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Ulam-Type Stability for a Boundary-Value Problem for Multi-Term Delay Fractional Differential Equations of Caputo Type

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Abstract: A boundary-value problem for a couple of scalar nonlinear differential equations with a delay and several generalized proportional Caputo fractional derivatives is studied. Ulam-type stability of the given problem is investigated. Sufficient conditions for the existence of the boundary-value problem with an arbitrary parameter are obtained. In the study of Ulam-type stability, this parameter was chosen to depend on the solution of the corresponding fractional differential inequality. We provide sufficient conditions for Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability for the given problem on a finite interval. As a partial case, sufficient conditions for Ulam-type stability for a couple of multi-term delay, Caputo fractional differential equations are obtained. An example is illustrating the results.

Keywords: Ulam-type stability; boundary value problem; generalized proportional Caputo fractional derivative

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1. Introduction

The Ulam-type stability concept is significant in many applications, such as in numerical analysis, optimization, biology and economics. Ulam-type stability has been discussed by a number of mathematicians, and this area has grown to be one of the main subjects in mathematical analysis. For more details on the recent study on the Ulam-type stability of differential equations, one can see for ordinary differential equations [1,2], for delay differential equations [3], for Caputo fractional differential equations [4,5], for Caputo fractional differential equations with impulses [6], for Riemann–Liouville fractional differential equations [7], for generalized fractional derivatives [8], for Caputo fractional differential equations with delays [9], and for Hadamard fractional equations [10]. However, to the best of our knowledge, the Ulam-type stability is rarely studied for boundary-value problems for fractional differential equations. It is because of difficulties caused by the applied type of fractional derivatives and also the connection between the solution of the boundary-value problem for fractional differential equations and the solutions of the corresponding fractional differential inequality. In some papers, the boundary-value problem is not changeable, and according to the proved existence results, the solutions of the studied boundary-value problem is unique and independent of the chosen solution of the corresponding fractional differential inequality ([11,12] for Caputo fractional derivative, [13] for Riemann–Liouville fractional derivative). It changes the meaning of the Ulam-type stability, and it is in contradiction with the definitions for this types of stability. In this paper, we consider the boundary-value problem as depending on a parameter, and this parameter is chosen in deep connection with the used solution of the corresponding fractional differential inequality. This idea is the basis of the study of Ulam-type stability for a couple of delay fractional differential equations with several generalized proportional Caputo fractional derivatives. As is known, there are many defined and applied types of fractional derivatives. Two typical kinds of fractional derivatives are the Rieman–Liouville type and the Caputo type of derivatives. There are several differences between Riemann–Liouville-type and Caputo-type fractional derivatives based on their definitions. The most notable difference is connected with the form of initial conditions and their known physical interpretation. Another difference is connected with the constants. (for more detailed comparisons, see [14]). Since the initial/boundary-value conditions for Caputo-type fractional differential equations and their physical interpretations are similar to those for ordinary differential equations in this paper, we use the Caputo-type fractional derivative. Additionally, to be more generalized, we consider the generalized proportional Caputo fractional derivative. This derivative was recently defined (see [15,16]). It is a generalization of the Caputo fractional derivative, and it provides wider possibilities for modeling more adequate complexity of real world problems. Additionally, in the development of many mathematical models through fractional differential equations, one can observe that a single operator is not utilized exclusively. For more adequate modeling of a certain situation, in some cases we need differential equations with several fractional derivatives of different orders called multi-term fractional differential equations (for example, the Basset equation [17] is an example of equations of such kind). Motivated by this, we consider a couple of multi-term, generalized, proportional, Caputo fractional differential equations. Based on the integral presentation of the studied problem and the existence result, obtained in [18], we define and study the Ulam–Hyers stability, the Ulam–Hyers–Rassias stability, and the generalized Ulam–Hyers–Rassias stability for the given problem. As a partial case, sufficient conditions for Ulam-type stability for a couple of multi-term delay, Caputo fractional differential equations are obtained. Some of the obtained results are illustrated with an example.

2. Preliminary Notes on Generalized Proportional Fractional Derivatives

We recall that the generalized proportional fractional integral and the generalized Caputo proportional fractional derivative of a function $u : [0, b] \to \mathbb{R}$, $(b \le \infty)$, are defined, respectively, by (as long as all integrals are well defined; see [15,16])

$$({}_0\mathcal{I}^{\alpha,\rho}u)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}u(s)\, ds, \quad t \in (0, b],\ \ \alpha > 0,\ \rho \in (0, 1],$$

and

$$\begin{aligned}({}_0^C\mathcal{D}^{\alpha,\rho}u)(t) = &\frac{1-\rho}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}u(s)\, ds \\ &+ \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}u'(s)\, ds, \ \ t \in (0, b],\ \ \alpha \in (0, 1),\ \rho \in (0, 1].\end{aligned} \tag{1}$$

Remark 1. *For $\rho = 1$, the generalized Caputo proportional fractional derivative $({}_0^C\mathcal{D}^{\alpha,\rho}u)(t)$ is reduced to the classical Caputo fractional derivative ${}_0^C D_t^\alpha u(t)$.*

We introduce the following classes of functions:

$$\begin{aligned}C^{\alpha,\rho}[0, b] &= \{u : [0, b] \to \mathbb{R} : \ ({}_0^C\mathcal{D}^{\alpha,\rho}u)(t) \text{ exists on } (0, b]\},\\ I^{\alpha,\rho}[0, b] &= \{u : [0, b] \to \mathbb{R} : \ ({}_0\mathcal{I}^{\alpha,\rho}u)(t) \text{ exists on } (0, b]\}.\end{aligned}$$

For $u \in C^{\alpha,\rho}[0, b]$, ${}_0^C\mathcal{D}^{\alpha,\rho}u(.) \in I^{\alpha,\rho}[0, b]$, we have the following result:

Lemma 1 (Theorem 5.3 [15])**.** *Let $\rho \in (0, 1]$ and $\alpha \in (0, 1)$. Then we have*

$$({}_0\mathcal{I}^{\alpha,\rho}({}_0^C\mathcal{D}^{\alpha,\rho}u))(t) = u(t) - u(0)e^{\frac{\rho-1}{\rho}t}, \ \ t \in (0, b].$$

For $u \in I^{\alpha,\rho}[0, b]$, ${}_0\mathcal{I}^{\alpha,\rho}u(.) \in C^{\alpha,\rho}[0, b]$, $\rho \in (0, 1]$ we have:

Corollary 1 ([15])**.** *Let $\alpha \in (0,1)$. Then,*

$$({}_0^C\mathcal{D}^{\alpha,\rho}({}_0\mathcal{I}^{\alpha,\rho}u))(t) = u(t), \quad t \in (0,b].$$

Lemma 2 (Theorem 5.2 [15])**.** *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, we have*

$$({}_0\mathcal{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{\beta-1})(\tau) = \frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\beta+\alpha)})e^{\frac{\rho-1}{\rho}\tau}\tau^{\beta-1+\alpha}, \quad \beta > 0.$$

Corollary 2 (Remarks 3.2 and 5.4 [15])**.** *The equality $({}_0{}^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}(.)})(t) = 0$, $t \in (0,b]$, $\rho \in (0,1]$, $\alpha \in (0,1)$, holds.*

3. Statement of the Boundary-Value Problem for Multi-Term Fractional Problem

3.1. Generalized Proportional Caputo Fractional Derivatives

Let $\rho \in (0,1]$ and the sequences of numbers $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $1 > \beta_1 > \beta_2 > \cdots > \beta_N > 0$ be given.

