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Modified Adomian Method through Efficient Inverse Integral Operators to Solve Nonlinear Initial-Value Problems for Ordinary Differential Equations

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Abstract: The present manuscript examines different forms of Initial-Value Problems (IVPs) featuring various types of Ordinary Differential Equations (ODEs) by proposing a proficient modification to the famous standard Adomian decomposition method (ADM). The present paper collected different forms of inverse integral operators and further successfully demonstrated their applicability on dissimilar nonlinear singular and nonsingular ODEs. Furthermore, we surveyed most cases in this very new method, and it was found to have a fast convergence rate and, on the other hand, have high precision whenever exact analytical solutions are reachable.

Keywords: standard Adomian decomposition method; modified Adomian decomposition method; ordinary differential equations; initial-value problems; inverse integral operators

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1. Introduction

Many real-life problems are modeled mathematically using Initial-Value Problems (IVPs) featuring various types of Ordinary Differential Equations (ODEs). Solutions to these problems are very important as they improve human lives. Thus, this importance is what necessitates various researchers to devise different mathematical methods to tackle countless models arising in, for instance, physics, chemistry, engineering, biology, economics, and social sciences, to mention a few. In particular, the literature is full of diverse studies of the competent method called the Adomian decomposition method (ADM) and its various modifications and extensions [1–3]. In [4,5] some modifications of the Adomian decomposition method are presented for solving initial value problems in Ordinary Differential Equations. However, in [6], a dependable semi-analytical method via the application of a modified Adomian decomposition method (ADM) to tackle the coupled system of Emden–Fowler-type equations has been proposed, and an effective differential operator together with its corresponding inverse is successfully constructed. The present study [7] investigates certain singular Initial-Value Problems (IVPs) featuring the classical and generalized inhomogeneous LaneEmden-type equations. This study proposes different forms of inverse integral operators that are based on the Adomian method to accelerate the convergence rate of the standard Adomian decomposition method (ADM), which includes some cases from the survey that we present in this work. This method and its variants have been comprehensively utilized to treat different forms of linear and nonlinear ODEs, including integral equations and together with the combination of the two [8–11].

However, it is the aim of the present study to examine different forms of IVPs portraying different types of ODEs by proposing a proficient modification to the famous standards of ADM. The method devises different forms of inverse integral operators based on the available literature and further successfully demonstrates their applicability to a class of ODEs of physical relevance. Furthermore, we will assess this new method by establishing a comparative examination with the standard ADM and, on the other hand, with the exact analytical solutions whenever they are reachable.

2. Standard Adomian Decomposition Method

To present the standard ADM methodology, we take into consideration the following generalized ODE

$$Lw(t) + Rw(t) + Fw(t) = g(t), \tag{1}$$

where L and R are linear operators with $R < L$, L is the highest linear operator, and R is an operator with a degree less than L, while F is a nonlinear operator from a Hilbert H. $g(t)$ is a given function in H, and we are looking for $w \in H$ satisfying (1). We assume that (1) has a unique solution for $g \in H$ [12], and $g(t)$ is an inhomogeneous or source term. Next, we rewrite the above equation as follows

$$Lw(t) = g(t) - Rw(t) - Fw(t), \tag{2}$$

such that when the inverse operator L^{-1} of L is applied to both sides of the later equation it yields

$$w(t) = \psi(t) + L^{-1}g(t) - L^{-1}Rw(t) - L^{-1}Fw(t), \tag{3}$$

where function $\psi(t)$ emanates from the prescribed initial data.

Therefore, the ADM decomposes the solution $w(x)$ and the nonlinear term $F(w)$ as series forms as follows

$$w(t) = \sum_{n=0}^{\infty} w_n(t), \qquad Fw(t) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

where A_n is the Adomian polynomials that are recurrently computed using the following relation [13,14]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i w_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{5}$$

Additionally, from the above Adomian polynomials relation, we express some of these components as follows

$$\begin{cases} A_0 = F(w_0), \\ A_1 = F'(w_0)w_1, \\ A_2 = F'(w_0)w_2 + \frac{1}{2}F''(w_0)w_1^2, \\ A_3 = F'(w_0)w_3 + F''(w_0)w_1w_2 + \frac{1}{3!}F'''(w_0)w_1^3, \\ \vdots \end{cases} \tag{6}$$

Therefore, substituting Equation (4) into Equation (3) yields the following

$$\sum_{n=0}^{\infty} w_n(t) = \psi(t) + L^{-1}g(t) - L^{-1}R\sum_{n=0}^{\infty} w_n(t) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{7}$$

Hence, from the above equation, the recurrent solution is thus obtained via the ADM process as follows

$$\begin{cases} w_0 = \psi(t) + L^{-1}g(t), \\ w_{k+1} = -L^{-1}Rw_k - L^{-1}A_k, \; k \geq 0, \end{cases} \tag{8}$$

such that the n-term approximation takes the form

$$\Phi_n = \sum_{k=0}^{n-1} w_k, \tag{9}$$

where the closed-form solution is finally revealed as

$$w(t) = \lim_{n\to\infty} \Phi_n(t) = \sum_{k=0}^{\infty} w_k(t). \tag{10}$$

The theoretical treatment of the convergence of ADM has been considered in [12,15,16]. Cherruault [12] has given the first proof of convergence of the Adomian decomposition method, and he used fixed-point theorems for abstract functional equations. Abbaoui and Cherruault [15] have given new formulae that easily calculate the Adomian's polynomials used in the decomposition methods. A simple proof of convergence of Adomian's technique is presented in [16].

