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The Synchronization of a Class of Time-Delayed Chaotic Systems Using Sliding Mode Control Based on a Fractional-Order Nonlinear PID Sliding Surface and Its Application in Secure Communication

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Abstract: A novel approach for the synchronization of a class of chaotic systems with uncertainty, unknown time delays, and external disturbances is presented. The control method given here is expressed by combining sliding mode control approaches with adaptive rules. A sliding surface of fractional order has been developed to construct the control strategy of the abovementioned sliding mode by employing the structure of nonlinear fractional PID (NLPID) controllers. The suggested control mechanism using Lyapunov's theorem developed robust adaptive rules in such a way that the estimation error of the system's unknown parameters and time delays tends to be zero. Furthermore, the proposed robust control approach's stability has been demonstrated using Lyapunov stability criteria and Lipschitz conditions. Then, in order to assess the performance of the proposed mechanism, the presented control approach was used to simulate the synchronization of two chaotic jerk systems with uncertainty, unknown time delays, and external distortion. The results of the simulation confirm the robust and desirable synchronization performance. Finally, a secure communications mechanism based on the proposed technique is shown as a practical implementation of the introduced control strategy, in which the message signal is disguised in the transmitter with high security and well recovered in the receiver with high quality, according to the mean squared error (MES) criteria.

Keywords: chaotic synchronization; sliding mode control; adaptive control; uncertainty; unknown time-delay; secure communication

MSC: 93D09; 93B51

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1. Introduction

Chaos is a phenomenon that occurs in nonlinear dynamic systems. The dynamic behavior of these systems is fully dependent on the initial conditions; therefore, even the tiniest change in these parameters generates big changes in their behavior. Many domains of study, including economics [1,2], chemistry [3,4], biology [5,6], and engineering [7,8], have discovered chaotic systems. Many scientists from numerous domains have been interested in the synchronization of chaotic systems in recent years [5]. Pecora and Carroll proposed chaos synchronization in 1990 [6].

Fractional order (FO) controllers were developed by combining fractional calculation with existing controllers. The increased number of configurable parameters in this type of controller creates more flexibility in the control process, which improves the performance of the controlled system. In this regard, the fractional PI controller [7,8], the fractional PD controller [9], the fractional lag–lead controller [10], the fractional CRONE controller [11,12], the adaptive FO PID controller [13], the reference adaptive control [14], the fractional model [15,16], and the fractional sliding mode control [17–19] have all been mentioned.

Adaptive sliding mode controllers have been shown to exhibit robust performance against system parameter changes and system uncertainties [20,21], with their greatest advantage being stability against system disturbances.

Adaptive control [22–24] is used to achieve control objectives when there is uncertainty in the system. Because of the benefits of stability against system parameter uncertainties, adaptive control has a faster convergence speed.

Sliding mode control, on the other hand, is a popular and successful strategy that is simple to implement [25,26]. The sliding mode controller is a variable structure controller that works well for nonlinear systems with model uncertainty. The primary aspect of sliding mode control is that it directs the system's states from the initial states to a suitable sliding surface that is provided and then maintains the states at the mentioned sliding surface for all subsequent iterations. Several articles have been provided in this area for the synchronization of chaotic systems utilizing sliding mode control. Qamati et al. [27] successfully synchronized identical Genesio–Tesi chaotic systems using adaptive sliding mode control. The integrated sliding mode control method was used to examine the control and synchronization of extremely chaotic Zhou systems, as discussed in [28].

The authors of [29] proposed a hybrid fractional-order sliding mode control method for finite-time synchronization of a chaotic class using the direct Lyapunov method. Synchronization of fractional order chaotic systems is introduced in [30]. In this design, a sliding surface based on non-linear fractional order PID is presented. In [31], Khan et al. have proposed an adaptive sliding mode control approach for synchronizing complicated chaotic systems with uncertainty and disturbance. The authors of [32] have investigated an adaptive sliding control with fuzzy logic for the synchronization of chaotic fractional order systems with uncertainty and exogenous shocks. In [33], a class of complicated fractional order systems with non-uniform order has been explored, and an adaptive sliding control has been proposed for synchronizing this class of systems. An adaptive controller has been devised in [34] for integer-order synchronization with uncertainty and unknown time delays.

The time delays in the system being unknown is one of the factors to consider for uncertain systems with time delays. This issue can provide significant challenges in the controller design process for a variety of purposes, including synchronization. The amount of time delays in the system is unknown which, on the other hand, increases the complexity of the system model, which can be considered in enhancing the security level of data transmission in the field of secure telecommunications. A sliding surface based on NLFPID is proposed to direct and keep all the states of the system to the sliding surface in order to synchronize a class of chaotic systems with uncertainty and unknown time delays.

A sliding surface based on NLFPID is proposed to direct and keep all the states of the system on the sliding surface in order to synchronize a class of chaotic systems with uncertainty and unknown time delays. Then, for robust synchronization of uncertain chaotic systems with unknown time delays, a fractional order adaptive control is adopted. Following that, updating and estimating laws for uncertain parameters are determined using an appropriate Lyapunov function and Lipschitz condition to ensure the system's stability. Finally, the synchronization of two chaotic jerk systems with uncertain and unknown time delays, as well as uncertainty and distortion, are examined and simulated in order to evaluate the performance of the proposed approach. The simulation results demonstrate the efficacy of the proposed adaptive-sliding control mechanism for synchronization that is robust against uncertainty, external distortions, and unknown time delays. Furthermore, based on the obtained results, the proposed control approach is effective in estimating uncertain parameters and unknown delays.

To implement the proposed control strategy, a secure telecommunication mechanism based on chaos masking is presented at the end, indicating optimal security in sending and high quality in retrieving information despite the presence of various uncertainties and uncertain parameters in the system structure.

The following innovations have been presented in the study that was carried out in order to synchronize two uncertain chaotic systems.

1. Using the NLFOPID sliding surface instead of conventional sliding surfaces.
2. The existence of unknown time delays.
3. The limits of uncertainty and disturbance are unknown.

Accordingly, based on the above concepts, using the appropriate Lyapunov function and update rules, a control mechanism has been proposed, which can overcome the problem of unknown time delays, uncertain uncertainty, and uncertain disturbances by properly adjusting the controller parameters.

This article is structured as follows. Section 2 presents the basic definitions used in the article. Section 3 describes the description of chaotic systems with uncertainty and unknown time delays. Section 4 presents an NLFPID-based sliding surface for directing and maintaining system states on the sliding surface. The adaptive robust control technique for the synchronization of uncertain chaotic systems with finite and uncertain time delays is detailed in Section 5. Section 6 investigates and simulates the synchronization of two chaotic jerk systems with unknown time delays, as well as uncertainty and distortion, and Section 7 presents results based on the stated ideas.

2. Basic Definitions of the Fractional-Order Derivative

Definition 1. *The fractional order integral and derivative are defined as follows [27]:*

$$D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases} \tag{1}$$

where D_t^α is the fractional order operator.

Definition 2. *The Riemann–Liouville fractional order integral order α of the function $f(t)$ is defined as follows [28]:*

$$_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2}$$

where t_0 is the initial time and $\Gamma(\alpha)$ is the Gamma function, which is defined as follows:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{3}$$

where is the Gamma function operator.

Definition 3. *Suppose $n-1 < \alpha \le n$. $n \in N$. The fractional Riemann–Liouville derivative of order α is defined for the function $f(t)$ as follows [26]:*

$$_{t_0}D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{4}$$

Remark 1. *The Riemann–Liouville fractional order derivative in Equation (4) is first integrated and then derived. Therefore, the derivative of a constant number in this definition is not equal to zero.*

Definition 4. *The Caputo fractional order derivative of order α in the continuous function $f(t)$ is defined as follows [29]:*

$$
{}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau & m-1 < \alpha < m \\ \frac{d^m f(t)}{dt^m} & \alpha = m \end{cases} \tag{5}
$$

where m is the first integer after α.

Definition 5. *If the function $f(x,t)$ is piecewise linear and satisfies the Lipschitz conditions, then [34]:*

$$
|f(t,x) - f(t,\hat{x})| \leq \gamma_f |x - \hat{x}| \qquad \forall\, x, \hat{x} \in R^n \tag{6}
$$

where $f(x,t)$ is Lipschitz at x and the positive constant γ_f, is called the Lipschitz constant.

3. The System Descriptor Equations

The equations characterizing a class of master–slave chaotic systems with uncertainty and unknown time delays in the presence of an unknown disturbance will be introduced in this section; the canonical form dynamics of the master system are as follows:

$$
\begin{cases} \dot{x}_i = x_{i+1}. & 1 \leq i \leq n-1 \\ \dot{x}_n = \sigma_0^T \mathrm{x} + f(x(t-\tau_1),t) + \Delta f(x(t),t) + d_1(t). \end{cases} \tag{7}
$$

Equations of the slave system are as follows:

$$
\begin{cases} \dot{y}_i = y_{i+1}. & 1 \leq i \leq n-1 \\ \dot{y}_n = \sigma_0^T \mathrm{y} + g(y(t-\tau_2),t) + \Delta g(y(t),t) + d_2(t) + u(t). \end{cases} \tag{8}
$$

The differential equations are expressed in the forms corresponding to a number of well-known chaotic systems, such as the Van der Pol oscillator, Duffing's oscillator, Genesio–Tesi's system, Arneodo's system, etc. [35], where $x(t), y(t) \in R^n$ describes the dynamic states of the master and slave systems, σ_0^T represents the constant coefficients in linear states of the system, and $f(x(t-\tau_1),t) \cdot g(y(t-\tau_2),t) \in R$ are the terms of nonlinear functions with unknown time delays with a τ_1 , τ_2 delay, and $\Delta f(x(t),t)$, $\Delta g(x(t),t)$ describes nonlinear bounded uncertainties of the master and slave systems. Additionally, $d_1(t), d_2(t)$ describes the external disturbances of the master and slave systems, and $u(t)$ is the control law applied to the slave system.