Remark 2. *The case $\rho = 1$, i.e., the case of application of Caputo fractional derivatives, will be considered in the next section.*

Consider the couple of *delay differential equations with several generalized proportional Caputo fractional derivatives*, or so called multi-term generalized proportional fractional delay differential equations (MDFE):

$$\begin{aligned} &\sum_{i=1}^{n} A_i({}_0^C\mathcal{D}^{\alpha_i,\rho}x)(t) = f(t,x(t),x(\lambda t),y(t)), \text{ for } t \in (0,1], \\ &\sum_{i=1}^{N} B_i({}_0^C\mathcal{D}^{\beta_i,\rho}y)(t) = g(t,y(t),y(\lambda t),x(t)), \text{ for } t \in (0,1], \end{aligned} \tag{2}$$

with the nonlocal boundary-value conditions

$$\gamma_1 x(0) + \eta_1 x(\xi_1) + \mu_1 x(1) = \Phi_1, \quad \gamma_2 y(0) + \eta_2 y(\xi_2) + \mu_2 y(1) = \Phi_2, \tag{3}$$

where $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in (0,1)$ are arbitrary points, the numbers $A_i, B_j, i = 1,2,\ldots,n$, $j = 1,2,\ldots,N$: $A_1 \neq 0$, $B_1 \neq 0$, $|\gamma_k| + |\eta_k| + |\mu_k| \neq 0, k = 1,2$, and the functions $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, $\Phi_i \in \mathbb{R}$, $i = 1,2$.

We shall need the following assumption:

Assumption 1. *Let the inequalities*

$$\begin{aligned} K_1 &= \gamma_1 + \sum_{k=1}^{n} \frac{A_k\left(\eta_1 e^{\frac{\rho-1}{\rho}\xi_1}\xi_1^{\alpha_1-\alpha_k} + \mu_1 e^{\frac{\rho-1}{\rho}}\right)}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \neq 0, \\ K_2 &= \gamma_2 + \sum_{k=1}^{N} \frac{B_k\left(\eta_2 e^{\frac{\rho-1}{\rho}\xi_2}\xi_2^{\beta_1-\beta_k} + \mu_2 e^{\frac{\rho-1}{\rho}}\right)}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)} \neq 0, \end{aligned} \tag{4}$$

hold.

Consider the space $W = \left(\cup_{k=2}^{n} I^{\alpha_1-\alpha_k,\rho}[0,1]\right) \times \left(\cup_{k=2}{}^{N} I^{\beta_1-\beta_k,\rho}[0,1]\right)$ with the norm

$$||z||_W = ||(x,y)||_W = \max\{\sup_{s\in[0,1]}|x(s)|, \sup_{s\in[0,1]}|y(s)|\}, \quad z = (x,y) \in W,$$

and define the fractional integral operator $\Omega = (\Omega_1, \Omega_2) : W \to \mathbb{R}^2$ by

$$\begin{aligned}
\Omega_1(x,y)(t) &= \frac{P(\xi_1,x,y)}{K_1} e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} t^{\alpha_1-\alpha_k} \\
&\quad - \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{f(s,x(s),x(\lambda s),y(s))}{(t-s)^{1-\alpha_1}} ds, \quad t \in [0,1], \\
\Omega_2(x,y)(t) &= \frac{Q(\xi_2,x,y)}{K_2} e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{N} \frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)} t^{\beta_1-\beta_k} \\
&\quad - \sum_{k=2}^{N} \frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}} ds \\
&\quad + \frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{g(s,y(s),y(\lambda s),x(s))}{(t-s)^{1-\beta_1}} ds, \quad t \in [0,1],
\end{aligned} \tag{5}$$

where the constants K_1, K_2 are defined by (4) and

$$\begin{aligned}
P(\xi_1,x,y) &= \Phi_1 + \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \Big(\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \mu_1 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}} ds \Big) \\
&\quad - \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \Big(\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{f(s,x(s),x(\lambda s),y(s))}{(\xi_1-s)^{1-\alpha_1}} ds \\
&\quad + \mu_1 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{f(s,x(s),x(\lambda s),y(s))}{(1-s)^{1-\alpha_1}} ds \Big), \\
Q(\xi_2,x,y) &= \Phi_2 + \sum_{k=2}^{N} \frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)} \Big(\eta_2 \int_0^{\xi_2} e^{\frac{\rho-1}{\rho}(\xi_2-s)} \frac{y(s)}{(\xi_2-s)^{1-\beta_1+\beta_k}} ds \\
&\quad + \mu_2 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{y(s)}{(1-s)^{1-\beta_1+\beta_k}} ds \Big) \\
&\quad - \frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)} \Big(\eta_2 \int_0^{\xi_2} e^{\frac{\rho-1}{\rho}(\xi_2-s)} \frac{g(s,y(s),y(\lambda s),x(s))}{(\xi_2-s)^{1-\beta_1}} ds \\
&\quad + \mu_2 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{g(s,y(s),y(\lambda s),x(s))}{(1-s)^{1-\beta_1}} ds \Big).
\end{aligned} \tag{6}$$

We will use the following definition for the mild solution:

Definition 1 ([18])**.** *The couple of functions $(x(t), y(t)) : t \in [0,1]$, such that $x \in \cup_{k=2}^{n} I^{\alpha_1-\alpha_k,\rho}[0,1]$ and $y \in \cup_{k=2}^{N} I^{\beta_1-\beta_k,\rho}[0,1]$, is called a mild solution of the boundary-value problem for MDFE (2), (3) if it is a fixed point of the fractional integral operator Ω, defined by (5).*

Remark 3. *In the definition of the mild solution of MDFE (2) and (3) the boundary condition is not used explicitly, but it is deeply included in the applied integral operator.*

The connection between the mild solution and the solution of the boundary-value problem for MDFE (2) and (3) is discussed in Theorems 1 and 2 [18]. In the same work, the following existence result is also proved:

Theorem 1 ([18])**.** *Let the following conditions be satisfied:*

1. *The constants $\alpha_i, \beta_k \in (0,1)$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,N$, $\rho \in (0,1)$, and condition A1 is satisfied.*
2. *There exist constants L_i, M_i, $i = 1,2,3$, such that for $t \in [0,1]$, $x_j, y_j, z_j \in \mathbb{R}$, $j = 1,2$, the inequalities*

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|,$$
$$|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)| \leq M_1|x_1 - x_2| + M_2|y_1 - y_2| + M_3|z_1 - z_2|.$$

   hold.
3. *The inequalities*

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{L}\Big[1 + \frac{(|\eta_1| + |\mu_1|)}{|K_1|}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\Big] \\
&\times \Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k) - \Gamma(\alpha_1-\alpha_k, \frac{1-\rho}{\rho}))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} + \frac{\Gamma(\alpha_1) - \Gamma(\alpha_1, \frac{1-\rho}{\rho})}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big) < 1, \\
\mathcal{P}_2 &= \mathcal{M}\Big[1 + \frac{(|\eta_2| + |\mu_2|)}{|K_2|}\sum_{k=1}^{N}\frac{|B_k|}{|B_1|\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\Big] \\
&\times \Big(\sum_{k=2}^{N}\frac{|B_k|(\Gamma(\beta_1-\beta_k) - \Gamma(\beta_1-\beta_k, \frac{1-\rho}{\rho}))}{|B_1|(1-\rho)^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)} + \frac{\Gamma(\beta_1) - \Gamma(\beta_1, \frac{1-\rho}{\rho})}{|B_1|(1-\rho)^{\beta_1}\Gamma(\beta_1)}\Big) < 1,
\end{aligned} \tag{7}$$

   hold, where $\mathcal{L} = \max\{1, L_1 + L_2, L_3\}$, $\mathcal{M} = \max\{1, M_1 + M_2, M_3\}$, and $\Gamma(.,.)$ is the incomplete Gamma function.

   Then, the boundary-value problem for MDFE (2) and (3) has a unique mild solution.

3.2. *Caputo Fractional Derivatives*

Now we will consider the case of $\rho = 1$, i.e., the case of Caputo fractional derivatives in (2).

Let the sequences of numbers $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $1 > \beta_1 > \beta_2 > \cdots > \beta_N > 0$ be given.

Consider a couple of *differential equations with several Caputo fractional derivatives*, or so called multi-term Caputo fractional differential equations (MFE):

$$\begin{aligned}
&\sum_{i=1}^{n} A_i\, {}_0^C D^{\alpha_i} x(t) = f(t, x(t), x(\lambda t), y(t)), \text{ for } t \in (0,1], \\
&\sum_{i=1}^{N} B_i\, {}_0^C D^{\beta_i} y(t) = g(t, y(t), y(\lambda t), x(t)), \text{ for } t \in (0,1],
\end{aligned} \tag{8}$$

with the nonlocal boundary-value conditions

$$\gamma_1 x(0) + \eta_1 x(\xi_1) + \mu_1 x(1) = \Phi_1, \quad \gamma_2 y(0) + \eta_2 y(\xi_2) + \mu_2 y(1) = \Phi_2, \tag{9}$$

where ${}_0^C D^{\alpha} x(t)$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $\lambda \in (0,1)$, ξ_1, and $\xi_2 \in (0,1)$ are arbitrary points; the numbers $A_i, B_j, i = 1,2,\ldots,n, j = 1,2,\ldots,N :\ A_1 \neq 0,\ B_1 \neq 0$, and $|\gamma_k| + |\eta_k| + |\mu_k| \neq 0, k = 1,2$; and the functions $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and $\Phi_i \in \mathbb{R},\ i = 1,2$.

