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Orbit Tracing Properties on Hyperspaces and Fuzzy Dynamical Systems

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Abstract: Let X be a compact metric space and a continuous map $f: X \to X$ which defines a discrete dynamical system (X, f). The map f induces two natural maps, namely $\overline{f}: \mathcal{K}(X) \to \mathcal{K}(X)$ on the hyperspace $\mathcal{K}(X)$ of non-empty compact subspaces of X and the Zadeh's extension $\hat{f}: \mathcal{F}(X) \to \mathcal{F}(X)$ on the space $\mathcal{F}(X)$ of normal fuzzy set. In this work, we analyze the interaction of some orbit tracing dynamical properties, namely the specification and shadowing properties of the discrete dynamical system (X, f) and its induced discrete dynamical systems $(\mathcal{K}(X), \overline{f})$ and $(\mathcal{F}(X), \hat{f})$. Adding an algebraic structure yields stronger conclusions, and we obtain a full characterization of the specification property in the hyperspace, in the fuzzy space, and in the phase space X if we assume that the later is a convex compact subset of a (metrizable and complete) locally convex space and f is a linear operator.

Keywords: specification property; shadowing property; hyperspaces; fuzzy sets

MSC: 47A16; 37B02; 54A40; 54B20

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1. Introduction and Preliminaries

One of the strongest versions of chaos for discrete dynamical systems is the specification property, in which we consider the strong periodic specification property. In a few words, the specification property (SP) means that, for any finite family of points, it is possible to approximate arbitrary long pieces of orbits by a single periodic orbit by allowing a certain "shift" time that only depends on the precision of the approximation. This property was introduced by Bowen [1] in the context of Axiom A diffeomorphisms, and it was used by Bauer and Sigmund in their early work [2]. Since then, this notion and its generalizations have been developed by many researchers and is now a well-established property in the theory of dynamical systems.

Another important property in discrete dynamical systems is the shadowing property. For a continuous map $f: X \to X$ on a metric space X, a (finite) δ-pseudo orbit is a (finite) sequence of points $(x_i)_i$ such that the distance between $f(x_i)$ and x_{i+1} is, for every i, less than δ, and a pseudo-orbit is said to be ε-shadowed if we can find a point $x \in X$, such that its orbit approximates the pseudo orbit within a distance ε. The map f has the (finite) shadowing property if, for any $\varepsilon > 0$, there is $\delta > 0$ such that any (finite) δ-pseudo orbit is ε-shadowed.

The relationship between the action of a single continuous map on its phase space and the (hyperspace) action of the corresponding induced map on compact subsets of the phase space has a natural generalization to the space of normal *fuzzy sets* (that is, upper semicontinuous functions with compact support defined on the phase space with values in $[0,1]$).

For a continuous map $f: X \to X$ on a metric space X, one of the most important associated dynamics is that of the induced map $\overline{f}$ on the hyperspace of all non-empty compact subsets with the Hausdorff distance. Actually, the interest on this interplay goes back to Bauer and Sigmund [3] in 1975. Since then, the studies of hyperspace dynamics experienced great development (see, e.g., [4–9] and references therein).

A refinement of this kind of *collective dynamics* consists of the system $(\mathcal{F}(X), \hat{f})$, where the space $\mathcal{F}(X)$ consists on all upper semicontinuous functions from X to $[0,1]$ with compact support. The induced map $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is called the fuzzyfication or Zadeh's extension of f. Jardón, Sánchez and Sanchís studied, in [10], the interplay of the topological transitivity between the systems (X, f) and $(\mathcal{F}(X), \hat{f})$. This work was extended in [11] by studying different chaotic properties, such as Devaney chaos [12], $\mathcal{A}$-transitivity for a Furstenberg family $\mathcal{A}$, Li-Yorke chaos [13,14] and distributional chaos [15] (see, e.g., [16] for a survey on chaotic properties), also extending results in [7] concerning Devaney chaos for linear operators on complete locally convex spaces; see also [17–20] for the study of dynamical properties of the Zadeh's extension on the space of fuzzy sets. Moreover, some applications in the computer science of fuzzy sets can be found in [21] and the references therein.

In [10], the authors analyze the dynamics of the fuzzy space $\mathcal{F}(X)$ with the sendograph and endograph metrics, apart from the most usual ones, which are the supremum and Skorokhod's metrics. Despite these metrics having interesting applications in fuzzy theory, we did not consider these cases. Here, we are concerned with the connection between the dynamics of (X, f), $(\mathcal{K}(X), \overline{f})$ and $(\mathcal{F}(X), \hat{f})$. Thus, the space $(\mathcal{F}(X)$ is endowed with the supremum metric d_∞ and Skorokhod's metric d_0, respectively, which represents the behavior in fuzzy dynamical systems. We recall that the topologies associated with the endograph and the sendograph metrics are coarser than the topology induced by d_∞. Therefore, many dynamical results can be extended as a consequence of this fact.

Our results are organized as follows: For a compact metric space X and a continuous map $f : X \to X$, we show in Section 2 the equivalence of the specification property, either for the induced hyperspace dynamical system $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$, or for the induced fuzzyfied dynamics $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ (Theorem 2). By adding certain structure to the compact set and to the map, namely convexity and linearity, we obtain a characterization of the specification property that includes the original system (X, f) (Theorem 3). In Section 3, we obtain a characterization of finite shadowing, on the space, hyperspace or fuzzy space, in Theorem 5.