Definition 6. *If the following criterion is met for the systems given in Equations (1) and (2) for all conditions influencing the system, including all initial conditions, uncertainties, and unknown time delays, as well as external disturbance, the system has robust synchronization:*

$$
\lim_{t\to\infty} |y_i(t) - x_i(t)| = lim_{t\to\infty} |e_i(t)| = 0.\ i = 1, \ldots, n, \tag{9}
$$

where $e_i(t)$ introduces synchronization errors in the master and slave systems.

As a result, the following are the dynamic equations proposing the synchronization error for the uncertain chaotic master and slave systems with unknown time delays, as specified in Equations (1) and (2):

$$\begin{cases} \dot{e}_i = e_{i+1}. & 1 \le i \le n-1 \\ \dot{e}_n = \sigma_0^T E + g(y(t-\tau_2),t) + \Delta g(x(t),t) + d_2(t) \\ -f(x(t-\tau_1),t) - \Delta f(x(t),t) - d_1(t) + u(t). \end{cases} \tag{10}$$

where $E = (e_1 \cdot e_2 \cdot \cdots \cdot e_n)^T$. As a result, by initially introducing a PI sliding surface and a fractional order non-linear derivative, all states of the system should be directed to and held on the sliding surface. The system's uncertainty bounds and unknown parameters should then be estimated and updated by creating an adaptive controller. In the continuation of the robust synchronization of chaotic systems (7) and (8) in the presence of external disturbances, bounded nonlinear uncertainties, and unknown time delays, it should be performed such that the dynamics of the slave system state in a finite time conforms to the dynamic behavior. The estimation error of the unknown parameters in both chaotic systems tend to zero in any state, and the robust stability of the system in finite time is guaranteed.

Assumption 1. *The unknown external disturbances $d_1(t), d_2(t)$ and unknown bounded nonlinear uncertainties $\Delta f(x(t),t)$ and $\Delta g(x(t),t)$ in the master and slave systems (7) and (8) satisfy the following conditions:*

$$\begin{gathered} |\Delta f(x(t),t)| \le \beta_1 \omega_1(x). \\ |\Delta g(y(t),t)| \le \beta_2 \omega_2(y). \\ |d_1(t)| \le \rho_1. \\ |d_2(t)| \le \rho_2. \end{gathered} \tag{11}$$

where $|\cdot|$ describes l_1 norm and $\beta_2,\ \beta_1, \rho_2, \rho_1$ are unknown real positive constants, and $\omega_2(\cdot)\ , \omega_1(\cdot)$ are known functions.

Assumption 2. *Unknown time delays introduced by non-linear functions $f(x(t-\tau_1),t)$, $g(y(t-\tau_2),t) \in R$ are represented in the general forms of (7) and (8) in the master and slave systems, for each $x(t), y(t) \in R$ and, according to (6), they satisfy the following Lipschitz condition:*

$$\begin{gathered} |f(x(t-\tau_1)) - f(x(t-\hat{\tau}_1))| \le k_1 |x(t-\tau_1) - x(t-\hat{\tau}_1)| \\ |x(t-\tau_1) - x(t-\hat{\tau}_1)| \le m_1 |(t-\tau_1) - (\mathrm{t}-\hat{\tau}_1)| = m_1 |\widetilde{\tau}_1| \\ \Rightarrow |f(x(t-\tau_1)) - f(x(t-\hat{\tau}_1))| \le l_1 |\tau_1 - \hat{\tau}_1| = l_1 |\widetilde{\tau}_1|,\ l_1 = k_1 m_1 \\ |g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2))| \le k_2 |y(t-\tau_2) - y(t-\hat{\tau}_2)| \\ |y(t-\tau_2) - y(t-\hat{\tau}_2)| \le m_1 |(t-\tau_2) - (t-\hat{\tau}_2)| = m_2 |\widetilde{\tau}_2| \\ \Rightarrow |g(y(t-\tau_2)) - g(y(t-\hat{\tau}_2))| \le l_2 |\tau_2 - \hat{\tau}_2| = l_2 |\widetilde{\tau}_2|,\ l_2 = k_2 m_2 \end{gathered} \tag{12}$$

where $\tau_1\ , \tau_2 \in R$ expresses unknown time delays, $\hat{\tau}_1\ , \hat{\tau}_2 \in R$ estimates unknown time delays, and l_1 and l_2 are positive and uncertain constants.

4. The Sliding Mode Control Approach Based on NLFPID Controllers

We will provide an NLFPID-based sliding surface to synchronize the chaotic system with unknown uncertainty and time delays presented in (7) and (8). The novel NLF sliding surface is presented in accordance with the NLFPID controller structure established in [35], which enhances tracking:

$$s(t) = h(e) \cdot \left[k_p e_n(t) + T_I D^{-\lambda} \sum_{i=1}^{n} k_{1i} e_i + T_d D^{\delta} \sum_{i=1}^{n} k_{2i} e_i(t) \right]. \tag{13}$$

where $h(e)$ is a nonlinear function, defined as follows:

$$h(e) = k_0 + (1 - k_0)\|E(t)\|. \quad k_0 \in (0,1) \tag{14}$$

where $\|\mathrm{E}\|$ is the first norm of the dynamic state of the system error, expressed as $\|\mathrm{E(t)}\| = \sum_{\mathrm{i=1}}^{\mathrm{n}} |\mathrm{e_i}|$. $\mathrm{T_I}$ and $\mathrm{T_d}$ are the time constants of integral and derivative phrases. The parameters $\mathrm{k_{1i}}$ and $\mathrm{k_{2i}}$ are positive constant values of the sliding surface that fulfill the intended system's stability. The following conditions must be met if the system is in sliding mode:

$$s(t) = 0 \quad , \quad \dot{s}(t) = 0. \tag{15}$$

Therefore, the fractional order derivative of the sliding surface in Equation (13) is as follows:

$$\begin{aligned}
\dot{s}(t) = {} & k_0 k_p \dot{e}_n(t) + k_0 T_i D^{1-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t) + k_0 T_d D^{1+\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t) \\
& + (1 - k_0)\Big[k_p \tfrac{d}{dt}(\|E(t)\| e_n(t)) \\
& + T_I \tfrac{d}{dt}\big(\|E(t)\| D^{-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t)\big) \\
& + T_d \tfrac{d}{dt}\big(\|E(t)\| D^{\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t)\big)\Big] = 0.
\end{aligned} \tag{16}$$

In this scenario, we will substitute the final dynamic of the system's integer order error specified in Equation (10) in Equation (16), yielding:

$$\begin{aligned}
\dot{s}(t) = {} & k_0 k_p (g(y(t - \tau_2), t) + \Delta g(x(t), t) + d_2(t) - f(x(t - \tau_1), t) \\
& - \Delta f(x(t), t) - d_1(t) + \sigma_0^T \cdot E(t) + u(t)) \\
& + k_0 T_i D^{1-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t) + k_0 T_d D^{1+\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t) \\
& + (1 - k_0)\Big[k_p \tfrac{d}{dt}(\|E(t)\| e_n(t)) \\
& + T_I \tfrac{d}{dt}\big(\|E(t)\| D^{-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t)\big) \\
& + T_d \tfrac{d}{dt}\big(\|E(t)\| D^{\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t)\big)\Big] = 0.
\end{aligned} \tag{17}$$

The control signal in this scenario is as follows:

$$\begin{aligned}
u(t) = {} & \tfrac{-1}{k_0 k_p}\big(k_0 T_i D^{1-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t) + k_0 T_d D^{1+\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t) \\
& + (1 - k_0)\Big[k_p \tfrac{d}{dt}(\|E(t)\| e_n(t)) \\
& + T_I \tfrac{d}{dt}\big(\|E(t)\| D^{-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t)\big) \\
& + T_d \tfrac{d}{dt}\big(\|E(t)\| D^{\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t)\big)\Big]\big) \\
& + f(x(t - \hat{\tau}_1), t) - g(y(t - \hat{\tau}_2), t) - \sigma_0^T \cdot E(t) - bs + \overline{u}(t).
\end{aligned} \tag{18}$$

In Equation (18), the phrase $\overline{u}(t)$ comprises the sentences arising from estimating the bounds of the system's uncertainties and disturbances, which are specified as follows:

$$\overline{u}(t) = -sgn(s)[\hat{\beta}_2 \omega_2(\mathrm{y}) + \hat{\beta}_1 \omega_1(\mathrm{x}) + \hat{\rho}_2 + \hat{\rho}_1)]. \tag{19}$$

5. Stability Analysis of the Proposed Mechanism

In this section, the robust adaptive controller is designed using the sliding surface based on NLFPID such that the synchronization of chaotic systems is guaranteed by the proposed control approach.