3. Modified Adomian Decomposition Method

Let us now present a modification methodology based on the standard ADM to solve certain classes of nonsingular and singular ODEs featuring IVPs, including, for instance, low- and high-order and systems of inhomogeneous ODEs.

3.1. First-Order IVPs

To present this procedure on ODEs of the first order, let us consider the following first-order IVP [17]

$$\begin{cases} w' + p(t)w + F(t,w) = g(t), \\ w(0) = A, \end{cases} \tag{11}$$

where $g(t)$ and $p(t)$ are given functions, $F(t,w)$ is the general nonlinear real term, and A is a supplied real constant.

Next, we rewrite Equation (11) using the operator denotation as follows

$$Lw = g(t) - F(t,w), \tag{12}$$

where L is the linear differential operator, together with its corresponding one-fold inverse integral operator L^{-1} defined as follows

$$L(.) = e^{-\int p(t)dt} \frac{d}{dt}\Big(e^{\int p(t)dt}(.)\Big), \qquad L^{-1}(.) = e^{-\int p(t)dt} \int_0^t e^{\int p(t)dt}(.)dt. \tag{13}$$

Now, we apply the inverse operator L^{-1} expressed in Equation (13) to the first two terms of Equation (11) as follows

$$L^{-1}(w' + p(t)w) = e^{-\int p(t)dt} \int_0^t e^{\int p(t)dt}(w' + p(t)w)dt, \tag{14}$$

$$= w - w(0)\psi(0)e^{-\int p(t)dt}, \tag{15}$$

where $\psi(t) = e^{\int p(t)dt}$. Furthermore, applying the inverse operator L^{-1} to Equation (12) yields the following

$$w(t) = w(0)\psi(0)e^{-\int p(t)dt} + L^{-1}g(t) - L^{-1}F(t,w). \tag{16}$$

Therefore, we make use of the modification of ADM by first decomposing $w(t)$ and $F(t,w)$ as suggested in Equation (4), and then we obtain

$$\sum_{n=0}^{\infty} w_n(t) = w(0)\psi(0)e^{-\int p(t)dt} + L^{-1}g(t) - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{17}$$

such that the overall recursive relation is acquired as follows

$$\begin{cases} w_0 = w(0)\psi(0)e^{-\int p(t)dt} + L^{-1}g(t), \\ w_{k+1} = -L^{-1}A_k, \quad k \geq 0. \end{cases} \tag{18}$$

Lastly, on using Equation (18) via Equation (4), the closed-form solution is finally revealed for computational purposes as follows

$$\Phi_n = \sum_{k=0}^{n-1} w_k,$$

where the closed-form solution is finally revealed as

$$w(t) = \lim_{n\to\infty} \Phi_n(t) = \sum_{k=0}^{\infty} w_k(t).$$

3.2. Second-Order IVPs

To present an efficient method based on the standard ADM to solve IVPs featuring singular ODEs of the second-order we refer to the well-known modification of ADM as suggested in [10]. In doing so, we take into consideration the following generalized second-order IVP

$$\begin{cases} w'' + p(t)w' + F(t,w) = g(t), \\ w(0) = A_1, \;\; w'(0) = A_2, \end{cases} \tag{19}$$

where $g(t)$ and $p(t)$ are given functions, $F(t,w)$ is the general nonlinear real term, and A_1 and A_2 are supplied real constants.

What is more, we rewrite the ODE given in Equation (19) using the operator denotation as given in Equation (12) and further employ the following differential linear operator L together with its corresponding two-fold integral inverse L^{-1} as [10]

$$L(.) = e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d(.)}{dt}\right), \qquad L^{-1}(.) = \int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dtdt. \tag{20}$$

Therefore, applying the inverse operator L^{-1} given above to the resulting operator equation gives

$$w(t) = \psi(t) + L^{-1}g(t) - L^{-1}F(t,w), \tag{21}$$

such that

$$L\psi(t) = 0.$$

Hence, on decomposing the solution $w(t)$ and the nonlinear term $F(t,w)$ via infinite series earlier defined in Equation (4), the recurrent solution is thus given as follows

$$\begin{cases} w_0 = \psi(t) + L^{-1}g(t), \\ w_{n+1} = -L^{-1}A_n, \; n \geq 0, \end{cases} \tag{22}$$

such that the n-term approximation takes the form

$$\Phi_n = \sum_{k=0}^{n-1} w_k,$$

where the closed-form solution is finally revealed as

$$w(t) = \lim_{n\to\infty} \Phi_n(t) = \sum_{k=0}^{\infty} w_k(t).$$

3.3. Second-Order Singular IVPs

More importantly, we mention here that the method presented in the above subsection for the second-order IVPs was generalized by Hosseini and Jafari [18] for singular IVPs. This generalization is very powerful as it tackles different forms of second-order IVPs, including, for instance, linear, nonlinear, singular, and nonsingular ODEs.