We introduce the following condition:

Assumption 2. *The inequalities*

$$\mathcal{K}_1 = \gamma_1 + \sum_{k=1}^{n}\frac{A_k\Big(\eta_1\xi_1^{\alpha_1-\alpha_2} + \mu_1\Big)}{A_1\Gamma(1+\alpha_1-\alpha_k)} \neq 0, \quad \mathcal{K}_2 = \gamma_2 + \sum_{k=1}^{N}\frac{B_k\Big(\eta_2\xi_2^{\beta_1-\beta_k} + \mu_2\Big)}{B_1\Gamma(1+\beta_1-\beta_k)} \neq 0 \tag{10}$$

hold.

Remark 4. *The boundary-value problem for MFE* (8) *and* (9) *is studied in [11], but in the boundary conditions there are functions with arguments equal to the unknown functions with undefined arguments. Additionally, in the definitions of Ulam-type stability, any solution of the corresponding inequalities is applied. At the same time, in the proof for Ulam-type stability, a special solution of these inequalities is taken—the solution which is satisfying the boundary conditions. Thus, the meaning of the Ulam-type stability is a misunderstanding.*

Consider the following classes of functions

$$\begin{aligned} C^{\alpha}[0,b] &= \{u:[0,b]\to\mathbb{R}:\ {}_0^CD^{\alpha}u(t)\ \text{ exists on } (0,b]\},\\ I^{\alpha}[0,b] &= \{u:[0,b]\to\mathbb{R}:\ {}_0I^{\alpha}u(t)\ \text{ exists on } (0,b]\}. \end{aligned}$$

where ${}_0I^{\alpha}u(t)$ is the Riemann–Liouville fractional integral of order $\alpha\in(0,1)$.

We introduce the following space $\mathcal{W}=\left(\cup_{k=2}^{n}I^{\alpha_1-\alpha_k}[0,1]\right)\times\left(\cup_{k=2}{}^{m}I^{\beta_1-\beta_k}[0,1]\right)$ with the norm

$$||z||_{\mathcal{W}}=||(x,y)||_{\mathcal{W}}=\max\{\sup_{s\in[0,1]}|x(s)|,\ \sup_{s\in[0,1]}|y(s)|\},\quad z=(x,y)\in\mathcal{W}$$

and define the operator $\Omega=(\Omega_1,\Omega_2):\mathcal{W}\to\mathbb{R}^2$ by

$$\begin{aligned} \Omega_1(x,y)(t) &= \frac{P(\xi_1,x,y)}{\mathcal{K}_1}\sum_{k=1}^{n}\frac{A_k}{A_1\Gamma(1+\alpha_1-\alpha_k)}t^{\alpha_1-\alpha_k}\\ &\quad-\sum_{k=2}^{n}\frac{A_k}{A_1\Gamma(\alpha_1-\alpha_k)}\int_0^t\frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}}ds\\ &\quad+\frac{1}{A_1\Gamma(\alpha_1)}\int_0^t\frac{f(s,x(s),x(\lambda s),y(s))}{(t-s)^{1-\alpha_1}}ds,\quad t\in[0,1],\\ \Omega_2(x,y)(t) &= \frac{Q(\xi_2,x,y)}{\mathcal{K}_2}\sum_{k=1}^{N}\frac{B_k}{B_1\Gamma(1+\beta_1-\beta_k)}t^{\beta_1-\beta_k}\\ &\quad-\sum_{k=2}^{N}\frac{B_k}{B_1\Gamma(\beta_1-\beta_k)}\int_0^t\frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}}ds\\ &\quad+\frac{1}{B_1\Gamma(\beta_1)}\int_0^t\frac{g(s,y(s),y(\lambda s),x(s))}{(t-s)^{1-\beta_1}}ds,\quad t\in[0,1], \end{aligned}\tag{11}$$

where the constants $\mathcal{K}_1,\ \mathcal{K}_2$ are defined by (10) and

$$\begin{aligned} P(\xi_1,x,y) &= \Phi_1+\sum_{k=2}^{n}\frac{A_k}{A_1\Gamma(\alpha_1-\alpha_k)}\Big(\eta_1\int_0^{\xi_1}\frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}}ds\\ &\quad+\mu_1\int_0^1\frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}}ds\Big)\\ &\quad-\frac{1}{A_1\Gamma(\alpha_1)}\Big(\eta_1\int_0^{\xi_1}\frac{f(s,x(s),x(\lambda s),y(s))}{(\xi_1-s)^{1-\alpha_1}}ds+\mu_1\int_0^1\frac{f(s,x(s),x(\lambda s),y(s))}{(1-s)^{1-\alpha_1}}ds\Big),\\ Q(\xi_2,x,y) &= \Phi_2+\sum_{k=2}^{N}\frac{B_k}{B_1\Gamma(\beta_1-\beta_k)}\Big(\eta_2\int_0^{\xi_2}\frac{y(s)}{(\xi_2-s)^{1-\beta_1+\beta_k}}ds\\ &\quad+\mu_2\int_0^1\frac{y(s)}{(1-s)^{1-\beta_1+\beta_k}}ds\Big)\\ &\quad-\frac{1}{B_1\Gamma(\beta_1)}\Big(\eta_2\int_0^{\xi_2}\frac{g(s,y(s),y(\lambda s),x(s))}{(\xi_2-s)^{1-\beta_1}}ds+\mu_2\int_0^1\frac{g(s,y(s),y(\lambda s),x(s))}{(1-s)^{1-\beta_1}}ds\Big). \end{aligned}\tag{12}$$

The definition of the mild solution of the boundary-value problem for multi-term, Caputo fractional differential Equations (8) and (9) is similar to Definition 1 while replacing the fractional operator Ω, defined by (5), with the fractional operator Ω, defined by the equalities (11).

The connection between the mild solution and the solution of (8) and (9) is discussed in Theorems 3 and 4 [18]. Additionally, in the same work, the following existence result is proved:

Theorem 2 ([18])**.** *Let the following conditions be satisfied:*

1. *The constants $\alpha_i, \beta_k \in (0,1)$, $i = 1,2,\dots,n$, $k = 1,2,\dots,N$, and $K_i \neq 0$, $i = 1,2$.*
2. *There exist constants L_i, M_i, $i = 1,2,3$, such that for $t \in [0,1], x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2$, the inequalities*

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|,$$
$$|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)| \leq M_1|x_1 - x_2| + M_2|y_1 - y_2| + M_3|z_1 - z_2|.$$

3. *The inequalities*

$$\begin{aligned} \mathcal{P}_1 &= \mathcal{L}\Big[1 + \frac{(|\eta_1| + |\mu_1|)}{|\mathcal{K}_1|}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)}\Big] \\ &\quad\times\Big(\sum_{k=2}^{n}\frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} + \frac{1}{|A_1|\Gamma(1+\alpha_1)}\Big) < 1, \\ \mathcal{P}_2 &= \mathcal{M}\Big[1 + \frac{(|\eta_2| + |\mu_2|)}{|\mathcal{K}_2|}\sum_{k=1}^{N}\frac{|B_k|}{|B_1|\Gamma(1+\beta_1-\beta_k)}\Big] \\ &\quad\times\Big(\sum_{k=2}^{N}\frac{|B_k|}{|B_1|\Gamma(1+\beta_1-\beta_k)} + \frac{1}{|B_1|\Gamma(1+\beta_1)}\Big) < 1, \end{aligned} \tag{13}$$

hold, where $\mathcal{L} = \max\{1, L_1 + L_2, L_3\}$ and $\mathcal{M} = \max\{1, M_1 + M_2, M_3\}$.