Theorem 1. *Synchronization of systems (6) and (7) is guaranteed by the definition of the controller $u(t)$ in spite of disturbances d_1 and d_2 and uncertainties Δf and Δg along with unknown time delays τ_1 and τ_2:*

$$\begin{aligned} u(t) = -g(y(t-\hat{\tau}_1)) + f(x(t-\hat{\tau}_2)) \\ & -\tfrac{1}{k_0 k_p}\big(k_0 T_I D^{1-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t) + k_0 T_d D^{1+\delta} \sum_{i=1}^{n} k_{2i} e_i(t) \\ & +(1-k_0)\Big[k_p \tfrac{d}{dt}(\|E(t)\| e_n(t)) \\ & +T_I \tfrac{d}{dt}\big(\|E(t)\| D^{-\lambda} \textstyle\sum_{i=1}^{n} k_{1i} e_i(t)\big) \\ & +T_d \tfrac{d}{dt}\big(\|E(t)\| D^{\delta} \textstyle\sum_{i=1}^{n} k_{2i} e_i(t)\big)\Big]\big) - \sigma_0^T \cdot E(t) - bs \\ & -sgn(s)(\hat{\beta}_2 \omega_2(y) + \hat{\beta}_1 \omega_1(x) + \hat{\rho}_2 + \hat{\rho}_1). \end{aligned} \tag{20}$$

where $\hat{\rho}_1$ and $\hat{\rho}_2$ are the estimations of input disturbances, $\hat{\tau}_1$ and $\hat{\tau}_2$ are estimates of time delays, and $\hat{\beta}_1\omega_1$ and $\hat{\beta}_2\omega_2$ are estimates of uncertainty in the master and slave systems. Therefore, in order to guarantee the stability of the system, we use the update laws to estimate the mentioned parameters as follows:

$$\begin{gathered} \widetilde{\tau}_i = \tau_i - \hat{\tau}_i.\ \widetilde{\beta}_i = \beta_i - \hat{\beta}_i.\ \widetilde{\rho}_i = \rho_i - \hat{\rho}_i. \\ \dot{\hat{\tau}}_i = -\dot{\widetilde{\tau}}_i = |s| sgn(\widetilde{\tau}_i) + b\widetilde{\tau}_i. \\ \dot{\hat{\rho}}_i = -\dot{\widetilde{\rho}}_i = k_0 k_p |s| + b\widetilde{\rho}_i. \\ \dot{\hat{\beta}}_2 = -\dot{\widetilde{\beta}}_2 = k_0 k_p |s| \omega_2(y) + b\widetilde{\beta}_2. \\ \dot{\hat{\beta}}_1 = -\dot{\widetilde{\beta}}_1 = k_0 k_p |s| \omega_1(x) + b\widetilde{\beta}_1. \end{gathered} \tag{21}$$

Proof. The following Lyapunov function is defined as follows:

$$v(t) = \frac{1}{2}[s^2(t) + \widetilde{\beta}_1^2 + \widetilde{\beta}_2^2 + l_1\widetilde{\tau}_1^2 + l_2\widetilde{\tau}_2^2 + \widetilde{\rho}_1^2 + \widetilde{\rho}_2^2]. \tag{22}$$

According to Equation (22), the derivative of the Lyapunov function is as follows:

$$\begin{aligned} \Rightarrow \dot{v}(t) &= \tfrac{1}{2}\tfrac{d}{dt}(s^2 + \widetilde{\beta}_1^2 + \widetilde{\beta}_2^2 + l_1\widetilde{\tau}_1^2 + l_2\widetilde{\tau}_2^2 + \widetilde{\rho}_1^2 + \widetilde{\rho}_2^2) \\ &= s\dot{s} + \sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right). \end{aligned} \tag{23}$$

By substituting (17) in (23), Equation (24) is obtained:

$$
\begin{aligned}
\dot{v}(t) = s\cdot\big[&k_0k_p(g(y(t-\tau_2),t) + \Delta g(x(t),t) + d_2(t)\\
&-(f(x(t-\tau_1),t) + \Delta f(x(t),t) + d_1(t)) + \sigma_0^T\cdot E(t) + u(t)\big)\\
&+k_0T_iD^{1-\lambda}\textstyle\sum_{i=1}^{n}k_{1i}e_i(t) + k_0T_dD^{1+\delta}\sum_{i=1}^{n}k_{2i}e_i(t)\\
&+(1-k_0)\Big[k_p\tfrac{d}{dt}(\|E(t)\|e_n(t))\\
&+T_I\tfrac{d}{dt}\big(\|E(t)\|D^{-\lambda}\textstyle\sum_{i=1}^{n}k_{1i}e_i(t)\big)\\
&+T_d\tfrac{d}{dt}\big(\|E(t)\|D^{\delta}\textstyle\sum_{i=1}^{n}k_{2i}e_i(t)\big)\Big]\big] + \sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right).
\end{aligned}
\tag{24}
$$

By substituting (20) in (24), Equation (25) is obtained:

$$
\begin{aligned}
\dot{v}(t) = s\cdot k_0k_p(&g(y(t-\tau_2),t) - g(y(t-\hat{\tau}_2),t) + \Delta g(x(t),t) + d_2(t)\\
&+f(x(t-\hat{\tau}_1),t) - f(x(t-\tau_1),t) - \Delta f(x(t),t) - d_1(t) - bs\\
&-sgn(s)[\hat{\beta}_2\omega_2(y) + \hat{\beta}_1\omega_1(x) + \hat{\rho}_2 + \hat{\rho}_1])\\
&+\sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right).
\end{aligned}
\tag{25}
$$

Thus, we have:

$$
\begin{aligned}
\dot{v}(t) \le |s|\cdot[k_0k_p(&|g(y(t-\tau_2),t) - g(y(t-\hat{\tau}_2),t)| + |\Delta g(x(t),t)|\\
&+|f(x(t-\hat{\tau}_1),t) - f(x(t-\tau_1),t)| - |\Delta f(x(t),t)|\\
&+|d_2(t) - d_1(t)|)] - k_0k_pbs^2\\
&+k_0k_ps(-sgn(s)[\hat{\beta}_2\omega_2(y) + \hat{\beta}_1\omega_1(x) + \hat{\rho}_2 + \hat{\rho}_1)])\\
&+\sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right).
\end{aligned}
\tag{26}
$$

By substituting (11) and (12) in (26), Equation (27) is obtained:

$$
\begin{aligned}
\dot{v}(t) \le |s|\cdot\big[&k_0k_p(l_2|\tau_2-\hat{\tau}_2| + \beta_2\omega_2(y) + l_1|\tau_1-\hat{\tau}_1| + \beta_1\omega_1(x) + \rho_1 - \rho_2)\big]\\
&-k_0k_pbs^2\\
&+k_0k_ps\big((-sgn(s)[\hat{\beta}_2\omega_2(y) + \hat{\beta}_1\omega_1(x) + \hat{\rho}_2 + \hat{\rho}_1)])\big)\\
&+\sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right).
\end{aligned}
\tag{27}
$$

The derivative of the Lyapunov function would be as follows:

$$
\begin{aligned}
\dot{v}(t) \le |s|[&k_0k_p(l_1|\widetilde{\tau}_1| + \widetilde{\beta}_2\omega_2(y) + l_2|\widetilde{\tau}_2| + \widetilde{\beta}_1\omega_1(y) + \hat{\rho}_2 + \hat{\rho}_1)] - bs^2\\
&+\sum_{i=1}^{2}\left(\widetilde{\beta}_i\dot{\widetilde{\beta}}_i + l_i\widetilde{\tau}_i\dot{\widetilde{\tau}}_i + \widetilde{\rho}_i\dot{\widetilde{\rho}}_i\right).
\end{aligned}
\tag{28}
$$

Using the update laws (21) and substituting them in (28), Equation (29) is obtained:

$$\Rightarrow \dot{v}(t) \le -b(s^2 + \tilde{\beta}_1^2 + \tilde{\beta}_2^2 + \tilde{\tau}_1^2 + \tilde{\tau}_2^2 + \tilde{\rho}_1^2 + \tilde{\rho}_2^2) = -bv(t). \tag{29}$$

Thus, the stability of the proposed method for synchronization of the chaotic system with uncertainty, disturbance, and unknown time delays is proved. □

6. Simulation Results

This section evaluates the accuracy of synchronizing uncertain chaotic systems with unknown time delays using the proposed control mechanism based on an NLF sliding surface, adaptive controller, and update rules that estimate the system's parameters. Two modified jerk chaotic systems with the aforementioned properties were used for this purpose. The equations governing the master system in the canonical form are as follows [33]:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\varepsilon_1 x_1(t) - x_2(t) - \varepsilon_2 x_3(t) + f_3(x_1(t-\tau_1), t). \end{cases} \tag{30}$$

where $\varepsilon_1 = \frac{3}{2}$, $\varepsilon_2 = 0.35$, and $f_3(x_1(t-\tau_1), t)$ is a piecewise linear function:

$$\begin{aligned} f_3(x_1(t-\tau_1), t) = \tfrac{1}{2}(v_0 - v_1)[|x_1(t-\tau_1) + 1| - |x_1(t-\tau_1) - 1|] \\ + v_1 x_1(t-\tau_1). \end{aligned} \tag{31}$$

where $v_0 < -1 < v_1 < 0$ and $v_0 = -2.5$, $v_1 = -0.5$.