We want to set the context of our study. For a metric space X, we consider the hyperspace $\mathcal{K}(X)$ of all non-empty compact subsets of X with Hausdorff metric

$$d_H(K_1, K_2) := \max\left\{ \max_{x_1 \in K_1} d(x_1, K_2), \max_{x_2 \in K_2} d(x_2, K_1) \right\},$$

where d is the metric of X. This metric turns $\mathcal{K}(X)$ into a compact space and, therefore, all non-empty closed subsets are compact. We also recall the corresponding Vietoris topology with a basis of open sets of the form

$$\mathcal{V}(U_1, \cdots, U_r) := \left\{ K \in \mathcal{K}(X) : K \subset \bigcup_{i=1}^{r} U_i \text{ and } K \cap U_i \neq \varnothing \text{ for all } i = 1, \cdots, r \right\},$$

for $r \geq 1$ and arbitrary non-empty open sets $U_1, \cdots, U_r$ of X. A continuous map $f : X \to X$ induces $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined as

$$\overline{f}(K) := f(K) = \{f(x) \,:\, x \in K\}, \quad K \in \mathcal{K}(X).$$

The induced map $\overline{f}$ is continuous, too. A thorough study of hyperspaces can be found in [22].

The framework for fuzzy sets is the following: A normal fuzzy set u on X is an upper semicontinuous function $u : X \to [0,1]$ with compact support. Given a normal fuzzy set u, we set (u_α) as the compact set defined by

$$u_\alpha = \{x \in X : u(x) \geq \alpha\},\ \alpha \in]0,1],\ \text{and}\ u_0 = \overline{\cup\{u_\alpha : \alpha \in]0,1]\}}.$$

We set $\mathcal{F}(X)$ as the family of all normal fuzzy sets on X. We define on it the metric

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} \{d_H(u_\alpha, v_\alpha)\}.$$

For simplicity, we denote by $\mathcal{F}_\infty(X)$ the space $(\mathcal{F}(X), d_\infty)$.

A second natural metric is also introduced on $\mathcal{F}(X)$: given a strictly increasing homeomorphism $\xi : [0,1] \to [0,1]$, we set

$$d_0(u,v) = \inf\{\varepsilon \ :\ d_\infty(u, \xi v) \leq \varepsilon \text{ and } \sup_{\alpha\in[0,1]} |\xi(\alpha) - \alpha| \leq \varepsilon\}$$

which is a metric on $\mathcal{F}(X)$, called Skorokhod's metric. We have that $d_0 \leq d_\infty$, which implies that the topology induced in $\mathcal{F}(X)$ by d_0 is coarser than the one induced by d_∞. For simplicity, we denote by $\mathcal{F}_0(X)$ the space $(\mathcal{F}(X), d_0)$.

A continuous map $f : X \to X$ naturally induces $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$, which is the fuzzyfication or Zadeh's extension of f, defined by

$$\hat{f}(u)(x) = \begin{cases} \sup\{u(z) : z \in f^{-1}(x)\} & \text{if } f^{-1}(x) \neq \varnothing \\ 0 & \text{if } f^{-1}(x) = \varnothing \end{cases}$$

We also need some basic properties of fuzzy sets, which can be found in [10,23,24].

Remark 1. *If $f : (X,d) \to (X,d)$ is a continuous map on a metric space X, then the following properties hold:*

1. *For each $u \in \mathcal{F}(X)$, and any $\alpha \in [0,1]$, we have $\left[\hat{f}(u)\right]_\alpha = \overline{f}(u_\alpha)$.*
2. *$(\hat{f})^n = \hat{f^n}$ for every $n \in \mathbb{N}$.*
3. *$\hat{f}(\chi_K) = \chi_{\overline{f}(K)}$ for each characteristic function χ_K, where $K \in \mathcal{K}(X)$.*
4. *For each $u \in \mathcal{F}(X)$, and any $K \in \mathcal{K}(X)$, we have $d_0(u,\chi_K) = d_\infty(u,\chi_K)$.*

Moreover, some basic facts about the Hausdorff metric are very useful for the following section.

Remark 2. *For any compact sets $A, B, C, D \subset X$, we have that*

$$d_H(A \cup B, C \cup D) \leq max\{d_H(A,C)\,,\, d_H(B,D)\}\,, \tag{1}$$

$$d_H(A,B) \leq d_H(A,C) + d_H(C,B)\}\,, \tag{2}$$

$$A \subseteq B \subseteq C \ \text{ implies that } \ d_H(A,B) \leq d_H(A,C) \text{ and } d_H(B,C) \leq d_H(A,C)\,. \tag{3}$$

The following useful lemma is presented with the formulation of [10].

Lemma 1. *Let (X,d) be a metric space. For any finite family $(u^i)_{i=0}^n \subset \mathcal{F}(X)$, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exist numbers $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$ such that, for each $0 \leq i \leq n$*

$$\begin{aligned} &d_H(u^i_\alpha, u^i_{\alpha_1}) < \varepsilon \ \text{ for } \ \alpha \in [\alpha_0, \alpha_1] \\ &d_H(u^i_\alpha, u^i_{\alpha_{j+1}}) < \varepsilon \ \text{ for } \ \alpha \in]\alpha_j, \alpha_{j+1}],\ \ j = 1, 2, \ldots, m-1. \end{aligned} \tag{4}$$

2. Specification Property

The specification property (SP) is a strong property of approximation of arbitrary pieces of orbits by parts of a single periodic orbit. A nice review on shadowing and specification-like properties is provided in [25]. Bowen defined the specification property for systems with shadowing, but subsequent generalizations were defined mostly for systems without the shadowing property. There are several versions of the specification property, and here we use one of the strongest, namely, the periodic one.