However, considering a nonlinear singular second-order IVP of the form given in Equation (19), we suppose that the function $p(t)$ is of the following singular form

$$p(t) = \frac{1}{t-a}h(t), \tag{23}$$

where Taylor's series expansion of $h(t)$ exists at $t = a$. Now, having already considered the differential linear operator and its inverse in Equation (20) based on the suggestion in [10], it will be very difficult to obtain a closed-form solution in the presence of such a singularity in the above equation. Thus, it is pertinent to make use of polynomials to approximate $e^{\int p(t)dt}$ and $e^{-\int p(t)dt}$ in order to swiftly obtain the components w_i's. Therefore, we further obtain Taylor's series expansion of $h(t)$ at $t = a$ (for $m \in \mathbb{N}$) and re-express Equation (23) as follows

$$p(t) = \frac{1}{t-a}\sum_{k=0}^{m}\frac{(t-a)^k}{k!}h^{(k)}(a). \tag{24}$$

Thus, we have

$$\int p(t)dt = \ln(t-a)^{h(a)} + (t-a)h'(a) + \ldots + \frac{(t-a)^m}{m \times m!}h^{(m)}(a), \tag{25}$$

and

$$e^{\int p(t)dt} = (t-a)^{h(a)}S(t), \tag{26}$$

where

$$S(t) = e^{(t-a)h'(a)+\cdots+\dfrac{(t-a)^m}{m \times m!}h^{(m)}(a)}.$$

Additionally, for any $v \in \mathbb{N}$, we substitute Taylor's series expansion of $S(t)$ into (26) to yield

$$e^{\int p(t)dt} = (t-a)^{h(a)}\left(S(a) + (t-a)S^{'}(a) + \cdots + \frac{(t-a)^v}{v!}S^{(v)}(a)\right), \tag{27}$$

such that in the same manner, we obtain

$$e^{-\int p(t)dt} = (t-a)^{-h(a)}\left(\bar{S}(a) + (t-a)\bar{S}^{'}(a) + \cdots + \frac{(t-a)^v}{v!}\bar{S}^{(v)}(a)\right), \tag{28}$$

where

$$\bar{S}(t) = e^{-(t-a)h'(a)-\cdots-\dfrac{(t-a)^m}{m \times m!}h^{(m)}(a)}. \tag{29}$$

Finally, the difficulty associated with the singular function $p(t)$ with regards to the operators given in Equation (20) is thus solved in line with the present development presented above. Thus, the recurrent solution follows by easily computing the components w_i's.

3.4. Higher-Order IVPs

As higher-order IVPs arise in many real-life applications, we present here a promising technique based on the standard ADM to solve higher-order IVPs, as asserted in [19]. Thus, we take into consideration the following generalized n-order IVP [19]

$$\begin{cases} w^{(n)} + p(t)w^{(n-1)} + F(w) = g(t), \\ w(0) = \beta_0,\ w'(0) = \beta_1, \cdots, w^{(n-1)}(0) = \beta_{n-1}, \end{cases} \tag{30}$$

where $g(t)$ and $p(t)$ are given functions, F is a nonlinear differential operator of the order less than $(n-1)$, and $\beta_0, \beta_1, \cdots, \beta_{n-1}$ are prescribed real constants.

Furthermore, we equally express the ODE in the above system using differential operator denotation as follows

$$Lw = g(t) - F(w), \tag{31}$$

such that the differential operator L and its corresponding n-fold inverse integral operator L^{-1} are defined by

$$\begin{aligned} L(.) &= e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d^{n-1}(.)}{dt^{n-1}}\right), \\ L^{-1}(.) &= \int_0^t\int_0^t\cdots\int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dt\cdots dt. \end{aligned} \tag{32}$$

Thus, the application L^{-1} on Equation (31) transforms the equation to the following

$$w(t) = \psi(t) + L^{-1}g(t) - L^{-1}F(w), \tag{33}$$

such that

$$L\psi(t) = 0.$$

As we proceed through the use of the ADM procedure, we receive the following equation

$$\sum_{n=0}^{\infty} w_n = \psi(t) + L^{-1}g(t) - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{34}$$

which yields the following recurrent solution

$$\begin{cases} w_0 = \psi(t) + L^{-1}g(t), \\ w_{k+1} = -L^{-1}A_k,\ k \geq 0, \end{cases} \tag{35}$$

and a closed-form solution of

$$w(t) = \lim_{n\to\infty}\Phi_n(t) = \sum_{k=0}^{\infty} w_k(t), \qquad \Phi_n = \sum_{k=0}^{n-1} w_k.$$

3.5. Nonlinear System of IVPs

Let us take into consideration the following generalized system of nonlinear IVPs of ODEs,

$$\begin{cases} w_1^{(n)} + p(t)w_1^{(n-1)} + F_1(t, w_1, \cdots, w_1^{(n-2)}, w_2, \cdots, w_2^{(n-2)}, w_n, \cdots, w_n^{(n-2)}) = g_1(t), \\ w_2^{(n)} + p(t)w_2^{(n-1)} + F_2(t, w_1, \cdots, w_1^{(n-2)}, w_2, \cdots, w_2^{(n-2)}, w_n, \cdots, w_n^{(n-2)}) = g_2(t), \\ \vdots \\ w_n^{(n)} + p(t)w_n^{(n-1)} + F_n(t, w_1, \cdots, w_1^{(n-2)}, w_2, \cdots, w_2^{(n-2)}, w_n, \cdots, w_n^{(n-2)}) = g_n(t), \\ \qquad w_1(0) = \beta_1,\ w_2(0) = \beta_2, \cdots, w_n(0) = \beta_n, \end{cases} \tag{36}$$

where $g(t)$ and $p(t)$ are prescribed nice functions, $F_1, F_2, \ldots, F_n$ are nonlinear real functions, and $\beta_1, \beta_2, \ldots, \beta_n$ are supplied real constants.

Thus, without a loss of generalization, the modification of ADM personated in the above subsections can be equally extended to successfully tackle the system of nonlinear IVPs given above. This is, of course, can be performed by suitably constructing a generalized differential operator, L, together with its corresponding n-fold-generalized integral operator.