Then, the boundary-value problem for multi-term generalized proportional Caputo fractional differential equations, Equations (8) and (9), has a unique mild solution.

4. Ulam-Type Stability for Boundary-Value Problems

4.1. Some Comments and Remarks about Ulam-Type Stability

In this section, we discuss some concepts of the Ulam-type stability. For this purpose, consider the simple fractional differential equation with a Caputo fractional derivative of order $\alpha \in (0,1)$:

$${}_0^C D^\alpha u(t) = F(t,u(t)), \quad t \in (0,1] \tag{14}$$

with $F \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Together with this equation, we could consider

- initial value condition
$$u(0) = a, \tag{15}$$
- boundary-value condition
$$g(u(0), u(1)) = 0, \tag{16}$$

where $a \in \mathbb{R}, g : \mathbb{R}^2 \to \mathbb{R}$.

We recall the classical definition for Ulam–Hyers stability of (14): there exists a real constant $\mathcal{C} > 0$ such that for each $\varepsilon > 0$ and each solution $y(t)$ of the fractional differential inequality

$$|{}_0^C D^\alpha y(t) - F(t,y(t))| \leq \varepsilon, \ t \in [0,1] \tag{17}$$

there exists a solution $u(t)$ of (14) with $|u(t) - y(t)| \leq \mathcal{C}\varepsilon$.

In the case of initial-value problem (14) and (15), for the particular solution $y(t)$ of inequality (17), we could consider the solution of (14) and (17) with $a = y(0)$.

However, how could we proceed in the case of a boundary-value problem? How could the solution of the fractional differential equation be connected with the solution $y(t)$ of inequality (17)? The answers will be based off the main definitions of Ulam-type stability and the further study in the paper.

Example 1. *Consider the simple equation*

$$ {}_0^C D^\alpha u(t) = -u(t), \quad t \in (0,1]. \tag{18}$$

with $\alpha \in (0,1)$.

First, consider the initial value condition

$$u(0) = u_0. \tag{19}$$

The initial value problem (18) and (19) has a unique solution $u(t) = u_0 E_\alpha(-t^\alpha)$ *for any initial value* u_0*, where* $E_\alpha(.)$ *is the Mittag–Leffler function with one parameter. Consider the fractional differential inequality*

$$\left| {}_0^C D^\alpha y(t) + y(t) \right| \le \varepsilon, \quad t \in (0,1] \tag{20}$$

where $\varepsilon > 0$ *is an arbitrary number. The function* $y(t) = AE_\alpha(-t^\alpha)$ *is a solution of (20) for any constant* A*. Let* $u_0 = A$ *in (19). Then, the difference* $|u(t) - y(t)| = 0 \le \varepsilon$.

Now consider (18) with the boundary-value condition

$$u(0) = \frac{1}{E_\alpha(-1)} u(1). \tag{21}$$

The boundary value problem (18) and (21) has a unique solution $u(t) = E_\alpha(-t^\alpha)$.

Let $y(t)$ *be a solution of the fractional differential inequality (20). Then the difference* $|u(t) - y(t)| = |1 - A|E_\alpha(-t^\alpha)$*, and there is no constant* $C > 0$ *such that* $|1 - A||1 - A|E_\alpha(-t^\alpha) \le C\varepsilon$ *for any* A *and* $t \in [0,1]$*. Since the boundary-value condition (21) is fixed, solution* $u(t)$ *does not depend on solution* $y(t)$ *of the fractional differential inequality (20) as it did in the case of the initial value condition.*

Note that in some papers about boundary-value problems for various types of fractional differential equations, appropriate fractional inequalities similar to (17) are applied. Additionally, instead of considering any solution of these inequalities, the authors consider only their solutions satisfying the boundary condition (see, for example, the proof of Theorem 12 [13], the inequality (2.1) in Definition 2.4 [19], and its application in (4.1) [19] where the boundary condition is added; the last line of (4.5) [20]; the applied function $h(t)$ in (16) [7], which depends on the boundary conditions; the proof of Theorem 3 [21]; the solutions of (25) [11] which do not satisfy the boundary-value problem). In the cited papers, the meaning of Ulam-type stability is changed from all solutions of the appropriately defined fractional differential inequalities to applying only to those satisfying the given boundary-value condition (if any).

4.2. Ulam-Type Stability for Multi-Term Couple of Generalized Proportional Caputo Fractional Derivatives

In this section, we consider the case of $\rho \in (0,1)$.

Let $\varepsilon > 0$ and $\Psi \in C([0,1], \mathbb{R})$, $\Psi(t) \geq 0$ for $t \in [0,1]$ and $\Psi(.)$ be nondecreasing. We consider the following fractional differential inequalities:

$$\begin{aligned} &\left| \sum_{i=1}^{n} A_i ({}_0^C\mathcal{D}^{\alpha_i,\rho} u)(t) - f(t, u(t), u(\lambda_1 t), v(t)) \right| \leq \varepsilon \\ &\left| \sum_{i=1}^{N} B_i ({}_0^C\mathcal{D}^{\beta_i,\rho} v)(t) - g(t, v(t), v(\lambda_2 t), u(t)) \right| \leq \varepsilon, \ \ t \in [0,1], \end{aligned} \tag{22}$$

or

$$\begin{aligned} &\left| \sum_{i=1}^{n} A_i ({}_0^C\mathcal{D}^{\alpha_i,\rho} u)(t) - f(t, u(t), u(\lambda_1 t), v(t)) \right| \leq \varepsilon\Psi(t) \\ &\left| \sum_{i=1}^{N} B_i ({}_0^C\mathcal{D}^{\beta_i,\rho} v)(t) - g(t, v(t), v(\lambda_2 t), u(t)) \right| \leq \varepsilon\Psi(t), \ \ t \in [0,1], \end{aligned} \tag{23}$$

or

$$\begin{aligned} &\left| \sum_{i=1}^{n} A_i ({}_0^C\mathcal{D}^{\alpha_i,\rho} u)(t) - f(t, u(t), u(\lambda_1 t), v(t)) \right| \leq \Psi(t) \\ &\left| \sum_{i=1}^{N} B_i ({}_0^C\mathcal{D}^{\beta_i,\rho} v)(t) - g(t, v(t), v(\lambda_2 t), u(t)) \right| \leq \Psi(t), \ \ t \in [0,1]. \end{aligned} \tag{24}$$

Lemma 3. *Let the conditions of Theorem 1 be satisfied and the couple $(u,v) \in W$ be a solution of the fractional differential inequalities (22). Then, there exist constants $C_1, C_2 \in \mathbb{R}$: $|C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\tilde{\Omega} = (\tilde{\Omega}_1, \tilde{\Omega}_2)$, defined by (5) with changing $f(s, x(s), x(\lambda_1 s), y(s))$ and $g(s, y(s), y(\lambda_2 s), x(s))$ by $C_1 + f(s, x(s), x(\lambda_1 s), y(s))$ and $C_2 + g(s, v(s), v(\lambda_2 s), u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (6).*

Proof. Since the couple (u,v) is a solution of (22), there exist constants $C_1, C_2 \in \mathbb{R}$: $|C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that

$$\begin{aligned} &\sum_{i=1}^{n} A_i ({}_0^C\mathcal{D}^{\alpha_i,\rho} u)(t) = C_1 + f(t, u(t), u(\lambda_1 t), v(t)) \\ &\sum_{i=1}^{N} B_i ({}_0^C\mathcal{D}^{\beta_i,\rho} v)(t) = C_2 + g(t, v(t), v(\lambda_2 t), u(t)), \ \ t \in [0,1], \end{aligned} \tag{25}$$

According to Theorem 1 applied to the boundary-value problem for (25) with $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$, the couple (u,v) is a mild solution of (25) with the corresponding boundary condition; i.e., $u(t) = \tilde{\Omega}_1(u,v)$, $v(t) = \tilde{\Omega}_2(u,v)$, $t \in [0,1]$. $\square$

Lemma 4. *Let the conditions of Theorem 1 be satisfied and the couple $(u,v) \in W$ be a solution of the fractional inequalities (24). Then, there exist constants $C_1, C_2 \in [-1,1]$ such that the couple (u,v) is a fixed point of the fractional integral operator $\hat{\Omega} = (\hat{\Omega}_1, \hat{\Omega}_2)$, defined by (5) with changing $f(s, x(s), x(\lambda_1 s), y(s))$ and $g(s, y(s), y(\lambda_2 s), x(s))$ by $C_1\Psi(t) + f(s, x(s), x(\lambda_1 s), y(s))$ and $C_2\Psi(t) + g(s, v(s), v(\lambda_2 s), u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (6).*

Lemma 5. *Let the conditions of Theorem 1 be satisfied and the couple $(u,v) \in W$ be a solution of the fractional inequalities (23). Then there exist constants $C_1, C_2 \in \mathbb{R}$: $|C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\bar{\Omega} = (\bar{\Omega}_1, \bar{\Omega}_2)$, defined by (5) with changing $f(s, x(s), x(\lambda_1 s), y(s))$ and $g(s, y(s), y(\lambda_2 s), x(s))$ by $C_1\Psi(t) +$*

$f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2\Psi(t)+g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1=\gamma_1 u(0)+\eta_1 u(\xi_1)+\mu_1 u(1)$ and $\Phi_2=\gamma_2 v(0)+\eta_2 v(\xi_2)+\mu_2 v(1)$ in (6).