Definition 1. *Given a continuous map $f : X \to X$ on a compact metric space (X, d), we say that it has the specification property (SP) if, for any $\delta > 0$, we can find $N_\delta \in \mathbb{N}$ such that, for any integer $s \geq 2$, any points $\{y_1, \ldots, y_s\} \subset X$, and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq N_\delta$ for $r = 1, \cdots, s-1$, there exists a point $x \in X$ such that the following two conditions hold:*

$$d(f^i(x), f^i(y_r)) < \delta, \text{ if } i_r \leq i \leq j_r, \text{ for any } r \leq s, \text{ and}$$
$$f^{N_\delta + j_s}(x) = x \qquad \text{(periodicity condition).}$$

There are some previous results connecting the specification in the case of individual dynamics (X, f) with the collective dynamical system $(\mathcal{K}(X), \overline{f})$. The early work of Bauer and Sigmund [3] is worth mentioning. Let us recall a result from it:

Theorem 1 (Proposition 4, [3])**.** *Given a continuous function $f : X \to X$ on a compact metric space X, if f has the specification property, then $\overline{f}$ has the specification property on $\mathcal{K}(X)$.*

However, the converse is not true. In [6], the authors construct an example of dynamical system (f, X) such that the induced map $\overline{f}$ has the specification property on $\mathcal{K}(X)$, and the continuous map f does not exhibit the SP.

Now, we study the interplay of the SP between the dynamical systems $(\mathcal{K}(X), \overline{f})$ and $(\mathcal{F}(X), \hat{f})$, which is the main purpose of this section.

Theorem 2. *Given a continuous map $f : X \to X$ on a metric space X, the following assertions are equivalent:*

(i) $(\mathcal{K}(X), \overline{f})$ has the specification property.
(ii) $(\mathcal{F}_\infty(X), \hat{f})$ has the specification property.
(iii) $(\mathcal{F}_0(X), \hat{f})$ has the specification property.

Proof. (i) $\Rightarrow$ (ii): By hypothesis, the map $\overline{f}$ satisfies the SP, therefore, for each $\delta > 0$, there is $\overline{N}_\delta \in \mathbb{N}$ such that for any integer $s \geq 2$, any finite family of compact sets $\{K_1, \ldots, K_s\} \subset \mathcal{K}(X)$, and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq \overline{N}_\delta$ for $r = 1, \cdots, s-1$, there exists a compact set $K \in \mathcal{K}(X)$ such that the following conditions hold:

$$d_H(\overline{f}^i(K), \overline{f}^i(K_r)) < \delta, \text{ if } i_r \leq i \leq j_r, \text{ for any } r \leq s \quad \text{and} \quad \overline{f}^{\overline{N}_\delta + j_s}(K) = K.$$

We must check that the map $\hat{f}$ also exhibits the SP. Fix $\delta > 0$ and take $\hat{N}_\delta = \overline{N}_{\delta/2}$. Let us consider an integer $s \geq 2$, a family of fuzzy sets $\{u^1, \ldots, u^s\} \subset \mathcal{F}_\infty(X)$, and integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq \hat{N}_\delta = \overline{N}_{\delta/2}$ for $r = 1, \cdots, s-1$.

By Lemma 1, there exists a partition of the interval $[0,1]$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m = 1$ such that, for every $i = 1, 2, \ldots, s$,

$$d_H(u^i_\alpha, u^i_{\alpha_{j+1}}) < \delta/2 \text{ for each } \alpha \in [\alpha_j, \alpha_{j+1}],\ j = 0, \ldots, m-1, \tag{5}$$

where $u^i_\alpha = \{x \in X : u^i(x) \geq \alpha\}$ for each $\alpha \in]0,1]$, and $u^i_0 = \overline{\cup\{u^i_\alpha : \alpha \in]0,1]\}}$. We can apply the specification property for $\delta/2$ to the families $\{u^1_{\alpha_j}, u^2_{\alpha_j}, \ldots, u^s_{\alpha_j}\} \subset \mathcal{K}(X)$, for each

$j = 1, 2, \ldots, m$. Then, there are compact sets $K_1, K_2, \ldots, K_m$ in $\mathcal{K}(X)$ such that the following assertions hold for each $j = 1, 2, \ldots, m$:

$$d_H(\overline{f}^i(K_j), \overline{f}^i(u^r_{\alpha_j})) < \delta/2, \text{ if } i_r \leq i \leq j_r, \text{ for any } r \leq s \quad \text{and} \quad \overline{f}^{\overline{N}_{\delta/2}+j_s}(K_j) = K_j\,.$$

We define the compact sets:

$$\omega_{\alpha_j} := \bigcup_{p \geq j} K_p \in \mathcal{K}(X)\,,\ j = 1, 2, \ldots, m\,.$$

They satisfy that $\omega_{\alpha_{j+1}} \subseteq \omega_{\alpha_j}$, then, for each $i_r \leq i \leq j_r$ with $r \leq s$ and every $j = 1, 2, \ldots, m$, we have that

$$d_H(\overline{f}^i(\omega_{\alpha_j}), \overline{f}^i(u^r_{\alpha_j})) = d_H(\overline{f}^i(\bigcup_{p \geq j} K_p)\,, \overline{f}^i(\bigcup_{p \geq j} u^r_{\alpha_p})) \leq \max_{p \geq j}\{d_H(\overline{f}^i(K_p), \overline{f}^i(u^r_{\alpha_p}))\} < \delta/2. \tag{6}$$

Consider the family $(\omega_\alpha)_{\alpha \in [0,1]} \in \mathcal{K}(X)$ defined by

$$\omega_\alpha = \begin{cases} \omega_{\alpha_1}\,, & 0 \leq \alpha \leq \alpha_1 \\ \omega_{\alpha_{j+1}}\,, & \alpha_j < \alpha \leq \alpha_{j+1}\,,\ 1 \leq j \leq m-1 \end{cases} \tag{7}$$

By using the triangular inequality for d_H, the definition of ω_α in each subinterval and Equations (5) and (6), it is easy to check that the elements ω_α satisfy for every $\alpha \in [\alpha_j, \alpha_{j+1}]$, $j = 0, 1, 2, \ldots, m-1$ and $i_r \leq i \leq j_r$ with $r \leq s$,

$$d_H(\overline{f}^i(\omega_\alpha), \overline{f}^i(u^r_\alpha)) \leq d_H(\overline{f}^i(\omega_{\alpha_j}), \overline{f}^i(u^r_{\alpha_j})) + d_H(\overline{f}^i(u^r_{\alpha_j}), \overline{f}^i(u^r_\alpha)) < \frac{\delta}{2} + \frac{\delta}{2} < \delta. \tag{8}$$

The inequality is also fulfilled for $\alpha = 0$.