4. Numerical Illustrations

The present section demonstrates the application of the proposed methods on a number of test problems featuring different forms of ODEs.

Example 1. *Let us consider the following inhomogeneous first-order nonlinear IVP [17]*

$$\begin{cases} w' + 2tw = 1 + t^2 + w^2, \\ \qquad w(0) = 1. \end{cases} \tag{37}$$

Standard Adomian decomposition method

First, we define a differential operator L together with its corresponding one-fold inverse integral operator L^{-1} as follows

$$L = \frac{d}{dt}, \qquad L^{-1}(.) = \int_0^t (.)dt. \tag{38}$$

Next, we express Equation (37) in operator form as follows

$$Lw = -2tw + 1 + t^2 + w^2, \tag{39}$$

such that after applying L^{-1} to both sides of Equation (39) yields

$$w = w(0) - 2L^{-1}(tw) + L^{-1}(1 + t^2) + L^{-1}(w^2). \tag{40}$$

Therefore, without a loss in generality, we obtain the following recurrent relation

$$\begin{cases} w_0 = w(0) + L^{-1}(1 + t^2), \\ w_{n+1} = -2L^{-1}(tw_n) + L^{-1}(A_n),\ n \geq 0, \end{cases} \tag{41}$$

where A_n is the Adomian polynomial corresponding to the nonlinear term w^2 with a few components as follows

$$\begin{cases} A_0 = w_0^2, \\ A_1 = 2w_0w_1, \\ A_2 = w_1^2 + 2w_0w_2, \\ A_3 = 2w_1w_2 + 2w_0w_3, \\ \vdots \end{cases} \tag{42}$$

Therefore, substituting the above polynomial components into the recurrent relation determined in Equation (41) gives

$$\begin{aligned} &w_0 = w(0) + L^{-1}(1 + t^2) = 1 + t + \tfrac{1}{3}t^3, \\ &w_1 = -2L^{-1}(tw_0) + L^{-1}(A_0) = t - \tfrac{1}{3}t^3 + \tfrac{1}{6}t^4 + \tfrac{1}{63}t^7, \\ &w_2 = -2L^{-1}(tw_1) + L^{-1}(A_1) = t^2 - \tfrac{1}{6}t^4 + \tfrac{1}{5}t^5 - \tfrac{2}{63}t^7 + \cdots, \\ &\vdots \end{aligned} \tag{43}$$

Finally, from the above iterates, we obtain the following series solution

$$w(t) = 1 + 2t + t^2 + t^3 + t^4 + t^5 + \cdots, \tag{44}$$

whose closed-form solution is

$$w(t) = t + \frac{1}{1-t}. \tag{45}$$

Modified Adomian decomposition method

Let us define a differential operator L together with its corresponding one-fold inverse integral operator L^{-1} as follows

$$L(.) = e^{-t^2}\frac{d}{dt}(e^{t^2}(.)), \qquad L^{-1}(.) = e^{-t^2}\int_0^t e^{t^2}(.)dt. \tag{46}$$

Then, Equation (37) in operator form becomes

$$Lw = 1 + t^2 + w^2, \tag{47}$$

such that after operating L^{-1} in the later equation reveals Equation (47)

$$w(t) = e^{-t^2} + L^{-1}(1 + t^2) + L^{-1}(w^2), \tag{48}$$

with

$$w_0(t) = e^{-t^2} + e^{-t^2} \int_0^t e^{t^2}(1 + t^2)dt. \tag{49}$$

Therefore, on making use of Taylor's series expansion on e^{-t^2} and e^{t^2} of order 6, the following solution iterates are obtained

$$\begin{aligned} w_0 &= 1 + t - t^2 - \tfrac{t^3}{3} + \tfrac{t^4}{2} + \cdots, \\ w_1 &= t + t^2 - t^3 - \tfrac{7t^4}{6} + \cdots, \\ w_2 &= t^2 + \tfrac{4t^3}{3} - t^4 + \cdots, \\ &\vdots \end{aligned} \tag{50}$$

Hence, we obtain the following series solution of the form given by

$$w(t) = 1 + 2t + t^2 + t^3 + t^4 + t^5 + \cdots, \tag{51}$$

whose closed-form solution is

$$w(t) = t + \frac{1}{1-t}. \tag{52}$$

Example 2. *Let us consider the following inhomogeneous second-order linear singular IVP [10]*

$$\begin{cases} w'' + \frac{\cos t}{\sin t} w' = -2\cos t, \\ w(0) = 1,\ w'(0) = 0. \end{cases} \tag{53}$$

Standard Adomian decomposition method

We define a differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = \frac{d^2(.)}{dt^2}, \qquad L^{-1}(.) = \int_0^t \int_0^t (.)dtdt. \tag{54}$$

Expressing Equation (53) in an operator form becomes

$$Lw = -\frac{\cos t}{\sin t} w' - 2\cos t, \tag{55}$$

such that after taking L^{-1} of the later equation yields

$$w = w(0) + tw'(0) - L^{-1}\left(\frac{\cos t}{\sin t} w'\right) - L^{-1}(2\cos t). \tag{56}$$

Accordingly, we obtain the following recurrent relation

$$\begin{cases} w_0 = w(0) + tw'(0) - L^{-1}(2\cos t) \\ w_{n+1} = -L^{-1}(\frac{\cos t}{\sin t} w_n'),\ n \geq 0, \end{cases} \tag{57}$$

where some of its iterates are expressed as follows

$$\begin{aligned} w_0 &= 2\cos t - 1, \\ w_1 &= -2\cos t + 2, \\ w_2 &= 2\cos t - 2, \\ w_3 &= -2\cos t + 2, \\ &\vdots \end{aligned} \tag{58}$$

We, therefore, conclude from the above components that the standard ADM fails as the obtained series solution is divergent.