Based on the well defined and studied Ulam-type stability for initial-value problems for ordinary differential equations by Rus [1], we define Ulam-type stability for the boundary-value problem for MDFE (2) and (3).

Definition 2. *The boundary-value problem for MDFE (2) and (3) is Ulam–Hyers-stable if there exists a real number $C>0$ such that for each $\varepsilon>0$ and for each solution $(u,v)\in W$ of the inequality (22) there exists a mild solution $(x,y)\in W$ of the boundary-value problem for MDFE (2) and (3) such that*

$$\|(x,y)-(u,v)\|_W \le C\varepsilon. \tag{26}$$

Definition 3. *The boundary-value problem MDFE (2) and (3) is Ulam–Hyers–Rassias-stable with respect to Φ if there exists a positive real number C such that for each $\varepsilon>0$ and for each solution $(u,v)\in W$ of the inequality (23), there exists a mild solution $(x,y)\in W$ of the boundary-value problem for MDFE (2) and (3) such that*

$$\|(x,y)-(u,v)\|_W \le C\varepsilon \sup_{t\in[0,1]} \Phi(t). \tag{27}$$

Definition 4. *The boundary-value problem for MDFE (2) and (3) is generalized Ulam–Hyers–Rassias-stable with respect to Φ if there exists a positive real number C such that for each solution $(u,v)\in W$ of the inequality (24), there exists a mild solution $(x,y)\in W$ of the boundary-value problem for MDFE (2) and (3) such that*

$$\|(x,y)-(u,v)\|_W \le C \sup_{t\in[0,1]} \Phi(t). \tag{28}$$

Remark 5. *Note that in inequalities (22), (24) and (23) the boundary condition of the type (3) is not applied, but for any solution (u,v) of them, we will define in the appropriate way the boundary-value condition (3) depending on the solution (u,v), and we will consider the unique solution of (2) with the appropriately changed (3).*

Theorem 3 (*Stability results*)**.** *Assume that the conditions of Theorem 1 are satisfied.*

(i) *Suppose for any $\varepsilon>0$, the inequalities (22) have at least one solution. Then, the boundary-value problem for MDFE (2) and (3) is Ulam–Hyers-stable.*

(ii) *The function $\Psi\in C([0,1],\mathbb{R})$, $\Psi(t)\ge 0$ for $t\in[0,1]$, $\Psi(.)$ is nondecreasing, and there exists a constant $\Lambda>0$:*

$$\int_0^t e^{\frac{1-\rho}{\rho}(t-s)}\frac{\Psi(s)}{(t-s)^{1-\alpha_1}}ds \le \Lambda\Psi(t),\quad \int_0^t e^{\frac{1-\rho}{\rho}(t-s)}\frac{\Psi(s)}{(t-s)^{1-\beta_1}}ds \le \Lambda\Psi(t),\ \ t\in[0,1]. \tag{29}$$

Let for any $\varepsilon>0$, inequalities (23) have at least one solution. Then the boundary-value problem for MDFE (2) and (3) is Ulam–Hyers–Rassias-stable with respect to Ψ.

(iii) *The function $\Psi\in C([0,1],\mathbb{R})$, $\Psi(t)\ge 0$ for $t\in[0,1]$, $\Psi(.)$ is nondecreasing and there exists a constant $\Lambda>0$ such that inequalities (29) are satisfied. Let inequalities (24) have at least one solution. Then the boundary-value problem for MDFE (2) and (3) is generalized Ulam–Hyers–Rassias-stable with respect to Ψ.*

Proof.

(i) Let $\varepsilon>0$ be an arbitrary number and the couple $(u,v)\in W$ be a solution of inequalities (22). According to Lemma 3, there exist constants $C_1,C_2\in\mathbb{R}$: $|C_1|\le\varepsilon, |C_2|\le\varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\tilde{\Omega}=(\tilde{\Omega}_1,\tilde{\Omega}_2)$, defined by (5) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1+f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2+g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (6).

Let $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in the boundary-value conditions (3). According to Theorem 1, there exists a mild solution $(x, y) \in W$ of the boundary-value problem for MDFE (2) and (3) on $[0, 1]$.

Let $t \in [0, 1]$ be a fixed point. From Equation (6), using $\int_0^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^{1-\alpha}} ds = \frac{\rho^\alpha}{(1-\rho)^\alpha}(\Gamma(\alpha) - \Gamma(\alpha, \frac{1-\rho}{\rho} t))$ for $t \in [0, 1]$, $\alpha \in (0, 1)$ $\rho \in (0, 1)$, where $\Gamma(\alpha, z) = \int_z^\infty s^{\alpha-1} e^{-s} ds$ is the incomplete gamma function, and $\Gamma(\alpha, ct)$ is a decreasing function for $c > 0$, $t \in [0, 1]$ and condition 2 of Theorem 1, we obtain

$$\begin{aligned}
|P(\xi_1, x, y) - P(\xi_1, u, v)| &\le \sum_{k=2}^{n} \frac{|A_k|}{|A_1| \rho^{\alpha_1 - \alpha_k} \Gamma(\alpha_1 - \alpha_k)} \Big(|\eta_1| \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1 - s)} \frac{|x(s) - u(s)|}{(\xi_1 - s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + |\mu_1| \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{|x(s) - u(s)|}{(1-s)^{1-\alpha_1+\alpha_k}} ds \Big) \\
&\quad + \frac{1}{|A_1| \rho^{\alpha_1} \Gamma(\alpha_1)} \Big(|\eta_1| \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1 - s)} \frac{|f(s, x(s), x(\lambda s), y(s)) - f(s, u(s), u(\lambda_1 s), v(s))|}{(\xi_1 - s)^{1-\alpha_1}} ds \\
&\quad + |\mu_1| \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{|f(s, x(s), x(\lambda s), y(s)) - f(s, u(s), u(\lambda_1 s), v(s))|}{(1-s)^{1-\alpha_1}} ds \Big) \\
&\quad + \frac{|C_1|}{|A_1| \rho^{\alpha_1} \Gamma(\alpha_1)} \Big(|\eta_1| \int_0^{\xi_1} \frac{e^{\frac{\rho-1}{\rho}(\xi_1 - s)}}{(\xi_1 - s)^{1-\alpha_1}} ds + |\mu_1| \int_0^1 \frac{e^{\frac{\rho-1}{\rho}(1-s)}}{(1-s)^{1-\alpha_1}} ds \Big) \\
&\le (|\eta_1| + |\mu_1|) \Big(\sum_{k=2}^{n} \frac{|A_k| (\Gamma(\alpha) - \Gamma(\alpha, \frac{1-\rho}{\rho} t))}{|A_1| (1-\rho)^{\alpha_1 - \alpha_k} \Gamma(\alpha_1 - \alpha_k)} + \frac{L_1 + L_2}{|A_1| \rho^{\alpha_1} \alpha_1 \Gamma(\alpha_1)} \Big) \sup_{s \in [0,1]} |x(s) - u(s)| \\
&\quad + \frac{|\mu_1| L_3 (\Gamma(\alpha_1) - \Gamma(\alpha_1, \frac{1-\rho}{\rho}))}{|A_1| (1-\rho)^{\alpha_1} \Gamma(\alpha_1)} \sup_{s \in [0,1]} |y(s) - v(s)| \\
&\quad + \frac{\varepsilon (\Gamma(\alpha_1) - \Gamma(\alpha_1, \frac{1-\rho}{\rho} \xi_1))}{|A_1| \rho^{\alpha_1} \Gamma(\alpha_1)} (|\eta_1| + |\mu_1|) \\
&\le \mathcal{L} (|\eta_1| + |\mu_1|) \Big(\sum_{k=2}^{n} \frac{|A_k| (\Gamma(\alpha_1 - \alpha_k) - \Gamma(\alpha_1 - \alpha_k, \frac{1-\rho}{\rho} \xi_1))}{|A_1| (1-\rho)^{\alpha_1 - \alpha_k} \Gamma(\alpha_1 - \alpha_k)} \\
&\quad + \frac{\Gamma(\alpha_1) - \Gamma(\alpha_1, \frac{1-\rho}{\rho} \xi_1)}{|A_1| (1-\rho)^{\alpha_1} \Gamma(\alpha_1)} \Big) ||(x, y) - (u, v)||_W + \varepsilon \frac{\Gamma(\alpha_1) - \Gamma(\alpha_1, \frac{1-\rho}{\rho} \xi_1)}{|A_1| \rho^{\alpha_1} \Gamma(\alpha_1)} (|\eta_1| + |\mu_1|).
\end{aligned} \tag{30}$$