Moreover, the decreasing family $(\omega_\alpha)_{\alpha \in [0,1]}$ fulfills the hypothesis of Proposition 4.9 in [24]. Thus, we can find $\overline{\omega} \in \mathcal{F}_\infty(X)$ such that $\overline{\omega}_\alpha = \omega_\alpha$ for any $\alpha \in [0,1]$.

Let us check that $\overline{\omega}$ is a fuzzy set satisfying the conditions of the definition of the specification property. By using (8), for each $i_r \leq i \leq j_r$ with $r \leq s$, we obtain

$$d_\infty(\hat{f}^i(\overline{\omega}), \hat{f}^i(u^r)) = \sup_{\alpha \in [0,1]}\{d_H([\hat{f}^i(\overline{\omega})]_\alpha, [\hat{f}^i(u^r)]_\alpha)\} = \sup_{\alpha \in [0,1]}\{d_H(\overline{f}^i(\overline{\omega}_\alpha), \overline{f}^i(u^r_\alpha))\} < \delta\,.$$

Finally, we need to show that $\overline{\omega}$ is periodic for $\hat{f}$. Given any $\alpha \in [0,1]$, such that $\alpha = 0$ or $\alpha \in [\alpha_j, \alpha_{j+1}]$, $j = 0, \ldots, m-1$, we obtain

$$\left[\hat{f}^{\hat{N}_\delta + j_s}(\overline{\omega})\right]_\alpha = \overline{f}^{\overline{N}_{\delta/2}+j_s}(\overline{\omega}_\alpha) = \overline{f}^{\overline{N}_{\delta/2}+j_s}(\overline{\omega}_{\alpha_j}) = \bigcup_{p \geq j} \overline{f}^{\overline{N}_{\delta/2}+j_s}(K_p) = \bigcup_{p \geq j} K_p = \overline{\omega}_{\alpha_j} = \overline{\omega}_\alpha\,.$$

Hence, $\hat{f}^{\hat{N}_\delta + j_s}(\overline{\omega}) = \overline{\omega}$ and, therefore, the map $\hat{f}$ on $\mathcal{F}_\infty(X)$ has the SP.

(ii) $\Rightarrow$ (iii) This implication is a consequence of the fact that $d_0 \leq d_\infty$ in $\mathcal{F}(X)$.

(iii) $\Rightarrow$ (i): By hypothesis $(\mathcal{F}_0(X), \hat{f})$ has the specification property. Then, for any $\delta > 0$, there exists $\hat{N}_\delta$ such that for any integer $s \geq 2$, any finite family of fuzzy sets $\{u^1, \ldots, u^s\} \subset \mathcal{F}_0(X)$, and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq \hat{N}_\delta$ for $r = 1, \cdots, s-1$, we can find $v \in \mathcal{F}_0(X)$ such that

$$d_0(\hat{f}^i(v), \hat{f}^i(u^r)) < \delta, \text{ if } i_r \leq i \leq j_r, \text{ for any } r \leq s \quad \text{and} \quad \hat{f}^{\hat{N}_\delta + j_s}(v) = v. \tag{9}$$

We show that this implies the specification property of $(\mathcal{K}(X), \overline{f})$. Given $\delta > 0$, set $\overline{N}_\delta = \hat{N}_\delta > 0$ and consider an integer $s \geq 2$, a set of non-empty compact sets $\{K^1, \ldots, K^s\} \subset \mathcal{K}(X)$ and integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ satisfying $i_{r+1} - j_r \geq \overline{N}_\delta = \hat{N}_\delta$ for $r = 1, \cdots, s-1$.

Consider the characteristic functions $u^i := \chi_{K^i}$, $i = 1, \ldots, s$, which satisfy that $u^i_\alpha = K^i$ for every $\alpha \in [0,1]$. Applying the specification property to the set $\{u^1, \ldots, u^s\}$ for the intervals above, there exists $v \in \mathcal{F}_0(X)$ satisfying (9).

We construct a compact set $K \in \mathcal{K}(X)$ such that the conditions of the specification property are fulfilled. To perform this, fix $\tilde{\alpha} \in [0,1]$ and define $K := v_{\tilde{\alpha}} \in \mathcal{K}(X)$. By using Remark 1 (property 4) and Equation (9), we check that this compact set K satisfies for each $i_r \le i \le j_r$ with $r \le s$

$$\begin{aligned} d_H(\overline{f}^i(K), \overline{f}^i(K^r)) &= d_H(\overline{f}^i(v_{\tilde{\alpha}}), \overline{f}^i(u^r_{\tilde{\alpha}})) = d_H([\hat{f}^i(v)]_{\tilde{\alpha}}, [\hat{f}^i(u^r)]_{\tilde{\alpha}}) \\ &\le d_\infty(\hat{f}^i(v), \hat{f}^i(u^r)) = d_0(\hat{f}^i(v), \hat{f}^i(u^r)) < \delta\,, \end{aligned}$$

and, finally,

$$\overline{f}^{\overline{N}_\delta + j_s}(K) = \overline{f}^{\overline{N}_\delta + j_s}(v_{\tilde{\alpha}}) = \left[\hat{f}^{\hat{N}_\delta + j_s}(v)\right]_{\tilde{\alpha}} = v_{\tilde{\alpha}} = K\,.$$

□

The following corollary is a direct consequence of the last theorem and Theorem 1 of Bauer and Sigmund:

Corollary 1. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. If f has the specification property, then the dynamical systems $(\mathcal{F}_\infty(X), \hat{f})$ and $(\mathcal{F}_0(X), \hat{f})$ have the specification property.*

Certainly, the ideal situation is the one in which the specification property is equivalent to happen in the original space X or in the hyperspace, or the space of fuzzy sets. Unfortunately, this ideal situation cannot be achieved since, by [6], there are dynamical systems (X, f) without the specification property such that the induced map $\overline{f}$ has the specification property on $\mathcal{K}(X)$.