Modified Adomian decomposition method

Let us define a differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = \frac{1}{\sin t}\frac{d(.)}{dt}\sin t\frac{d}{dt}(.), \qquad L^{-1}(.) = \int_0^t \frac{1}{\sin t}\int_0^t \sin t(.)dtdt. \tag{59}$$

Expressing Equation (53) in an operator form becomes

$$Lw = -2\cos t, \tag{60}$$

while making use of L^{-1} on the above equations gives

$$L^{-1}Lw = -2\int_0^t \frac{1}{\sin t}\int_0^t \sin t(\cos t)dtdt. \tag{61}$$

Without a loss in generality, the proposed modified ADM reveals the following exact solution

$$w(t) = w(0) + tw'(0) + \cos t - 1 = \cos t. \tag{62}$$

In fact, this shows the power of the proposed method over the standard ADM.

Example 3. *Let us consider the following inhomogeneous second-order nonlinear IVP [10]*

$$\begin{cases} w'' + tw' + t^2w^3 = (2+6t^2)e^{t^2} + t^2e^{3t^2}, \\ w(0) = 1, \ \ w'(0) = 0, \end{cases} \tag{63}$$

with the exact solution

$$w(t) = e^{t^2}.$$

Standard Adomian decomposition method

Let us define a differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = \frac{d^2}{dt^2}, \qquad L^{-1}(.) = \int_0^t\int_0^t (.)dtdt. \tag{64}$$

Accordingly, we obtain the following recurrent relation

$$\begin{cases} w_0 = w(0) + tw'(0) + L^{-1}(g(t)), \\ w_{n+1} = -L^{-1}(tw_n') - L^{-1}(A_n), \ n \geq 0, \end{cases} \tag{65}$$

where

$$g(t) = (2+6t^2)e^{t^2} + t^2e^{3t^2},$$

and the Adomian polynomials, A_n, of nonlinear term t^2w^3 are given as follows

$$\begin{cases} A_0 = t^2 w_0^3, \\ A_1 = t^2(3w_0^2 w_1), \\ A_2 = t^2(3w_0^2 w_2 + 3w_0 w_1^2), \\ A_3 = t^2(3w_0^2 w_3 + 6w_0 w_1 w_2 + w_1^3), \\ \vdots \end{cases} \tag{66}$$

We mention here that Taylor's series of order 10 was utilized on $g(t)$ for the computation of w_0. Thus, the solution becomes

$$\begin{array}{l} w_0 = 1 + t^2 + \frac{3}{4}t^4 + \frac{1}{3}t^6 + \cdots, \\ w_0 + w_1 = 1 + t^2 + \frac{1}{2}t^4 + \frac{2}{15}t^6 + \frac{1}{96}t^8 + \cdots, \\ w_0 + w_1 + w_2 = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{19}{420}t^8 + \cdots, \\ w_0 + w_1 + w_2 + w_3 = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \frac{101}{12600}t^{10} + \cdots, \\ \vdots \end{array} \tag{67}$$

Clearly, this solution converges to the exact solution $w(t) = e^{t^2}$, as Taylor's series expansion of order 10 of e^{t^2} is expressed as

$$e^{t^2} = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \cdots$$

Modified Adomian decomposition method

Let us define a differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = e^{-t^2/2}\frac{d}{dt}e^{t^2/2}\frac{d(.)}{dt}, \qquad L^{-1}(.) = \int_0^t e^{-t^2/2}\int_0^t e^{t^2/2}(.)dtdt. \tag{68}$$

Without a loss in generality, we obtain the following recurrent relation

$$\left\{ \begin{array}{l} w_0 = w(0) + tw'(0) + L^{-1}(g(t)), \\ w_{n+1} = -L^{-1}(A_n),\ n \geq 0. \end{array} \right. \tag{69}$$

Further, making use of Taylor's series expansion of order 10 on $g(t)$, $e^{t^2/2}$ and $e^{-t^2/2}$, and coupling with obtaining the Adomian polynomials of the given nonlinearity terms in the original equation expressed in Equation (63), we receive the following solution

$$\begin{array}{l} w_0 = 1 + t^2 + \frac{7}{12}t^4 + \frac{23}{90}t^6 + \cdots, \\ w_0 + w_1 = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{25}{672}t^8 + \cdots, \\ w_0 + w_1 + w_2 = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \frac{1}{120}t^{10} + \cdots, \\ \vdots \end{array} \tag{70}$$

It is also obvious that the proposed modified ADM converges faster than the standard ADM; this can clearly be observed by comparing the two solutions.