When the initial conditions are selected as $(x_1(0); x_2(0); x_3(0))^T = (-0.52; 0.52; 0.87)^T$, the system's chaotic behavior would be as shown in Figure 1.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -\varepsilon_1 x_1(t) - x_2(t) - \varepsilon_2 x_3(t) + f_3(x_1(t-\tau_1), t) + \Delta f(x(t), t) + d_1(t). \end{cases} \tag{32}$$

The dynamic of the slave system follows the equations below:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_3 \\ \dot{y}_3 = -\varepsilon_1 y_1(t) - y_2(t) - \varepsilon_2 y_3(t) + g_3(y_1(t-\tau_2), t) + \Delta g(y(t), t) + d_2(t) + u(t). \end{cases} \tag{33}$$

where the nonlinear terms of the slave system are as follows:

$$\begin{aligned} g_3(y_1(t-\tau_2), t) = \tfrac{1}{2}(v_0 - v_1)[|y_1(t-\tau_2) + 1| - |y_1(t-\tau_2) - 1|] \\ + v_1 y_1(t-\tau_2). \end{aligned} \tag{34}$$

Accordingly, the dynamic of the synchronization error for the chaotic jerk master and slave systems would be as follows:

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = -\varepsilon_1 e_1(t) - e_2(t) - \varepsilon_2 e_3(t) - g(y_1(t-\tau_2)) + f(x_1(t-\tau_1)) \\ \quad + \Delta g(y(t), t) - \Delta f(x(t), t) + d_2(t) - d_1(t) + u(t). \end{cases} \tag{35}$$

In this step, we apply the robust adaptive control signal, which is designed by combining the sliding surface based on the NLFPID controllers and described in Equation (20), to the slave system.

The following figures show the behavior of the chaotic system synchronized with the above dynamic equations before and after applying the proposed control signal. Figures 1 and 2 respectively show the phase diagram and the behavior of the jerk system without applying the controller.

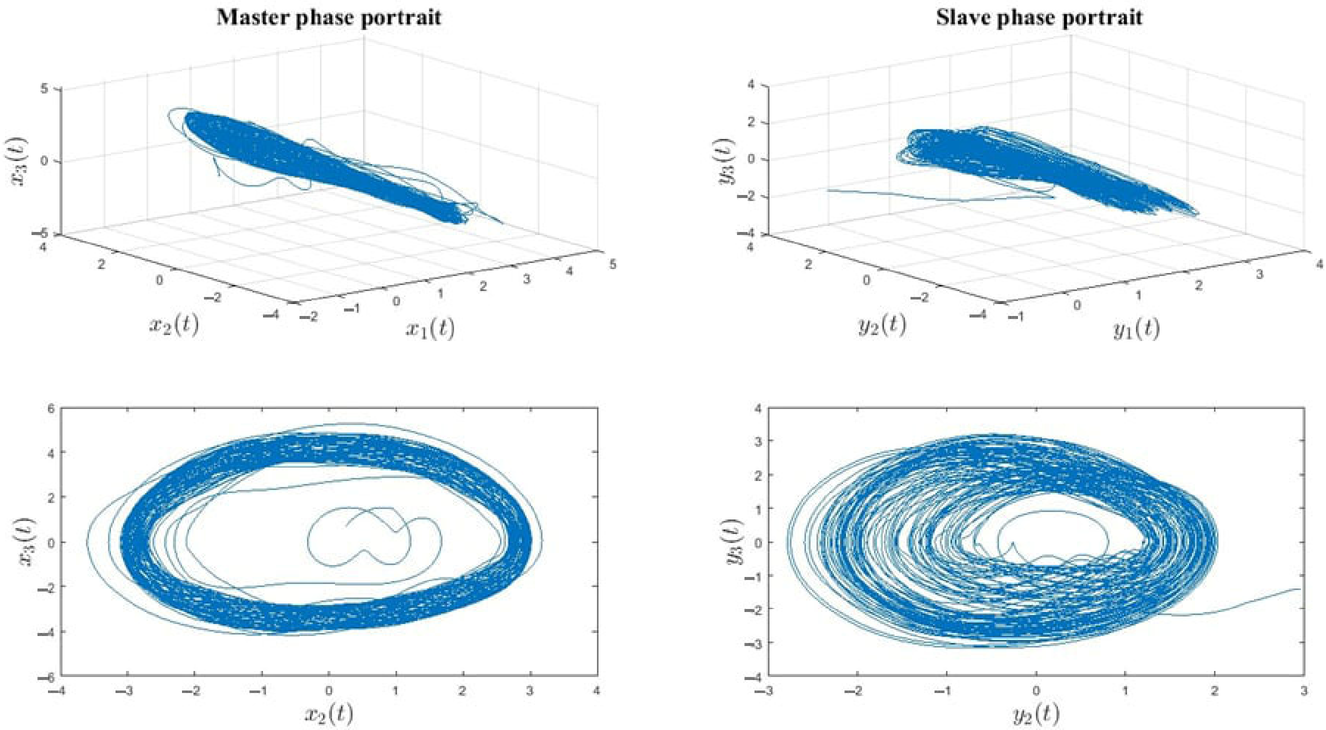

Figure 1. Phase diagram of the jerk master and slave systems without applying the controller.

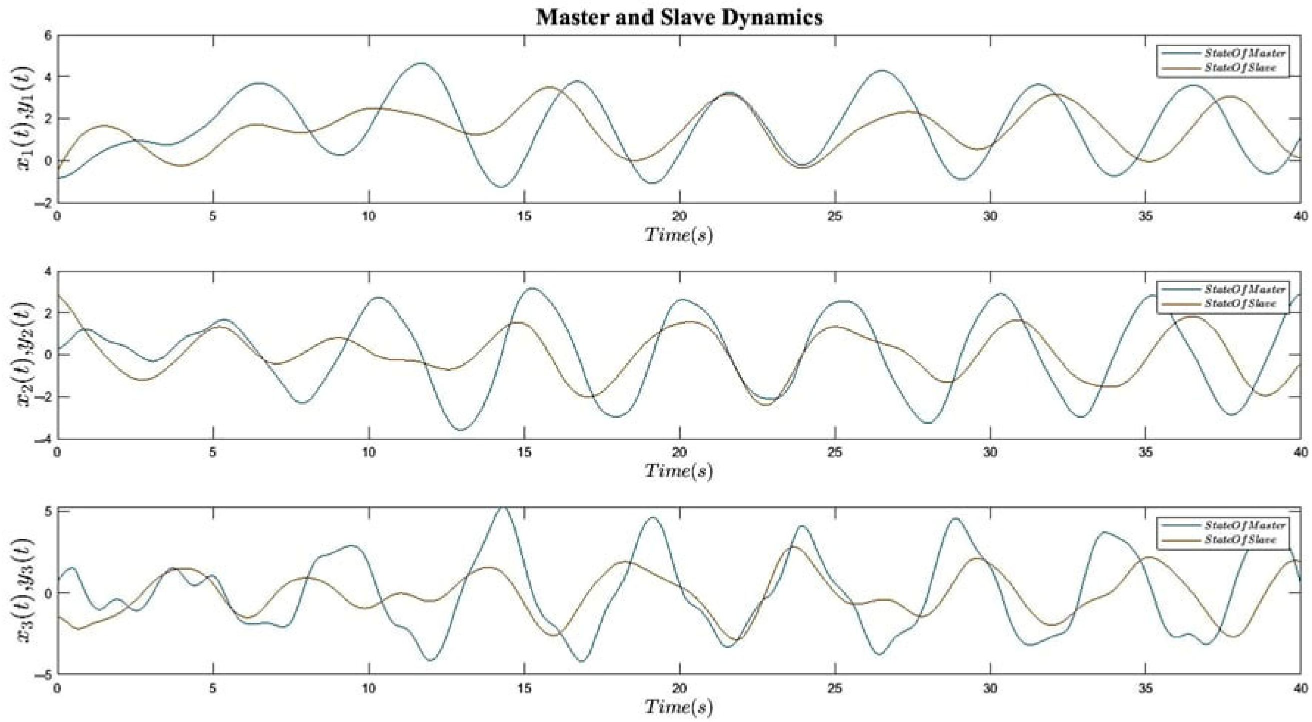

Figure 2. Behavior of master and slave system states without applying the control signal.

Figure 3 shows the behavior of the synchronized system. The synchronization error of the jerk system is shown in Figure 4. The control signal based on the proposed mechanism is shown in Figure 5. The estimation of the system parameters is shown in Figure 6. The disturbance and uncertainty of the master and slave system are shown in Figure 7.