Assuming some algebraic structure on the compact set, and on the map, we can obtain the desired equivalence. Actually, we were inspired by the result of [7] for Devaney chaos in order to obtain it.

If E is a complete and metrizable locally convex space (in short, a Fréchet space), then it is convenient to consider the hyperspace $\mathcal{C}(E)$ of convex compact subsets of E. Within this framework, the closed convex envelope $\overline{\text{co}}(K)$ is compact if $K \subset E$ is compact (see, e.g., Theorem 3.20(c) in [26]). Thus, the map

$$S : \mathcal{K}(E) \to \mathcal{C}(E), \quad K \mapsto \overline{\text{co}}(K),$$

is well-defined. Moreover, if $f = T \in L(E)$, a continuous and linear operator, then $S(T(K)) = T(S(K))$ for any $K \subset E$ compact (see ([7], Lemma 2.1)), a fact that is key in the last result of this section.

Moreover, another property is very useful is that, under the same assumptions on E and T, if $K \subset E$ is a compact and T-invariant ($T(K) \subset K$), then the specification property of $(K, T|_K)$ implies the one of $(\overline{\text{co}}(K), T|_{\overline{\text{co}}(K)})$ (see ([27], Proposition 10 ii)).

Now, we are in conditions to obtain the final equivalence of the specification property in this section.

Theorem 3. *Let T be a continuous and linear operator on a Fréchet space E, and let $X \subset E$ be a convex T-invariant compact set. We set $f = T|_X$, and the following assertions are equivalent:*

(i) $(\mathcal{K}(X), \overline{f})$ *has the specification property.*
(ii) $(\mathcal{F}_\infty(X), \hat{f})$ *has the specification property.*
(iii) $(\mathcal{F}_0(X), \hat{f})$ *has the specification property.*
(iv) (X, f) *has the specification property.*

Proof. By Theorems 1 and 2, we just need to prove that (i) implies (iv). Given $\delta > 0$ there is $N_\delta \in \mathbb{N}$ such that, for any integer $s \geq 2$, any collection $\{K_1, \dots, K_s\} \subset \mathcal{K}(X)$, and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq N_\delta$ for $r = 1, \cdots, s-1$, there exists $K \in \mathcal{K}(X)$ such that

$$d_H(f^i(K), f^i(K_r)) < \delta, \text{ if } i_r \leq i \leq j_r, \text{ for any } r \leq s, \text{ and}$$
$$f^{N_\delta + j_s}(K) = K.$$

Thus, given any integer $s \geq 2$, any finite collection $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ with $i_{r+1} - j_r \geq N_\delta$ for $r = 1, \cdots, s-1$, we set $K_i = \{y_i\}$, $i = 1, \dots, s$. We obtain $K \in \mathcal{K}(X)$, satisfying the above properties. Let $K' := \overline{\mathrm{co}}(K)$. Since $f^m(K') = S(f^m(K)) = K'$ for $m = N_\delta + j_s$, and K' is a convex compact set, by the Schauder–Tychonoff fixed point theorem (Theorem 5.28 in [26]) there exists $x \in K'$ such that $f^m(x) = x$. By the above properties, we obtain

$$d(f^i(x), f^i(y_r)) < \delta, \text{ with } i_r \leq i \leq j_r, \text{ for every } r \leq s, \text{ and}$$
$$f^{N_\delta + j_s}(x) = x,$$

and we conclude the specification property for (X, f). □

Example 1. *To illustrate the previous result, let us consider the weighted ℓ^p-space, $1 \leq p < \infty$, defined by*

$$\ell^p(v) = \{x = (x_i)_i \in \mathbb{R}^{\mathbb{N}} \ / \ \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p v_i\right)^{1/p} < \infty\},$$

where $v = (v_i)_i$ is a sequence of strictly positive weights so that $\sum_{i=1}^{\infty} v_i < \infty$. We know (see ([27], Theorem 5)) that, for the backward shift $T = B : \ell^p(v) \to \ell^p(v)$, $B(x_1, x_2, \dots) = (x_2, x_3, \dots)$, the following convex compact set is T-invariant

$$X := \{x = (x_i)_i \in \mathbb{R}^{\mathbb{N}} \ / \ |x_i| \leq 1 \ \forall i \in \mathbb{N}\},$$

and $T|_X$ has the specification property. This is a very natural example in which Theorem 3 applies to obtain the specification property on the hyperspace and on the fuzzy spaces.

3. Shadowing

Shadowing is an important dynamical property which was motivated by questions such as if an approximate trajectory can be fitted by a real trajectory, and it was originated in the works of Anosov, Bowen, and others.

Definition 2. *Let $f : X \to X$ be a continuous map on a compact metric space (X, d). Given $\delta > 0$, a sequence $(x_i)_{i=0}^n$ ($n \in \mathbb{N}_+$ or $n = \infty$) is a δ-pseudo orbit if it satisfies*

$$d(f(x_i), x_{i+1}) < \delta \,, \; i = 0, 1, \dots, n-1 \,.$$

Definition 3. *We say that (X, f) has the shadowing property if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for every δ-pseudo orbit $(x_i)_{i=0}^n$ ($n \in \mathbb{N}_+$ or $n = \infty$), there exists a point $x \in X$ with*

$$d(x_i, f^i(x)) < \varepsilon \text{ for all } 0 \leq i \leq n \,.$$

In other words, the dynamical system has shadowing if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit can be ε-shadowed by a real orbit. If only finite pseudo-orbits are shadowed, we say that (X, f) has the finite shadowing property. However, if X is compact, then f has the full shadowing property if and only if f has the finite shadowing property (see, e.g., [28], Remark 1).