Example 4. *Consider the following inhomogeneous second-order nonlinear singular IVP [18]*

$$\begin{cases} w'' + \frac{1+5t}{2t(t+1)}w' + w^3 = 5t + 11t^2 + \frac{8}{27}t^9 + \frac{2}{3}t^{10} + \frac{1}{2}t^{11} + \frac{1}{8}t^{12}, \\ w(0) = 0 = w'(0), \end{cases} \tag{71}$$

which admits the following exact solution

$$w(t) = \frac{2}{3}t^3 + \frac{1}{2}t^4. \tag{72}$$

Standard Adomian decomposition method

Let us define a differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = \frac{d^2(.)}{dt^2}, \qquad L^{-1}(.) = \int_0^t \int_0^t (.)dtdt. \tag{73}$$

As preceded, the following recurrent relation is obtained

$$\begin{cases} w_0 = w(0) + tw'(0) + L^{-1}(g(t)), \\ w_{n+1} = -L^{-1}(f(t)w'_n) - L^{-1}(A_n), \; n \geq 0, \end{cases} \tag{74}$$

where

$$g(t) = 5t + 11t^2 + \frac{8}{27}t^9 + \frac{2}{3}t^{10} + \frac{1}{2}t^{11} + \frac{1}{8}t^{12},$$

$f(t)$ is determined through the application of Taylor's series by expanding $\frac{1+5t}{2t(t+1)}$ at $t = 0$ with order 9 to yield

$$f(t) = \frac{1}{t}\left(\frac{1}{2} + 2t - 2t^2 + 2t^3 + \cdots - 2t^8\right), \tag{75}$$

while the Adomian polynomials, A_n, of the nonlinear term w^3 are expressed for some terms as follows

$$\begin{cases} A_0 = w_0^3, \\ A_1 = 3w_0^2 w_1, \\ A_2 = 3w_0^2 w_2 + 3w_0 w_1^2, \\ A_3 = 3w_0^2 w_3 + 6w_0 w_1 w_2 + w_1^3, \\ \vdots \end{cases} \tag{76}$$

In this case, we obtain

$$\begin{aligned} & w_0 = \tfrac{5}{6}t^3 + \tfrac{11}{12}t^4 + \tfrac{4}{1485}t^{11} + \cdots \\ & w_0 + w_1 = \tfrac{5}{8}t^3 + \tfrac{25}{72}t^4 - \tfrac{7}{60}t^5 + \cdots \\ & \vdots \\ & w_0 + w_1 + w_2 + \cdots + w_5 = \tfrac{1365}{2048}t^3 + \tfrac{372575}{746496}t^4 - \tfrac{41797}{19906560}t^5 + \cdots \end{aligned} \tag{77}$$

It is easy to see that the standard Adomian decomposition method converges to the exact solution (72) very slowly.

Modified Adomian decomposition method

We consider the following differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d(.)}{dt}\right), \qquad L^{-1}(.) = \int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dtdt, \tag{78}$$

such that

$$\int p(t)dt = \int \frac{1+5t}{2t(t+1)}dt = \ln\left(\sqrt{t}(1+t)^2\right), \tag{79}$$

and

$$e^{\int p(t)dt} = \sqrt{t}(1+t)^2 = t^{\frac{1}{2}} + 2t^{\frac{3}{2}} + t^{\frac{5}{2}}, \qquad e^{-\int p(t)dt} = \frac{1}{\sqrt{t}(1+t)^2}. \tag{80}$$

Therefore, with the application of Taylor's series expansion on $\frac{1}{(1+t)^2}$ of order 9 with regards to Equation (80), we acquire

$$e^{-\int p(t)dt} = t^{-\frac{1}{2}}\left(1 - 2t + 3t^2 + \cdots + 9t^8\right), \tag{81}$$

of which the recurrent relation is finally obtained as follows

$$\begin{cases} w_0 = L^{-1}(g(t)), \\ w_{n+1} = -L^{-1}A_n,\ n \geq 0. \end{cases} \tag{82}$$

Thus, through substituting Equations (79) and (80) into Equation (78), we have

$$\begin{cases} w_0 = \frac{2}{3}t^3 + \frac{1}{2}t^4 + \frac{16}{6237}t^{11} + \cdots, \\ w_0 + w_1 = \frac{2}{3}t^3 + \frac{1}{2}t^4 + O(t^{12}), \end{cases} \tag{83}$$

where $w \approx w_0 + w_1$ is pretty close to the exact analytical solution earlier stated.

Example 5. *Let us consider the following inhomogeneous third-order nonlinear IVP [19]*

$$\begin{cases} w''' + e^t w'' + 4t^2 w' + t^2 w^3 = g(t), \\ w(0) = w'(0) = w''(0) = 0, \end{cases} \tag{84}$$

where $g(t)$ is compatible with the following exact solution

$$w(t) = t^3 e^t.$$

What is more, expressing function $g(t)$ using Taylor's series expansion of order 9 yields

$$g(t) = 6 + 30t + 48t^2 + 45t^3 + \frac{171}{4}t^4 + \frac{164}{5}t^5 + \frac{529}{30}t^6 + \frac{243}{35}t^7 + \frac{2881}{1344}t^8. \tag{85}$$

Standard Adomian decomposition method

We consider the following differential operator L together with its corresponding three-fold inverse integral operator L^{-1} as follows

$$L(.) = \frac{d^{3(.)}}{dt^3}, \qquad L^{-1}(.) = \int_0^t \int_0^t \int_0^t (.)dtdtdt. \tag{86}$$

As preceded, the following recurrent relation is obtained

$$\begin{cases} w_0 = L^{-1}(g(t)), \\ w_{n+1} = -L^{-1}(f(t)w_n'') - 4L^{-1}(t^2 w_n') - L^{-1}(t^2 A_n),\ n \geq 0, \end{cases} \tag{87}$$

where $f(t)$ is determined through the application of Taylor's series by expanding e^t at $t = 0$ as follows

$$f(t) \approx 1 + t + \frac{t^2}{2} + \cdots + \frac{t^8}{8!}, \tag{88}$$

while the Adomian polynomials, A_n, corresponding to the nonlinear term w^3 are given for some components as follows

$$\begin{cases} A_0 = w_0^3, \\ A_1 = 3w_1 w_0^2, \\ A_2 = 3w_2 w_0^2 + 3w_0 w_1^2, \\ \vdots \end{cases} \tag{89}$$