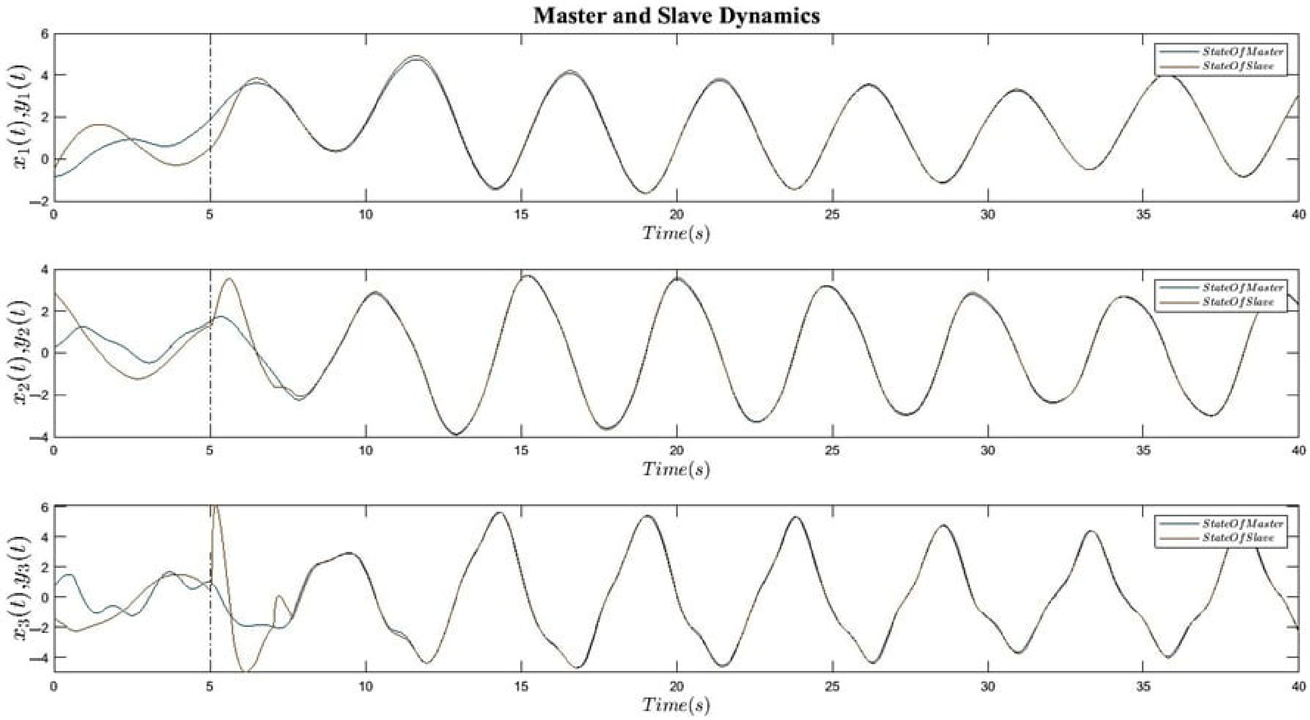

Figure 3. Synchronization of jerk systems using the proposed mechanism and applying the control signal at t = 5 s.

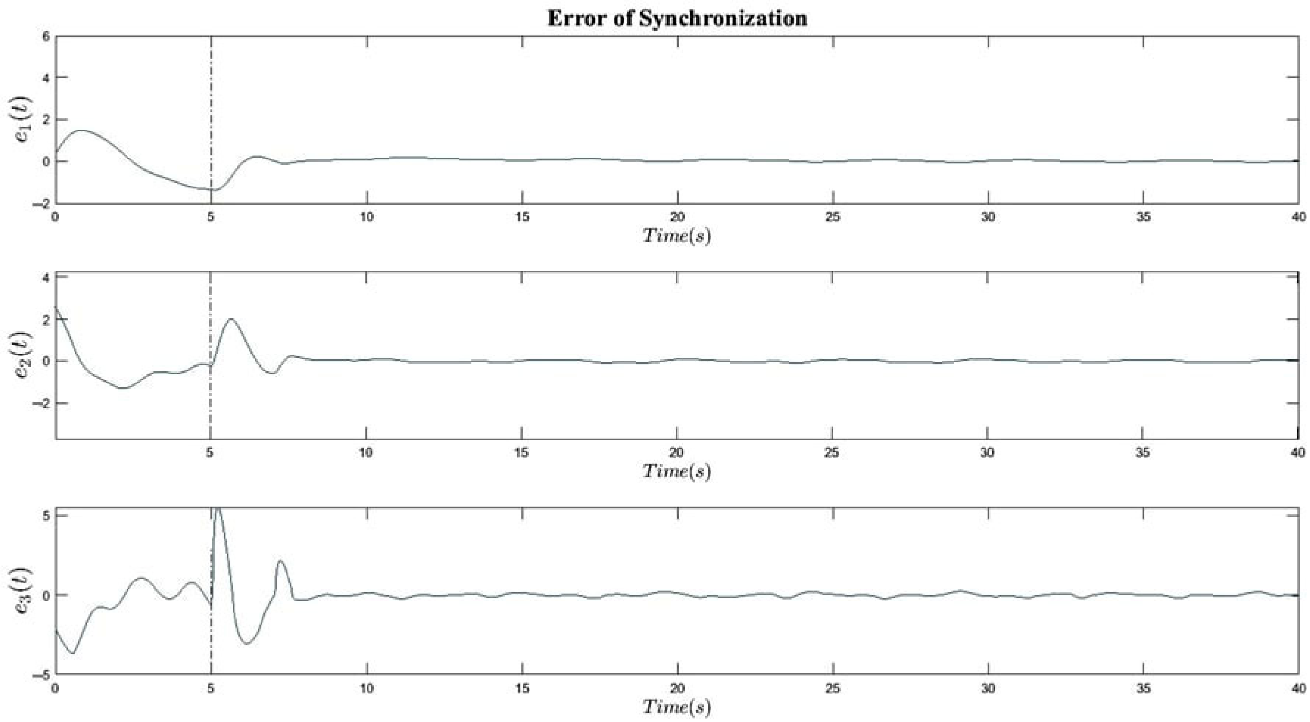

Figure 4. Synchronization error of the master and slave systems.

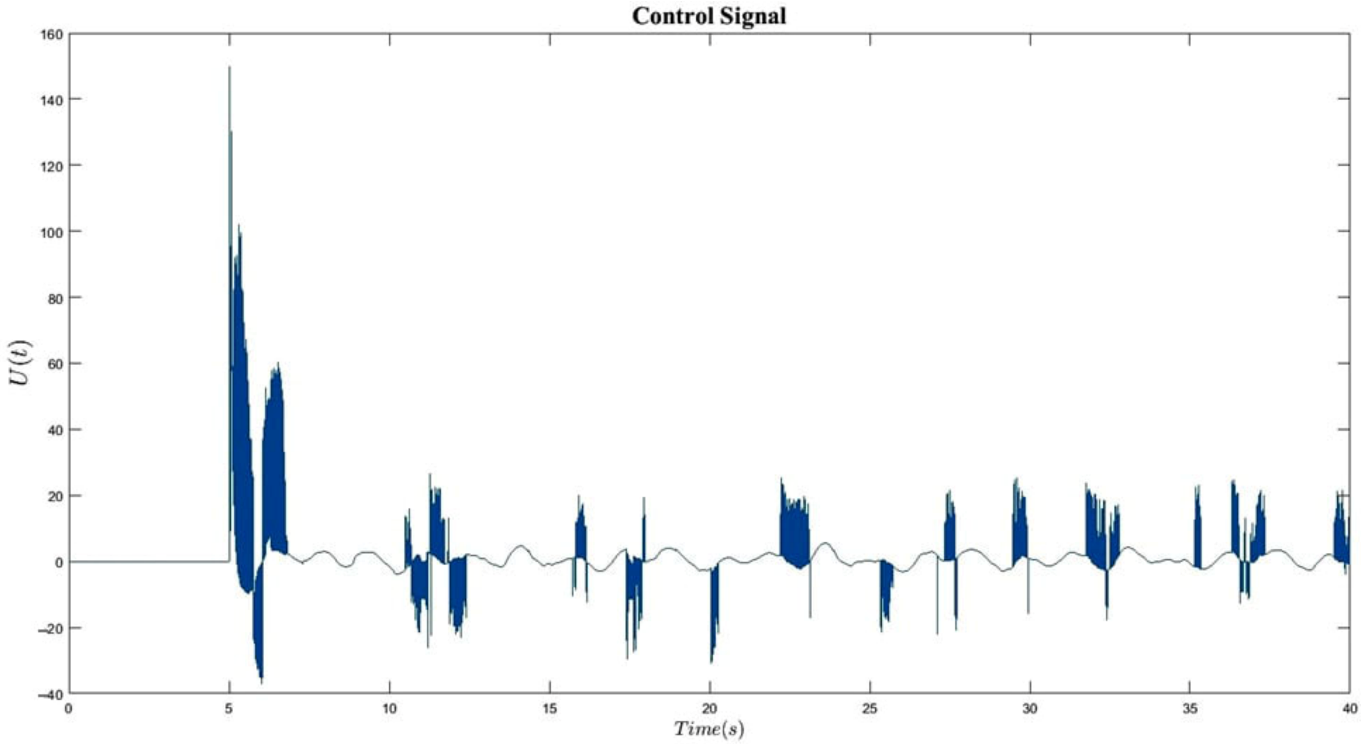

Figure 5. The control signal based on the proposed adaptive sliding mode control.