Thus, we get

$$
\begin{aligned}
&|u(t)-x(t)| = |\tilde{\Omega}_1(u,v)(t)-\Omega_1(x,y)(t)| \\
&\le |P(\xi_1,u,v)-P(\xi_1,x,y)|\, e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{n}\frac{|A_k|}{|K_1|\,|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}t^{\alpha_1-\alpha_k} \\
&+\sum_{k=2}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{|u(s)-x(s)|}{(t-s)^{1-\alpha_1+\alpha_k}}ds \\
&+\frac{1}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{|f(s,x(s),x(\lambda s),y(s))-f(s,u(s),u(\lambda s),v(s))|}{(t-s)^{1-\alpha_1}}ds \\
&+\frac{\varepsilon}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}\int_0^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^{1-\alpha_1}}ds \\
&\le |P(\xi_1,u,v)-P(\xi_1,x,y)|\sum_{k=1}^{n}\frac{|A_k|}{|K_1|\,|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \\
&+\Bigg\{\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}t))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \\
&\quad+\frac{\mathcal{L}(\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}t))}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Bigg\}||(u,v)-(x,y)||_W+\varepsilon\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}t)}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}.
\end{aligned}
\tag{31}
$$

Hence,

$$
\begin{aligned}
&|u(t)-x(t)| \le \varepsilon\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}t)}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)} \\
&+\varepsilon(|\eta_1|+|\mu_1|)\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}t)}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\sum_{k=1}^{n}\frac{|A_k|}{|K_1|\,|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \\
&\quad+\mathcal{P}_1||(u,v)-(x,y)||_W,
\end{aligned}
\tag{32}
$$

and therefore,

$$\sup_{s\in[0,1]}|x(s)-u(s)| \le \mathcal{G}\epsilon+\mathcal{P}_1||(u,v)-(x,y)||_W,$$

where $\mathcal{G}=\frac{1}{|A_1|(1-\rho)^{\alpha_1}}\left(1+(|\eta_1|+|\mu_1|)\sum_{k=1}^{n}\frac{|A_k|}{|K_1|\,|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\right)$.

Similarly,

$$\sup_{s\in[0,1]}|v(t)-y(t)| \le \mathcal{H}\varepsilon+\mathcal{P}_2||(u,v)-(x,y)||_W,$$

where $\mathcal{H}=\frac{1}{|B_1|(1-\rho)^{\beta_1}}\left(1+(|\eta_2|+|\mu_2|)\sum_{k=1}^{N}\frac{|B_k|}{|K_2|\,|B_1|\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\right)$.

Therefore, inequality (26) holds with $C=\frac{\max\{\mathcal{G},\mathcal{H}\}}{\max\{\mathcal{P}_1,\mathcal{P}_2\}}$.

(iii) Let the couple $(u,v)\in W$ be a solution of the inequalities (24). According to Lemma 4, there exist constants $C_1,C_2\in[-1.1]$ such that the couple (u,v) is a fixed point of the fractional integral operator $\bar{\Omega}=(\bar{\Omega}_1,\bar{\Omega}_2)$, defined by (5) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1\Psi(t)+f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2\Psi(t)+g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1=\gamma_1 u(0)+\eta_1 u(\xi_1)+\mu_1 u(1)$ and $\Phi_2=\gamma_2 v(0)+\eta_2 v(\xi_2)+\mu_2 v(1)$ in (6).

Let $\Phi_1=\gamma_1 u(0)+\eta_1 u(\xi_1)+\mu_1 u(1)$ and $\Phi_2=\gamma_2 v(0)+\eta_2 v(\xi_2)+\mu_2 v(1)$ in the boundary-value conditions (3). According to Theorem 1, there exists a mild solution $(x,y)\in W$ of the boundary-value problem for MDFE (2) and (3) on $[0,1]$.

Similarly to the proof of inequality (30), we obtain

$$
\begin{aligned}
|P(\xi_1,x,y)-P(\xi_1,u,v)| \le & \sum_{k=2}^{n} \frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\Big(|\eta_1|\int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)}\frac{|x(s)-u(s)|}{(\xi_1-s)^{1-\alpha_1+\alpha_k}}ds \\
& +|\mu_1|\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}\frac{|x(s)-u(s)|}{(1-s)^{1-\alpha_1+\alpha_k}}ds\Big) \\
& +\frac{1}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}\Big(|\eta_1|\int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)}\frac{|f(s,x(s),x(\lambda s),y(s))-f(s,u(s),u(\lambda_1 s),v(s))|}{(\xi_1-s)^{1-\alpha_1}}ds \\
& +|\mu_1|\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}\frac{|f(s,x(s),x(\lambda s),y(s))-f(s,u(s),u(\lambda_1 s),v(s))|}{(1-s)^{1-\alpha_1}}ds\Big) \\
& +\frac{|C_1|}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}\Big(|\eta_1|\int_0^{\xi_1}\frac{e^{\frac{\rho-1}{\rho}(\xi_1-s)}\Psi(s)}{(\xi_1-s)^{1-\alpha_1}}ds+|\mu_1|\int_0^1\frac{e^{\frac{\rho-1}{\rho}(1-s)}\Psi(s)}{(1-s)^{1-\alpha_1}}ds\Big) \\
\le & \mathcal{L}(|\eta_1|+|\mu_1|)\Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}\xi_1))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \\
& +\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}\xi_1)}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big)||(x,y)-(u,v)||_W+\sup_{s\in[0,1]}\Psi(s)\frac{\Lambda(|\eta_1|+|\mu_1|)}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}
\end{aligned} \tag{33}
$$

and similarly to inequality (31), we have

$$\sup_{s\in[0,1]}|x(s)-u(s)| \le \mathcal{G}\sup_{s\in[0,1]}\Psi(s)+\mathcal{P}_1||(u,v)-(x,y)||_W,$$

where $\mathcal{G}=\frac{\Lambda}{|A_1|(1-\rho)^{\alpha_1}}\Big(1+(|\eta_1|+|\mu_1|)\sum_{k=1}^{n}\frac{|A_k|}{|K_1|\ |A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\Big)$, and

$$\sup_{s\in[0,1]}|v(t)-y(t)| \le \mathcal{H}sup_{\sigma\in[0,1]}\Psi(s)+\mathcal{P}_2||(u,v)-(x,y)||_W,$$

where $\mathcal{H}=\frac{\Lambda}{|B_1|(1-\rho)^{\beta_1}}\Big(1+(|\eta_2|+|\mu_2|)\sum_{k=1}^{N}\frac{|B_k|}{|K_2|\ |B_1|\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\Big)$.

Therefore, inequality (28) holds with $C=\frac{\max\{\mathcal{G},\mathcal{H}\}}{\max\{\mathcal{P}_1,\mathcal{P}_2\}}$.