Hence, substituting Equations (88) and (89) into Equation (87) gives

$$\begin{aligned} & w_0 = t^3 + \tfrac{5}{4}t^4 + \cdots, \\ & w_0 + w_1 = t^3 + t^4 + \cdots, \\ & \vdots \\ & w_0 + w_1 + w_2 + \cdots + w_7 = t^3 + t^4 + \tfrac{1}{2}t^5 + \tfrac{1}{6}t^6 + \cdots, \end{aligned} \tag{90}$$

Additionally, we further affirm the obtained series solution in the above equation by applying Taylor's series expansion of order 9 to the exact solution as follows

$$w(t) = t^3 + t^4 + \frac{1}{2}t^5 + \frac{1}{6}t^6 + \cdots \quad (91)$$

Certainly, the obtained series solution gradually progresses to the exact closed-form solution, but slowly. Thus, the convergence rate of standard ADM is slow; this can clearly be seen in the proposed scheme.

Modified Adomian decomposition method

Accordingly, we consider the following differential operator L together with its corresponding three-fold inverse integral operator L^{-1} as follows

$$L(.) = e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d^2(.)}{dt^2}\right), \qquad L^{-1}(.) = \int_0^t\int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dtdtdt \quad (92)$$

such that

$$\int p(t)dt = \int e^t dt = e^t \quad (93)$$

and

$$e^{-\int p(t)dt} = e^{-e^t}, \qquad e^{\int p(t)dt} = e^{e^t}. \quad (94)$$

Therefore, with the application of Taylor's series expansion of order 9 on e^{e^t} and e^{-e^t} with regards to Equation (92), we obtain

$$\begin{aligned} a = e^{-e^t} &= e^{-1}\left(1 - t + \frac{1}{6}t^3 + \cdots + \frac{5}{4032}t^8\right), \\ b = e^{e^t} &= e\left(1 + t + t^2 + \frac{5}{6}t^3 + \cdots + \frac{23}{224}t^8\right). \end{aligned} \quad (95)$$

In addition, we rewrite the above inverse operator in terms of a and b as follows

$$L^{-1}(.) = \int_0^t\int_0^t (a)\int_0^t (b)(.)dtdtdt. \quad (96)$$

Thus, the resulting recurrent relation is obtained based on Equation (35) as follows

$$\begin{cases} w_0 = L^{-1}g(t), \\ w_{n+1} = -L^{-1}(t^2A_n),\ n \geq 0, \end{cases} \quad (97)$$

such that some of its component sums are as follows

$$\begin{cases} w_0 = t^3 + t^4 + \frac{1}{2}t^5 + \frac{1}{6}t^6 + \frac{83}{840}t^7 + \cdots, \\ w_0 + w_1 = t^3 + t^4 + \frac{1}{2}t^5 + \frac{1}{6}t^6 + \cdots \end{cases} \quad (98)$$

Therefore, it is obvious that the proposed modified ADM converges faster than the standard ADM, as earlier demonstrated; this can clearly be seen by comparing the two series solutions in Equations (90) and (98), respectively. The modified ADM solution of Φ_2 equals the standard ADM solution of Φ_8; in fact, the convergence rate of this method is higher by far.

Example 6. *Consider the following nonlinear system of inhomogeneous second-order IVPs [19]*

$$\begin{cases} u'' + \tan(t)u' + v^2 = g(t), \\ \quad u(0) = 0,\ u'(0) = 0, \\ \\ v'' + 100v' + u^2 = h(t), \\ \quad v(0) = 0,\ v'(0) = 0, \end{cases} \quad (99)$$

where $g(t)$ and $h(t)$ are compatible with the following exact solution set

$$u(t) = t\sin t, \quad v(t) = t\tan t.$$

More so, expressing functions $g(t)$ and $h(t)$ using Taylor's series expansion order 9 yields

$$\begin{aligned} g(t) &= 2 + \frac{5}{4}t^4 + \frac{3}{4}t^6 + \cdots, \\ h(t) &= 2 + 200t + 4t^2 + \frac{400}{3}t^3 + 5t^4 - 80t^5 + \frac{121}{45}t^6 + \cdots. \end{aligned} \tag{100}$$

Standard Adomian decomposition method

Without a loss in generality, the system admits the following recurrent relation

$$\begin{cases} u_0 = L^{-1}(g(t)) = t^2 + \frac{1}{24}t^6 + \frac{3}{224}t^8, \\ v_0 = L^{-1}(h(t)) = t^2 + \frac{100}{3}t^3 + \frac{1}{3}t^4 + \frac{20}{3}t^5 + \frac{1}{6}t^6 - \frac{40}{21}t^7 + \frac{121}{2520}t^8 + \cdots, \end{cases} \tag{101}$$

and

$$\begin{cases} u_{n+1} = -L^{-1}(f(t)u_n') - L^{-1}A_n, \ \ n \geq 0, \\ v_{n+1} = -L^{-1}(100v_n') - L^{-1}B_n, \ \ n \geq 0, \end{cases} \tag{102}$$

where A_n and B_n are the Adomian polynomials corresponding to the nonlinear terms v^2 and u^2, respectively. Additionally, function $f(t)$ represents Taylor's series expansion of $\tan t$ of order 9. Thus, by considering Equation (101) and (102), we obtain