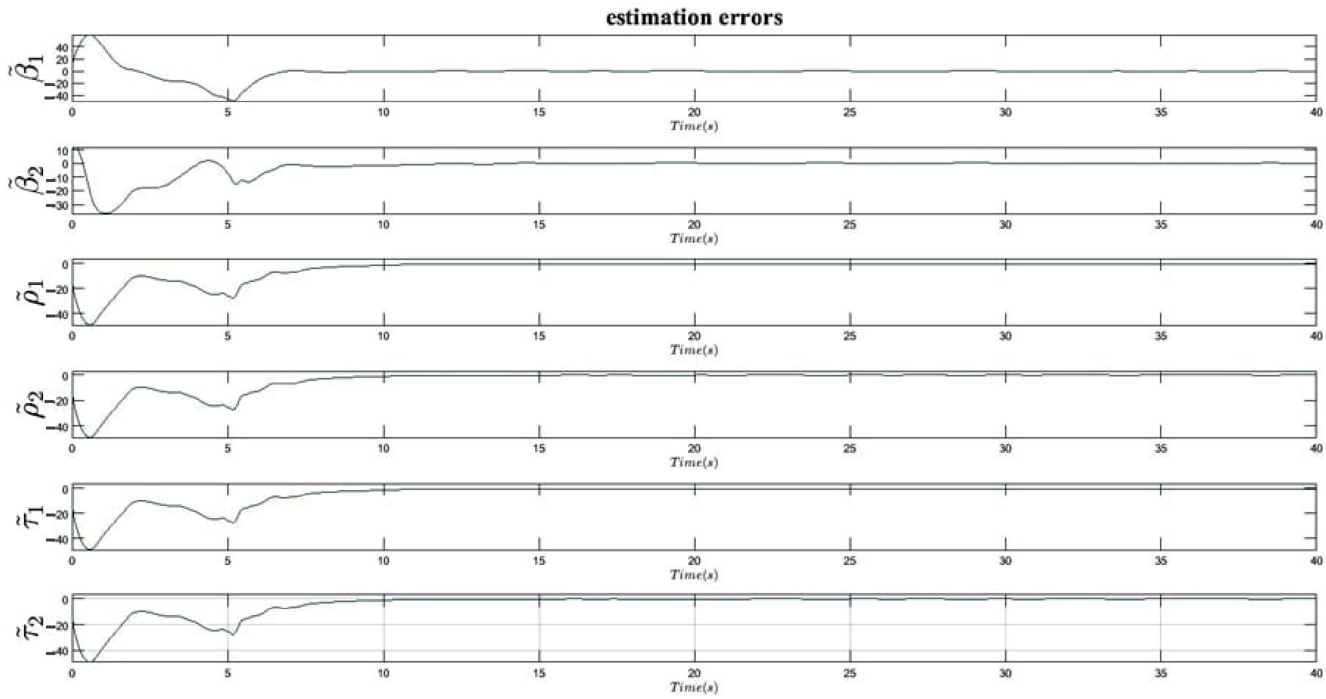

Figure 6. Estimation error of the system parameters.

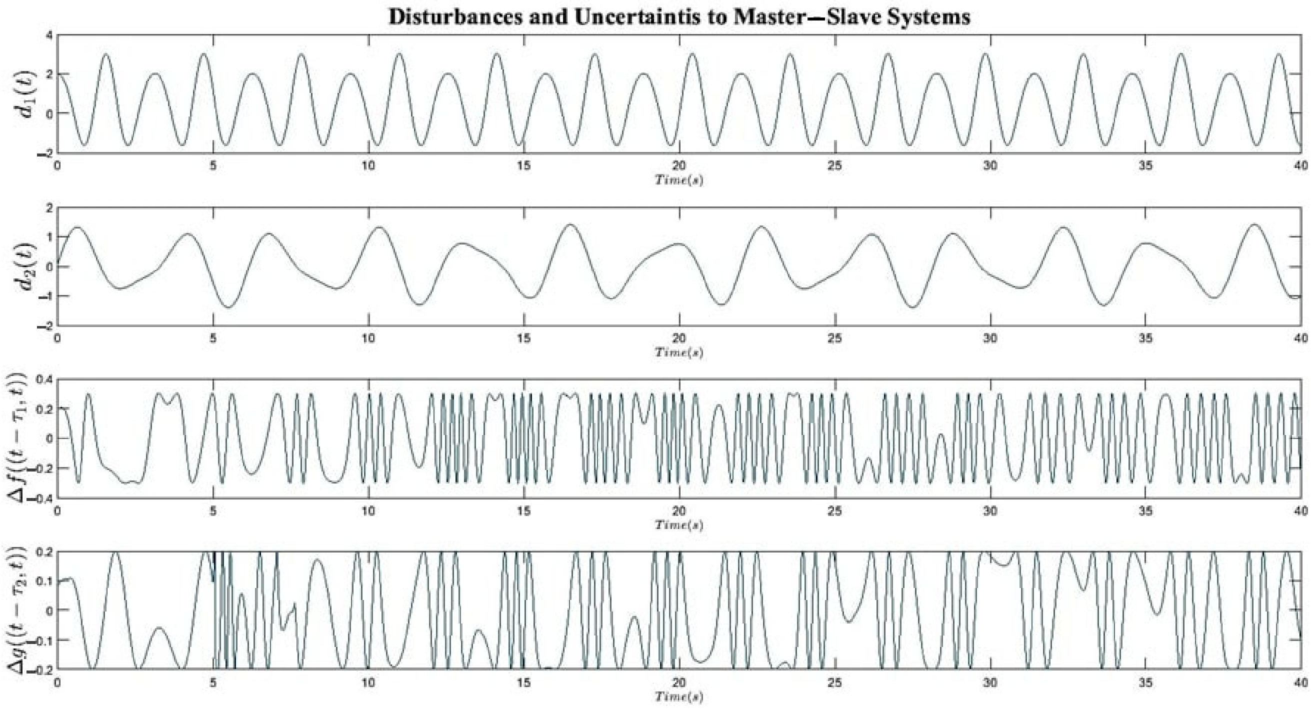

Figure 7. Uncertainty and disturbances of the master and slave systems.

In this article, simulations have been performed for $t = 40$ S, where $k_{11} = k_{22} = 10$ and $k_{12} = k_{21} = 20$ have been selected. Additionally, the gain and time constants of the non-linear fractional order PID sliding surface are $k_p = 1.5$, $T_i = 0.75$ and $T_d = 0.5$. The fractional order of the integral and derivative part of the sliding surface is defined as $\delta = 0.4$ and $\lambda = 0.75$. The parameters of the proposed robust controller are $b = 2$. The uncertain time delays of the system are $\tau_1 = 0.65$ and $\tau_2 = 0.35$. Unknown disturbances applied to both systems are as follows:

$$d_1(t) = sin^2 3t + 2cos4t, \;\; d_2(t) = sin2t + 0.4sin\pi t$$

7. Application in Secure Communication

The synchronization of the chaotic system with uncertainty and unknown time delays was discussed in the preceding section. The proposed robust control strategy is placed in the framework of a secure communication mechanism in this part such that the message signal is sent by the master system after the encryption process. The chaotic masking method was utilized for this aim. Figure 8 depicts how to convey the primary message signal using the proposed mechanism. A wireless communication channel is employed in this block diagram. The communication channel might be either wired or wireless.

If $M(t)$ is the original message signal that is coupled with the master system and $S(t)$ is the sent message, then [36]:

$$S(t) = M(t) + \sum_{j=1}^{n} \gamma_j x_j. \tag{36}$$

where x_j is the state of the chaotic system and γ_j is the constant number. $S(t)$ is the masked chaotic signal that is sent by the created communication channel. Using the proposed chaotic synchronization mechanism, the system states are integrated with the message signal in a weighted manner to perform the masking process. The receiver side can receive the message signal as follows:

$$R(t) = S(t) - \sum_{j=1}^{n} \gamma_j y_j. \tag{37}$$

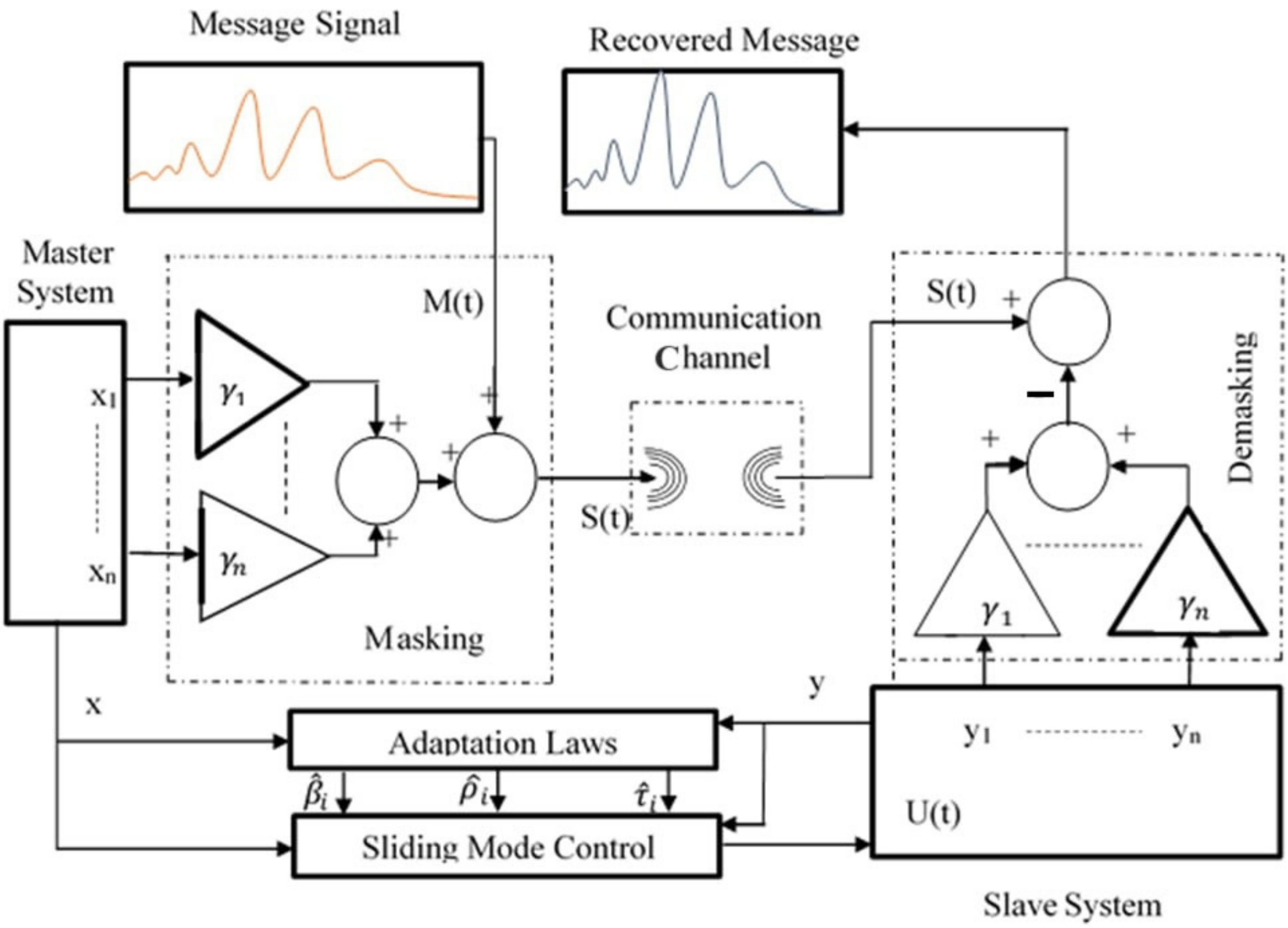

Figure 8. Chaotic secure communication structure based on the proposed approach.

Using the concept of synchronization, the following signal can be reconstructed in the receiver as follows:

$$\begin{gathered} R(t) = M(t) + \sum_{j=1}^{n} \gamma_j x_j - \sum_{j=1}^{n} \gamma_j y_j = M(t) + \sum_{j=1}^{n} \gamma_j (x_j - y_j) \\ \Rightarrow \;\; x_j - y_j = \; e_j \cong 0 \;\Rightarrow\; R(t) \to M(t). \end{gathered} \tag{38}$$

In this step, two message signals are generated, which are applied to the synchronized system via the chaotic masking mechanism mentioned above. In this phase, the control signal is applied to the slave system, and the message signal is applied to the synchronized system after achieving the established time to reduce the mean square error. The transition time in the simulations ends at $t = 4.5$ s. At this point, the encrypted message's signal is applied, and the signal $S(t)$ is received on the receiver's side and decoded to extract the signal $R(t)$. The simulation results in the Figures 9 and 10 demonstrate a good performance of the proposed system.

Table 1 shows the mean square error of the original message signal and the recovered signal.

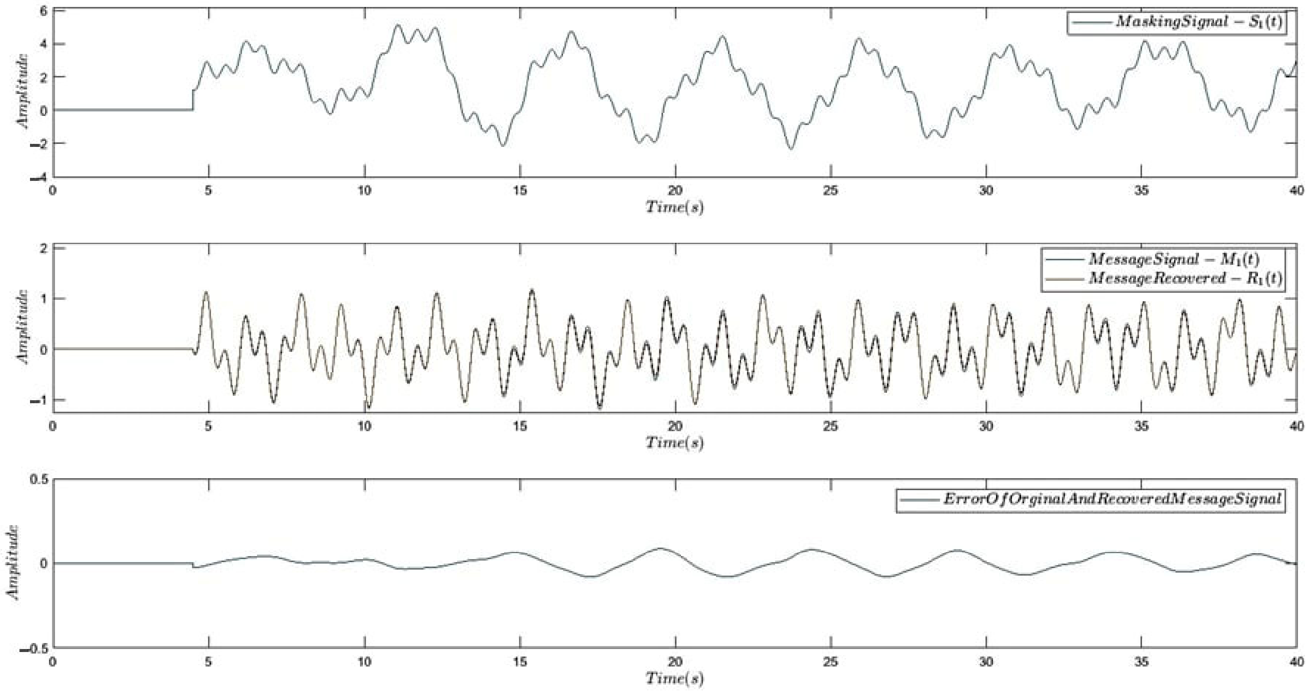

Figure 9. Message 1, masked and retrieved, and the error between these two signals.

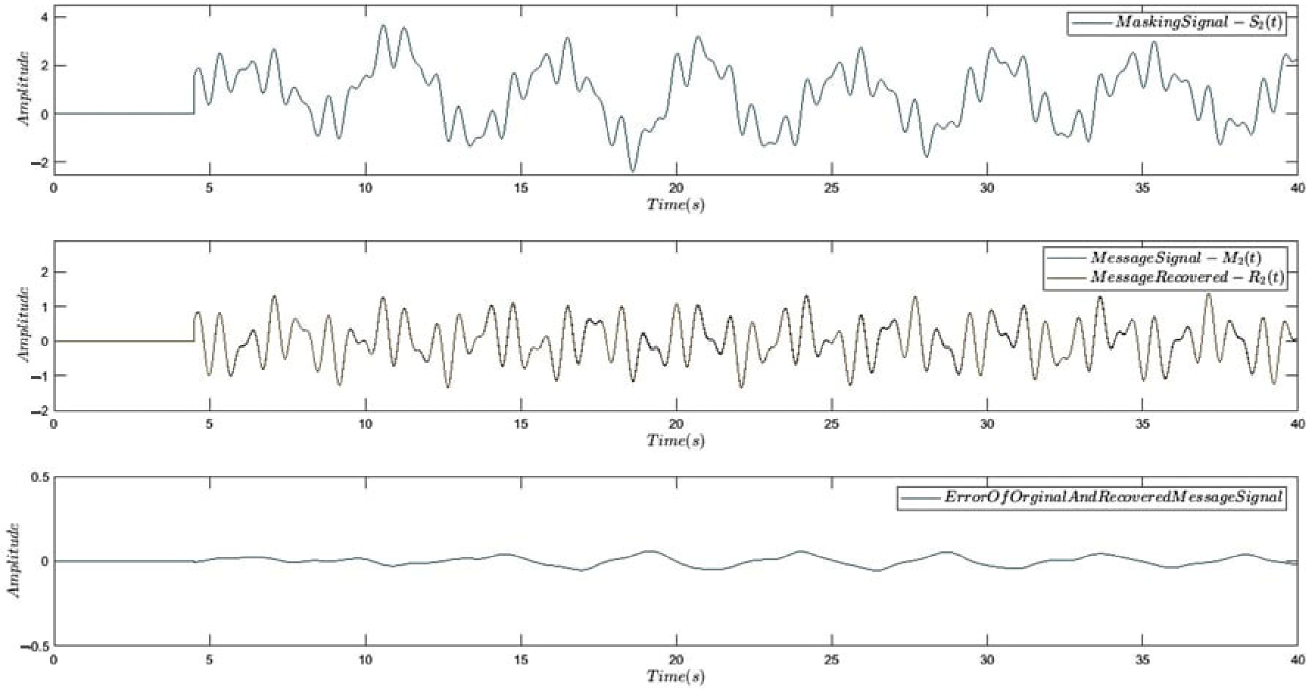

Figure 10. Message 2, masked and retrieved, and the error between these two signals.

Table 1. Represents the mean square error of the main message signal and the retrieved signal.