There are some previous results connecting shadowing in the case of individual dynamics (X, f) with the collective dynamical system $(\mathcal{K}(X), \overline{f})$.

Theorem 4 (Theorem 3.4, [29])**.** *Given a compact metric space X and a continuous map $f : X \to X$, then f has the shadowing property if and only if $\overline{f}$ has the shadowing property on $\mathcal{K}(X)$.*

We refer to [30] for an extension of the above result.

Example 2. *A typical example of a continuous map with the shadowing property is the tent map $T : [0,1] \to [0,1]$, $T(x) = 2x$ for $x \in [0,1/2]$ and $T(x) = 2 - 2x$ for $x \in [1/2,1]$.*

By using a similar construction to the Theorem 2, it is possible to obtain an analogous result for shadowing on the induced dynamical systems.

Theorem 5. *Given a continuous map $f : X \to X$ on a compact metric space X, the following assertions are equivalent:*

(i) (X, f) *has the (finite) shadowing property.*
(ii) $(\mathcal{K}(X), \overline{f})$ *has the (finite) shadowing property.*
(iii) $(\mathcal{F}_\infty(X), \hat{f})$ *has the finite shadowing property.*
(iv) $(\mathcal{F}_0(X), \hat{f})$ *has the finite shadowing property.*

Proof. (i) $\Leftrightarrow$ (ii): By Theorem 4, since X is a compact metric space, the hyperspace $\mathcal{K}(X)$ of all non-empty compact subsets is identical to the hyperspace 2^X of all non-empty closed subsets.

(ii) $\Rightarrow$ (iii): By hypothesis, the dynamical system $(\mathcal{K}(X), \overline{f})$ has the *shadowing property*, which means that for each $\varepsilon > 0$, there exists $\overline{\delta}_\varepsilon > 0$ such that for every $\overline{\delta}_\varepsilon$-pseudo-orbit $(K_i)_{i=0}^n$ ($n \in \mathbb{N}_+$ or $n = \infty$) one can find $K \in \mathcal{K}(X)$ with

$$d_H(K_i, \overline{f}^i(K)) < \varepsilon \text{ for all } 0 \le i \le n\,.$$

Fixed $\varepsilon > 0$, take $\delta = \overline{\delta}_{\varepsilon/2} > 0$ and a finite δ-pseudo-orbit of the map $\hat{f}$, $(u^i)_{i=1}^n$, $n \in \mathbb{N}+$, $u^i : X \to [0,1]$, such that

$$d_\infty(u^{i+1}, \hat{f}(u^i)) < \delta\,,\; i = 1,2,\ldots,n-1\,.$$

We have to show if there is a $\omega \in \mathcal{F}_\infty(X)$ such that $d_\infty(u^i, \hat{f}^i(\omega)) < \varepsilon\,,\; i = 1,2,\ldots,n.$

Consider the families of compact sets $(u^i_\alpha)_{\alpha\in[0,1]}$, $i = 1,2,\ldots,n$, in $\mathcal{K}(X)$. For each $\alpha \in [0,1]$, they satisfy

$$d_H(u^{i+1}_\alpha, \overline{f}(u^i_\alpha)) \le d_H(u^{i+1}_\alpha, [\hat{f}(u^i)]_\alpha) \le d_\infty(u^{i+1}, \hat{f}(u^i)) < \delta\,,\; i = 1,2,\ldots,n-1\,.$$

Hence, for each $\alpha \in [0,1]$, $(u^i_\alpha)_{i=1}^n \subset \mathcal{K}(X)$ is a finite δ-pseudo-orbit of the map $\overline{f}$.

By Lemma 1, we can find a partition of $[0,1]$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$, such that

$$d_H(u^i_\alpha, u^i_{\alpha_{j+1}}) < \varepsilon/2\,,\; \alpha \in]\alpha_j, \alpha_{j+1}]\,,\; j = 0,1,2,\ldots,m-1\,,\; i = 1,2,\ldots,n\,. \tag{10}$$

For each $j = 1,2,\ldots,m$, the set $\{u^1_{\alpha_j}, u^2_{\alpha_j}, \ldots, u^n_{\alpha_j}\}$ is a finite δ-pseudo-orbit of the map $\overline{f}$. By hypothesis, the dynamical system $(\mathcal{K}(X), \overline{f})$ has the finite shadowing property, there exist m compact sets $K_1, K_2, \ldots, K_m$ in $\mathcal{K}(X)$ such that ($\delta = \overline{\delta}_{\varepsilon/2} > 0$)

$$d_H(u^i_{\alpha_j}, \overline{f}^i(K_j)) < \varepsilon/2\,,\; j = 1,2,\ldots,m\,,\; i = 1,2,\ldots,n\,. \tag{11}$$

We define the collection of compact sets

$$\omega_{\alpha_j} := \bigcup_{r\geq j} K_r \in \mathcal{K}(X) \ , \ j = 1,2,\ldots,m \, ,$$

and we have, for each $i = 1,2,\ldots,n$,

$$d_H(u^i_{\alpha_j}, \overline{f}^i(\omega_{\alpha_j})) = d_H(\bigcup_{s\geq j} u^i_{\alpha_s} \, , \, \overline{f}^i(\bigcup_{s\geq j} K_s)) \leq d_H(\bigcup_{s\geq j} u^i_{\alpha_s} \, , \, \bigcup_{s\geq j} \overline{f}^i(K_s))$$

$$\leq \max_{s\geq j}\{d_H(u^i_{\alpha_s}, \overline{f}^i(K_s))\} \, .$$

Hence,

$$d_H(u^i_{\alpha_j}, \overline{f}^i(\omega_{\alpha_j})) < \varepsilon/2 \, , \ j = 1,2,\ldots,m \ , \ i = 1,2,\ldots,n \, . \tag{12}$$

The family ω_α for each $\alpha \in [0,1]$, defined analogously to the proof of Proposition (2) (Equation (7)), defines a unique $\overline{\omega} \in \mathcal{F}_\infty(X)$ such that $\overline{\omega}_\alpha = \omega_\alpha$ for each $\alpha \in [0,1]$. We show that the orbit of $\overline{\omega}$ ε-traces the δ-pseudo-orbit in $\mathcal{F}_\infty(X)$.

