)!(m+n-2 r+1)!} U\left(-r, \frac{3}{2}, 1\right) B_{m+n-2 r+1}\right) .
\end{aligned}
$$

Proof. Direct application to Theorem 8 making use of the connection formula (28) yields the desired result.

### 5.2. Definite Integrals for the Product of Euler Polynomials with Non-Symmetric Polynomials

This section focuses on developing a new closed expression for a definite integral for the product of the Euler polynomial of any degree with any non-symmetric polynomial of any degree. Furthermore, it focuses on some specific definite integrals for the product of Euler polynomials with some celebrated non-symmetric polynomials. In this regard, the following theorem will be stated and proved.

Theorem 9. Let $\phi_{n}(x)$ by any non-symmetric polynomial that is connected with Euler polynomials by the following formula:

$$
\begin{equation*}
\phi_{n}(x)=\sum_{r=0}^{n} S_{r, n} E_{n-r}(x) . \tag{44}
\end{equation*}
$$

The following integral formula applies:

$$
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x=4 m!\sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right)(n-r)!}{(m+n-r+2)!} B_{m+n-r+2} S_{r, n} .
$$

Proof. Based on the connection Formula (44), one has the following integral formula:

$$
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x=\sum_{r=0}^{n} S_{r, n} F_{m, n-r},
$$

where $F_{m, n}$ are given by (37). This leads to the formula

$$
\int_{0}^{1} \phi_{n}(x) E_{m}(x) d x=4 m!\sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right)(n-r)!}{(m+n-r+2)!} B_{m+n-r+2} S_{r, n} .
$$

Remark 6. As a consequence of Theorem 9, along with the connection formulas in Section 4.2, several new definite integral formulas for the product of some non-symmetric polynomials with the Euler polynomials can be obtained. The following corollaries exhibit some of these integral formulas.

Corollary 16. For all positive integers $m$ and $n$, the following integral formulas hold:

$$
\begin{align*}
& \int_{0}^{1} \tilde{V}_{n}^{(\lambda, \delta)}(x) E_{m}(x) d x=\frac{2 m!n!\Gamma(\lambda+1)}{\Gamma(n+\lambda+1) \Gamma(n+\delta+1+\lambda)} \times \\
& \sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right) \Gamma(2 n-r+\delta+\lambda+1)}{r!(m+n+2-r)!\Gamma(n-r+\delta+1) \Gamma(n-r+\lambda+1)} \times  \tag{45}\\
& \left(\Gamma(n-r+\delta+1) \Gamma(n+\lambda+1)+(-1)^{r} \Gamma(n+\delta+1) \Gamma(n-r+\lambda+1)\right) B_{m+n-r+2} .
\end{align*}
$$

Proof. The proof is based on utilizing Theorem 9 along with the connection Formula (29). The following two corollaries give six special formulas of Formula (45).

Corollary 17. For all positive integers $m$ and $n$, the following integral formulas hold:

$$
\begin{align*}
\int_{0}^{1} \tilde{G}_{n}^{(\delta)}(x) E_{m}(x) d x= & \frac{4(-1)^{n} m!n!\Gamma\left(\delta+\frac{1}{2}\right)}{\Gamma(n+2 \delta)} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+2}-1\right) \Gamma(2(n-r+\delta))}{(2 r)!(m+n-2 r+2)!\Gamma\left(n-2 r+\delta+\frac{1}{2}\right)} \times  \tag{46}\\
& B_{m+n-2 r+2,} \\
\int_{0}^{1} \tilde{P}_{n}(x) E_{m}(x) d x= & 4 m!(-1)^{n} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+2}-1\right)(2 n-2 r)!}{(2 r)!(n-2 r)!(m+n-2 r+2)!} B_{m+n-2 r+2,}  \tag{47}\\
\int_{0}^{1} \tilde{T}_{n}(x) E_{m}(x) d x= & 4(-1)^{n} n \sqrt{\pi} m!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n-2 r+2}-1\right)(2 n-2 r-1)!}{(2 r)!\Gamma\left(n-2 r+\frac{1}{2}\right)(m+n-2 r+2)!} B_{m+n-2 r+2,}  \tag{48}\\
\int_{0}^{1} \tilde{U}_{n}(x) E_{m}(x) d x= & 2(-1)^{n} m!\sqrt{\pi} \sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(2^{m+n+2-2 r}-1\right)(2 n-2 r+1)!}{(2 r)!\Gamma\left(n-2 r+\frac{3}{2}\right)(m+n-2 r+2)!} B_{m+n-2 r+2 .} . \tag{49}
\end{align*}
$$