$$\begin{cases} u_0 = t^2 + \dfrac{1}{24}t^6 + \dfrac{3}{224}t^8 \\ u_0 + u_1 = t^2 - \dfrac{1}{6}t^4 - \dfrac{1}{72}t^6 + \cdots, \\ \vdots \\ u_0 + u_1 + u_2 + \cdots + u_6 = t^2 - \dfrac{1}{6}t^4 + \dfrac{1}{120}t^6 + \cdots, \end{cases} \tag{103}$$

and

$$\begin{cases} v_0 = t^2 + \dfrac{100}{3}t^3 + \dfrac{1}{3}t^4 + \cdots, \\ v_0 + v_1 = t^2 - 883t^4 - \dfrac{4994}{45}t^6 + \cdots, \\ \vdots \\ v_0 + v_1 + v_2 + \cdots + v_6 = t^2 + \dfrac{1}{3}t^4 + \dfrac{2}{15}t^6 + \cdots \end{cases} \tag{104}$$

Therefore, the obtained standard ADM solution converges to the exact solution; this could be seen clearly by expanding the exact solution using Taylor's expansion and thereafter comparing the two solutions.

Modified Adomian decomposition method

Accordingly, we consider the following differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$L(.) = e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d(.)}{dt}\right), \qquad L^{-1}(.) = \int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dtdt \tag{105}$$

such that

$$e^{-\int p(t)dt} = \cos(t), \qquad e^{\int p(t)dt} = \frac{1}{\cos(t)}. \tag{106}$$

Therefore, with the application of Taylor's series expansion of order 9 on $\cos(t)$ and $\frac{1}{\cos(t)}$ with regards to Equation (105), we obtain

$$
\begin{aligned}
a &= \cos(t) = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots + \frac{1}{40320}t^8, \\
b &= \frac{1}{\cos(t)} = 1 + \frac{1}{2}t^2 + \frac{5}{24}t^4 + \cdots + \frac{277}{8064}t^8.
\end{aligned} \tag{107}
$$

In addition, we rewrite the above inverse operator in terms of a and b as follows

$$
L^{-1}(.) = \int_0^t (a) \int_0^t (b)(.)dtdt. \tag{108}
$$

Thus, the resulting recurrent relation is obtained as follows

$$
\begin{cases}
u_0 = L^{-1}(g(t)) = t^2 - \frac{1}{6}t^4 + \frac{1}{24}t^6 + \cdots, \\
u_{n+1} = -L^{-1}A_n, \ n \geq 0,
\end{cases} \tag{109}
$$

Now, we consider the second differential operator L together with its corresponding two-fold inverse integral operator L^{-1} as follows

$$
L(.) = e^{-\int p(t)dt}\frac{d}{dt}\left(e^{\int p(t)dt}\frac{d(.)}{dt}\right), \qquad L^{-1}(.) = \int_0^t e^{-\int p(t)dt}\int_0^t e^{\int p(t)dt}(.)dtdt \tag{110}
$$

such that

$$
e^{-\int p(t)dt} = e^{-\int 100dt} = e^{-100t}, \qquad e^{\int p(t)dt} = e^{\int 100dt} = e^{100t}. \tag{111}
$$

Therefore, with the application of Taylor's series expansion of order 9 on e^{-100t} and e^{100t} with regards to Equation (105), we obtain

$$
\begin{aligned}
c &= e^{-100t} = 1 - 100t + 5000t^2 + \cdots + \frac{15625000000000}{63}t^8, \\
d &= e^{100t} = 1 + 100t + 5000t^2 + \cdots + \frac{15625000000000}{63}t^8.
\end{aligned} \tag{112}
$$

In addition, we rewrite the above inverse operator in terms of c and d as follows

$$
L^{-1}(.) = \int_0^t (c) \int_0^t (d)(.)dtdt. \tag{113}
$$

Thus, the resulting recurrent relation is obtained as follows

$$
\begin{cases}
v_0 = L^{-1}(h(t)) = t^2 + \frac{1}{3}t^4 + \frac{1}{6}t^6 + \cdots \\
v_{n+1} = -L^{-1}B_n, \ n \geq 0,
\end{cases} \tag{114}
$$

such that some components are expressed as follows

$$
\begin{cases}
u_0 = t^2 - \frac{1}{6}t^4 + \frac{1}{24}t^6 + \cdots, \\
u_0 + u_1 = t^2 - \frac{1}{6}t^4 + \frac{1}{120}t^6 + \cdots,
\end{cases} \tag{115}
$$

and

$$
\begin{cases}
v_0 = t^2 + \frac{1}{3}t^4 + \frac{1}{6}t^6 + \cdots, \\
v_0 + v_1 = t^2 + \frac{1}{3}t^4 + \frac{2}{15}t^6 + \cdots,
\end{cases} \tag{116}
$$

where the present method also outperforms the standard ADM. This could undoubtedly be noted by comparing the two solutions where the modified ADM solution of Φ_2 matches the standard ADM solution of Φ_7. This further affirms the higher convergence rate of the proposed method.

5. Conclusions

In conclusion, the present manuscript examined various forms of IVPs by proposing a proficient modification to the famous standard ADM. The proposed method collected different forms of inverse integral operators and successfully applied them to dissimilar inhomogeneous nonlinear singular and nonsingular ODEs. The efficiency of the method was further assessed, taking into account its faster convergence rate and, on the other hand, its higher precision with the available exact analytical solutions. Thus, we finally recommend that the proposed method should be utilized to solve physical models arising in different nonlinear sciences.

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