The proof of claim (ii) is similar to the one of (iii), and we omit it. □

4.3. Ulam-Type Stability for a Multi-Term Couple of Caputo Fractional Derivatives

Let $\varepsilon>0$ and $\Psi\in C([0,1],(0,\infty))$, $\Psi(.)$ be nondecreasing. We consider the inequalities:

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}A_i({}_0^C D^{\alpha_i}u(t)-f(t,u(t),u(\lambda_1 t),v(t))\right|\le\varepsilon \\
&\left|\sum_{i=1}^{N}B_i({}_0^C D^{\beta_i}v(t)-g(t,v(t),v(\lambda_2 t),u(t))\right|\le\varepsilon,\quad t\in[0,1],
\end{aligned} \tag{34}
$$

or

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}A_i({}_0^C D^{\alpha_i}u(t)-f(t,u(t),u(\lambda_1 t),v(t))\right|\le\varepsilon\Psi(t) \\
&\left|\sum_{i=1}^{N}B_i({}_0^C D^{\beta_i}v(t)-g(t,v(t),v(\lambda_2 t),u(t))\right|\le\varepsilon\Psi(t),\quad t\in[0,1],
\end{aligned} \tag{35}
$$

or

$$\left|\sum_{i=1}^{n} A_i({}_0^C D^{\alpha_i} u(t) - f(t, u(t), u(\lambda_1 t), v(t))\right| \leq \Psi(t)$$
$$\left|\sum_{i=1}^{N} B_i({}_0^C D^{\beta_i} v(t) - g(t, v(t), v(\lambda_2 t), u(t))\right| \leq \Psi(t), \ \ t \in [0,1]. \tag{36}$$

Similarly to Lemmas 3–5 for inequalities (34)–(36), the following results are true:

Lemma 6. *Let the conditions of Theorem 2 be satisfied and the couple $(u,v) \in \mathcal{W}$ be a solution of the fractional differential inequalities (34). Then there exist constants $C_1, C_2 \in \mathbb{R} : |C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\tilde{\Omega} = (\tilde{\Omega}_1, \tilde{\Omega}_2)$, defined by (11) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1 + f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2 + g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (12).*

Lemma 7. *Let the conditions of Theorem 2 be satisfied and the couple $(u,v) \in \mathcal{W}$ be a solution of the fractional inequalities (36). Then there exist constants $C_1, C_2 \in [-1,1]$ such that the couple (u,v) is a fixed point of the fractional integral operator $\hat{\Omega} = (\hat{\Omega}_1, \hat{\Omega}_2)$, defined by (11) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1\Psi(t) + f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2\Psi(t) + g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (12).*

Lemma 8. *Let the conditions of Theorem 2 be satisfied and the couple $(u,v) \in \mathcal{W}$ be a solution of the fractional inequalities (35). Then there exist constants $C_1, C_2 \in \mathbb{R} : \ |C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\bar{\Omega} = (\bar{\Omega}_1, \bar{\Omega}_2)$, defined by (11) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1\Psi(t) + f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2\Psi(t) + g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (12).*

The definitions of Ulam-type stability of (8) and (9) are similar to Definitions 2–4, only replacing inequalities (22)–(23) with (34)–(36), respectively.

Theorem 4. *Assume that the conditions of Theorem 2 are satisfied.*

(i) *Suppose for any $\varepsilon > 0$, the inequalities (34) have at least one solution. Then, problems (8) and (9) is Ulam–Hyers-stable with the constant $C = \frac{\max\{\mathcal{G},\mathcal{H}\}}{\max\{\mathcal{P}_1,\mathcal{P}_2\}}$, where $\mathcal{P}, \mathcal{P}_2$ are defined by (13), the constants $\mathcal{K}_1, \mathcal{K}_2$ are defined by (10), and*

$$\mathcal{G} = \frac{1}{|A_1|\Gamma(1+\alpha_1)}\Big(1 + (|\eta_1| + |\mu_1|)\sum_{k=1}^{n} \frac{|A_k|}{|\mathcal{K}_1| \ |A_1|\Gamma(1+\alpha_1-\alpha_k)}\Big) \tag{37}$$

$$\mathcal{H} = \frac{1}{|B_1|\Gamma(1+\beta_1)}\Big(1 + (|\eta_2| + |\mu_2|)\sum_{k=1}^{N} \frac{|B_k|}{|\mathcal{K}_2| \ |B_1|\Gamma(1+\beta_1-\beta_k)}\Big). \tag{38}$$

(ii) *The function $\Psi \in C([0,1], \ \Psi(t) \geq 0$ for $t \in [0,1]$, $\Psi(.)$ is nondecreasing, and there exists a constant $\Lambda > 0$:*

$$\int_0^t \frac{\Psi(s)}{(t-s)^{1-\alpha_1}} ds \leq \Lambda\Psi(t), \quad \int_0^t \frac{\Psi(s)}{(t-s)^{1-\beta_1}} ds \leq \Lambda\Psi(t), \ \ t \in [0,1]. \tag{39}$$

Let any $\varepsilon > 0$ inequalities (35) have at least one solution. Then, problems (8) and (9) is Ulam–Hyers–Rassias-stable with respect to Ψ.

(iii) *The function* $\Psi \in C([0,1], \Psi(t) \geq 0$ *for* $t \in [0,1]$, $\Psi(.)$ *is nondecreasing and there exists a constant* $\Lambda > 0$ *such that* (39) *hold. Let inequalities (36) have at least one solution. Then, problems* (8) *and* (9) *is generalized- Ulam–Hyers–Rassias-stable with respect to* Ψ.

Proof.

(i) Let $\varepsilon > 0$ be an arbitrary number and the couple $(u,v) \in \mathcal{W}$ be a solution of inequalities (34). According to Lemma 6, there exist constants $C_1, C_2 \in \mathbb{R}$: $|C_1| \leq \varepsilon, |C_2| \leq \varepsilon$ such that the couple (u,v) is a fixed point of the fractional integral operator $\tilde{\Omega} = (\tilde{\Omega}_1, \tilde{\Omega}_2)$, defined by (11) with changing $f(s,x(s),x(\lambda_1 s),y(s))$ and $g(s,y(s),y(\lambda_2 s),x(s))$ by $C_1 + f(s,x(s),x(\lambda_1 s),y(s))$ and $C_2 + g(s,v(s),v(\lambda_2 s),u(s))$, respectively, and $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in (12).

Let $\Phi_1 = \gamma_1 u(0) + \eta_1 u(\xi_1) + \mu_1 u(1)$ and $\Phi_2 = \gamma_2 v(0) + \eta_2 v(\xi_2) + \mu_2 v(1)$ in the boundary-value conditions (9). According to Theorem 2, there exists a mild solution $(x,y) \in \mathcal{W}$ of the boundary-value problems (8) and (9).

Let $t \in [0,1]$ be a fixed point. From Equation (12), $\int_0^t \frac{1}{(t-s)^{1-\alpha}} ds = \frac{t^\alpha}{\alpha}$ for $t \in [0,1]$, $\alpha \in (0,1)$ and $\alpha\Gamma(\alpha) = \Gamma(1+\alpha)$, we obtain

$$\begin{aligned}
|P(\xi_1,x,y) - P(\xi_1,u,v)| &\leq (|\eta_1| + |\mu_1|)\Big(\sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} + \frac{L_1+L_2}{|A_1|\Gamma(1+\alpha_1)}\Big) \sup_{s\in[0,1]} |x(s)-u(s)| \\
&+ \frac{|\mu_1|L_3}{|A_1|\Gamma(1+\alpha_1)} \sup_{s\in[0,1]} |y(s)-v(s)| \\
&+ \frac{\varepsilon}{|A_1|\Gamma(1+\alpha_1)}(|\eta_1| + |\mu_1|) \\
&\leq \mathcal{L}(|\eta_1| + |\mu_1|)\Big(\sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} + \frac{1}{|A_1|\Gamma(1+\alpha_1)}\Big)||(x,y)-(u,v)||_{\mathcal{W}} \\
&+ \varepsilon\frac{1}{|A_1|\Gamma(1+\alpha_1)}(|\eta_1| + |\mu_1|).
\end{aligned} \tag{40}$$

Thus, we get

$$\begin{aligned}
|u(t)-x(t)| &= |\tilde{\Omega}_1(u,v)(t) - \Omega_1(x,y)(t)| \\
&\leq |P(\xi_1,u,v) - P(\xi_1,x,y)| \sum_{k=1}^{n} \frac{|A_k|}{|\mathcal{K}_1|\,|A_1|\Gamma(1+\alpha_1-\alpha_k)} \\
&+ \Big\{\sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} + \frac{\mathcal{L}}{|A_1|\Gamma(1+\alpha_1)}\Big\}||(u,v)-(x,y)||_{\mathcal{W}} \\
&+ \varepsilon\frac{1}{|A_1|\Gamma(1+\alpha_1)},
\end{aligned} \tag{41}$$

where the constant $\mathcal{K}_1$ is defined by (10).