By using the triangular inequality for d_H, relations (10) and (12) and the definition of $\overline{\omega}_\alpha$ in each subinterval,

$$d_H(u^i_\alpha, \overline{f}^i(\overline{\omega}_\alpha)) < d_H(u^i_\alpha, u^i_{\alpha_j}) + d_H(u^i_{\alpha_j}, \overline{f}^i(\overline{\omega}_{\alpha_j})) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \, ,$$

for $\alpha \in]\alpha_j, \alpha_{j+1}]$, $0 \leq j \leq m-1$ and $i = 1,2,\ldots,n$. Since $\alpha_0 = 0$ and $u^i_{0^+} = u^i_0$, $i = 1,2,\ldots,n$, the last expression is also fulfilled for $\alpha = 0$.

Finally, the orbit of $\overline{\omega}$ ε-traces the pseudo-orbit $(u^i)_{i=1}^n$,

$$d_\infty(u^i, \hat{f}^i(\overline{\omega})) = \sup_{\alpha\in[0,1]} \{d_H(u^i_\alpha, [\hat{f}^i(\overline{\omega})]_\alpha)\} = \sup_{\alpha\in[0,1]} \{d_H(u^i_\alpha, \overline{f}^i(\overline{\omega}_\alpha))\} < \varepsilon \ , \ i = 1,2,\ldots,n \, .$$

(iii) $\Rightarrow$ (iv): It is a consequence of the fact that in $\mathcal{F}(X)$, we have $d_0 \leq d_\infty$.

(iv) $\Rightarrow$ (ii): Given $\varepsilon > 0$, take $\delta > 0$ such that every finite δ-pseudo-orbit in $(\mathcal{F}_0(X), \hat{f})$ can be ϵ-shadowed by a true orbit. We consider a finite δ-pseudo-orbit given by the compact sets $(K_i)_{i=1}^n \subset \mathcal{K}(X)$. They satisfy

$$d_H(K_{i+1}, \overline{f}(K_i)) < \delta \ , \ i = 1,\ldots,n \, .$$

Let us consider the characteristic function of each (K_i)

$$u^i := \chi_{K_i} \in \mathcal{F}_0(X) \ , \ i = 1,\ldots,n \, .$$

Notice that, for $i = 1,\ldots,n$, and for every $\alpha \in [0,1]$, we have that $u^i_\alpha = K_i$.

It is easy to check that the finite sequence $(u^i)_{i=1}^n$ is a δ-pseudo orbit:

$$d_0(u^{i+1}, \hat{f}^i(u^i)) = d_\infty(u^{i+1}, \hat{f}(u^i)) = \sup_{\alpha\in[0,1]} \{d_H(u^{i+1}_\alpha, \overline{f}(u^i_\alpha))\} = d_H(K_{i+1}, \overline{f}(K_i)) < \delta \, ,$$

for $i = 1,\ldots,n$, by hypothesis, there exists $v \in \mathcal{F}_0(X)$ such that

$$d_0(u^i, \hat{f}^i(v)) < \varepsilon \ , \ i = 1,\ldots,n \, .$$

By Remark 1, $d_0(u,v) = d_\infty(u,v) = \sup_{\alpha\in[0,1]}\{d_H(u_\alpha, v_\alpha)\}$, which implies

$$d_H(u^i_\alpha, [\hat{f}^i(v)]_\alpha) < \varepsilon \text{ for each } \alpha \in [0,1] \ , \ i = 1,\ldots,n \, . \tag{13}$$

We now fix $\tilde{\alpha} \in [0,1]$ and set $K = v_{\tilde{\alpha}} \in \mathcal{K}(X)$. By using (13), we have that the orbit of K ε-traces the δ-pseudo-orbit $(K_i)_{i=1}^n$,

$$d_H(K^i, \overline{f}^i(K)) = d_H(u^i_{\tilde{\alpha}}, \overline{f}^i(v_{\tilde{\alpha}})) = d_H(u^i_{\tilde{\alpha}}, [\hat{f}^i(v)]_{\tilde{\alpha}}) < \varepsilon\,.$$

□

4. Conclusions

Summarizing our results, for a compact metric space X and a continuous map $f : X \to X$, we were able to prove in Section 2 the equivalence of the specification property, either for the induced hyperspace dynamical system $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ or for the induced fuzzyfied dynamics $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ (Theorem 2).

Moreover, we saw that it was necessary to add a certain structure to the compact set and to the map, namely convexity and linearity, in order to obtain a characterization of the specification property that includes the original system (X, f) (Theorem 3). The notion of convexity can be generalized to topological groups X by defining $K \subset X$ convex if $K + K = 2K$. In principle, it sounds reasonable that, under some general conditions, Theorem 3 can be extended to topological groups if we can apply a generalized Schauder–Tychonoff fixed point theorem (see, e.g., [31]). It would be interesting to know which conditions on the topological group ensure such generalization.

Finally, in Section 3, we obtained the full characterization of finite shadowing, on the space, hyperspace or fuzzy space, in Theorem 5. As for the shadowing, the fuzzy spaces are not compact, and we do not know if we can inherit the full shadowing on the fuzzy spaces when we have shadowing for (X, f), X being a compact metric space. This is also an interesting future direction of work.