Proof. Formula (46) can be obtained from the general Formula (45) if both $\lambda$ and $\delta$ are replaced by $\left(\delta-\frac{1}{2}\right)$. Formulas (47), (48) and (49) are special ones of Formula (46) for the cases $\delta=\frac{1}{2}, 0,1$, respectively.

Corollary 18. For all positive integers $m$ and $n$, the following integral formulas hold:

$$
\begin{align*}
& \int_{0}^{1} \tilde{V}_{n}(x) E_{m}(x) d x=\sqrt{\pi} m!\times \\
& \sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right)\left(1+(-1)^{r}+2\left(1+(-1)^{r}\right) n-2 r\right)(2 n-r)!}{r!\Gamma\left(n-r+\frac{3}{2}\right)(m+n+2-r)!} B_{m+n-r+2},  \tag{50}\\
& \int_{0}^{1} \tilde{W}_{n}(x) E_{m}(x) d x=\sqrt{\pi} m!\times \\
& \sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right)\left(1+2 n+(-1)^{r}(2 n-2 r+1)\right)(2 n-r)!}{r!\Gamma\left(n-r+\frac{3}{2}\right)(m+n+2-r)!} B_{m+n-r+2} . \tag{51}
\end{align*}
$$

Proof. Formulas (50) and (51) can be obtained as direct special cases of Formula (45) for the three cases $\lambda=-\frac{1}{2}, \delta=\frac{1}{2}$, and $\lambda=\frac{1}{2}, \delta=-\frac{1}{2}$, respectively .

Corollary 19. For all positive integers $m$ and $n$, the following integral formula holds:

$$
\begin{aligned}
& \int_{0}^{1} L_{n}^{(\lambda)}(x) E_{m}(x) d x= 2 m!\Gamma(n+\lambda+1) \sum_{r=0}^{n} \frac{\left(2^{m+n-r+2}-1\right)\left(1+{ }_{1} F_{1}(-r ; n-r+\lambda+1 ; 1)\right)}{r!(m+n-r+2)!\Gamma(n-r+\lambda+1)} \times \\
& B_{m+n-r+2}
\end{aligned}
$$

Proof. Direct application to Theorem 9, taking into consideration the connection Formula (32), will yield the desired result.

Corollary 20. For all positive integers $m$ and $n$, the following integral formula holds:

$$
\begin{aligned}
& \int_{0}^{1} S_{n}(x) E_{m}(x) d x=2 m!\times \\
& \sum_{r=0}^{n} \frac{(-1)^{n-r}\left(2^{m+n-r+2}-1\right)\left((n+1)!(2 n-r)!+(2 n)!(n-r+1)!{ }_{2} F_{1}\left(\left.\begin{array}{c}
-r,-n-1 \\
-2 n
\end{array} \right\rvert\,-1\right)\right)}{(n+1)!r!(n-r+1)!(m+n+2-r)!} \times \\
& B_{m+n-r+2} .
\end{aligned}
$$

Proof. Direct application to Theorem 9 taking into consideration the connection Formula (33) will yield the desired result.

## 6. Concluding Remarks

In this article, we developed new identities involving the Euler polynomials. We established new derivative expressions for different polynomials in terms of Euler polynomials. Connection formulas between various polynomials and the Euler polynomials. We proved that the connection coefficients are in many cases simple and free of any hypergeometric functions, but in other cases, they involve certain hypergeometric functions. An interesting application is provided where various definite integrals involving Euler polynomials are computed exactly in closed forms to highlight the significance of the derived connection formulas. We intend to derive further identities and integrals involving Euler polynomials in the near future based on other formulas between different polynomials and Euler polynomials. We think that the majority of the findings in this work are novel, and they might be applicable to other areas of mathematics.

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