Hence,

$$\sup_{s\in[0,1]} |x(s)-u(s)| \leq \mathcal{G}\epsilon + \mathcal{P}_1||(u,v)-(x,y)||_W,$$

where $\mathcal{P}_1$ is defined by (13), $\mathcal{G}$ is defined by (37).

Similarly,

$$\sup_{s\in[0,1]} |v(t)-y(t)| \leq \mathcal{H}\varepsilon + \mathcal{P}_2||(u,v)-(x,y)||_W,$$

where $\mathcal{P}_2$ is defined by (13), $\mathcal{H}$ is defined by (38), the constant $\mathcal{K}_2$ is defined by (10).

Therefore, inequality (26) holds with $C = \frac{\max\{\mathcal{G},\mathcal{H}\}}{\max\{\mathcal{P}_1,\mathcal{P}_2\}}$.

The proof of (ii) and (iii) is similar to that of Theorem 3 and the case (i), and we omit it. □

5. Example

We give an example to illustrate the main results of this paper.

Consider the following boundary-value problem for MFE:

$$\begin{aligned} &3\,{}_0^C D^{0.3}x(t) + 0.005\,{}_0^C D^{0.01}x(t) = e^{-|x(t)|} + e^{0.5t-|y(t)|}, \\ &4\,{}_0^C D^{0.4}y(t) + 0.006\,{}_0^C D^{0.01}y(t) = e^{-|x(t)|} + e^{1.2t-|y(t)|}, \end{aligned} \tag{42}$$

with boundary-value condition

$$3x(0) - 0.1x(0.4) + 2x(1) = \Phi_1, \quad 3y(0) + 0.7y(0.6) - 0.5y(1) = \Phi_2. \tag{43}$$

Clearly, $f(t,x,y,z) = e^{-|x|} + e^{0.5t-|z|}$, $g(t,x,y,z) = e^{-|x|} + e^{0.5t-|z|}$ and hence $L_1 = M_1 = 1, L_2 = M_2 = 0, L_3 = e^{0.5}, M_3 = e^{1.2}$, $A_1 = 3, A_2 = 0.005, \alpha_1 = 0.3$, $\alpha_2 = 0.01, \gamma_1 = \gamma_2 = 3, \eta_1 = -0.1, \mu_1 = 2, \xi_1 = 0.4, B_1 = 4, B_2 = 0.006, \beta_1 = 0.4$, $\beta_2 = 0.01, \eta_2 = 0.7, \mu_2 = -0.5, \xi_2 = 0.6$ and

$$\begin{aligned} \mathcal{K}_1 &= 3 + \frac{0.005\big(-0.1 * 0.4^{0.3-0.01} + 2\big)}{3\Gamma(1+0.3-0.01)} + (-0.1+2) = 4.90357 \neq 0, \\ \mathcal{K}_2 &= \gamma_2 + \frac{B_2\Big(\eta_2\xi_2^{\beta_1-\beta_2} + \mu_2\Big)}{B_1\Gamma(1+\beta_1-\beta_k)} + (\eta_2+\mu_2) = 3.35012 \neq 0 \end{aligned} \tag{44}$$

$$\begin{aligned} \mathcal{P}_1 &= e^{0.5}\Big[1 + \frac{(0.1+2)}{|K_1|}(1 + \frac{0.005}{3\Gamma(1+0.3-0.01)})\Big]\Big(\frac{0.005}{3\Gamma(1+0.3-0.01)} + \frac{1}{3\Gamma(1+0.3)}\Big) \\ &= 0.615903 < 1, \\ \mathcal{P}_2 &= \mathcal{M}\Big[1 + \frac{(|\eta_2| + |\mu_2|)}{|\mathcal{K}_2|}\Big(1 + \frac{|B_2|}{|B_1|\Gamma(1+\beta_1-\beta_k)}\Big)\Big] \\ &\quad \times \Big(\frac{|B_2|}{|B_1|\Gamma(1+\beta_1-\beta_k)} + \frac{1}{|B_1|\Gamma(1+\beta_1)}\Big) = 0.941699 < 1, \end{aligned} \tag{45}$$

Additionally,

$$\mathcal{G} = \frac{1}{3\Gamma(1.3)}\Big(1 + \frac{0.1+2}{4.90357}\Big(1 + \frac{0.005}{3\Gamma(1.3-0.01)}\Big)\Big) = 0.530771, \tag{46}$$

$$\mathcal{H} = \frac{1}{4\Gamma(1.4)}\Big(1 + (\frac{0.7+0.5}{3.35012}\Big(1 + \frac{0.006|}{4\Gamma(1.4-0.01)}\Big)\Big) = 0.382863, \tag{47}$$

and $C = \frac{0.530771}{0.941699} = 0.563631$.

Therefore, the conditions of Theorem 2 are satisfied, and therefore, the boundary-value problem for MFE (42) and (43) has a unique solution for any Φ_1, Φ_2, and according to Theorem 4, the problems (42) and (43) is Ulam–Hyers-stable. We will illustrate it.

For example, let $\varepsilon = 2.9$. Consider the functions $u(s) = s$ and $v(s) = s$, $s \in [0,1]$. Then, applying ${}_0^C D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,ds = \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}$, we obtain the inequalities

$$\begin{aligned} &\left|3\frac{t^{0.7}}{0.7\Gamma(0.7)} + 0.005\frac{t^{0.99}}{0.99\Gamma(0.99)} - e^{-t} - e^{0.5t-t}\right| \leq 2.9, \\ &\left|4\frac{t^{0.6}}{0.4\Gamma(0.6)} + 0.006\frac{t^{0.99}}{0.99\Gamma(0.99)} - e^{-t} - e^{1.2t-t}\right| \leq 2.9; \end{aligned} \tag{48}$$

i.e., the couple (u,v) is a solution of the inequalities (34) with $\varepsilon = 2.9$.

Consider the solution $(x(t), y(t))$ in (42) with the boundary condition (43) with $\Phi_1 = 3u(0) - 0.1u(0.4) + 2u(1) = 1.96$ and $\Phi_2 = 3v(0) + 0.7v(0.6) - 0.5v(1) = -0.08$. Then, the inequality $\max\{|x(s) - s|, |y(s) - s|\} \leq 2.9C = 1.63453$, $s \in [0,1]$ holds; i.e., $x(t), y(t) \in [t - 1.63453, t + 1.63453]$ for $t \in [0,1]$.

Remark 6. *As is mentioned above the problem MFE* (8) *and* (9) *with Caputo fractional derivatives is studied in* [11]*. If we apply the results of* [11] *to the above example, then* $\mathcal{L}_1 = 3.15314 > 1$ *and the conditions of Theorem 4* [11] *are not satisfied. This is because of the mistaken integral presentation of the solution of MFE* (8) *and* (9)*.*

6. Conclusions

In this paper, the concept of Ulam-type stability was applied to a couple of nonlinear delay fractional differential equations with several generalized proportional Caputo fractional derivatives, and the nonlocal boundary-value condition was discussed and studied. Various types of Ulam stability were defined and investigated. The solution of the appropriate fractional differential inequality is deeply connected with the boundary condition of the given system and its solution, respectively. Then, the closeness between both solutions, the solution of the inequality, and the solution of the corresponding boundary-value problem were proved. Additionally, as a partial case, some sufficient conditions for Ulam-type stability for a boundary-value problem for a couple of fractional differential equations with delay and several Caputo fractional derivatives were provided.

Further, the ideas for Ulam-type stability, the connection between the solutions of the corresponding fractional inequalities, and the boundary-value conditions could be applied to various types of differential equations with different types of boundary-value conditions.

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