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References

1. Bowen, R. Topological entropy and axiom A. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*; American Mathematical Society: Providence, RI, USA, 1970; pp. 23–41.
2. Sigmund, K. On dynamical systems with the specification property. *Trans. Am. Math. Soc.* **1974**, *190*, 285–299. [CrossRef]
3. Bauer, W.; Sigmund, K. Topological dynamics of transformations induced on the space of probability measures. *Monatsh. Math.* **1975**, *79*, 81–92. [CrossRef]
4. Banks, J. Chaos for induced hyperspace maps. *Chaos Solitons Fractals* **2005**, *25*, 681–685. [CrossRef]
5. Peris, A. Set-valued discrete chaos. *Chaos Solitons Fractals* **2005**, *26*, 19–23. [CrossRef]
6. Guirao, J.L.G.; Kwietniak, D.; Lampart, M.; Oprocha, P.; Peris, A. Chaos on hyperspaces. *Nonlinear Anal.* **2009**, *71*, 1–8. [CrossRef]
7. Bernardes, N.C., Jr.; Peris, A.; Rodenas, F. Set-valued chaos in linear dynamics. *Integral Equ. Oper. Theory* **2017**, *88*, 451–463. [CrossRef]
8. Wang, Y.; Wei, G. Characterizing mixing, weak mixing and transitivity of induced hyperspace dynamical systems. *Topol. Appl.* **2007**, *155*, 56–68. [CrossRef]
9. Wang, Y.; Wei, G.; Campbell, W.H. Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems. *Topol. Appl.* **2009**, *156*, 803–811. [CrossRef]
10. Jardón, D.; Sánchez, I.; Sanchis, M. Transitivity in Fuzzy Hyperspaces. *Mathematics* **2020**, *8*, 1862. [CrossRef]
11. Martínez-Giménez, F.; Peris, A.; Rodenas, F. Chaos on Fuzzy Dynamical Systems. *Mathematics* **2021**, *9*, 2629. [CrossRef]

12. Devaney, R.L. *An Introduction to Chaotic Dynamical Systems*, 2nd ed.; Addison-Wesley Studies in Nonlinearity, Addison-Wesley Publishing Company Advanced Book Program: Redwood City, CA, USA , 1989; pp. xviii+336.
13. Li, T.Y.; Yorke, J.A. Period three implies chaos. *Am. Math. Mon.* **1975**, *82*, 985–992. [CrossRef]
14. Kolyada, S.F. Li-Yorke sensitivity and other concepts of chaos. *Ukr. Math. J.* **2004**, *56*, 1043–1061. [CrossRef]
15. Schweizer, B.; Smítal, J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Am. Math. Soc.* **1994**, *344*, 737–754. [CrossRef]
16. Forti, G.L. Various notions of chaos for discrete dynamical systems. A brief survey. *Aequ. Math.* **2005**, *70*, 1–13. [CrossRef]
17. Kupka, J. On Devaney chaotic induced fuzzy and set-valued dynamical systems. *Fuzzy Sets Syst.* **2011**, *177*, 34–44. [CrossRef]
18. Wu, X.; Ding, X.; Lu, T.; Wang, J. Topological dynamics of Zadeh's extension on upper semi-continuous fuzzy sets. *Int. J. Bifur. Chaos Appl. Sci. Eng.* **2017**, *27*, 1750165. [CrossRef]
19. Wu, X.; Chen, G. Sensitivity and transitivity of fuzzified dynamical systems. *Inf. Sci.* **2017**, *396*, 14–23. [CrossRef]
20. Li, N.; Wang, L.; Lei, F. Sensitivity and specification property of fuzzified dynamical systems. *Mod. Phys. Lett. B* **2018**, *32*, 1850268. [CrossRef]
21. Ebrahimnejad, A. An effective computational attempt for solving fully fuzzy linear programming using MOLP problem. *J. Ind. Prod. Eng.* **2019**, *36*, 59–69. [CrossRef]
22. Illanes, A.; Nadler, S.B., Jr. *Hyperspaces—Monographs and Textbooks in Pure and Applied Mathematics*; Fundamentals and Recent Advances; Marcel Dekker, Inc.: New York, NY, USA, 1999; Volume 216, pp. xx+512.
23. Román-Flores, H.; Chalco-Cano, Y. Some chaotic properties of Zadeh's extensions. *Chaos Solitons Fractals* **2008**, *35*, 452–459. [CrossRef]
24. Jardón, D.; Sánchez, I.; Sanchis, M. Some questions about Zadeh's extension on metric spaces. *Fuzzy Sets Syst.* **2020**, *379*, 115–124. [CrossRef]
25. Kwietniak, D.; Łącka, M.; Oprocha, P. A panorama of specification-like properties and their consequences. In *Dynamics and Numbers*; American Mathematical Society: Providence, RI, USA, 2016; Volume 669, pp. 155–186. [CrossRef]
26. Rudin, W. *Functional Analysis*, 2nd ed.; International Series in Pure and Applied Mathematics; McGraw-Hill, Inc.: New York, NY, USA, 1991; pp. xviii+424.
27. Bartoll, S.; Martínez-Giménez, F.; Peris, A. Operators with the specification property. *J. Math. Anal. Appl.* **2016**, *436*, 478–488. [CrossRef]
28. Barwell, A.D.; Good, C.; Oprocha, P.; Raines, B.E. Characterizations of ω-limit sets in topologically hyperbolic systems. *Discret. Contin. Dyn. Syst.* **2013**, *33*, 1819–1833. [CrossRef]
29. Fernández, L.; Good, C. Shadowing for induced maps of hyperspaces. *Fund. Math.* **2016**, *235*, 277–286. [CrossRef]
30. Good, C.; Mitchell, J.; Thomas, J. Preservation of shadowing in discrete dynamical systems. *J. Math. Anal. Appl.* **2020**, *485*, 123767. [CrossRef]
31. Ardanza-Trevijano, S.; Chasco, M.J.; Domínguez, X. The role of real characters in the Pontryagin duality of topological abelian groups. *J. Lie Theory* **2008**, *18*, 193–203.