Root Mean Squre Error	
M_1 **and** R_1	M_2 **and** R_2
0.041544	0.025857

The following are the message signals:

$$M_1(t) = 1.8sin(1.7t) + 4.5cos(10.2t) + 5.4sin(1.33\pi t).$$

$$M_2(t) = 2.25sin(1.9t) + 2.88cos(10.66t) + 4.05sin(2.33\pi t).$$

The masked parameters of the message signal are $\gamma_1 = 1$, $\gamma_2 = 0.5$, and $\gamma_3 = 0.33$. Thus, encoding is as follows:

$$S_1(t) = M_1 + \gamma_1 x_1$$

$$S_2(t) = M_2 + \gamma_2 x_1 + \gamma_3 x_2$$

8. Conclusions

This study investigates the robust synchronization of a class of chaotic systems with uncertainty, external disturbances, and unknown parameters, such as unknown time delays, by introducing a new adaptive sliding mode control technique. First, an NLFPID-based sliding surface is proposed in the suggested robust control mechanism. The adaptive laws are then defined in order to estimate the uncertain parameters of the system using Lyapunov theory and Lipschitz conditions in chaotic systems, and ultimately the stability of the proposed robust control system is proven. The synchronization of two uncertain jerk chaotic systems with unknown time delays based on the proposed controller is simulated using MATLAB, and the results express the capability and desired performance of the proposed approach in the robust synchronization of the mentioned systems. Finally, the proposed adaptive sliding mode control approach has been used in a robust secure chaotic communication mechanism, and the simulation results indicate favorable quality in the secure transmission and reception of information despite uncertain parameters in the master and slave systems of the communication mechanism.

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References

1. Peters, E.E. *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*; John Wiley & Sons: Hoboken, NJ, USA, 1994.
2. Rössler, O.E. Chaos and chemistry. In *Nonlinear Phenomena in Chemical Dynamics*; Springer: Berlin/Heidelberg, Germany, 1981; pp. 79–87.
3. Rapp, P.E. Chaos in the neurosciences: Cautionary tales from the frontier. *Biologist* **1993**, *40*, 89–94.
4. Chen, Y.; Leung, A.Y. *Bifurcation and Chaos in Engineering*; Spfinge-Vereag: London, UK, 1998; pp. 169–180.
5. Nijmeijer, H. Control of Chaos and Synchronization. Available online: https://www.sciencedirect.com/science/article/abs/pii/S016769119700042X?via%3Dihub (accessed on 1 December 2022).
6. Pecora, L.M.; Thomas, L.; Carroll, T.L. Synchronization in chaotic systems. *Phys. Rev. Lett.* **1990**, *64*, 821. [CrossRef] [PubMed]
7. Maheswari, C.; Priyanka, E.; Meenakshipriya, B. Fractional order $PI^{\lambda}D^{\mu}$ controller tuned by coefficient diagram method and particle swarm optimization algorithms for SO_2 emission control process. *Proc. Inst. Mech. Eng. Part I J. Syst. Control. Eng.* **2017**, *231*, 587–599.
8. Podlubny, I. Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers. *IEEE Trans. Autom. Control.* **1999**, *44*, 208–214. [CrossRef]
9. Rahimian, M.; Tavazoei, M. Stabilizing fractional-order PI and PD controllers: An integer-order implemented system approach. *Proc. Inst. Mech. Eng. Part I J. Syst. Control. Eng.* **2010**, *224*, 893–903. [CrossRef]

10. Luo, Y.; Chen, Y. Fractional order [proportional derivative] controller for a class of fractional order systems. *Automatica* **2009**, *45*, 2446–2450. [CrossRef]
11. Monje, C.A.; Calderon, A.J.; Vinagre, B.M.; Feliu, V. The fractional order lead compensator. In Proceedings of the Second IEEE International Conference on Computational Cybernetics, 2004. ICCC 2004, Vienna, Austria, 30 August–1 September 2004; pp. 347–352.
12. Monje, C.A.; Vinagre, B.M.; Calderón, A.J.; Feliu, V.; Chen, Y.Q. Auto-tuning of fractional lead-lag compensators. *IFAC Proc. Vol.* **2005**, *38*, 319–324. [CrossRef]
13. Oustaloup, A.; Sabatier, J.; Moreau, X. From fractal robustness to the CRONE approach. *ESAIM Proc.* **1998**, *5*, 177–192. [CrossRef]
14. Yaghooti, B.; Salarieh, H. Robust adaptive fractional order proportional integral derivative controller design for uncertain fractional order nonlinear systems using sliding mode control. *Proc. Inst. Mech. Eng. Part I J. Syst. Control. Eng.* **2018**, *232*, 550–557. [CrossRef]
15. Abedini, M.; Nojoumian, M.A.; Salarieh, H.; Meghdari, A. Model reference adaptive control in fractional order systems using discrete-time approximation methods. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *25*, 27–40. [CrossRef]
16. Shi, B.; Yuan, J.; Dong, C. On fractional model reference adaptive control. *Sci. World J.* **2014**, *2014*, 521625. [CrossRef] [PubMed]
17. Dumlu, A. Design of a fractional-order adaptive integral sliding mode controller for the trajectory tracking control of robot manipulators. *Proc. Inst. Mech. Eng. Part I J. Syst. Control. Eng.* **2018**, *232*, 1212–1229. [CrossRef]
18. Binazadeh, T.; Shafiei, M. Output tracking of uncertain fractional-order nonlinear systems via a novel fractional-order sliding mode approach. *Mechatronics* **2013**, *23*, 888–892–892. [CrossRef]
19. Chen, D.; Liu, Y.; Ma, X.; Zhang, R. Control of a class of fractional-order chaotic systems via sliding mode. *Nonlinear Dyn.* **2012**, *67*, 893–901. [CrossRef]
20. Yahyazadeh, M.; Noei, A.R.; Ghaderi, R. Synchronization of chaotic systems with known and unknown parameters using a modified active sliding mode control. *ISA Trans.* **2011**, *50*, 262–267. [CrossRef]
21. Wang, X.; Zhang, X.; Ma, C. Modified projective synchronization of fractional-order chaotic systems via active sliding mode control. *Nonlinear Dyn.* **2012**, *69*, 511–517. [CrossRef]
22. Yuan, W.; Zhou, J.; Nadal, I.; Boccaletti, S.; Wang, Z. Adaptive control of dynamical synchronization on evolving networks with noise disturbances. *Phys. Rev. E* **2018**, *97*, 022211. [CrossRef]
23. Jajarmi, A.; Hajipour, M.; Baleanu, D. New aspects of the adaptive synchronization and hyperchaos suppression of a financial model. *Chaos Solitons Fractals* **2017**, *99*, 285–296.
24. Cho, S.; Baek, J.; Han, S. Adaptive control using time delay control for synchronization of chaotic systems. In Proceedings of the 2016 16th International Conference on Control, Automation and Systems (ICCAS), Gyeongju, Republic of Korea, 16–19 October 2016; pp. 763–766.
25. Li, W.; Liang, W.; Chang, K. Adaptive sliding mode control for synchronization of unified hyperchaotic systems. In Proceedings of the 2019 24th International Conference on Methods and Models in Automation and Robotics (MMAR), Międzyzdroje, Poland, 26–29 August 2019; pp. 93–98.
26. Luo, J.; Qu, S.; Xiong, Z. Finite-time increased order chaotic synchronization using an adaptive terminal sliding mode control. In Proceedings of the 2019 Chinese Control and Decision Conference (CCDC), Nanchang, China, 3–5 June 2019; pp. 1258–1263.
27. Ghamati, M.; Balochian, S. Design of adaptive sliding mode control for synchronization Genesio–Tesi chaotic system. *Chaos Solitons Fractals* **2015**, *75*, 111–117. [CrossRef]
28. Toopchi, Y.; Wang, J. Chaos control and synchronization of a hyperchaotic Zhou system by integral sliding mode control. *Entropy* **2014**, *16*, 6539–6552. [CrossRef]
29. Mohadeszadeh, M.; Pariz, N. Hybrid control of synchronization of fractional order nonlinear systems. *Asian J. Control.* **2021**, *23*, 412–422. [CrossRef]
30. Mirrezapour, S.Z.; Zare, A.; Hallaji, M. A new fractional sliding mode controller based on nonlinear fractional-order proportional integral derivative controller structure to synchronize fractional-order chaotic systems with uncertainty and disturbances. *J. Vib. Control* **2022**, *28*, 773–785. [CrossRef]
31. Mahmoud, M.S. Disturbance observer-based robust control and its applications: 35th anniversary overview. *IEEE Trans. Ind. Electron.* **2019**, *67*, 2042–2053.
32. Khan, A.; Nasreen; Jahanzaib, L.S. Synchronization on the adaptive sliding mode controller for fractional order complex chaotic systems with uncertainty and disturbances. *Int. J. Dyn. Control.* **2019**, *7*, 1419–1433. [CrossRef]
33. Shirkavand, M.; Pourgholi, M. Robust fixed-time synchronization of fractional order chaotic using free chattering nonsingular adaptive fractional sliding mode controller design. *Chaos Solitons Fractals* **2018**, *113*, 135–147. [CrossRef]
34. Sun, Z. Synchronization of fractional-order chaotic systems with non-identical orders, unknown parameters and disturbances via sliding mode control. *Chin. J. Phys.* **2018**, *56*, 2553–2559. [CrossRef]
35. Zare, A.; Mirrezapour, S.Z.; Hallaji, M.; Shoeibi, A.; Jafari, M.; Ghassemi, N.; Alizadehsani, R.; Mosavi, A. Robust adaptive synchronization of a class of uncertain chaotic systems with unknown time-delay. *Appl. Sci.* **2020**, *10*, 8875. [CrossRef]
36. Petráš, I. *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, 1st ed.; Springer Science & Business Media: London, UK, 2011.